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Prepared for
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TOPICAL REPORT

ALGEBRAIC GEOMETRIC METHODS FOR THE STABILIZABILITY AND RELIABILITY OF MULTIVARIABLE AND OF MULTIMODE SYSTEMS

by

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Abstract

This report consists of a series of papers which concentrate on the basic problem of understanding the extent to which feedback can alter the dynamic characteristics (e.g. instability, oscillations) of a control system, possibly operating in one or more modes (e.g. failure versus non-failure of one or more components). One problem studied here is to determine the existence of, and the order q of, a $m \times p$ compensator which can stabilize a given r-tuple of $m \times p$ plants $G_1(s),\ldots,G_r(s)$ of orders $n_1,\ldots,n_r$. The classical case, $r = 1$, remains one of the most challenging problems of linear system theory and is studied in several of these papers, for the case $q = 0$ and the case $q \geq 0$ and the case $q \geq 1$, in a geometric setting, viz. Schubert's calculus of enumerative geometry. This algebraic geometric approach yields both sufficient and necessary conditions which improve, sometimes vastly, on the results obtained by more traditional methods. The development of algebraic formulae or numerical algorithms for finding such a compensator, when it exists, is considered in the context of Galois Theory and the "homotopy continuation method".

These geometric methods are also extended to the multimode case, $r \geq 1$. Among the results obtained are the assertion that provided $r \leq \max(m,p)$, the generic r-tuple may be arbitrarily pole-assigned and, a fortiori, stabilized. This generalizes the only known results, due to Murray-Saeks $(m = p = 1)$ and to Vidyasagar-Viswanadham, which were obtained in the case $r = 2$ and improves on these results, even when $r = 2$, by giving an upper bound on the order q when the condition $r \leq \max(m,p)$. In the case $r = 1$, this implies the celebrated Brasch-Pearson Theorem, while if $\min(m,p) = 1$ both the condition $r \leq \max(m,p)$ and the estimate on q are sharp.

Subsequent work will be directed toward closing the gap between the necessary and the sufficient conditions obtained here, for both the classical case $r = 1$ and the multimode case $r \geq 2$, by more sophisticated algebraic geometric techniques, and towards analyzing the problem of stabilizing a parameterized family $G_\lambda(s)$ of systems where λ is a slowly-varying parameter, not assumed to be independent of time as in the $\lambda \in \{1,\ldots,r\}$, representing a degradation of a component in the plant $G_o(s)$.

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ALGEBRAIC AND GEOMETRIC ASPECTS OF THE ANALYSIS OF FEEDBACK SYSTEMS

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§1. INTRODUCTIONS, NOTATION, AND THE STATEMENTS OF THE PROBLEMS

This manuscript represents two of the three lectures which I gave at this Advanced Study Institute and, for this reason, I shall give two introductions. (The third lecture is historical and may be found in "Introductory Chapter," this volume.) In the first four sections, I shall discuss recent work in algebraic and geometric system theory which centers around the question, "What can be done using state or output feedback." To fix the ideas, it is at least initially sufficient to consider a system σ as being defined by the <u>state-space</u> equations

$$\dot{x}(t) = Fx(t) + Gu(t) \qquad y(t) = Hx(t) \tag{1.1}$$

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or by the transfer function (the zero initial state Laplace transform)

$$\hat{y}(x) = T(s)\hat{u}(s), \quad T(s) = H(sI-F)^{-1}G \qquad (1.1)'$$

which relates the input vector $u \in U \simeq \underline{k}^m$ to the corresponding output $y \in Y \simeq \underline{k}^p$, without explicit mention of the (internal notions of) state $x \in X \simeq \underline{k}^n$. Thus (1.1)' is an external description of σ, as one might see in Ohm's law, where (1.1) is an internal description (i.e., involving states) of σ, as one might see in the non-autonomous differential equations for an RLC network being driven by an applied current $u(t)$ and generating a voltage $y(t)$.

Now, feedback engineering is perhaps the second or third oldest profession and needs little introduction. Indeed, any list of well-known examples of feedback systems should include the oil lamp of Philon, the water clock of Gaza, Christiaan Huygens' construction of a regulator for clock mechanisms, and the centrifugal governor for steam engines, developed by James Watt, followed by a plethora of more sophisticated modern systems. In each of these example, some output--or function of the state--of the system is used to control the evolution of the state in future time and a rather basic question is to determine how much control over the state one can obtain by feeding back the output as an input. For a vehicle driven by a steam engine, one would like to produce a uniform motion in the vehicle by such a feedback law and this is where the mathematics begins to play a role. In an often cited paper [34], J. C. Maxwell linked the intrinsic deviation, of some feedback systems, from uniform motion to the instability of the corresponding differential equations. Now, a feedback law in the linear context is just a linear map

$$K : Y \to U$$

and the corresponding closed loop system has dynamics given by

$$\dot{x}(t) = (F - GKH)x(t) + Gu(t)$$

$$y(t) = Hx(t) , \qquad (1.2)$$

or, in external terms, by

$$T(s)(I + KT(x))^{-1} = N(s)(D(s) + KN(s))^{-1} \qquad (1.2)'$$

where $N(s)D(s)^{-1}$ is a coprime factorization of $T(s)$ into polynomial matrices. The instability, or rather stability, question is thus whether

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$$\chi_\sigma(K) = \det(sI - F + GKH) \qquad (1.3)$$

has its roots in the left-half plane. Naturally, the inverse problem is deeper and more applicable: can one find K so that (1.3) has it roots in the left-half plane? More generally, can one adjust arbitrarily via output feedback, the natural frequencies of (1.1)? Since the eigenvalues of F are the poles of $T(s)$ (provided n is minimal), one refers to this problem as pole-placement. For the sake of completeness, this is stated as:

Problem A. Analyze as explicitly as possible the algebraic map

$$\chi_\sigma : \underline{k}^{mp} \to \underline{k}^n$$

defined by regarding the right hand side of (1.3), via its coefficients in s, as a point in k^n. In particular, is χ_σ surjective (pole placement), what are the topological or geometric properties of χ_σ, or of image χ_σ?

In §2, I shall given an exposition of the infinitesimal analysis of χ_σ, viz. a calculation of the Jacobian $d\chi_\sigma$ on k^{mp} and on a certain submanifold $M \subset k^{mp}$. This uses the fact that χ_σ is a polynomial but, except over algebraically closed ground fields, makes more use of differential calculus than of algebraic geometry. However, one of the new results is a proof and sharpening of Kimura's generic pole-placement theorem [29]. This is a simple, geometric proof (taken from [5]) of an honest output pole-placement theorem under the hypothesis $m + p - 1 \geq n$ used by Kimura. The final topic in this section is a classification, due to Brockett [3], of the Lie algebras $\{F, GH\}_{\mathcal{L}}$ associated to a transfer function $T(s)$ for $m = p = 1$, as well as a multi-input-output generalization, with application to Problem A even in the case of time-varying feedback $K(t)$.

In §3, the geometric foundation for §4 is developed, the starting point being the interpretation of graph $T(s)$ as a curve of m-dimensional subspaces of $\mathbb{C}^{m+p}$; i.e., as an algebraic curve in the Grassmannian variety Grass(m,m+p)--due to Hermann-Martin. This geometric approach is actually very close in spirit to Kimura's original proof of this theorem. In this setting, the degree of the curve so obtained is the intersection of this curve with a hyperplane, as in Bézout's Theorem, and the Theorem of Hermann-Martin asserts that the points of one such intersection are precisely the poles of $T(s)$.

In §4, the output feedback group is brought into play, whereas in §3 only the identity element is considered. From this point of view, placing poles by output feedback is the same as prescribing points of intersection of the curve with a hyperplane. This inverse problem in geometry has a long history, making contact with several basic themes in algebraic geometry, and in this context χ_σ may be regarded as the restriction of a central projection, about which several important facts are known. From this "central projection lemma", much in the way of Problem A can be deduced, containing in particular some rather surprising results--especially in view of negative results previously obtained. For example, although over $\mathbb{C}$ all is well, Willems and Hesselink [45] have shown that over $\mathbb{R}$, for $m = p = 2$ and $n = 4$, for generic σ it is a fact that χ_σ misses an open set. Using the Schubert calculus in the case $mp = n$ Brockett and the author [5] have shown that, for $m = 2$, $p = 2^r - 1$ (a Mersenne number), χ_σ is generically onto (over $\mathbb{R}$), although these may be the only such cases (up to symmetries and excluding the scalar cases).

In the remaining sections, I consider linear systems depending on parameters and the corresponding questions of pole placement and stabilization by state feedback. Such parameter dependence arises quite often, through dependence on physical parameters such as altitude or attitude of an aircraft or as the value of a resistor, etc. In these cases, (F,G,H) have entries in an appropriate ring of functions on the parameter space Λ and conversely linear systems defined over rings can be viewed as linear systems depending on parameters--in a slightly more general sense. Two remarkable examples are: first, the representation of linear delay-differential systems, via convolution with finite measures on $\mathbb{R}$, as linear systems defined over a polynomial ring [27] and second, the representation of half-plane digital filters as linear systems defined over ℓ^1, also due to Kamen [28]. Thus, one may pose the problem of pole-placement over a ring R, commutative with identity, such as a ring of functions.

In section 5, I review some of the known positive results, starting with Morse's theorem for P.I.D.'s, a result of the author's for very special systems defined over polynomial rings (or, more generally, projective free rings), and in §6 turn to the recent counterexamples to the general question for certain rings, linking the arithmetic aspects of R with pole-placement.

In section 7, I turn to the more modest question, which is however sufficient for applications:

<u>Problem B</u>. If $(F(\lambda),G(\lambda))$ is defined over an algebra of functions and is stabilizable for each fixed λ, does there exist $K(\lambda)$ defined over the same algebra, such that the closed loop system (1.2) is stable for each λ?

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In that section, I use a lemma of D. Delchamps' on smoothness of solutions to a smooth family of Riccati equations to obtain an affirmative answer to Problem B in case where $(F(\lambda),G(\lambda))$ is C^k, $k \geq 0$, and controllable for each λ.

In closing, I would like to apologize for having omitted, primarily for reasons of time and space, recent work which might belong under such a title. Some of the work by Rosenbrock, Fuhrmann, et al. on dynamic conpensation is reported in their lectures, while related work has recently appeared in the thesis of T. Djaferis [15], Djaferis and Mitter [35], and in Emre [16]. It is my intention to report elsewhere on the work of Postlethwaite-MacFarlane [37], et al., which develops root-locus techniques for square multi-input, multi-output systems with respect to scalar gain $K = \lambda I$. One should also mention recent work by Sastry-Desoer [43], which evaluates the asymptotic values of the unbounded root loci, for generic systems.

§2. KIMURA'S THEOREM, INFINITESIMAL ANALYSIS OF χ_σ, LIE ALGEBRAIC INVARIANTS OF χ_σ.

Now, in order to compute the rank of

$$d\chi_\sigma : T_0(\underline{k}^{mp}) \to T_\chi(\underline{k}^n), \quad \text{where } \chi = \det(sI - F),$$

it is efficient to change coordinates by use of the frequency domain. Thus, if $N(s)D(s)^{-1} = T(s) = H(sI-F)^{-1}G$ is a coprime factorization of the transfer function $T(s)$ and if $-K:Y \to U$ is the output gain, the closed loop transfer function is as given in (1.2)':

$$T^{-K}(s) = T(s)(I - KT(s))^{-1} = N(s)(D(s) - KN(s))^{-1} . \tag{2.0}$$

Thus, to solve $p(s) = \chi_\sigma(K)$, with $K \in M$ a subset of matrices, is to solve for rational functions

$$p(s)/\det D(s) = \det(I - KT(s)), \tag{2.1}$$

with $K \in M$. With this change of coordinates on $\underline{k}^n$, χ_σ takes the form:

$$K \mapsto 1 + \sum_{i=1}^{n} c_i(-KT(s)) , \tag{2.2}$$

where the $c_i(R)$ are the characteristic coefficients of R.

Ignoring the constant term, χ_σ is given to first order as

$$\chi_\sigma(K) = tr(-KT(s)) = \langle -K, T(s) \rangle$$

and, since $T(s)$ is rational, the Jacobian is given by the formula

$$d\chi_\sigma(K) = (\langle -K, HF^iG \rangle)_{i=0}^{n-1} . \qquad (2.3)$$

From (2.3) one recovers the calculation

> $d\chi_\sigma$ *is surjective whenever the Hankel matrices*
> $HG, HFG, \ldots, HF^{n-1}G$ *are independent*,

which (since the Hankel matrices are vectors in $\underline{k}^{mp}$) refines the necessary condition, $mp \geq n$, for surjectivity of χ_σ. Indeed, R. Hermann and C. Martin combined this calculation with the dominant morphism theorem to obtain, for $k = \mathbb{C}$,

<u>Theorem</u> ([23]). *If* $mp \geq n$, *then for almost all* (F,G,H) *the image of* χ_σ *is open and dense.*

Several remarks are in order. First, in any such theorem, the "almost all (F,G,H)" hypothesis is necessary. Above, the affine algebraic set which must be excluded is contained in the variety defined by the vanishing of all minors of order n of the $mp \times n$ matrix $(HG, \ldots, HF^{n-1}G)$. But this is as it should be, for in general such conditions must in particular exclude systems which are equivalent to lower order systems, e.g., rank $G = 1$, where image χ_σ is a line. Second, it is in fact true that, for almost all (F,G,H), χ_σ is closed. And finally, over $\mathbb{R}$, J. Willems and W. Hesselink [45] have proved that, for $m = p = 2$, for almost all (F,G,H), image χ_σ is not dense, which illustrates the absence of the "fundamental openness principle" over $\mathbb{R}$.

There is, however, a similar result over $\mathbb{R}$, under stronger hypothesis, due to H. Kimura.

<u>Theorem</u> ([29]). *If* $m + p - 1 \geq n$, *then for generic* (F,G,H) *image* χ_σ *is open and dense.*

In the latter part of this lecture, I shall turn to Kimura's proof, which is quite long. Here, I shall follow a geometric line of reasoning [5] starting from (2.3). First of all, notice that $m + p - 1 = \dim M$, where $M \subset \mathbb{R}^{mp}$ is the submanifold of rank 1 matrices. Surprisingly, it's enough to restrict χ_σ to

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M and we wish to compute $d\chi_\sigma$ acting on $T_K M$. In (2.3), this has the effect of restricting K to be of the form ${}^t(x+\varepsilon)(y+\delta)$, for $x \in \mathbb{R}^m - \{0\}$, $y \in \mathbb{R}^p - \{0\}$, $K = {}^t xy$ rank 1, and we therefore consider the vectors

$$d\chi_\sigma({}^t x_j y_j) = (y_j H \tilde{F}^i G x_j)_{i=0}^{n-1} \tag{2.4}$$

where $\tilde{F} = F + GKH$. As before, one sees that, if $m + p - 1 \geq n$, then generically in (F,G,H) there exists matrices ${}^t x_j y_j$, $j = 1,\ldots,n$ such that the vectors (2.4) are linearly independent. In particular, χ_σ is surjective to first order and hence, by the implicit function theorem, image χ_σ contains an open set. Moreover, since $c_i(KT(s)) = 0$ for $i \geq 2$ whenever K has rank one, χ_σ is equal, along M, to $1 + d\chi_\sigma$ and is therefore surjective! Note that, by combining this observation with (2.1)-(2.2), one can develop an algorithm for the solution of (2.1).

More recently, H. Kimura [30] has improved the bound to $m + p + \kappa_1 - 1 \geq n$, where κ_1 is the largest Kronecker index, subject to the constraint $m \geq \mu_1$ (the largest observability index), $p \geq \kappa_1$. This, too, has an amplification to a pole-placement theorem.

Now, as an example of an invariant of χ_σ, which plays a role in the output feedback problem but which is not captured by our previous calculations, we consider a Lie algebra determined by σ. Explicitly, by choosing a minimal realization (F,G,H) of a scalar transfer function $T(s)$, one may form $\mathscr{L}_\sigma = \{F,GH\}_L$--the Lie subalgebra of $g\ell(n,\mathbb{R})$ generated by F and GH. In this way, one obtains not only $\mathscr{L}_\sigma$ but also a representation, $\rho: \mathscr{L}_\sigma \to g\ell(n,\mathbb{R})$, and by the state-space isomorphism theorem, any other realization (F',G',H') give rise to an equivalent representation ρ'. Of course, $\mathscr{L}_\sigma$ is also invariant under output feedback, since $F + KGH$ is contained in $\mathscr{L}_\sigma$ for any scalar K, and this accounts for its importance in the output feedback problem. And symmetries in the representation $\rho: \mathscr{L}_\sigma \to g\ell(n,\mathbb{R})$ reflect symmetries in the closed-loop characteristic equation. For example, if $T(s) = 1/s^2$, then it's not hard to see that $\rho\mathscr{L}_\sigma = s\ell(2,\mathbb{R}) = sp(1,\mathbb{R})$, which reflects the equivalent facts that $\operatorname{tr}(F + KGH) = 0$, for all K, and that the closed-loop characteristic polynomial is always an even function. These ideas were developed by R. Brockett in [3], [6] leading to his classification of those $\mathcal{L}_\sigma$ which can occur:

Theorem ([3]). *The following is a list of the* $\mathcal{L}_\sigma$ *which can occur together with the corresponding symmetry properties:*

$\mathcal{L}_\sigma$	symmetries
1. $sp\ \frac{n}{2}, \mathbb{R}$	$T(s) = T(-s)$
2. $sp\ \frac{n}{2}, \mathbb{R} + \mathbb{R}$	$T(s) = T(-s+\alpha)$, for some α
3. $s\ell(n, \mathbb{R})$	$tr(F) = tr(GH) = 0$, and none of the above hold
4. $g\ell(n, \mathbb{R})$	none of the above hold

One should also note that this classification gives the same information for time-varying gains $K(t)$. Now, the multivariable case is handled, in part, by a reduction to the scalar case by a lemma (see [2]) reminiscent of Heymann's Lemma. That is, for (F,G,H) minimal, there exists a gain K and input and output channels g and h such that $(F + GKH,g,h)$ is a minimal triple. And, if one defines $\mathcal{L}_\sigma$ to be the smallest Lie subalgebra of $g\ell(n, \mathbb{R})$ containing $\{F + GKH:K \text{ arbitrary}\}$, this reduction enables one to prove:

Theorem ([4]). *If* $\text{rank } T(s) \geq 2$, *then* $\mathcal{L}_\sigma$ *is either* $s\ell(n, \mathbb{R})$ *or* $g\ell(n, \mathbb{R})$, *depending on the vanishing of* trF *and* $tr(HG)$.

§3. BEZOUT'S THEOREM, THE THEOREM OF HERMANN-MARTIN.

There are important external symmetries too, which arise as subgroups of the (output) feedback group. Now, as far as I am aware, the applications of algebraic geometry to linear system theory arise from Laplace transform techniques, from the existence of algebraic groups actions in the form of symmetry groups, and from the interrelation between these 2 points of view. Indeed, perhaps one of the least understood contributions in the Hermann-Martin series is the recognition of the Laplace transform as an intertwining map between the actions of the state and output feedback groups at the state-space level and the classical actions of these groups as linear fractional transformations. This This observation is the starting point for our global analysis of

To fix the ideas, I shall begin with a review of Kimura's proof of his theorem, in the case $m = 1$, $p = 2 = n$. Here, one has

$$T(s) = \begin{pmatrix} \dfrac{q_1(s)}{p(s)} \\ \dfrac{q_2(s)}{p(s)} \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} q_1(s) \\ q_2(s) \\ \hline p(s) \end{pmatrix} = \begin{pmatrix} N(s) \\ \hline D(s) \end{pmatrix} \qquad (3.1)$$

a coprime factorization of $T(s)$. If $\lambda_1 \neq \lambda_2$ are complex numbers, then the method of proof is to select a non-zero vector from each of the lines

$$\begin{pmatrix} N(\lambda_i) \\ D(\lambda_i) \end{pmatrix} .$$

Geometrically, one has the set-up in Fig. 3.1 where we denote the line through

$$\begin{pmatrix} N(\lambda) \\ D(\lambda) \end{pmatrix}$$

by $T(\lambda)$. This is as it should be, for a choice of coprime factors is only unique up to multiplication by a non-zero scalar. I claim that if one takes the plan π spanned by $T(\lambda_1)$ and $T(\lambda_2)$, then $\pi = \text{graph}(-K)$, where K is a gain for which the closed-loop poles equals $\{\lambda_1,\lambda_2\}$. Notice that to say $\{\lambda_1,\lambda_2\}$ is the polar set of T is to say λ_1,λ_2 are the roots of p in (3.1). Thus, in this case, the linear $T(\lambda_i)$ lie in the Y-plane (Fig. 3.1) and so $K = 0$. This, however, is even far from explaining the minus sign, which occurs for group-theoretic reasons. Since the geometry of lines in $\mathbb{C}^3$ is at issue, it's more efficient to rephrase the observation made above in terms of projective geometry. That is, the transfer function gives a map,

$$T:\mathbb{C}^* \to \mathbb{P}^2 ,$$

of the extended complex line $(T(\infty) = 0)$ to the projective plane. The lines in the Y-plane form the projective line $\mathbb{P}^1$, embedded in $\mathbb{P}^2$, while $T(\mathbb{C}^*)$ is a curve in $\mathbb{P}^2$. Moreover, since p has degree 2, T is a curve of degree 2 and intersects the line $\mathbb{P}^1$ twice, as it should according to Bézout's Theorem (see Fig. 3.2).

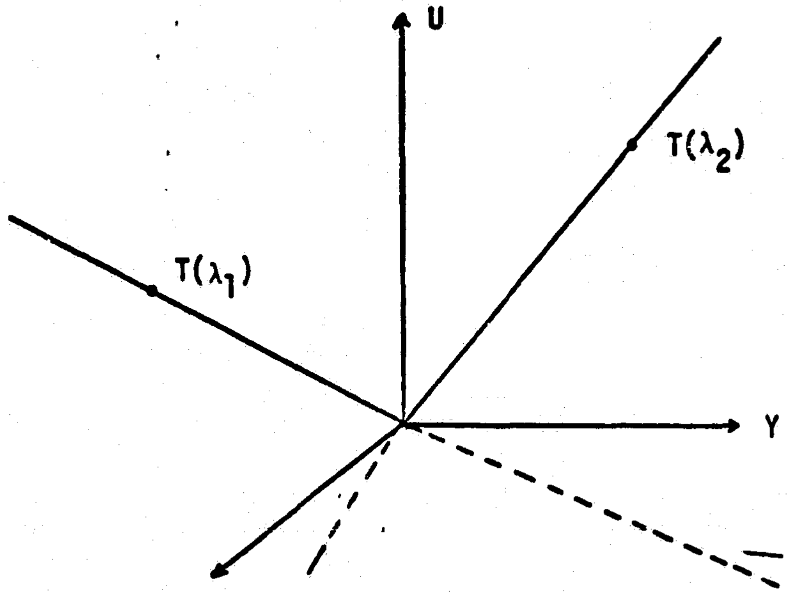

Figure 3.1

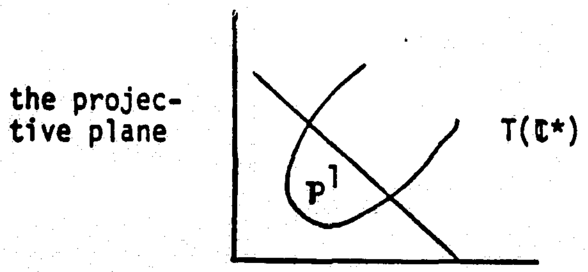

Figure 3.2

In fact, the same reasoning shows that, for any such T with n arbitrary, *the McMillan degree of* T *equals the degree of the curve* T, with

$$T^{-1}(T(\mathbb{C}^*) \cap \mathbb{P}^1) = \mathrm{sing}(T) \quad . \tag{3.2}$$

If we now choose any other plane Y_1 in $\mathbb{C}^3$, complementary to U, then Y_1 determines another line in $\mathbb{P}^2$ and by Bézout's Theorem (or by a little algebra), $T(\mathbb{C}^*)$ intersects this line in n-points. On the other hand, such a plane Y_1 is the graph of a linear map $-K:Y \to U$ (and conversely) and from this point of view, Bézout's Theorem asserts that the McMillan degree is preserved under output feedback, assuming the claim made above. However, the claim is now fairly easy to see. For, any K may be regarded as an element of $GL(U \oplus Y)$ via the representation,

$$\rho:K \to \begin{pmatrix} I_Y & 0 \\ K & I_U \end{pmatrix} \in GL(U \oplus Y) \quad , \tag{3.3}$$

and $GL(U \oplus Y)$ acts on $\mathbb{P}^2$ (the points in $\mathbb{P}^2$ regarded as lines in $U \oplus Y$). One therefore has two possibly distinct actions of K on T: the first is the standard output feedback transformation $T \to T^K$ given in (1.2)', and the second is obtained by composing the map $T:\mathbb{C}^* \to \mathbb{P}^2$ with the classical action $\rho K:\mathbb{P}^2 \to \mathbb{P}^2$. As one can see, ρK leaves the line U fixed and therefore $\rho K \circ T$ is a rational function, vanishing at ∞; i.e., $\rho K \circ T$ is a transfer function. Explicitly, by combining (3.3) with (1.2)' one has

$$\rho K \circ T = \begin{pmatrix} I_Y & 0 \\ K & I_U \end{pmatrix} \begin{pmatrix} N \\ D \end{pmatrix} = \begin{pmatrix} N \\ D+KN \end{pmatrix} = T^K . \tag{3.4}$$

In particular, one may now compute (3.2) acted on by K in two ways:

$$\mathrm{sing}(T^K) = (T^K)^{-1}(T^K(\mathbb{C}^*) \cap \mathbb{P}^1 = T^{-1}(T(\mathbb{C}^*) \cap (-K)\,\mathbb{P}^1) \tag{3.5}$$

where $(-K)\,\mathbb{P}^1$ is the linear $\mathbb{P}^1$(= the plane Y in $\mathbb{C}^3$) acted on by $\rho(-K)$ (= graph of $-K:Y \to U$ in $\mathbb{C}^3$)! Thus, in order to compute the closed loop poles, $\mathrm{sing}(T^K)$, one can alternatively keep the curve T (see Fig. 3.2) fixed and, instead, move $\mathbb{P}^1$ through the inverse "rotation" $-K$.

This proves the claim but in a more general setting, viz. in (3.3) and (3.4) U and Y could just as well be an m-plane and a p-plane, with $GL(U \oplus Y)$ acting as linear transformation on the space of m-planes = Grass(m,$U \oplus Y$). In this setting, the generalization of (3.2) is due to R. Hermann and C. Martin, who interpreted the McMillan degree as an intersection number ([24]). The codimension 1 subvariety of Grass(m,U + Y) which plays the role of the line $\mathbb{P}^1$ in the plane $\mathbb{P}^2$ is the Schubert variety of m-planes:

$$\sigma(Y) = \{W : W \cap Y \supsetneq (0)\} \quad . \tag{3.6}$$

That is, $W \in \sigma(Y)$ if, and only if, W meets Y in at least a line. The beautiful (and useful for our purposes) theorem of R. Hermann-C. Martin is

$$T^{-1}(T(\mathbb{C}^*) \cap \sigma(Y)) = \mathrm{sing}(T) \quad . \tag{3.7}$$

[Alternatively, the extended plane $\mathbb{C}^*$ is the Riemann sphere S^2 (or $\mathbb{P}^1$) and

$$[T(\mathbb{P}^1)] \in \pi_2(\mathrm{Grass}(m, U \oplus Y) \simeq \mathbb{Z}$$

corresponds to the McMillan degree, where the isomorphism is canonical, by virtue of the Hurewicz isomorphism and a choice of complex structure.] As before we can act on (3.6) with K, in two ways, to obtain

$$T^{-1}(T(\mathbb{C}^*) \cap \sigma(-KY)) = \mathrm{sing}(T^K). \tag{3.7}$$

Now, as an illustration of these geometric ideas and in order to return to some of Kimura's algebraic techniques, I shall prove a little pole-placement theorem for state-feedback, i.e., for the case $p = n$. [This combinatorial theorem is a special case of a theorem of Rado ([38]) which also generalizes Ph. Hall's Theorem. Moreover, an elegant application of Rado's Theorem to pole-placement appears, for the first time in Hautus's proof of pole-assignment by state feedback ([19]), published in 1970. I was mistaken in my lectures in ascribing it solely to Kimura.] What I wish to give is a proof of the Wonham-Simon-Mitter-Heymann-Kalman Theorem for distinct poles $\{\lambda_1, \ldots, \lambda_n\}$.

The principal lemma in [29] is in fact a celebrated theorem in combinatorics, in disguise. Kimura calls a collection $V_1, \ldots, V_n$, of subspaces of a vector space V, normal just in case one can select vectors $v_i \in V_i$ such that $\{v_1, \ldots, v_n\}$ is independent. The lemma asserts that a collection of subspaces is normal if, and only if, the (general position) condition (*) is satisfied:

for each selection $V_{i_1}, \ldots, V_{i_k}$ of distinct V_i's,

$$\dim\left(V_{i_1} + \ldots + V_{i_k}\right) \geq k \; . \tag{*}$$

Notice, however, that (*) is precisely the diversity condition in Ph. Hall's theorem on distinct representatives, modified to include the set (or subspace) function $\dim(\cdot)$--which, after all, does satisfy a form of the inclusion-exclusion principle.

In order to apply this result to multivariable state-feedback consider, for $\lambda_1,\dots,\lambda_n$ distinct, the subspaces $T(\lambda_1),\dots,T(\lambda_n)$ of $U \oplus X$, where $T(s) = (sI-F)^{-1}G$. By Lemma 2 of [29],

$$\dim(T(\lambda_{i_1}) + \dots + T(\lambda_{i_k})) \geq \ell_k ,$$

where $\ell_k = \dim sp(G, FG, \dots, F^{k-1}G)$ are the dual Kronecker indices and hence $\ell_k \geq k$. Following Kimura, we may select independent vectors $v_i \in T(\lambda_i)$ and as before define the gain K by the equation

$$gr(K) \doteq sp\{v_1,\dots,v_n\} \subset U \oplus X .$$

Then,

$$\det(sI - F + GK) = \prod_{i=1}^{n} (s - \lambda_i) .$$

And if $\{\lambda_1,\dots,\lambda_n\}$ is self-conjugate, $gr(K)$ can be taken to be self-conjugate.

§4. GLOBAL ANALYSIS OF χ_σ, THE CENTRAL PROJECTION LEMMA, POLE PLACEMENT BY OUTPUT FEEDBACK OVER $\mathbb{R}$ AND $\mathbb{C}$.

<u>Theorem</u>. *If* $mp \leq n$, *then generically* χ_σ *is a proper map. In particular, over* $\mathbb{C}$ *(or any algebraically closed field) image* χ_σ *is a subvariety of* $\mathbb{C}^n$. *Over* $\mathbb{R}$, χ_σ *extends to a map* $\bar{\chi}_\sigma : S^{mp} \to S^n$ *of spheres and image* χ_σ *is Euclidean closed in* $\mathbb{R}^n$.

If $mp > n$, then χ_σ is no longer proper--i.e., $C \subset \underline{k}^n$ a compact set implies $\chi_\sigma^{-1}(C) \subset \underline{k}^{mp}$ is compact, although one can still prove that image χ_σ is (generically) closed.

<u>Proof</u>. The proof begins with a study of the map

$$T : \mathbb{C}^* \to \mathrm{Grass}(m, U \oplus Y) .$$

Now, $GL(U \oplus Y)$ acts transitively on m-planes in $U \oplus Y$ and so parameterizes $\mathrm{Grass}(m, U \oplus Y)$, i.e., there is a map

$$\pi : GL(U \oplus Y) \to \mathrm{Grass}(m, U \oplus Y)$$

$$\pi : g \to gU$$

corresponding to the choice of the m-plane U. π, however, is an overparameterization since there are many g's which fix U. In fact, in terms of the decomposition

$$\underline{k}^{m+p} = U \oplus Y \quad ,$$

the subgroup of g's which fix U has the form,

$$g \in \begin{pmatrix} GL(Y) & 0 \\ \mathrm{Hom}(Y,U) & GL(U) \end{pmatrix} = \mathcal{F}, \qquad (4.1)$$

the output feedback group! By dividing out by $\mathcal{F}$, we get an honest (i.e., 1-1) parameterization of $\mathrm{Grass}(m, U \oplus Y)$,

$$\mathrm{Grass}(m, U + Y) \simeq GL(U \oplus Y)/\mathcal{F} \quad .$$

This extends the picture in (3.4) quite a bit. In fact, the main idea of the proof is to extend χ_σ by evaluating the left-hand side of (3.7) for all $g \in GL(U \oplus Y)$; that is, we keep T fixed, as a curve in $\mathrm{Grass}(m, U \oplus Y)$, and intersect it will all $\sigma(gY)$, for $g \in GL(U \oplus Y)$. Now, when $g \in \mathcal{F}$, gY is complementary to U and is, in fact, the graph of some linear map $K:Y \to U$. In particular, for such a g

$$T^{-1}(T(\mathbb{C}^*) \cap \sigma(gY)) = \mathrm{sing}\, T^{g^{-1}} = \mathrm{spec}(F\text{-}GKH) \qquad (4.2)$$

is an unordered set $\{\lambda_1, \ldots, \lambda_n\}$ of points in $\mathbb{C}^* = \mathbf{P}^1$. That is $\{\lambda_1, \ldots, \lambda_n\} \in \mathbf{P}^1 \times \ldots \times \mathbf{P}^1/S_n \simeq \mathbf{P}^n$, the so-called symmetric product. For $g \in \mathcal{F}$, each λ_i is finite, by virtue of (4.2), and

$$\{\lambda_1, \ldots, \lambda_n\} \in \mathbb{C}^n \subset \mathbf{P}^n \quad .$$

Here $\mathbb{C}^n/S_n \simeq \mathbb{C}^n$, where the isomorphism is simply

$$\{\lambda_1, \ldots, \lambda_n\} \to (c_i(\lambda))_{i=1}^n$$

with c_i the elementary symmetric functions. In summary, we have our old picture in this new setting,

$$\chi_\sigma : \sigma(gY) \to T^{-1}(T(\mathbb{C}^*) \cap \sigma(gY)) \in \mathbb{C}^n \subset \mathbf{P}^n \quad ,$$

via

$$g \to \mathrm{sing}(T^{g^{-1}}) = (c_i(F + GKH))_{i=1}^n \quad .$$

By conservation of difficulty, one needs more than this restatement of the problem and it is at this point that we consider $\sigma(gY)$ for any $g \in GL(U \oplus Y)$ or, what is the same $\sigma(Y')$ for a p-plane Y' not necessarily complementary to U.

Lemma A. $\sigma(Y')$ *either contains* $T\mathbb{C}^*$ *or intersects it (counting multiplicity) in exactly* n *points. In the latter case, such a point is infinite if, and only if,* Y' *is not complementary to* U.

Proof. The first part of the lemma is an elementary application of value distribution theory and can be found in Chern, "Complex Manifolds with Potential Theory," D. van Nostrand under the topic: "Holomorphic Curves in a Grassmannian." The second part is, in fact, the condition $U \in \sigma(Y')$ and follows from the definition (3.6).

To facilitate the discussion, I shall refer to a p-plane Y' as a generalized feedback (law) while a p-plane Y' complementary to U will be referred to as a classical feedback (law). The idea is therefore to extend χ_σ in (4.3) to all "generalized feedbacks," i.e., to all points in the dual Grassmannian, Grass(p,U ⊕ Y). That is, we wish to define

$$\bar{\chi}_\sigma : \mathrm{Grass}(p, U \oplus Y) \to \mathbb{P}^n \tag{4.4}$$

via

$$Y' \to T^{-1}(T\mathbb{C}^* \cap \sigma(Y')) .$$

Remark. Consider the scalar case and restrict attention to real gains K. Then, the real Grassmannian is the space of lines in $\mathbb{R}^2$, i.e., the circle S^1 and

$$\bar{\chi}_\sigma : S^1 \to \mathbb{P}^n$$

is precisely the root-locus map!

Now the fact that χ_σ is defined at $\infty \in S^1$ is just the fact that $\chi_\sigma(\infty) = \{\text{zeros of } T(s)\}$. For m,p arbitrary the recent formula of Kailath et al. [25], which computes the difference between the number of closed loop poles of $T(s)$ and the number of open-loop zeroes in terms of the left and right Kronecker indices of $T(s)$, shows that there may not be enough open loop zeroes to account for the asymptotic root loci $\bar{\chi}_\sigma(K)$ as $K \to \infty \in S^{mp}$, although this may be the case if $m = p$. In the Grassmannian compactification $K \to \infty$ takes on an entirely new meaning, as ∞ is replaced by the whole subvariety

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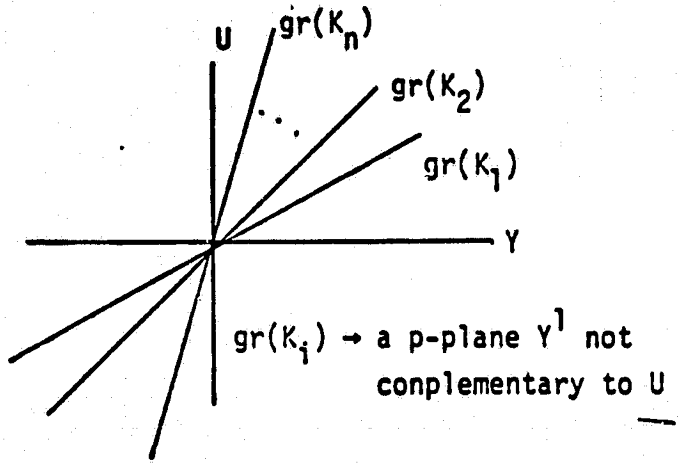
$\sigma(U) \subset$ Grass(p,U ⊕ Y). This gives much more freedom in the manner in which K "becomes infinite (and this is important for potential applications, allowing for various channels in the gain to grow at various rates) and Lemma B shows that as K "becomes infinite" the root locus $\chi_\sigma(K)$ (still) apporaches an n-tuple of points in the extended complex plane, as in the classical case, for generic systems provided $mp \leq n$. The case $mp \leq n$ is illustrated below



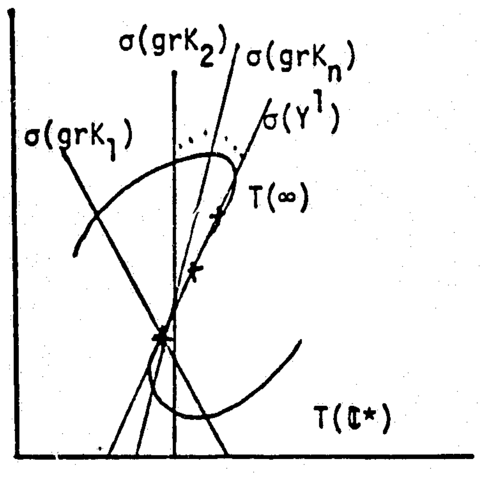


Figure 4.1 Depicting Asymptotic Root Loci as Points of Intersection

The explicit obstruction to this asymptotic extension is in fact explained in the lemma, there can exist p-planes Y' such that $\sigma(Y') \supset T\mathbb{C}^*$. Indeed, if $mp > n$, examples exist in great profusion.

Lemma B. *If* $mp \leq n$, *then for almost all* (F,G,H), $\sigma(Y') \cap T(\mathbb{C}^*)$ *in* n *points, for all* p-*planes* Y'.

Proof. First of all, the set of (F,G,H) for which there exists a Y' with $\sigma(Y') \supset T(\mathbb{C}^*)$ is closed in the variety of all (F,G,H). In fact, if $[(F,G,H)]$ is the corresponding point in the moduli space $\Sigma^n_{m,p}$ then the subset

$$V \subset \Sigma^n_{m,p} \times \mathrm{Grass}(p, U \oplus Y)$$

defined via

$$V = \{([F,G,H],Y') : \sigma(Y') \supset T(\mathbb{C}^*)\}$$

is a subvariety. Since $\mathrm{Grass}(p,U \oplus Y)$ is projective (compact) and $\Sigma^n_{m,p}$ is quasi-projective ([13],[22]) $\pi_1: \Sigma^n_{m,p} \times \mathrm{Grass}(p,U \oplus Y) \to \Sigma^n_{m,p}$ is closed. In particular, $\pi_1(V)$ is closed in $\Sigma^n_{m,p}$, but $\pi_1(V)$ is precisely the variety we wish to delete in the lemma. To show that, if $mp \leq n$, $\pi_1(V)$ is a proper subvariety, one may appeal to the duality

$$\mathrm{Grass}(m,U \oplus Y) \simeq \mathrm{Grass}(p,U \oplus Y) ,$$

which is related to the duality between inputs and outputs. By Lemma A, to say $\sigma(Y') \supset T(\mathbb{C}^*)$ is to say in particular

$$T(\lambda_1),\ldots,T(\lambda_n),\ T(\infty) \in \sigma(Y') \tag{4.5}$$

for λ_i distinct, finite points. However, to say $T(\lambda) \in \sigma(Y')$ in $\mathrm{Grass}(m,U \oplus Y)$ is to say $Y' \in \sigma(T(\lambda))$ in $\mathrm{Grass}(p,U \oplus Y)$, by the symmetry of the definition (3.6). But $\sigma(T(\lambda))$ is codimension 1 in $\mathrm{Grass}(p,U \oplus Y)$ and hence of dimension $mp-1$. And (4.5) is the assertion

$$Y' \in \bigcap_{i=1}^{n} \sigma(T(\lambda_i)) \cap \sigma(T(\infty)) .$$

Since the Schubert varieties σ are hyperplane sections via the

Plücker imbedding, $\dim \cap_{i=1}^{\ell} \sigma(U_i) \geq mp - \ell$ (this is not true for arbitrary varieties, see Section I of this introductory chapter). On the other hand, generically one has $\dim \cap_{i=1}^{\ell} \sigma(U_i) \leq mp - \ell$. Therefore Y' must lie, generically, on a subvariety of dimension $mp - (n+1)$, which is impossible unless $mp > n$. ⌑

It is worth remarking that this gives an independent proof that, in case $mp > n$, χ_σ is generically almost onto, assuming the field is algebraically closed. Indeed, using an output feedback invariant version of (2.3) one can give explicitly, the equations defining the generic properly given above.

Returning to the proof of the theorem, by our lemmata, we can generically extend the map χ_σ to the root-locus map

$$\overline{\chi_\sigma}:\mathrm{Grass}(p,U \oplus Y) \to \mathbb{P}^n \quad .$$

In particular, $\overline{\chi_\sigma}$ is a proper map and by the second part of Lemma A, χ_σ is also proper, i.e., since

$$\text{image } \chi_\sigma = \text{image } \overline{\chi_\sigma} \cap \mathbb{C}^n \quad . \tag{4.6}$$

Furthermore, since the real Grassmannian is canonically imbedded in the complex Grassmannian as a compact submanifold, χ_σ remains proper over $\mathbb{R}$. Since the ℓ-sphere is the 1-point compactification of $\mathbb{R}^\ell$, χ_σ extends to a map of spheres. And, by virtue of (4.6), image χ_σ is a subvariety of $\mathbb{C}^n$ and its real points form a closed subspace of $\mathbb{R}^n$. ⌑

In case $mp > n$, one cannot extend χ_σ to the root-locus map as above. However, one can replace the Grassmannian by the closure of the graph of the rational function χ_σ, viz.

$$\overline{\chi_\sigma} = \overline{\text{graph } \chi_\sigma} \subset \mathrm{Grass}(p,U \oplus Y) \times \mathbb{P}^n \quad .$$

And one replaces the map $\overline{\chi_\sigma}$ by the proejction onto the 2nd factor,

$$\pi_2:\overline{\chi_\sigma} \to \mathbb{P}^n \quad .$$

The analog of (4.6) still holds, although one must work a bit harder. In this case, one may still deduce the Theorem, except for the statement that χ_σ is proper. This no longer is valid, as $\chi_\sigma^{-1}(p)$ is a subvariety of positive dimension in affine space and therefore admits unbounded analytic functions.

Remarks. Both cases can be treated in a more unified fashion, but relying on slightly more sophisticated ideas. Denoted by P the Plücker imbedding

$$P:\text{Grass}(m,U \oplus Y) \to \mathbb{P}^N$$

of the Grassmannian and suppose for simplicity that, if $\{\lambda_1,\ldots,\lambda_n\} = \text{Sing } T$, for a transfer function $T:\mathbb{P}^1 \to \text{Grass}(m,U \oplus Y)$ then $T(\lambda_i)$ are distinct points in $\text{Grass}(m,U \oplus Y)$ and hence so are the points $V_i = P(T(\lambda_i))$ in $\mathbb{P}^N$. By duality, each point V_i corresponds to linear functional L_i on $\mathbb{C}^{N+1}$ and hence to a hyperplane H_i in $\hat{\mathbb{P}}^N$, the dual projective space. Finally, denote by H_{n+1} (and L_{n+1}) the hyperplane (and functional) corresponding to the point $P(G(\infty))$ and by B, the variety

$$B = \bigcap_{i=1}^{n+1} H_i \quad \text{in} \quad \hat{\mathbb{P}}^N .$$

Following Lemma A, we consider the central projection with base locus B,

$$\phi:\hat{\mathbb{P}}^N - B \to \mathbb{P}^n,$$

defined via

$$\phi(x) = [L_i(x)] , \quad \text{in homogeneous coordinates.}$$

Restricted to $\hat{P}(\text{Grass}(p,U \oplus Y)) - \hat{P}(\text{Grass}(p,U \oplus Y)) \cap B$, one recovers

$$\chi:\text{Grass}(p,U \oplus Y) - B \cap \text{Grass}(p,U \oplus Y) \to \mathbb{P}^n ,$$

as a central projection, with Lemma B asserting that in the correct dimension range χ has no base lower on $\text{Grass}(p,U \oplus Y)$. This admits a particularly nice exploitation of Schubert calculus, especially in the case $mp = n$ (see [5]). For in that case, generically

$$\chi:\text{Grass}(p,U \oplus Y) \to \mathbb{P}^n$$

is globally defined and $\dim \text{Grass}(P,U + Y) = mp = n = \dim \mathbb{P}^n$. The degree of χ is the degree of the subvariety

$$\hat{P}(\text{Grass}(p, U + Y)) \subset \hat{\mathbb{P}}^N,$$

since χ is a central projection, and this is well-known ([31]) to be

$$\deg \chi = \frac{1!\, 2! \cdots (p-1)!\, (mp)!}{m!\, (m+1)! \cdots (m+p-1)!} \quad . \tag{4.7}$$

Briefly, what this entails is first the observation that the system of n equations in mp unknowns,

$$\chi(K) = p(s) \ ,$$

can be expressed as an intersection of n hypersurfaces in mp space and these hypersurfaces are well-studied. That is, regarding p, via its roots, as a point $(\lambda_1,\ldots,\lambda_n) \in \mathbb{P}^1)^{(n)}$, one can view p as $\bigcap_{i=1}^{n} H_{\lambda_i}$ of n hyperplanes, H_λ denoting the hyperplane of $(\lambda_1,\ldots,\lambda_n)$ such that $\lambda_i = \lambda$, for some i. In this setting, $\chi^{-1}(p)$ has the form

$$\chi^{-1}\left(\bigcap_{i=1}^{n} H_{\lambda_i}\right) = \bigcap_{i=1}^{n} \chi^{-1}(H_{\lambda_i}) = \bigcap_{i=1}^{n} \sigma(T(s_i))$$

provided $\lambda_i = T(s_i)$ lies on the curve $T(\mathbb{P}^1)$. Now, the Schubert calculus enables one to express such intersections in terms of basic, or Schubert, varieties. In particular, if $mp = n$ then a repeated use of one such expression, Pieri's formula, allows us to count the number of points in

$$\bigcap_{i=1}^{n} \sigma(T(s_i)),$$

counting multiplicities, as $\deg \chi$ in (4.7). Thus, the main point of all this is that the output feedback map as a system of equations is actually a well-studied, classical system of equations--about which much is known (see [31]). As a corollary, one can show that when $\deg \chi$ is odd the map

$$\chi : \text{Grass}_{\mathbb{R}}(p, U \oplus Y) \to \mathbb{RP}^n$$

is surjective, hence we can place poles with real gains! It has been shown ([1]) that $\deg \chi$ is odd if, and only if, either $\min(m,p) = 1$ or $\min(m,p) = 2$ and $\max(m,p) = 2^r - 1$, a Mersenne number.

<u>Theorem</u> ([5]). *Assume* $mp = n$. *It is possible generically in* σ *to place any self-conjugate set of poles by real output feedback provided either* $\min(m,p) = 1$ *or* $\min(m,p) = 2$ *and*

$\max(m,p) = 2^r-1$.

We note that Willems and Hesselink ([45]) have shown that, generically in σ, in χ_σ misses an open set in $\mathbb{R}^4$, if $m = p = 2$. This is in harmony with our result but whether the surprisingly combinatorial conditions in this theorem are necessary is at present an open question. On the more positive side, extensions and corollaries of the central projection lemma give rise to sufficient conditions for generic stabilizability in the more general setting $mp \geq n$. These take the form of inequalities

$$C_{m,p} \geq n$$

where $C_{m,p}$ is a function of m and p. These are, however, too complicated to give here.

§5. POLE PLACEMENT OVER RINGS, SOME POSITIVE RESULTS: MORSE'S THEOREM, AND FEEDBACK INVARIANTS

In this section and in the last two, I shall be concerned with the general question of what can be done, particularly in the way of stabilizability and pole-placement, with state feedback where the coefficients of the system lie in a commutative ring R with 1. It is very easy to motivate the study of stabilizability of several classes of systems defined over rings and this has, in turn, motivated the study of general pole-assignability questions over an arbitrary R. This route to stabilizability has the potential advantage (at least over Noetherian R) of providing finite procedures for obtaining a desired gain. However, it is fair to say that, at this point in time, general pole-assignability questions are, in all honesty, primarily mathematical questions about the algebraic structure of dynamical systems, and the reader who wishes to may skip to §7, which deals with the more modest question of stabilizability. On the other hand, pole-assignability questions over a ring are of theoretical interest in their own right and, as recent work has shown, such questions are much harder than anyone had first suspected--even the elementary examples involve non-trivial topological and arithmetic obstructions. We begin with a quick review of the main problems and motivating examples, for the realization theory of such systems we should refer to Professor Rouchaleau's lectures ([40]).

R is a commutative ring with 1, $X \simeq R^{(n)}$ and $U \simeq R^{(m)}$ are free R-modules. It is meaningful to distinguish between two verions of the question: First, the problem of solving the system of equations, for $K \in \mathrm{Hom}_R(X,U)$,

$$\det(sI - F - GK) = p(s), \quad p(s) \in R[s] \text{ monic of deg } n, \tag{5.1}$$

where $F:X \to X$, $G:U \to X$ are fixed, is referred to as coefficient assignability (over R). Second, if

$$p(s) = \prod_{i=1}^{n} (s-r_i)$$

with $r_i \in R$, then to solve 5.1 for K is to solve a problem of pole, or zero, assignability over R.

In the general situation, one may think of (X,U,F,G) as the data defining the discrete-time system,

$$x(t+1) = Fx(t) + Gu(t) \quad . \tag{5.2}$$

On the other hand, in specific situations, this data may represent continuous-time systems. Explicitly, if one considers a controlled heat equation

$$\frac{\partial u}{\partial t} = \Delta u + f \tag{*}$$

on the n-torus $\mathbb{T}^n$, then it is rather natural (and frequently done) to discretize (*) in the spatial variable relative to a grid in T^n, obtaining a "lumped approximation." If one chooses the grid G_ℓ of points of order ℓ on $\mathbb{T}^n$, then (*) reduces to a linear control system with coefficients in the group algebra $\ell^1(G_\ell)$ of the group $G_\ell \subset \mathbb{T}^n$ (see [7]).

One important class of examples is the class of systems depending on parameters, say in a C^k-fashion,

$$\dot{x}(t) = F(\lambda)x(t) + G(\lambda)u(t), \quad \lambda \in U \subset \mathbb{R}^N \tag{5.2)'$$

where λ is the value of a resistor, or an altitute or attitude. Another class of examples arises in the algebraic theory of delay-differential systems, where a system,

$$\dot{x}(t) = F*x(t) + G*u(t) \quad , \tag{5.2)"$$

is regarded as a system defined over a ring of convolution operators ([24],[36]). Explicitly, consider the system

$$\dot{x}(t) = F_0x(t) + F_1x(t-1) + Gu(t) \quad . \tag{5.3}$$

Introducing the convolution operator,

$$(\delta*x)(t) = x(t-1) \quad ,$$

one may regard (5.3) as

$$\dot{x}(t) = F(\delta)*x(t) + G*u(t) \qquad (5.3)'$$

where $F = F_0 + F_1\delta$ is defined over $R = \mathbb{R}[\delta]$, as is G. More generally, a delay-differential system, involving only the commensurate delays by 1,2,... seconds, may be regarded as a system

$$\dot{x}(t) = F(\delta)*x(t) + G(\delta)*u(t) \qquad (5.4)$$

defined over $R = \mathbb{R}[\delta]$. The non-commensurate case, of course, leads to a change of scalars $\mathbb{R} \subset \mathbb{R}[\delta_1,\ldots,\delta_N]$.

In both of these latter cases, it is important to know whether there exists a gain K, preferably defined over R, such that the closed loop system $(F+GK,G)$ is asymptotically stable in an appropriate functional analytic sense. In the first case, since ideally K ought to depend on the system (F,G), it is clear that if $(F(\lambda),G(\lambda))$ is C^k for $0 \le k \le \omega$, in $\lambda \in U$ then K ought to be C^k in λ as well; i.e., K ought to be defined over the ring $R = C^k(U)$. Now, in the second case, to ask that K be defined over $R = \mathbb{R}[\delta_1,\ldots,\delta_N]$ is natural from the point of view that K should be constructible from the same components as the system (F,G). And, if one can place the poles of (F,G) over R, then for each $c > 0, \exists K$ defined over R such that $(F+GK,G)$ exponentially stable (in L^1) with order c ([27], Theorem 8). On the other hand, one should remark that, especially in light of §6, the functional approach has been far more successful ([14],[32]) in obtaining pole-placement results, at the expense of using more general operators K (eg. convolution with continuous measures).

Now, motivated by work of A. S. Morse ([36]) on delay-differential systems, one is led to the

Definition 5.1. *The system* $\sigma = (F,G)$, *defined over* R, *is reachable over* R *just in case the controllability operator*

$$C = (B,AB,\ldots) \quad \bigoplus_{i=1}^{\infty} U \to X \qquad (5.5)$$

is surjective.

As observed in [44], if σ is coefficient assignable, then σ is reachable. To see this, setting max(R) = M:M a maximal ideal of R}, recall (see Introductory Chapter) that C is surjective if, and only if,

$$C: \bigoplus_{i=1}^{\infty} U \to X \qquad \mathrm{mod}(M)$$

is surjective for all $M \in \max(R)$. For σ defined over R, denote by $\sigma(M)$ the system (F,G) over the field R/M, obtained by reducing (X,U,F,G) modulo $M \in \max(R)$. Now, if σ is coefficient assignable over R, then $\sigma(M)$ is coefficient assignable over R/M and therefore reachable over R/M for any $M \in \max(R)$, as we wished to show. Similarly, zero-assignability implies reachability and the converse, for R a P.I.D., is due to A. S. Morse.

<u>Theorem 5.2</u> ([36]) *A reachable system* $\sigma = (F,G)$, *defined over* $\mathbb{R}[x]$, *is zero assignable.*

Indeed, Morse's proof applies to reachable systems defined over a P.I.D. and, in this setting, is the best result know at present.

In order to study the coefficient assignability question over R, it is useful to bring in the group of symmetries for state feedback and the Rosenbrock pencil. Explicitly, consider the pencil of equations

$$sx(x) = Fx(s) + Gu(s) \tag{5.6}$$

where $x(s) = \Sigma x_i s^i$, $u(s) = \Sigma u_i s^i$, are polynomial "vectors" with coeffieicnets in the modules X,U, respectively. Thus, the pencil (5.6) is a "formal Laplace transform" of (5.2) and, once again, this transform intertwines the action on systems of the state feedback group with a classical action. It is more precise to regard $x(s)$ as an element of the $R[s]$ module $X[s] \simeq X \otimes \mathbb{Z}[s]$ and $u(s)$ as an element of $U[s]$. Then, the Rosenbrock pencil takes the form

$$\mathcal{R}:(X \oplus U)[s] \to X[s] \quad ,$$

$$(x(s),u(s)) \to (Fx(s) - sIx(s) + Gu(s)) \; , \tag{5.7}$$

and $\mathcal{R}$ is surjective if, and only if, (F,G) is reachable over R. In this case, we are led to the exact sequence,

$$0 \to \ker \mathcal{R} \to (X \oplus U)[s] \to X[s] \to 0 \; , \tag{5.8}$$

where the submodule $\ker \mathcal{R}$ is, at least formally, the Laplace transform of solutions to (5.2) with zero initial data. Now, just as in the case $R = k$, one may show that the strict equivalence of 2 such pencils $\mathcal{R}_1$ and $\mathcal{R}_2$, in the sense of linear algebra, implies the equivalence (under state feedback) of the systems σ_1 and σ_2. That is, to say $\mathcal{R}_1 \sim \mathcal{R}_2$ is to say there exists

$$C:(X \oplus U)[s] \to (X \oplus U)[s]$$

and $D:X[s] \to X[s]$,

invertible maps of $R[s]$-modules, such that

$$DR_1C = R_2 , \tag{5.9}$$

and such that C,D are independent of s (i.e. extended from R-module maps). Since R_1 (and R_2) has block diagonal form, it follows from comparing degrees (with respect to s) in (5.9) that C decomposes, with respect to $(X \oplus U)[s] \simeq X[s] \oplus U[s]$, as

$$C = \begin{pmatrix} D^{-1} & 0 \\ KD^{-1} & B \end{pmatrix} \tag{5.10}$$

where $B \in GL(U)$ and $K \in Hom(X,U)$. In particular, B,D, and K give the desired equivalence of σ_1 and σ_2 modulo the state feedback group $\mathcal{F}_s(R)$. Conversely, this familiar triangular matrix representation of $\mathcal{F}_s(R)$ shows that an equivalence mod $\mathcal{F}_s(R)$ induces a strict equivalence of R_1 with R_2 (see [26]). This is summarized in the classical and well-known proposition:

<u>Theorem 5.3</u> *The Rosenbrock pencil* R *of* σ, *up to strict equivalence, is a complete invariant for* σ *modulo the state feedback group* $F_s(R)$.

Now, over a field, Kronecker has given a classification for matrix pencils in terms of the degrees of minimal basis vectors for the submodule $\ker R \subset (X \oplus U)[s]$. These degrees constitute a partition

$$\sum_{i=1}^{m} n_i = n, \quad n_1 \geq n_2 \geq \ldots \geq n_m \geq 0 \tag{5.11}$$

of dim(X), and in this way one obtains a complete set of invariants for σ modulo $\mathcal{F}_s(\underline{k})$, see Professor Rosenbrock's lectures ([39], esp. §5). Now, for general R, one may replace the arithmetic data (5.11) by the isomorphism class of the submodules,

$$0 \to \ker R \to (X \oplus U)[s], \tag{5.12}$$

and in some cases this isomorphism class is expressible in a more intrinsic form.

Suppose $R = \underline{k}[\lambda_1,\ldots,\lambda_N]$, with $\underline{k}$ algebraically closed. Since R in (5.7) is surjective R-module map of free R-modules, it is not too hard to see that $\ker \mathbb{R}$ is projective as an R-module and has finite rank, since X and U have finite rank. Recalling the connection between projective modules and vector bundles (see Introductory Chapter, Section III), it is quite plausible to seek a vector bundle characterizing the data (5.12). Indeed, consider the m-vector bundle on $\mathbb{A}^N \times \mathbb{A}^1$ whose fiber over (λ, s) is the vector space

$$\ker[F(\lambda) - sI, G(\lambda)] \subset \underline{k}^n \oplus \underline{k}^m \quad . \qquad (5.13)$$

Now, since all m-vector bundles on $\mathbb{A}^N \times \mathbb{A}^1$ are isomorphic (by the Quillen-Suslin Theorem) provided we allow isomorphisms depending on $s \in \mathbb{A}^1$, (5.13) is not fine enough. However, the notion of strict equivalence (i.e., independence of s) suggests an order of growth at $s = \infty$ reminiscent of Lionville's Theorem. That is, by homogenizing all of the above we construct a bundle W_σ on $\mathbb{A}^N \times \mathbb{P}^1$, whose fiber over $(\lambda, [s,t])$ is given by

$$\ker[tF(\lambda) - sI, t\,G(\lambda)] \subset \underline{k}^n \oplus \underline{k}^m. \qquad (5.13)'$$

<u>Theorem 5.4</u> ([10]) *The bundle* W_σ *is a complete invariant for* σ *modulo state feedback.*

<u>Remarks</u>. For $N = 0$, this was studied by R. Hermann and C. Martin, who related the Kronecker invariants (5.11) to the Grothendieck invariants of W_σ, thereby proving Theorem 5.4. For $N > 0$, if one forms the transfer function for the triple $(F(\lambda), G(\lambda), I)$, then just as in the earlier sections one obtains a map

$$T: \mathbb{A}^N \times \mathbb{P}^1 \to \mathrm{Grass}(m, \underline{k}^n \oplus \underline{k}^m)$$

exhibiting (5.13)' as the pullback along the transfer function of the (topological) universal bundle, and Theorem 5.4 follows from Riemann-Roc (see Professor Martin's Lectures, [ ]).

Now, if σ is independent of λ then W_σ is independent of λ, i.e. W_σ is a pullback along the second projection $p_2: \mathbb{A}^N \times \mathbb{P}^1 \to \mathbb{P}^1$ of a bundle on $\mathbb{P}^1$. And, Theorem 5.4 asserts that the converse in true. Thus, if W_σ is a pullback, then σ is coefficient assignable--since this result is valid for σ defined over the field k. On the other hand, one can express this condition more explicitly. For each λ, σ gives rise to a system $\sigma(\lambda)$, by evaluation of λ, over the field $\underline{k}$ and therefore to pointwise Kronecker indices,

$$\sum_{i=1}^{m} n_i(\lambda) = n(\lambda) = n \ .$$

Corollary 5.5 *If the* $n_i(\lambda) = n_i$ *are constant in* λ, *then* σ *is feedback equivalent to a constant system. In particular,* σ *is coefficient assignable.*

Proof. It follows from the main theorem of C. C. Hanna's thesis [18], that constancy in λ of the (Kronecker)-Grothendieck indices implies

$$W_\sigma \simeq \sum_j p_1^*(V_j) \otimes p_2^*(W_j)$$

where V_j is a vector bundle on $\mathbb{A}^N$ and W_j is a bundle on $\mathbb{P}^1$. By the Quillen-Suslin Theorem, V_j is trivial and therefore

$$W_\sigma \simeq \sum_j p_2^*(W_j) \simeq p_2^*(\sum_j W_j) \tag{5.14}$$

Moreover, $\sum_j W_j \simeq \sum_{i=1}^m \mathcal{O}(n_i)$. (5.14), however, is enough for our purposes. ♮

Example 5.6 The use of the Quillen-Suslin Theorem is, in fact, essential. Consider the following readable pair, defined over $R = C^\infty(S^2)$. Define $U \simeq R^{(3)}$ as the module of smooth sections of a rank 3, trivial vector bundle on S^2--viz., the restriction of the tangent bundle $T(\mathbb{R}^3)$ to S^2. If X is the R-module of smooth sections of $T(S^2)$, i.e. smooth vector fields on S^2, then $X \subset U$. In fact,

$$U \simeq X \oplus R^{(1)}$$

where $R^{(1)}$ is the module of sections of the normal bundle to $S^2 \subset \mathbb{R}^3$. In particular, we are led to the reachable pair $\sigma = (F,G)$ defined by $G = \mathrm{Proj}_1 : U \to X$, and $F = \mathrm{Id}: X \to X$. One easily checks that for $p \in S^2$, the pointwise Kronecker indices of σ are given by

$$(n_1(x), n_2(x)) = (1,1), \quad \text{for } x \in S^2,$$

noting that $\max C^\infty(S^2) \simeq S^2$ in the canonical way. However, the spectrum of (F,G) is not arbitrarily assignable, supples $FK: X \to U$ such that $\mathrm{spec}(F + GK) = \{0,1\}$. Then $F + GK$ is a projection on X and its kernel and image give rise to a decomposition, $X = M_1 \oplus M_2$, corresponding to a decomposition

$$T(S^2) = L_1 \oplus L_2$$

of the tangent bundle into line subbundles. Now, since $\dim L_i = 1$, each L_i is an integrable distribution and, by Frobenius' theorem, L_i forms a codimension one folition on S^2, as it were, implying that $\chi(S^2) = 0$, contrary to fact.

On the other hand, if one supposes that X and its projective submodules are free, then it becomes harder (see §6) to construct a "counterexample" to pole-assignability for reachable pairs. Moreover, as several authors ([11],[21b],[41]) have noted since Corollary 5.5 appeared, under these conditions on the state module X, a reachable pair (F,G) is coefficient assignable whenever the pointwise Kronecker indices $(n_1(M),\ldots,n_m(M))$ of $\sigma(M)$ are constant in $M \in \max(R)$. More generally, if $\max(R)$ is given the Stone-Jacobson-Zariski topology, where the basic closed sets have the form

$$h(I) = \{M : I \subset M\}, \text{ for } I \text{ and ideal of } R,$$

then σ is coefficient assignable whenever the pointwise Kronecker indices are locally constant. Recall that a ring R is said to be "projective-free" just in case each finitely generated projective module over R is free, thus the Quillen-Suslin Theorem asserts that $R = \underline{k}[x_1,\ldots,x_N]$ is projective free.

<u>Proposition 5</u> *Suppose* R *is projective-free and* $\sigma = (F,G)$ *is a reachable system with free state module and locally constant Kronecker invariants, then* σ *is coefficient assignable.*

<u>Remarks</u>.

1. The basic idea in the proof is to note first that constancy of the pointwise Kronecker indices $(n_1(M),\ldots,n_m(M))$ is equivalent to constancy of the rank

$$r_j(M) = \operatorname{rank}(G(M),F(M)G(M),\ldots,F^{j-1}(M)G(M))$$

(as the $r_j(M)$ form the dual partition to the partition $\Sigma n_i(M) = n$ of n) and that hypothesis on R now implies the freeness of the projective modules

$$(0) \quad \operatorname{span}(G) \quad \operatorname{span}(G,FG) \quad \ldots \quad \operatorname{span}(G,GF,\ldots,F^nG) = X.$$

A careful choice of basis now puts $\sigma = (F,G)$ in a standard canonical form, in which form coefficient assignability is immediate. For a very careful proof, see [21b].

2. This is also the route taken in [11]. However, the proposed extension (by working locally and then trying to patch local solutions) of this argument to cover arbitrary R and projective X is incorrect--as Example 5.6 amply demonstrates.

3. I would like to raise the question: Can Proposition 5.7 be improved upon by assuming only that

$$\sum_{i=1}^{m} in_i(M) = \text{constant} \quad ?$$

This is much more applicable, and I know of no counterexample.

We close this section with a few corollaries to Proposition 5.7.

Corollary 5.8 (Sontag) *If* R *is semilocal, then to say* σ *is reachable is to say* σ *is coefficient assignable.*

Proof ([41]) A semi-local ring has only finitely many maximal ideals, by definition. Thus, max(R) is discrete and every function is locally constant.

Corollary 5.9 (Brockett and Willems) *If* R *is the group algebra of a finite abelian group, then to say* σ *is reachable is to say* σ *is coefficient assignable.*

This follows, although not historically ([17]), from Corollary 5.8 or from somewhat deeper considerations. That is, one measure of the complexity of calculations in R is the structure of set of primes of R,

$$\text{spec}(R) = \{P : P \text{ a prime ideal of } R\} ,$$

and in particular of the Krull dimension of R--i.e., the least upper bound of the lengths of chains of prime ideals of R (see Introductory Chapter, Section VI). Note that any P.I.D. has dimension 1, whereas a field has dimension 0.

Corollary 5.10 *If* R *has Krull dimension* 0, *then to say* σ *is reachable over* R *is to say* σ *is coefficient assignable.*

One example of such a ring, in addition to the group algebras of Corollary 5.9, is furnished by the class of Boolean rings. Indeed, for any ring of dimension 0, max(R) is a Boolean space and based on the general (sheaf-theoretic) structure theory for such rings, we may apply Proposition 5.7.

The present state of affairs is rather intriguing. Morse's Theorem suggests that zero assignability is perhaps related to reachability for rings of Krull dimension 1, while Example 6.1 shows that reachability does not imply zero assignability in dimension 2.

§6. THE COUNTEREXAMPLES OF BUMBY, SONTAG, SUSSMANN AND VASCONCELOS.

In this section we present recent counterexamples ([8]) to zero assignability (over $\mathbb{R}[x,y]$) and to coefficient assignability (over $\mathbb{Z}$, or $\mathbb{R}[x]$). Indeed, all that we do here is for systems σ with rank $X = 2$. Note that, in this case, as an easy consequence ([10], Corollary 4.2) of the results in §5, one knows for R a polynomial ring or the ring $\mathbb{Z}$:

> Let $n = 2$ or 3, and suppose $G(M)$ has constant rank for all $M \in \max(R)$. Then $\sigma = (F,G)$ is reachable if, and only if, σ is coefficient assignable. (*)

<u>Example 6.1</u> Let $R = \mathbb{R}[\lambda_1,\lambda_2]$ and consider $\sigma = (F,G)$:

$$F(\lambda_1,\lambda_2) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \; G(\lambda_1,\lambda_2) = \begin{pmatrix} \lambda_1+\lambda_2 & \lambda_1+\lambda_2-1+\lambda_1^2+\lambda_2^2 \\ \lambda_2-\lambda_k & \lambda_2-\lambda_1+1-\lambda_1^2-\lambda_2^2 \end{pmatrix} \tag{6.1}$$

Notice that σ is reachable over $R_{\mathbb{C}} = \mathbb{C}[\lambda_1,\lambda_2]$. Indeed, it is easy to compute the Kronecker indices $(n_1(\lambda),n_2(\lambda))$ of σ. Consider the algebraic sets in $\mathbb{C}^2$,

$$V_1^{\mathbb{C}} = \{(\lambda_1,\lambda_2):\lambda_1^2 + \lambda_2^2 = 1\} ,$$

$$V_2^{\mathbb{C}} = \{(\lambda_1,\lambda_2):\lambda_2 = 0\} .$$

With this notation,

$$(n_1(\lambda),n_2(\lambda)) = \begin{cases} (2,0) & \text{if } \lambda \in V_1 \cup V_2 , \\ (1,1) & \text{otherwise} . \end{cases} \tag{6.2}$$

In either case, $n(\lambda) = n_1(\lambda) + n_2(\lambda) = 2$, so σ is reachable over $R_{\mathbb{C}}$. However, one cannot find $K(\lambda_1,\lambda_2)$ defined over R such that

$$\det(sI - (F+GK)) = s(s+1) .$$

For then, as it were, the submodule

$$\ker(F + GK) \subset R^{(2)}$$

is complemented and hence gives rise to a splitting of R-modules

$$R^{(2)} \simeq M_0 \oplus M_{-1} \,, \tag{6.3}$$

with each M_i free of rank 1. But, to say M_0 is free is to say there exists $u \in R^{(2)}$, a unimodular element a posteriori, such that $V = F^{-1}Gu$ is a generator for M_0. Computing along the real points of $V_1^{\mathbb{C}}$, viz. S^1,

$$v(\lambda) = \begin{pmatrix} v_1(\lambda) \\ v_2(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda_2 & \lambda_2 \\ -\lambda_1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} u_1(\lambda_1,\lambda_2) \\ u_2(\lambda_1,\lambda_2) \end{pmatrix} \tag{6.4}$$

is a non-zero tangent vector field to S^1, extendable throughout $\mathbb{R}^2$. By the Poincaré-Bendixson Theorem, $v(\lambda)$ has a zero inside the unit disc, contradicting unimodularity.

Next, consider the question of coefficient assignability for 2×2 systems over a P.I.D. As an example, consider the following system σ defined over $R = \mathbb{Z}$.

Example 6.2

$$F = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \,, \qquad G = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$$

For p a prime, the Kronecker indices of $\sigma(p)$ are given by

$$(n_1(p), n_2(p)) = \begin{cases} (2,0) & \text{if } p \text{ is even,} \\ (1,1) & \text{if } p \text{ is odd.} \end{cases}$$

Consider the monic polynomial, $p(s) = s^2 + 1$, noting that neither Theorem 5.2 nor Corollary 5.5' apply. In fact, the system of Diophantine equations

$$\mathrm{tr}(F + GK) = 1 \tag{6.4a}$$

$$\det(F + GK) = 1 \tag{6.4b}$$

has no solution

$$K = \begin{pmatrix} x & y \\ w & v \end{pmatrix} \in M_2(\mathbb{Z}) \,.$$

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To see this, substitute a solution of (6.4a) into (6.4b), obtaining

$$100w^2 + 10w - 1 = (-y)(3 + 10v) \tag{6.4)'$$

or

$$(100w^2 + 10w - 1) \equiv 0 \bmod (3 + 10v) \tag{6.5}$$

Now (6.5) has a solution if and only if the discriminant $\Delta = 500$ is a square modulo $|3 + 10v|$, if and only if 5 is a square modulo $|3 + 10v|$. It can be shown using Quadratic Reciprocity that this occurs if, and only if, 3 is a square modulo 5, contrary to fact.

One may construct a similar counterexample over $\mathbb{R}[x]$; in contrast, all reachable 2×2 systems over $\mathbb{C}[x]$ are coefficient assignable ([8]). More generally, if R is a P.I.D., then following the matrix operations in [36], (F,G) may be taken (modulo state feedback) in the form

$$F = \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$$

where $(b,c) = 1$. Now following [8], if $f(s) = s^2 - \alpha s + \beta$, then arguing as above leads to the condition: if there exists $K \in M_2(F)$ satisfying

$$\det(sI - F - GK) = f(s), \tag{6.6}$$

then $\alpha^2 - 4\beta$ is a square mod(p), for any irreducible p dividing $b + cv$. That is, the solvability of (6.6) implies

the monic $f(s)$ splits modulo p, for each prime $p|b + cv$, (6.7)

This is in harmony with Morse's Theorem--where $f(s)$ is assumed to split over R. And, if R has the property for $p \in \max(R)$, $\mathrm{char}(R/(p)) \neq 2$, then (6.7) is also sufficient, giving a refinement of Morse's Theorem in the 2×2 case. It appears that the general case lies much deeper. Moreover, the criterion (6.7) involve the unknown quantity v and for this reason is not always easy to apply. As a final remark, we may include $\mathbb{Z}$

but, of course, special care must be taken in including the prime $p = 2$.

§7. STABILIZABILITY OF PARAMETERIZED FAMILIES, DELCHAMP'S LEMMA

Consider a parameterized family of linear systems,

$$\dot{x}(t) = F(\lambda)s(t) + G(\lambda)u(t), \quad x(0,\lambda) = x_0(\lambda), \tag{7.1}$$

real analytic in $\lambda \in \Lambda$, an open subset of $\mathbb{R}^N$ (although we could, of course, take Λ to be a real analytic submanifold of $\mathbb{R}^N$). We seek a real analytic $K(\lambda)$ such that the force-free closed-loop system

$$\dot{x}(t) = (F(\lambda) + G(\lambda)K(\lambda))x(t) \tag{7.2}$$

is asymptotically stable, for all λ. It is natural to find such a $K(\lambda)$ by solving a variational problem, in this case a quadratic optimal control problem leading to an algebraic Riccati equation for $K(\lambda)$. Now, a lemma of D. Delchamps' applies the implicit function theorem--on the manifold of controllable pairs (F,G)--to show that such a $K(\lambda)$ can be chosen real analytic in λ. This also applies to C^k-families, for $k \geq 1$, and by a little global reasoning we extend this to continuous families as well. We begin by giving an exposition of these ideas.

First, suppose $\Lambda = \mathbb{R}^N$ and $\sigma = (F(\lambda),G(\lambda))$ is controllable for all λ. If the Kronecker indices $(n_1(\lambda),\ldots,n_m(\lambda))$ of σ are constant, then we can place the spectrum of $F(\lambda)$ arbitrarily (modulo the constraint that the eigenvalues form a self-conjugate set) and thus, in particular, find a stabilizing $K(\lambda)$ as in (7.2). In order to motivate what follows, we offer another proof of this fact. Set

$$C(n,m) = \{(F,G):(F,G) \text{ is controllable}, \ F^{n\times m}, G^{n\times m}\},$$

and denote the state feedback group of $F_s = F_s(\mathbb{R})$. Thus, one has a real algebraic group action

$$\mathcal{F}_s \times C(n,m) \to C(n,m) \tag{7.3}$$

with finitely many orbits

$$\mathcal{O}_{(F,G)} = \{g(F,G): g \in \mathcal{F}_s\},$$

parameterized by the partition

$$\sum_{i=1}^{m} n_i = n, \quad n_1 \geq n_2 \geq \ldots \geq n_m \geq 0$$

of the McMillan degree n into the Kronecker indices. Now, a real analytic family $\sigma = (F(\lambda), G(\lambda))$, for $\lambda \in \Lambda$, is given by a real analytic function,

$$f_\sigma : \Lambda \to C(n,m) \ , \tag{7.4}$$

and conversely. In this context, to say that the Kronecker indices are constant is to say that the function

$$f_\sigma : \Lambda \to \mathcal{O}_{(F,G)} \subset C(n,m)$$

has its range in a single orbit of the action (7.3). Thus, if

$$\mathcal{H}_{(F,G)} = \{g \in \mathcal{F}_s : g(F,G) = (F,G)\}$$

one has a real analytic map, for (F,G) a point on $\overline{\mathcal{O}}$,

$$f_\sigma : \Lambda \to \mathcal{F}_s / \mathcal{H}_{(F,G)} \ . \tag{7.5}$$

A study of the topology of $\mathcal{F}_s / \mathcal{H}_{(F,G)}$ was begun in [4], where (for example) formulae, in terms of the Kronecker indices, for the number of connected components and for the dimension of $\mathcal{F}_s / \mathcal{H}_{(F,G)}$ are given. Here, we need only know that $\mathcal{O}$ is a homogeneous space which is the base of an $\mathcal{H}_{(F,G)}$ fiber bundle $\mathcal{F}_s \to \mathcal{F}_s / \mathcal{H}_{(F,G)}$. In particular, f_σ in (7.5) induces an $\mathcal{H}_{(F,G)}$-bundle on Λ, viz.

$$\mathcal{L}_\sigma^* \mathcal{F}_s \to \Lambda \tag{7.6}$$

and to say (7.6) is trivial is, of course, to say that by using real analytic state feedback one can put $(F(\lambda), G(\lambda))$ into a canonical form which is independent of λ. For example, by choosing $\mathcal{H}$ to be the isotropy subgroup of the Brunovsky normal form one obtains a global Brunovsky form. In any case, coefficient assignability follows from the result over a field. Assuming $\Lambda = \mathbb{R}^N$, such a bundle is trivial and therefore $(F(\lambda), G(\lambda))$ is coefficient assignable over the ring $R = C^\omega(\mathbb{R}^N)$.

With this notation in mind, consider the variational problem on $C(n,m)$, where $L(\sigma)$ is an arbitrary real analytic, positive definite form in $\sigma \in C(n,m)$: minimize the functional

$$\eta = \int_0^\infty (s'(t)L(\sigma)s(t) + u(t)u'(t))dt$$

along trajectories $(s(t),u(t))$ of the system σ,

$$\frac{dx}{dt}(t) = Fx(t) + Gu(t) \quad ,$$

initialized at some (fixed) real analytic state vector, $x_0 = x_\sigma$.

It is well-known that for a single system $\sigma = (F,G)$ the minimizing control is given by

$$u_\sigma(t) = -G'K(\sigma)\exp\{(F - GG'K(\sigma))t\}\, x_\sigma \quad ,$$

where K_σ is the unique positive definite solution to algebraic Riccati equation

$$F'K(\sigma) + K(\sigma)F - K(\sigma)GG'K(\sigma) + 4(\sigma) = 0 \ . \qquad (7.7)$$

D. Delchamps has shown me a proof that $K(\sigma)$ is real analytic in σ, we only need consider the case $L(\sigma) = I$. In this case, V = {positive definite symmetric forms on $\mathbb{R}^n$} and consider the real analytic map π

$$C(n,m) \times V \to C(n,m)$$

restricted to the subvariety $X = \{(\sigma,K)$ satisfying (7.7)$\}$.

<u>Lemma 7.1</u> (Delchamps) X *is a submanifold and* $\pi|_X$ *is a submission, which* 0-dimensional *fibers. In particular,* π *is a real analytic diffeomorphism with inverse* $K(\sigma) = (\sigma, K_\sigma)$.

Now consider the universal family of systems, $\sigma = (F(\sigma),G(\sigma)) \in C(n,m)$, parameterized by the real analytic manifold $C(n,m)$. Since for each fixed σ, the choice of state feedback $\tilde{K}(\sigma) = G'(\sigma)K(\sigma)$ renders the closed-loop system

$$\dot{x}(t) = (F(\sigma) + G(\sigma)\tilde{K}(\sigma))$$

asymptotically stable Delchamps Lemma implies the existence of a stabilizing gain, for all σ, analytic in σ. In particular, if $\Lambda \subset C(n,m)$ is a submanifold then restricting $\tilde{K}$ one obtains a stabilizing gain for $(F(\lambda),G(\lambda))_{\lambda\in\Lambda}$ analytic in λ. More generally, __Proposition 7.2__ *If* $(F(\lambda),G(\lambda))$ *is* C^k *in* Λ, $k \geq 0$, *then there exists a gain* $\tilde{K}(\lambda)$, C^k *in* Λ, *for which the closed-loop system*

$$\dot{x}(t) = (F(\lambda) + G(\lambda)\tilde{K}(\lambda))\, x(t) \tag{7.8}$$

is asymptotically stable, for all λ. *In fact,* $\tilde{K}$ *is a function of the system* $(F(\lambda),G(\lambda))$.

__Proof.__ $(F(\lambda),G(\lambda))$ defines a C^k-function

$$f:\Lambda \to C(n,m)$$

as in (7.4). By composing the real analytic gain $\tilde{K}(\sigma)$ with f one obtains a C^k-gain, rendering (7.8) stable for all λ.

__Remarks.__

1. What is surprising here is that the C^0 case comes out so easily, indeed much more is true--for example, similar conclusions hold for Lipschitz continuous functions or L^∞ functions on a finite measure space, by applying the Gel'fand representation to the Banach algebra $L^\infty(X)$ ([12]). In fact, similar cuestions for ℓ^1 arise in recent work by E. Kamen on half-plane digital filters.

2. D. Delchamps proved a more general form of Lemma 7.1 in order to construct a metric, the Riccati metric, on the state bundle of the moduli space $\{(F,G,H)\}/GL(n,\mathbb{R})$ and to study its properties. Some of his work will appear in the proceedings of this conference, published by the AMS.

3. Constancy of the Kronecker indices is, of course, a very stringent assumption, and it is interesting to study the limiting behavior of the $(n_i(\lambda))$. Thus, if the $(n_i(\lambda))$ are generically constant, then

$$\mathcal{F}_\sigma:\Lambda \to \text{Closure}(\mathcal{O}_1) \subset C(n,m)$$

where $\mathcal{O}_1$ is the orbit corresponding to the generic value of the $(n_i(\lambda))$. As one can see by examining the matrices $(F(\lambda),G(\lambda))$, the partition $(\tilde{n}_i)$ occurs as a limit, or specialization, of the partition (n_i) only if $(\tilde{n}_i) \geq (n_i)$ in the Rosenbrock ordering

$$\tilde{n}_1 \geq n_1,\ \tilde{n}_1 + \tilde{n}_2 \geq n_1 + n_2, \ldots, \tag{θ}$$

as one may observe in Example 6.1. From the vector bundle point

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of view, this illustrates the result of Shatz ([43]) that the Grothendieck decomposition of W_σ rises in the Harder-Narasimhan ordering under specialization ([43]). It has been proven (independently) by Hazewinkel, Kalman, and Martin that $\mathcal{O}_{\tilde{n}}$ Closure $(\mathcal{O}_n)$ iff $\tilde{n} \geq n$ in the ubiquitous, or preferably the natural, ordering (*).

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MULTIVARIABLE NYQUIST CRITERIA, ROOT LOCI, AND POLE PLACEMENT: A GEOMETRIC VIEWPOINT

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Abstract

Classical control theory is concerned with the topics in our title in the context of single-input/single-output systems. There is now a large and growing literature on the extension of these ideas to the multi-input/multi-output case. This development has posed certain difficulties, some due to the intrinsic nature of the problem and some, we would argue, due to an inadequate reflection on what the multivariable problem calls for. In this paper we describe what seems to us to be the natural multivariable analogues of these concepts from classical control theory. A rather satisfactory generalization of the Nyquist Criterion will be described, and a clear analog of the asymptotic properties of the root locus will be obtained in the "multi-parameter" case. However, an example is given which illustrates the quite surprising fact that the root locus map is not always continuous at infinite gains. This calls for a new ingredient, a compactification of the space of gains, and perhaps the most interesting new feature in this circle of ideas comes in the area of pole placement. This problem is difficult in the multivariable case, but by establishing a correspondence with a classical set of problems in geometry we are able to understand its main aspects and to derive results on pole-placement by output feedback over the real field.

1. Introduction

It is commonly felt that the standard ideas involved in the classical techniques of root locus and Nyquist stability do not have natural generalizations to the multivariable case and that although partial generalizations of various kinds exist, those methods which are most useful are often the most ad hoc. In this paper we want to develop this circle of ideas emphasizing on one hand that there does exist one, rather natural (from a mathematical point of view), generalization of the Nyquist stability criterion and one rather natural generalization of root locus. Unfortunately, these mathematical generalizations require constructions in higher dimensional spaces and so some of the graphical appeal is lost. However, by starting with the right "pure" generalization it is easier to understand what is being gained and lost when one adopts one or more of the somewhat specialized techniques such as one finds in the literature of multivariable design.

What we will show here is that the natural analog of the Nyquist locus is a curve which is plotted not in the complex plane (or the Riemann sphere) but rather in a certain "Grassmann space" consisting of m-dimensional subspaces of an $m+p$ dimensional space where m is the number of inputs and p the number of outputs. Thus, this curve is a curve in an mp dimensional space. A feedback gain k corresponds to a choice of p-dimensional subspace, i.e., a point in a dual Grassmann space. The analog of the Nyquist locus passing through the point $-1/k$ is that of the curve of m-dimensional subspaces intersecting the p-dimensional subspace gr(k) in a nontrivial way. Using some elementary ideas from algebraic geometry, we are able to state a generalized Nyquist theorem relating the number of poles in the right half-plane after feedback to the number before feedback and to a winding number.

Root-locus theory also fits into this analysis in a natural way. With m and p as above, the gain is thought of as a point in the space of p-dimensional subspaces of an $(m+p)$-dimensional space. The root locus itself is the point set consisting of all possible closed loop poles. We show that if the number of gains $(=mp)$ is less than or equal to the number of poles $(=n)$ then there is--generically--a version of the asymptotic $(k \to \infty)$ analysis which one does in single variable root locus theory. However, even in this case there is not just one "infinite gain" and this, in part, explains the controversial nature of what a zero of a multivariable system should be.

Both the Nyquist ideas and the root-locus ideas have recently proved useful in understanding the pole placement problem for output feedback. We discuss this problem from the point of view outlined above and give some new results based on earlier work done by algebraic geometers in connection with the Schubert calculus.

2. The Nyquist Locus

The point of view we use here results in a very natural and clean generalization of the Nyquist criterion. However, it does involve the geometry of the so-called Grassmann manifolds in an essential way. Since this is essential, and at the same time, not yet too familiar to many people working in automatic control, we begin with some background notation and ideas.

Let $\mathbb{C}^n$ denote the set of all complex n-tuples, regarded as a vector space in the obvious way. The set of all complex lines in $\mathbb{C}^n$ can be thought of as equivalence classes $[x_1,\ldots,x_n]$ of points in $\mathbb{C}^n$-{0} with $(x_1,x_2,\ldots x_n)$ being equivalent to $(y_1,y_2,\ldots,y_n)$ if and only if there exists $\alpha \in \mathbb{C}$-{0} such that $\alpha x = y$. This is called n-1 dimensional *projective space* and is denoted by $\mathbb{P}^{n-1}$. Likewise, we can consider the set of all complex two-dimensional subspaces in $\mathbb{C}^n$. These can be identified with equivalence classes of pairs of linearly independent vectors in $\mathbb{C}^n$, $\{x,x'\}$ whereby two pairs $\{x,x'\}$ and $\{y,y'\}$ are regarded as being equivalent if they span the same two-dimensional subspace. More generally, we may consider equivalence classes of p linearly independent vectors in $\mathbb{C}^n$, say $\{x^1,x^2,\ldots,x^p\}$, with the understanding that two such sets of vectors are equivalent if and only if they span the same subspace of $\mathbb{C}^n$. This set of p-planes in n-space will be denoted by Grass(p,n). Of course, if $p = 1$ we recover our previous construction, Grass(1,n) $\simeq \mathbb{P}^{n-1}$. Grass(p,n) actually admits the structure of an analytic manifold and also that of a nonsingular algebraic variety.

This is the Grassmann manifold of p planes in n-space and we will need to concern ourselves with its geometry.

To being with, the dimension of Grass(p,n) is $np-p^2$. To see this, think of Grass(p,n) as being a collection of equivalence classes in the space of n by p matrices of rank p with two such being equivalent if for some nonsingular p by p matrix we have

$$M_1 P = M_2 \quad .$$

Since the n by p matrices of full rank form an np-dimensional manifold, and since the equivalence relation identifies with one point a p^2-dimensional family of points, we see that Grass(p,n) is a manifold of dimension $np-p^2$. A second important point is the inherent duality between Grass(p,n) and Grass(n-p,n). The formula

$$\dim \text{Grass}(n-p,n) = n(n-p) - (n-p)^2 = np-p^2 = \dim \text{Grass}(p,n)$$

suggests a possible identification between these spaces. In fact, they do define the same abstract analytic manifold and Grass(n-p,n) is called the dual of Grass(p,n). The reason for this terminology will become apparent: it turns out that a point $x \in$ Grass(p,n) canonically determines a hypersurface $\sigma(x)$ in Grass(n-p,n).

In ordinary calculus a curve is usually a mapping from some interval $I \subset \mathbb{R}$ to some real manifold M. For our present purposes it seems best to take the viewpoint found in algebraic geometry and to call an analytic mapping of any Riemann surface into a (complex) analytic manifold a *curve*. Of course, this means that a curve is an object of real dimension 2 (or complex dimension one). For example, any rational function gives us a curve since it maps the Riemann sphere into itself. Complex, nonsingular, algebraic varieties are special kinds of complex analytic manifolds which enjoy the property that they can be covered by coordinate charts such that coordinates in overlapping meighborhoods are related by rational maps. An *algebraic curve* is then a mapping of some Riemann surface into an algebraic variety which can be defined by rational functions.

Classically, the Nyquist locus is defined as the image of the imagainary axis in $\mathbb{C}$ under the mapping ϕ_g: $\mathbb{C} \to \mathbb{C}$ defined by

$$\phi_g(s) = g(s)$$

There is, however, an alternative way of thinking about this which more readily suggests the generalization we want to use. Consider the fact that $g(s)u = y$, which can be written as

$$[g(s),-1] \begin{bmatrix} u \\ y \end{bmatrix} = 0$$

In this form, it defines a mapping of the complex plane $\mathbb{C}$ into the set of (complex) lines in $\mathbb{C}^2$ according to the rule

$$s \mapsto \mathrm{Ker}[g(s),-1]$$

Moreover, if we add a point at infinity to $\mathbb{C}$ to get the Riemann sphere then this same relation gives a mapping of the Riemann sphere into the space of (complex) lines in $\mathbb{C}^2$, i.e., a mapping of the Riemann sphere into $\mathbb{P}^1$. Incidentally, and we will make use of this point of view later, this makes it clear that the Riemann sphere and the complex projective space $\mathbb{P}^1$ are the same space.

The basic multivariable feedback equations

$$G(s)u = y \quad ; \quad u = -Ky$$

which we prefer to write as

$$\begin{bmatrix} G(s) & -I_p \\ I_m & K \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

have a solution if and only if the kernels of $[G(s),-I_p]$ and $[I_m,K]$ intersect in a nontrivial way. Following the innovative paper of Hermann and Martin [11], we now define the *Nyquist locus* of a p by m transfer function $G(s)$ to be the points, on the algebraic curve in Grass$(m,m+p)$ given by

$$s \mapsto \mathrm{Ker}[G(s),-I_p] \quad ,$$

which are the images of points on the imaginary axis, Re $s = 0$. This conclusion has the distinct advantage that the Nyquist contour is the image of the closed imaginary axis, which is a circle in $\mathbb{P}^1$, in a space where the transfer function takes its values, viz., the Grassmannian Grass$(m,m+p)$. It is important to notice that in general this space is formally different from, but dual to, the space Grass$(p,m+p)$ where the gains live (i.e., except when $m = p$).

Remark. Note that the curve in Grass(m,m+p) defined by

$$s \mapsto \text{Image} \begin{bmatrix} I_m \\ G(s) \end{bmatrix}$$

is the same as the curve defined above, since

$$\begin{bmatrix} u \\ y \end{bmatrix} \in \text{Image} \begin{bmatrix} I_m \\ G(s) \end{bmatrix} \Leftrightarrow [G(s), -I_p] \begin{bmatrix} u \\ y \end{bmatrix} = 0$$

As in many aspects of linear algebra, we find it convenient for some purposes to work with kernels and for other purposes to work with spans. In particular, to get our definition to specialize to the usual Nyquist locus we need only choose to represent lines in $\mathbb{C}^2$ by

$$\text{span} \begin{bmatrix} 1 \\ g(s) \end{bmatrix} .$$

In the paper cited above [11], Hermann and Martin interpreted the McMillan degree and the Kronecker indices of systems in terms of this algebraic curve. In the next section we show how these ideas lead to a clean generalization of the Nyquist criterion, and in Section 6 we will use a geometric reinterpretation of the pole placement problem based on these ideas.

Finally, these ideas also play an important role in our study of root loci since, at a very primitive level, the Grassmannian setting allows us to precisely define what we mean by an infinite gain and to study the corresponding questions about asymptotic and limiting behavior in the presence of compactness. This description of an infinite gain is new in the control theory literature and is important in all that follows. Therefore, we will devote the remainder of this section to an exposition (see [4]) of this circle of ideas.

Since a real gain K may be regarded as a point $K \in \mathbb{R}^{mp}$, the real vector space of matrices, it is perhaps at first glance tempting to use the usual Euclidean picture of a single point at infinity, with the convention that whenever $||K|| \to \infty$, K approaches infinity. If $mp > 1$, this is somewhat unnatural, since for example,

$$K_\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} , \quad \tilde{K}_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$$

both tend to infinity in this sense but, as relations $u = Ky$ between outputs and inputs impose different limiting behavior on the closed loop systems.

Explicitly, $u = K_\lambda y$ is the linear relation

$$\begin{matrix} u_1 = y_1 \\ u_2 = y_2 \end{matrix} \quad \text{or} \quad \begin{matrix} (1/\lambda)u_1 = y_1 \\ (1/\lambda)u_2 = y_2 \end{matrix}$$

and, therefore, as $\lambda \to \infty$, K_λ approaches the linear constraint

$$y_1 = y_2 = 0$$

among the input-output variables u_1, u_2, y_1, and y_2. In Section 5, a physical interpretation of such limiting constraints is offered, but for now it is important to observe that, as linear constraints among the input-output variables, the families K_λ and $\tilde{K}_\lambda$ have different asymptotic behavior. For, as the reader can check, as $\lambda \to \infty$, $\tilde{K}_\lambda$ approaches the linear relation

$$u_2 = y_2 = 0 \quad .$$

It is therefore desirable to allow for many points at infinity, precisely so as to account for differences in the limiting behavior of the constrained relations $u = K_\lambda y$. With this basic consideration in mind, it is natural to study the set--or as it turns out, the manifold--of linear relations between u and y. More geometrically, setting $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$, the gain relation $u = Ky$ is represented by its graph, viz., the p-plane

$$gr(K) = \{(Ky, y) : y \in Y\} \subset U \oplus Y \quad .$$

Of course, not every p-plane in $U \oplus Y$ space can be represented as a graph of a linear function $K: Y \to U$, since such a p-plane is necessarily complementary to U. Thinking of a p-plane V as a point in the real Grassmannian $\text{Grass}_{\mathbb{R}}(p, m+p)$, we have two alternatives:

(i) V is the graph of a linear function (or gain) $K: Y \to U$,

(ii) $V \in \sigma(U) = \{W | \dim(W \cap U) \geq 1, \ W \in \text{Grass}_{\mathbb{R}}(p, m+p)\}$.

In the first case $V \in \mathbb{R}^{mp}$ and in the second case V lies in $\sigma(U)$ --the set of infinite gains. It should be clear geometrically that $\mathbb{R}^{mp} = \text{Grass}(p,m+p)-\sigma(U)$ is dense and that every limiting value $V \in \sigma(U)$ represents a linear constraint between u and y, just as in the example above.

For any m-plane W in (m+p)-space we can define the subset $\sigma(W)$ of $\text{Grass}_{\mathbb{R}}(p, m+p)$ defined by

$$\sigma(W) = \{V \mid \dim(W \cap V) \geq 1 \ ; \ W \in \text{Grass}_{\mathbb{R}}(m, m+p)\} \ .$$

Such a hypersurface is known as a Schubert hypersurface (associated to W). In this language $\sigma(U)$ is the Schubert hypersurface of infinite gains [4], [15]. Similar remarks apply to complex gains in the setting of complex p-planes in $\mathbb{C}^{m+p}$, i.e., in $\text{Grass}_{\mathbb{C}}(p,m+p)$.

We shall now eliminate the subscripts $\mathbb{R}$ and $\mathbb{C}$, since whether we are working with real planes or complex planes should be clear from the context.

Example. If $m=p=1$, we are proposing the consideration of $\text{Grass}(1,2) = \mathbb{P}^1$, the space of lines through 0 in $\mathbb{R}^2$, which is a circle since each line is parameterized by the angle it makes with the Y-axis. Note that in this special case $\sigma(U) = \{U\}$ is just the line $\{U\}$ itself, as is shown in the figure below.

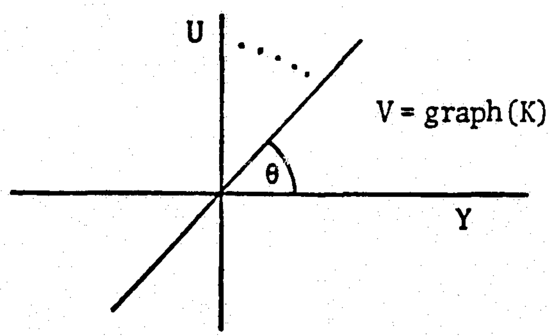

Figure 2.1

Therefore, if $m = p = 1$, $\mathbb{P}^1 - \sigma(U) \approx \mathbb{R}$ and $\sigma(U)$ represents the point at infinity, in the standard euclidean sense.

More generally, if $m = p$ and K is invertible then it is easy to check that the graph of the function K_λ: $\lambda K y = u$, approaches {U} as λ goes to infinity. This implies that the closed loop poles approach the open loop zeroes, as is well known. However, $\dim \sigma(U) = mp-1$ and therefore there exists (if $mp > 1$) a continuum of possible limiting values in $\sigma(U) - \{U\}$, each corresponding to limiting values of more general 1-parameter families K_λ. As we shall see in Sections 4 and 5, for non-degenerate G(s) to each of these infinite gains one can still assign an unordered n-tuple of limit points on the Riemann sphere and obtain asymptotic expressions for the corresponding root loci.

3. The General Nyquist Criterion

The basic equality which lies at the heart of Nyquist's application of the principle of the argument to study stability is

$$\mu = \rho + \nu$$

where μ is the number of closed-loop poles in the right half-plane, ν is the number of open loop poles in the right half-plane, and ρ is a winding number -- the number of times the Nyquist locus encircles the $-1/k$ point in the clockwise direction. Although special techniques are readily developed to enable one to handle the case where the Nyquist locus is unbounded, in its primitive form one assumes that the Nyquist locus avoids two points, the point at ∞ and the point $-1/k$. In the multivariable version we present here μ and ν will have virtually the same meaning; the Nyquist locus itself will be as defined above. The definition of the winding number ρ will now be given.

As we have seen the feedback stability problem is concerned with the pair of equations

$$\begin{bmatrix} G(s) & -I_p \\ I_m & K \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3.1)$$

Since ambiguity arises when for some value of $s=i\omega$ this pair has a nontrivial solution we must eliminate this possibility. Regarding the kernel of $[I_m, K]$ as a point in Grass$(p,m+p)$ suggests that we introduce the Schubert hypersurface

$$\sigma(\mathrm{Ker}[I_m,K]) = \{W \mid W \in \mathrm{Grass}(m,m+p) : W \cap \mathrm{Ker}[I_m,K] \neq 0\} \quad ,$$

since in order to have a nontrivial solution of the above equations we need to have an intersection between the Nyquist locus and $\sigma(\mathrm{Ker}[I_m,K])$.

Our first result describes pole-placement by output feedback in terms of intersection properties.

Theorem 3.1: The closed-loop poles corresponding to a feedback gain matrix K occur at the points s_i where the algebraic curve $\mathrm{Ker}(G(s),-I_p) \subset \mathrm{Grass}(m,m+p)$ intersects the Schubert hypersurface $\sigma(\mathrm{Ker}[I_m,K])$.

Proof: This is simply a reinterpretation of equation (3.1) above. The closed-loop poles are values of s where (3.1) has a nontrivial solution but this occurs where $\mathrm{Ker}[G(s),-I_p]$ and $\sigma[\mathrm{Ker}(I_m,K)]$ intersect.

We now turn to the Nyquist criterion itself. In order to make sense out of the concept of a winding number we need to be sure we are working in a space in which the equivalence classes of homotopic closed curves can be put in one to one correspondence with the integers in such a way as to preserve the basic idea that traversing a closed curve Γ and then a closed curve Γ' should result in a winding number which is the sum of the respective winding numbers $\nu(\Gamma)+\nu(\Gamma')$ and that traversing a closed curve Γ in the opposite direction should result in the negative of the winding number associated with the original orientation. More precisely, the fundamental group of the space should be isomorphic to $\mathbb{Z}$, the additive group of the integers. We use π_1 to denote the fundamental group of a space (see [23], [9]).

Lemma 3.2: For i=1,2, let $\sigma_i \subset \mathrm{Grass}(p,p+m)$ be Schubert hypersurfaces of the form described above derived from m-planes W and W' which satisfy $\dim(W \cap W') = \min(0,p-m)$. (That is, they intersect on a subspace of smallest possible dimension.) Then the fundamental group of $\mathrm{Grass}(p,p+m)-\sigma_1 \cup \sigma_2$ is isomorphic to $\mathbb{Z}$.

This lemma is, except for language, the same as the following one which is the form we will actually use in the proof of the Nyquist Criterion. The notation $G\ell(n,\mathbb{C})$ refers to the set of all nonsingular n by n complex matrices--the general linear group of dimension n.

Lemma 3.2 Let K be a fixed m by p complex matrix of rank r. Then the set of all p by m matrices G such that

$$\det \begin{bmatrix} G & -I_p \\ I_m & K \end{bmatrix} \neq 0$$

is a connected complex manifold diffeomorphic to $G\ell(r,\mathbb{C}) \times \mathbb{C}^{mp-r^2}$. In particular, if $r \geq 1$, the fundamental group of this manifold is isomorphic to $\mathbb{Z}$.

Proof of Lemma 3.2: The proof follows from the fact that the above matrix is invertible if and only if $KG + I_m$ is invertible or, equivalently, $GK + I_p$ is invertible. Let P and Q be invertible matrices such that

$$PKQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \overset{\text{def}}{=} \Sigma$$

Then $\det(GK + I_p) = \det(Q^{-1}GP^{-1}\Sigma + I_p)$. Thus we see that the upper left r by r minor of $Q^{-1}GP^{-1}$ is such that when added to I_r the result is a nonsingular matrix, the other components of $Q^{-1}GP^{-1}$ being completely arbitrary. Thus we see that the admissible G's are in one to one correspondence with the choice of a nonsingular r by r matrix and a point in a $mp-r^2$-dimensional vector space. It is well known [7] that π_1 of $G\ell(r,\mathbb{C})$ is isomorphic to $\mathbb{Z}$ for every $r \geq 1$. In fact, the determinant mapping of $G\ell(m,\mathbb{C})$ into the nonzero complex numbers $\mathbb{C}^*$, given by det: $G\ell(r,\mathbb{C}) \to \mathbb{C}^*$, gives a means of determining when two closed curves in $G(m,\mathbb{C})$ are homotopic--they are if the image curves encircle zero in $\mathbb{C}^*$ the same number of times.

We adopt the following convention. If Γ is a curve in $\text{Grass}(m,m+p)-\sigma_1 \cup \sigma_2$ as in the lemma, if $\sigma_1 = \sigma[\text{Ker}(I_m,0)]$ and $\sigma_2 = \sigma[\text{Ker}(I_m,K)]$, then we define the winding number of Γ with respect to σ_2 to be the net increase in argument of

$$\det \begin{bmatrix} G & -I_p \\ I_m & K \end{bmatrix} = \det(GK + I_p) = \det(KG + I_m)$$

as ω increases from $-\infty$ to ∞, divided by 2π.

With these preliminaries in hand, we now give a graphical test for stability which generalizes the classical result of Nyquist [18]. Of course it is important to observe that when counting poles in the right half-plane one counts the multiplicity of a given pole according to its contribution to the McMillan degree of $G(s)$.

Theorem 3.3: Suppose that $G(s)$ is a proper matrix valued rational function with no poles on Re $s = 0$ and suppose that the Nyquist locus does not intersect the Schubert hypersurface $\sigma[\text{Ker}(I_m,K)]$ defined by the gain matrix K. Let μ be the number of closed loop poles in Re $s > 0$ and let ν be the number of poles of $G(s)$ in Re $s > 0$. Then

$$\mu = \rho + \nu$$

where ρ is the number of times the Nyquist locus encircles the Schubert hypersurface $\sigma[\text{Ker}(I_m,K)]$ in the positive direction.

Proof: To begin with we observe that for a given minimal triple (A,B,C) and a given K we have ($G(s) = C(Is-A)^{-1}B$)

$$\det(G(s)K+I_p) = \det(KG(s)+I_m)$$
$$= \det(Is-A-BKC)/\det(Is-A)$$

Moreover, by the principle of the argument

$$\#\text{ zeros } [\det(GK+I)] - \#\text{ poles } [\det(GK+I)] = \text{net change in argument of } \det(GK+I)/2\pi \quad (3.2)$$

along the $j\omega$ axis, where on the left hand side only contributions arising in the right half-plane. Thus we see that the number closed-loop poles in Re s > 0 minus the number of open loop poles in Re s > 0 is the change in argument in det(GK + I). But from the proof of Lemma 3.2', this is the number of encirclements the Nyquist locus makes around the Schubert hypersurface $\sigma(\text{Ker}[I_m,K])$.

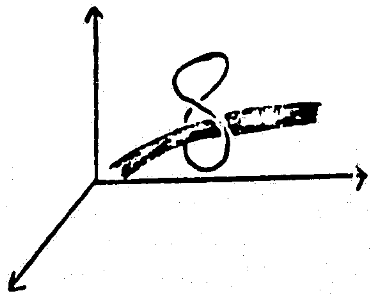

Figure 3.1: Illustrating the Nyquist Criterion

Remark: This result retains what is in our opinion the basic assets of the classical Nyquist criterion, viz. the result involves a fixed curve obtained from the open loop transfer function which does not need to be changed with changes in the gain. However, formula (3.2) does represent the easiest route in calculating the appropriate winding number and has been used extensively in the square case with scalar gain $K_\lambda = \lambda I$ (see [1], [9], [20]) and in the nonsquare case by Callier and Desoer ([5]).

4. Feedback, Nondegeneracy, and Zeros

Changes of basis in input space, change of basis in output space, and the operation $u \mapsto u - Ky$ (feedback), taken together yield a group of operations on transfer functions. We call this the *feedback group*. Triples (T,S,K) act on $G(s)$ to give

$$TG(s)(KG(s)+S)^{-1} \quad .$$

We denote the group by $\mathcal{F}$ and note that it may be represented as a subgroup of $G\ell(\mathbb{R}^{m+p}) \subset G\ell(\mathbb{C}^{m+p})$, given in block form by

$$\mathcal{F} = \begin{bmatrix} T & 0 \\ K & S \end{bmatrix} \quad ; \quad T \in G\ell(\mathbb{R}^p),\ S \in G\ell(\mathbb{R}^m)$$

Now, any transformation in $G\ell(\mathbb{R}^{m+p})$ acts by change of basis on an m plane W to give another m plane W', and in this way the subgroup $\mathcal{F}$ acts on Grass(m,m+p)--as the subgroup which fixed the m-plane $U \subset Y \oplus U$. It is important to note [8] that,

$$\begin{bmatrix} I_p & 0 \\ K & I_m \end{bmatrix} \begin{bmatrix} G(s) \\ I_m \end{bmatrix} = \begin{bmatrix} G(s) \\ I_m + KG(s) \end{bmatrix} , \qquad (4.1)$$

so that the two actions:

(i) K acting on G(s) by output feedback

$$G(s) \mapsto G(s)(KG(s)+I)^{-1}$$

(ii) K acting on G(s) by composition (as in (4.1))

$$G: \mathbb{P}^1 \to \text{Grass}(m,m+p) \xrightarrow{K} \text{Grass}(m,m+p)$$

are the same. That is, given a transfer function G(s) and a gain K, we can either apply the output gain, obtaining the closed loop plant

$$G^K(s) \overset{\text{def}}{=} G(s)(KG(s)+I)^{-1} = (G(s)K+I)^{-1}G(s)$$

and hence a new curve in Grass(m,m + p) corresponding to G^K, or we can take the curve G(s) in Grass(m,m + p), apply the "rotation" (4.1) corresponding to K, and get a new "rotated" curve in Grass(m,m + p). This new curve is the curve corresponding to G^K. In particular, to describe the closed loop poles, we can either rotate first obtaining $G^K(s)$ and intersect with the fixed Schubert hypersurface $\sigma[\text{Ker}(0,I_m)]$, or fix the curve G(s) and intersect with the Schubert hypersurface $\sigma[\text{Ker}(-K,I_m)]$, which we get by "rotating" $\sigma[\text{Ker}(0,I_m)]$ through the inverse "rotation" defined by -K as in Equation 4.1. This gives an alternative proof of Theorem 3.1.

It is also important to notice that F acts on the Schubert hypersurface in Grass(m,m + p) as well; that is, each $F \in F$ transforms the Schubert hypersurface $\sigma(W)$ to the Schubert hypersurface $\sigma(FW)$. For this reason, it is clear that the following definition is invariant under output feedback.

<u>Definition 4.1</u> G(s) is nondegenerate if, and only if, no Schubert hypersurface $\sigma(W)$ in Grass(m,m + p) contains the curve G(s).

Since nondegeneracy plays an important role in what follows, it is therefore worthwhile to derive alternate forms for nondegeneracy. Suppose W is a p-plane in $\mathbb{C}^{m+p}$, so that W is defined as the common zeros of independent linear functionals $\phi_1,\ldots,\phi_m$ on $\mathbb{C}^{m+p}$. Let $g_j(s)$ denote j-th column of the matrix

$$\mathcal{G}(s) = \begin{bmatrix} G(s) \\ I_m \end{bmatrix}$$

Now, to say that the span of $\mathcal{G}(s_0)$ intersects W non-trivially is to say that some non-trivial linear combination, $\sum_{i=1}^{m} a_i g_i(s_0)$, lies in W. That is,

$$\phi_j\left(\sum_{i=1}^{m} a_i g_i(s_0)\right) = 0, \qquad \text{for } j = 1,\ldots,m$$

or, the rational function

$$\Phi(s_0) = \det(\phi_j(g_i))(s_0) = 0 \tag{4.2}$$

Now, a different choice of defining equations changes $\Phi(s)$ by a non-zero multiplicative factor. Since s_0 is arbitrary, nondegeneracy of $G(s)$ is equivalent to

$$\text{for any } \Phi(s), \qquad \Phi(s) \not\equiv 0 \quad . \tag{4.3}$$

A third equivalent form is that for no matrix $[K_1, K_2]$ of rank $\min(m,p)$ is

$$\det \begin{bmatrix} G(s) & -I_p \\ K_1 & K_2 \end{bmatrix} \equiv 0$$

For example, any scalar transfer function is nondegenerate, since for α, β not both 0,

$$\Phi(s) = \alpha G(s) + \beta \equiv 0$$

only if $G(s)$ is constant. However, if $m > 1$, any $m \times m$ diagonal (or even block diagonal) transfer function is degenerate. And, since nondegeneracy is feedback-invariant, any system which may be decoupled is degenerate. For example,

$$G(s) = \begin{bmatrix} 1/(s^2-1) & 0 \\ 0 & 1/(s^2-4) \end{bmatrix}$$

is degenerate. If y_1, y_2, u_1, u_2 denote the coordinates on a column vector in $\mathbb{C}^4$, the choice $\phi_1(v) = y_2, \phi_2(v) = u_2$ leads to

$$\phi(s) = \det \begin{bmatrix} 0 & 0 \\ 1/(s^2-4) & 1 \end{bmatrix} \equiv 0 .$$

On the other hand, the generic 2×2 transfer function with McMillan degree 4 is non-degenerate. Indeed, part of our interest in non-degeneracy stems from this property. If n is the McMillan degree of a system, we have

<u>Theorem 4.2</u> If $mp \leq n$, non-degeneracy is generic. If $mp > n$, then every transfer function is degenerate.

By generic we mean that the set of non-degenerate systems is the complement of a set defined by algebraic equations in the space of minimal realizations $\{(A,B,C)\} \subset \mathbb{R}^{n^2+nm+np}$. Since $G(s) \subset \sigma(V)$ is an algebraic constraint and the space Grass $(p, m+p)$ is compact, degeneracy is defined by algebraic conditions, so in order to prove genericity it is enough to find one $G(s)$ which is non-degenerate. Now suppose V or, equivalently, $\Phi = \{\phi_1, \ldots, \phi_m\}$ is given. To say $G(\mathbb{P}^1) \subset \sigma(V)$ is to say, in particular, that

$$G(s_1), \ldots, G(s_n), G(\infty) \in \sigma(V), \text{ for } s_i \in \mathbb{C}.$$

We may assume the m-planes $G(s_i)$ are in general position, so that each condition

$$\det(\phi_i(g_j))(s_i) = 0$$

places a non-trivial constraint on V. These are, dually, n independent condition on V which lies in a variety, Grass(p,m+p), of dimension mp. The additional constraint,

$$\det(\phi_i(g_j))(\infty) = 0 \quad ,$$

then constrains V to lie on an algebraic subset X of Grass(p,m+p) of dimension,

$$\dim X \leq mp - (n+1) \quad , \tag{4.5}$$

so that X is empty if $mp \leq n$. However, by a special property of Schubert hypersurfaces, which should not be interpreted as a general fact (see [22] p. 57), (4.5) is an equality for planes in general position. Therefore, if $mp > n$, $X \neq \emptyset$ which, together with the following lemma, proves the last statement. In case $m = p = 1$, the lemma asserts that every nonconstant rational function takes on any given value the same number of times.

Lemma 4.3 $\sigma(V)$ either contains the algebraic curve G(s) or intersects it (counting multiplicity) in exactly n points. In the latter case, at least one such point is infinite if, and only if, V is not complementary to U.

Proof: The first part of the lemma can be found in [6]. As for the second part, if V is complementary to U, then V = gr(K), for some $K: Y \to U$, and the points of intersection are the poles of $G^{-K}(s)$ which are all finite frequencies. The converse follows from a duality between

Grass(m,m+p) and Grass(p,m+p) and the inherent duality in the statement "V is not complementary to U." That is, to say

$$V \in \sigma(U) \subset \text{Grass}(p,m+p)$$

is to say $U \in \sigma(V) \subset \text{Grass}(m,m+p)$. Therefore, if V is not complementary to U, then $G(\infty) = U \in \sigma(V)$ is in the intersection $G(\mathbb{P}^1) \cap \sigma(V)$. Q.E.D.

This concept is of great importance in studying asymptotic root loci and pole-placement by output feedback.

Preparatory to our treatment of root-locus, we discuss the question of how to define what one might mean by the zeroes of a multivariable system. Our approach will agree with most other authors in the case where G(s) is square, but in the rectangular case we argue that it is best to focus attention on a locus of all "potential zeroes" rather than a finite set of points. To state our results in a clean way we need a little further notation. If p(s) is a nonzero polynomial of degree $\leq n$, then by $[p(s)]_n$ we understand an equivalence class consisting of all polynomials of the form $\alpha p(s)$ with $\alpha \neq 0$. Since multiplying p(s) by α does not change its zeroes, we see that $[p(s)]_n$ defines a set of n unordered points in the Riemann sphere. (Note that $[0s^2+s+1]_2$, for example, defines the point set $\{\infty,-1\}$.) On the other hand, each equivalence class defines a line in $\mathbb{R}^{n+1}$ (or $\mathbb{C}^{n+1}$) and hence a point in a projective space $\mathbb{P}^n$, namely the line $\text{span}(p_n,p_{n-1},\ldots,p_0)$.

<u>Definition 4.3</u> Let (A,B,C) be a minimal triple. Let $G(s) = C(Is-A)^{-1}B$ be a nondegenerate p by m matrix of proper rational functions having McMillan degree n. If $m \geq p$, then we say that a point $\{[q(s)]_n\}$ is a *right zero polynomial* of G(s) if there exists an m by p matrix K of rank p such

that $q(s) = \alpha \det(G(s)K) \det(Is-A)$ for some $\alpha \neq 0$. We say that $\{[q(s)]_r\}$ is a *left zero polynomial* of G(s) if q(s) divides det(Is-A)det(G(s)K) for all m by p matrices of rank p and q(s) is of maximal degree relative to all polynomials with this property. If $m \leq p$, then we say that $\{[q(s)]_n\}$ is a *left zero polynomial* for G(s) if it is a right zero polynomial for G'(s); we say that $\{[q(s)]_r\}$ is a *right zero polynomial* if it is a left zero polynomial for G'(s).

As remarked above, $[q(s)]_n$ is the same as a point in $\mathbb{P}^n$ and we know that the set of m by p matrices K having rank $p \leq m$ is associated with an equivalence class $[K] = \{KP: P \in G\ell(p)\}$ which defines a point in Grass(p,m). Thus the above definition of left zeros (right zeros) gives us for $m \leq p$ $(m \geq p)$ a mapping of Grass(m,p) to $\mathbb{P}^n$ (Grass(p,m) to $\mathbb{P}^n$). Thus we see that the set of zero polynomials is, except in the case $m = p$, a whole locus of points and not just a finite set.

One desirable property that our definition has is revealed by the consideration of feedback. Suppose $p \leq m$, we say that G(s) is feedback equivalent to $(G(s)K_1 + I)^{-1}G(s)$. Now, if for some K we have

$$\det(G(s)K) \det(Is-A) = \alpha g(s)$$

then an easy calculation shows that the polynomial

$$\det(G(s)K_1 + I)^{-1}\det((Is-A) + BK_1C)\cdot \det G(s)K\cdot =$$
$$\det(G(s)K\cdot\det(Is-A) = \alpha g(s) \quad .$$

<u>Theorem 4.4</u> Let G(s) be nondegenerate, then each left (respectively, right) zero polynomial is invariant with respect to output feedback.

5. Root Loci

In this section it is our aim to lay some of the foundations for a general study of multi input, multi output root loci. Quite specifically, we are interested in the asymptotic behavior of closed loop poles with respect to high gain feedback and, in particular, in making sense of the concept of infinite gain. We note that, through the work of many authors, the situation where one studies a (generic) square transfer function G(s) and the 1-parameter family of gains $(\lambda I: \lambda \in \mathbb{R})$ is well understood and indeed reflects many of the properties familiar in the scalar input-output setting, e.g., the closed loop poles approach the open loop zeros. However, this family of gains is rather special and it is rather widely appreciated that for obvious practical considerations one would prefer a theory of high gain feedback which allows more flexibility in the choice of gain.

To fix the ideas, if G(s) is a transfer function and K is a gain then $\chi(K)$ denotes the unordered n-tuple of closed loop poles, or what is the same the n-vector of coefficients of the closed loop characteristic polynomial. One general question we have in mind is, given a 1-parameter family K_λ of gains describe the limit $\chi(K_\lambda)$ as an unordered n-tuple, and describe the asymptotic behavior of the closed loop poles as $\lambda \to \infty$. In particular, in what sense does this limit exist? For example, if $K_\lambda \to K_\infty \in \sigma(U)$ and $\tilde{K}_\lambda \to K_\infty$, does $\lim_{\lambda\to\infty} \chi(K_\lambda) = \lim_{\lambda\to\infty} \chi(\tilde{K}_\lambda)$?

Explicitly, consider

$$G(s) = \begin{bmatrix} 1/(s^2-1) & 0 \\ 0 & 1/(s^2-4) \end{bmatrix}$$

and the 1-parameter family

$$K_\lambda = \begin{bmatrix} -1 & \lambda \\ 0 & 1 \end{bmatrix} = K_0 + K_1\lambda$$

Now, K_λ and $\tilde{K}_\lambda = K_1\lambda$ approach the same infinite gain K_∞, viz., the linear relation $y_2 = u_2 = 0$ (reflecting the fact that for $\lambda >> 0$ each frequency s behaves approximately like a pole for any input with $u_2(s) \neq 0$). However, the asymptotics are not determined by the highest order term! Indeed,

$$\chi(K_\lambda) \rightarrow [(s^2-2)(s^2-3)], \quad \chi(\tilde{K}_\lambda) \rightarrow [(s^2-1)(s^2-4)] ,$$

while the zeros of G(s) are all infinite. We also note that the 1-parameter families

$$L_\lambda = \begin{bmatrix} 0 & \lambda \\ 0 & 1 \end{bmatrix}, \quad \tilde{L}_\lambda = \begin{bmatrix} -\lambda & \lambda^2 \\ 0 & 0 \end{bmatrix}$$

also approach K_∞, yet

$$\chi(L_\lambda) \rightarrow [s^4-4s^2-3] \quad , \quad \chi(\tilde{L}_\lambda) = [(s^2-4)(s^2-1-\lambda)] \rightarrow [s^2-4]_4 .$$

Roughly speaking, this discontinuity reflects the fact that, in the limit, each $s_0 \in \mathbb{P}^1$ deserves to be called a pole of the "closed loop system with infinite gain." More precisely, given any frequency $s_0 \in \mathbb{C}$, any $\varepsilon > 0$, and any $c > 0$, there exists an input u with $||u(s_0)|| < \varepsilon$ and an $N(c,\varepsilon) >> 0$, such that $||y(s_0)|| = ||G_\lambda u(s_0)|| > c$, for $\lambda > N(c,\varepsilon)$. One cannot satisfy this condition unless $u_2(s_0) \neq 0$, and for this reason the discontinuity of χ at K_∞ is tied up with the degeneracy of G(s). In fact, the equations $u_2 = y_2 = 0$ defining K_∞ in $U \oplus Y$ are precisely the equations used in Section 4 to show that G(s) is degenerate. Moreover, these constraints reflect that fact that, only under the condition $u_2 = 0$ (and therefore $y_2 = 0$), will the limiting "transfer function" $\lim_{\lambda\to\infty} G_\lambda(s)$ yield a finite output from a finite input.

We shall say that asymptotic root loci exist for G(s) provided we can assign unordered n-tuple $\chi(K)$ of points on the Riemann sphere to any gain K (finite or infinite, as in Section 2) so that χ is continuous at each infinite gain.

Theorem 5.1 G(s) is nondegenerate if and only if asymptotic root loci exist, for all infinite gains K_∞. In this case, as $K_\lambda \to K_\infty$ at least one closed loop pole becomes infinite.

Proof: The key to this proof is to assign to each p-plane V in (m+p) space an unordered n-tuple of points on $\mathbb{P}^1$ in an unambiguous way. If $V = \text{graph}(-K)$ for a finite gain K, Theorem 3.1 asserts that

$$\chi(K) = G^{-1}(G(\mathbb{P}^1) \cap \sigma(V))$$

so it is natural to define, for any $V \in \text{Grass}(p, m+p)$,

$$\chi(V) = G^{-1}(G(\mathbb{P}^1) \cap \sigma(V))$$

where V may represent an infinite gain. According to Lemma 4.3, either $G(\mathbb{P}^1) \cap \sigma(V)$ in exactly n points, or $G(\mathbb{P}^1) \subset \sigma(V)$, in which case G is degenerate, by definition.

Therefore, if K_∞ is an infinite gain, then $K_\infty = V$ for some p-plane V, and we may define

$$\chi(K_\infty) = G^{-1}(G(\mathbb{P}^1) \cap \sigma(V)) \quad .$$

And, with this definition, the continuity of χ follows from the continuous dependence of roots on the coefficients of a defining equation. Conversely, it is easy to see that the continuity of χ implies nondegeneracy. The second claim in the theorem follows the second statement in Lemma 4.3. Q.E.D.

In particular, the asymptotic root loci corresponding to the polynomial family of gains, $K_\lambda = K_0 + K_1\lambda +,\ldots,+K_d\lambda^d$, is determined by the asymptotics of the highest order term. This is not, of course, true when G(s) is degenerate. In order to compute this asymptotic value, one must compute $\lim_{\lambda\to\infty} \text{graph}(\lambda K_d))$ in Grass(p,m+p). Let V_1 = image $K_d \subset U$, and V_2 = ker $K_d \subset Y$, thus

$$V(K_d) = V_1 \oplus V_2 = \lim_{\lambda\to\infty} \text{graph}(\lambda K_d) \quad .$$

Corollary 5.3: If $K_\lambda = K_o+K_1\lambda+\ldots+K_d\lambda^d$ is a polynomial family of gains, and if $G(s)$ is non-degenerate, then the asymptotic root loci are given by

$$\lim_{\lambda\to\infty} \chi(K_\lambda) = \lim_{\lambda\to\infty} \chi(\lambda K_d) = \chi(V(K_d)) \quad . \qquad (5.1)$$

In particular, the asymptotic values of the root loci are determined by the highest order term.

Now suppose K_d is maximal rank. If $m = p$, then as we have seen in §2, $\mathrm{graph}(\lambda K_d) \to U \subset U \oplus Y$, so that

$$\lim_{\lambda\to\infty} \chi(K_\lambda) = \chi(U) = G^{-1}(G(\mathbb{P}^1) \cap \sigma(U)) \quad .$$

But, to say $G(s_o) \in \sigma(U)$ is to say

$$\mathrm{span}\begin{bmatrix} G(s_o) \\ I_m \end{bmatrix} \cap \mathrm{span}\begin{bmatrix} 0 \\ I_m \end{bmatrix} \supsetneq (0)$$

or, that there exists $u \neq 0$ so that the equations

$$N(s_o)w = 0 \quad , \quad u = D(s_o)w$$

are satisfied. That is, G has a zero at s_o. This is the well-known fact that the closed loop poles approach the open loop zeroes, as $\lambda \to \infty$, provided $\det G(s) \not\equiv 0$. However, to say $\det G(s) \equiv 0$ is to say G is degenerate, in fact it is to say that $G(s) \subset \sigma(U)$. More generally if $m \geq p, V(K_d) \subset U$, i.e. $V(K_d) \in \mathrm{Grass}(p,m)$, and $\chi(K_\lambda)$ approaches a right zero of $G(s)$!

Corollary 5.4: If $G(s)$ is nondegenerate and if $K_\lambda = K_o+K_1\lambda+\ldots+K_d\lambda^d$ with K_d maximal rank, then $\chi(K_\lambda)$ approaches the right zero $[\det(G(s)K_d)(\det(sI-A)]_n$ if $m > p$, and $\chi(K_\lambda)$ approaches the left zero $[\det(sI-A)\det(K_dG(s))]_n$ if $p < m$. In either case, the closed loop poles approach the open loop zeroes.

Now, by a straight forward modification of the square case ([17], [19], or [24]), the number n_∞ of closed loop poles which approach infinity is therefore generically given by n-r where r is the rank of the matrix

$$Z(K_d) = \begin{bmatrix} K_dCA^{n-1}B & . & \cdots & K_dCAB & K_dCB \\ \vdots & & & & \\ K_dCAB & & & & 0 \\ K_dCB & & & 0 & 0 \end{bmatrix}$$

and one may ask for the rate of growth of these n_∞ poles, with an eye toward sketching the root-locus plots corresponding to the family $K_\lambda = K_0+K_1\lambda+\ldots+K_d\lambda^d$. According to Corollary 5.3 the most rapidly increasing asymptotic arise from the highest order term in K_λ and there are at most rank CB of these branches.

Indeed, one may obtain all of the asymptotic expansions of $s(\lambda)$, $\lambda \to \infty$ in terms of Puiseux expansions at (∞,∞) on the Riemann surface defining $s(\lambda)$--this is essentially the technique employed in [20] in the case of scalar gain λI. That is, consider the subset $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ defined as the closure $\overline{X}_0$ of the subvariety

$$X_0 = \{(\lambda,s): \det(sI-A-BK_\lambda C) = 0\} \subset \mathbb{C}\times\mathbb{C}. \qquad (5.2)$$

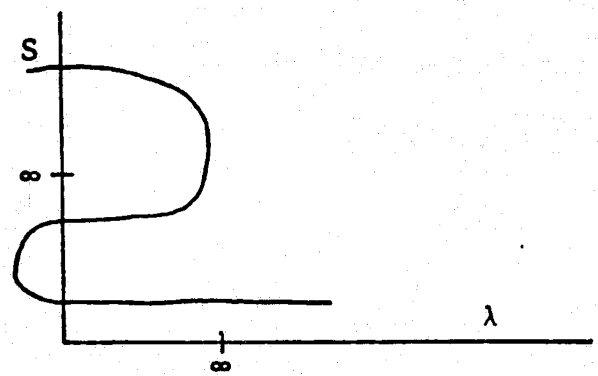

Figure 5.1 Depicting $X \subset \mathbb{P}^1 \times \mathbb{P}^1$

We shall assume that X is a manifold near (∞,∞) and as usual we shall use coordinates $\frac{1}{\lambda}$ and $\frac{1}{s}$ near $\infty \in \mathbf{P}^1$. Consider the projection $\rho_1 \colon X \to \mathbf{P}^1$ defined on X_0 via

$$\rho_1 \colon (\lambda,s) \to \lambda \quad . \tag{5.3}$$

By the implicit function theorem, if the tangent line to $(\infty,\infty) \in X$ is not vertical, there exists an analytic inverse to ρ_1 near (∞,∞). Equivalently, there exists an absolutely convergent local expansion

$$\frac{1}{s} = \sum_{i=0}^{\infty} a_i \left(\frac{1}{\lambda}\right)^i \tag{5.4}$$

for each local branch of X at (∞,∞). If the tangent to (∞,∞) is vertical (as depicted in Figure 5.1), then such an expansion is no longer possible. However, since X is nonsingular at (∞,∞), one cannot have a horizontal tangent, and applying similar reasoning to the second projection, there exists an absolutely convergent local expansion

$$\frac{1}{\lambda} = \sum_{i=0}^{\infty} b_i \left(\frac{1}{s}\right)^i \tag{5.4'}$$

for every local branch of X.

Choosing one such branch, let n_j denote the order at ∞ of $\frac{1}{\lambda}$ in (5.4)'. Solving for $\frac{1}{s}$, one obtains the absolutely convergent Puiseux expansion

$$\frac{1}{s} = \sum_{i=1}^{\infty} c_i \left(\frac{1}{\lambda}\right)^{i/n_j}$$

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or equivalently, the asymptotic expression (if $c_1 \neq 0$)

$$s = d_1 \sqrt[n_j]{\lambda} + d_2 \sqrt[n_j]{\lambda^2} + \dots$$

where (if $c_1 \neq 0$) one has n_j such expressions for this branch. In particular, the unbounded closed loop poles are given asymptotically as

$$s \sim d \sqrt[N]{\lambda} \quad , \quad \text{for some integer } N$$

and, taken together, approach infinity in several superimposed Butterworth patterns. Such an expansion is also valid when (∞,∞) is a singular point.

Now, Puiseux also derived (1850) a method, based on Newton's polygon, for explicitly determining the leading exponents i/n_j which appear in (5.5) for each branch. In the case at hand, we consider

$$F(\lambda,s) = \det(sI - A + BK_\lambda C) = \Sigma a_{ij}\lambda^i s^j \tag{5.6}$$

and construct the Newton polygon of F. That is, in the (λ,s)-plane construct the smallest convex polygon containing each (i,j) for which $a_{ij} \neq 0$, as in the figure below.

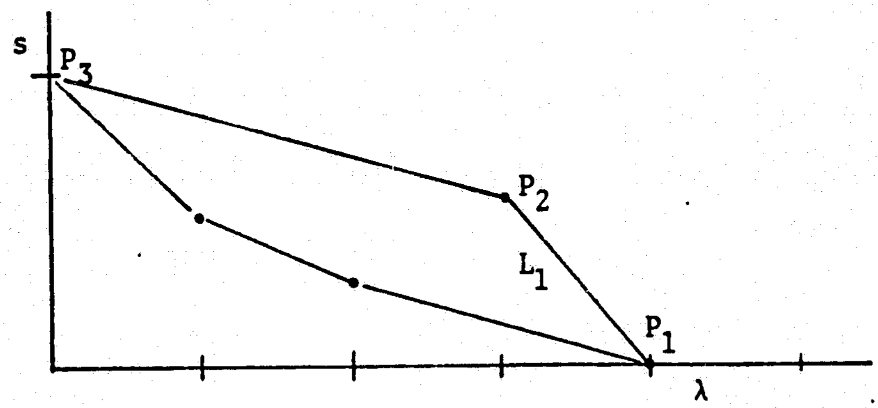

Figure 5.2

Newton Polygon for $F(s,\lambda) = s^4 + \lambda^3 s^3 + \lambda s^2 + \lambda^2 s + \lambda^4$

Now, each edge of this polygon which faces infinity will contribute to a family of branches in the following fashion: we shall be rather explicit here since the literature usually treats the derivation of the Puiseux expansions in a neighborhood of (0,0), using the edges of the Newton polygon which face the origin. Indeed, choose the points P_1, P_2 where $P_1 = (\lambda_0, 0)$ is the point on the Newton polygon closest to the origin, and P_2 is the remaining vertex on the edge L_1, facing infinity, issuing from P_1. Let $-p_1/q_1$ be, in lowest common form, the slope of L_1, and define m_1 by the equation

$$\Delta_y(P_2-P_1)/\Delta_x(P_2-P_1) = -m_1p_1/m_1q_1 \tag{5.7}$$

Then, there are m_1 "cycles" of branches of $s(\lambda)$ each giving rise to an expansion, for $j = 1,\ldots,p_1$

$$s = \sqrt[p_1]{\lambda^{q_1}}\left(d_{1j} + d_{2j}\sqrt[p_1]{\lambda^{q_1-1}} + \ldots\right)$$

where each $d_{1j} \neq 0$ and differ from one another by a p_1-th root of unity, and may be obtained from (5.6) by substitution.

Next, one may continue in exactly the same way with the vertices P_2, P_3 of the next edge L_2 (as in Figure 5.2) which faces infinity, eventually obtaining all branches of $s(\lambda)$ which tend to ∞ with λ. We note that this algorithm, viz., the method of Puiseux, does not require eigenvalue/eigenvector calculations, and in fact requires only rational operations.

It is, of course, generically the case that $m_i = 1$ for each edge L_i and, in the literature ([17],[19],[21]), if $q_i = 1$ the root loci are said to be of integer order. For $K_\lambda = \lambda I$, the integer order case has been studied in great detail and the powers p_i are known, generically, to equal the Morse structural invariants [19] of the system or, again generically, the Smith-McMillan invariants of G(s) at ∞ [24]. In such cases, the leading coefficient of the asymptotic expansion (5.5)' is easily expressible in terms of the Markov parameters of G(s) and these results generalize to the non-square case and to polynomial gains

$$K_\lambda = K_0 + \dots + K_d\lambda^d \quad .$$

As an example, we calculate the leading term of the highest order asymptotic by appealing to the return-difference determinant

$$\det(I + K_\lambda G(s)) = 0 \quad , \qquad (5.8)$$

which also defines the algebraic curve (5.6). Developing G(s)

$$G(s) = \sum_{i=1}^{\infty} G_i/s^i$$

in a Laurent expansion and equating terms, we find that the leading coefficients of the highest order asymptotic are given as

$$d_{ij} = -\text{ eigenvalue of } K_d G_1 \qquad (5.9)$$

and in particular p_{min} = rank $(K_d CB)$, generalizing the Owens and Sastry-Desoer formulae (see esp. [21], VI).

6. Pole Placement by Output Feedback

The inverse problem of placing the eigenvalues of a linear system by output feedback has been studied by many people. The literature includes the work by Kimura [13], [14] and Willems and Hesselink [26] which we will have occasion to refer to below. Other references to the literature will be found in these papers. Here we undertake a systematic general study of the pole placement problem with a view toward clarifying the geometrical content of the problem. In doing so we are lead to the rather astounding formula

$$d(m,p) = \frac{1!2!\ldots(p-1)!1!2!\ldots(m-1)!(mp)!}{1!2!\ldots(m+p-1)}$$

giving the number of different (in general complex) gains which yield the same set of poles. This, rather unexpectedly large, number emphasizes the nonlinear nature of the pole placement problem and suggests that it is probably rather difficult to solve algorithmically. It turns out that $d(m,p)$ is odd if and only if either $\min(m,p) = 1$ or $\min(m,p) = 2$ and $\max(m,p) = 2^k-1$ and in these cases we are able to show that typically there exists at least one real solution to the pole placement problem.

We also give a new and insightful proof (and strengthening) of a result of Kimura [13] on placing poles in the case where the number of inputs plus outputs exceeds the number of poles to be placed. This new proof is completely transparent and yields a set of relatively simple equations which define the desired gain.

Let $G(s)$ be a p by m matrix of real proper rational functions. If K is an m by p matrix of real or complex numbers then

$$G^K(s) = (G(s)K+I)^{-1}G(s) \tag{6.1}$$

is said to be obtained from G(s) by *output feedback*. In this section we are concerned with the question of finding K so that the numerator coefficients of the rational function

$$\det(G(s)K+I) = \chi(K)/\chi(0) \tag{6.2}$$

take on prescribed values. The map of the space of m by p matrices K into the space of monic polynomials $\chi(s)$ will be called the *pole placement map*.

By counting dimensions, it is clear that $mp \geq n$ is a necessary condition for pole-placement, over either the real or complex field, and throughout this section we consider the first non-trivial case, $mp = n$.

In what follows, G(s) is a non-degenerate transfer function and K denotes a gain matrix, k represents the corresponding p-plane in (m+p)-space, i.e. $k \in \mathrm{Grass}(p,m+p)$. We begin our study over the complex numbers.

By virtue of Theorem 3.1, the pole-placement problem in the present setting is the inverse problem of passing a Schubert hypersurface $\sigma(k)$ through the curve G(s) at the prescribed set of points $G(\lambda_1),\ldots,G(\lambda_n)$.

That is, given a set of n points $\{\lambda_1,\lambda_2,\ldots\lambda_n\}$ in $\mathbb{C}$ find k, a p-dimensional subspace in $\mathbb{C}^{m+p}$, such that k intersects the n, m-dimensional subspaces in Grass(m,m+p) defined by evaluating the map of $\mathbb{P}^1(\mathbb{C})$ into Grass(m,p+m) defined by the transfer function G(s) at $(\lambda_1,\lambda_2,\ldots\lambda_n)$. To see this, note that $\{(u,y)\,|\,-u+Ky = 0\}$ defines a p-dimensional subspace in $\mathbb{C}^{m+p}$. It intersects the m-dimensional subspace $\{(u,y)\,|\,Gu+y = 0\}$ if and only if there exists z such that $GKz+z = 0$. A moments thought shows that this is the same as finding K such that

$$\det(G(\lambda_i)K+I) = 0; \quad i = 1,2,\ldots n \quad . \tag{6.3}$$

This tacitly assumes that λ_i are regular points of G(s); otherwise one must use a coprime factorization of G(s).

As we mentioned above, the collection of all m dimensional subspaces in an (m+p)-dimensional space is an mp-dimensional space. If we are given n points in this space when can we expect to find a p-plane which intersects all n m-planes? Schubert invented a calculus to solve such intersection theoretic questions and his ideas subsequently came to play a sizeable role in algebraic geometry; e.g. Hilbert's 15th problem is devoted to the Schubert calculus [15]. For our present purposes it is enough to know that given mp m-dimensional subspaces in $\mathbb{C}^{m+p}$ there exists (generically)

$$d(m,p) = \frac{(1!2!\ldots(p-1)!(1!2!\ldots(m-1)!(mp)!}{1!2!\ldots(m+p-1)!} \tag{6.4}$$

p-dimensional subspaces which intersect them all. (See Chapter IV, Section 7 of [12] or [15].)

At this point it is important to note that, by the second statement in Lemma 5.2, if $\lambda_1,\ldots,\lambda_n$ are all finite then the only p-planes V which intersect all the m-planes $G(\lambda_1),\ldots,G(\lambda_n)$ are finite, i.e. V = k for some gain K. Thus, for a generic choice of finite $\lambda_1,\ldots,\lambda_n$, there exist d(m,p) distinct complex gains K for which

$$\chi(K) = \prod_{i=1}^{n} (s-\lambda_i) \quad .$$

Now, by Theorem 5.1 one can define χ at infinite gains V as well, obtaining $\chi(V) \in \mathbb{P}^n$ = {unordered n-tuples of points on $\mathbb{P}^1$}. This latter identification was treated in some detail in Section 4, and it extends the identification

$$\mathbb{C}^n = \{\text{unordered n-tuples of points in } \mathbb{C}\}$$

obtained by factoring monic polynomials

$$s^n+c_1s^{n-1}+\ldots+c_n = \prod_{i=1}^{n} (s-\lambda_i) \quad .$$

In particular, χ extends to a continuous (by Theorem 5.1) function

$$\chi : \text{Grass}(p,m+p) \to \mathbb{P}^n \quad . \tag{6.5}$$

Now, we know χ has a dense image by the Schubert calculus and, since Grass(p,m+p) is compact, χ is onto. But then

$$\chi : \mathbb{C}^{mp} \to \mathbb{C}^n$$

is also onto, by the second part of Lemma 5.2 . Since χ is algebraic we have shown

Theorem 6.1: Let G(s) be a non-degenerate p by m transfer function of McMillan degree $n = mp$. For all choices $(\lambda_1,\ldots,\lambda_n)$ we can find d(m,p) solutions (counted with multiplicity) to the pole-placement problem,

$$\det(G(\lambda_i)K+I) = 0, \qquad i = 1,\ldots,n \quad . \tag{6.6}$$

Moreover, for the generic n-tuple, the solutions to (6.6) are distinct.

It is known in the literature ([10],[26]) that for $mp \geq n$ and for generic G(s), one can place almost all poles over $\mathbb{C}$ - by the dominant morphism theorem. The full surjectivity of χ, as well as the formula (6.4) for the "degree" of χ, are both new. Notice that if $m = p = 2$, $n = 4$, then $d(m,p) = 2$ so that in some sense the problem is quadratic and one might expect real solutions "only half of the time". Indeed, this was shown in [26] and we give an independent proof here, based on the following result.

Theorem 6.2 Necessary and sufficient conditions to be able to solve the real system of equations

$$Ax+bQ(x) = v \tag{6.7}$$

for all $v \in \mathbb{R}^n$ where $Q(s)$ is quadratic is that A is invertible, $Q(A^{-1}b) = 0$ and $\left.\frac{\partial Q}{\partial x}\right|_{x=A^{-1}b} = 0$.

Proof: Clearly to be able to solve for all v the columns of (A,b) must span $\mathbb{R}^n$. If A is singular, then we can change coordinates to $z = Px$ in such a way that the equations take the form

$$z_i + b_i Q(z) = v_i \qquad i=1,2,\ldots,n-1$$
$$b_n Q(z) = v_n$$

If these were to be solvable for all v, then b_n would be nonzero and we could use the last equation to eliminate $Q(z)$ from the others yielding $z_i^1 = v_i^1$ $i=1,2,\ldots,n-1$ and $b_n Q(z) = v_n$. This set fixes the values of z_i^1 $i=1,\ldots,n-1$, but $Q(z)$ for $Q(z)$ a quadratic does not map $\mathbb{R}^1$ onto $\mathbb{R}^1$ for all values of z_i, $i=1,2,\ldots,n-1$ as one easily sees.

Now suppose that A is invertible. In this case a linear transformation reduces the quation to the form

$$x^1 + e_1\tilde{Q}(x^1) = v^1$$

where e_1 is the first standard basis vector in $\mathbb{R}^n$. If $v^1 = \alpha e_1$, then x^1 must be of the form βe_1 and we see that to solve $\beta + \tilde{Q}(\beta e_1) = \alpha$ for all values of α, we must have $Q(e_1) = 0$. In terms of the original coordinates, this means $Q(A^{-1}b) = 0$ is a necessary condition. Now consider $v^1 = \alpha e_1 + v_0^1$. In this case, $x^1 = \beta e_1 + v_0^1$, and we must solve

$$\beta + \tilde{Q}(\beta e_1 + v_0^1) = \alpha$$

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since $\tilde{Q}(e_1) = 0$, if

$$\left.\frac{\partial\tilde{Q}}{\partial x}\right|_{e_1} \neq 0$$

then we can choose v_0 so that $\tilde{Q}(\beta e_1 + v_0^1)$ is independent of β and hence $\beta + Q(\beta e_1 + v_0) = \alpha$ can not be solvable for all v_0 and α. Again, in terms of the original coordinates this means

$$\left.\frac{\partial Q}{\partial x}\right|_{A^{-1}b} = 0$$

If $\left.\frac{\partial Q}{\partial x}\right|_{e_1}$ is not zero, we can choose δ so that $\gamma + Q(\gamma e_1 + \delta v_0)$ is independent of γ. This establishes the necessity of the conditions. Sufficing is easily seen and we leave it to the reader.

Corollary 6.3 If $Q(x)$ is non-degenerate, then (6.7) is solvable for all v if, and only if, $b = 0$.

This result, together with the formula

$$(G,-I)^{(2)} \begin{pmatrix} K \\ I \end{pmatrix}^{(2)} = \det(GK + I) \tag{6.8}$$

means we cannot place the closed-loop poles of $G(s)$ arbitrarily unless $G(s)$ satisfies certain conditions. In fact since

$$<[\det G(s), g_1, g_2, g_3, g_4, 1], [k_1k_4 - k_3k_2], k_1, k_2, k_3, k_4, 1> = \det(G(s)R + I) \quad (6.8)'$$

the quadratic form $Q(w,z,y,z) = wz - yz$ in (6.8) is non-degenerate we must have $\det G(s) \equiv 0$. Thus solvability of (6.8) for all monic polynomials of degree 4, amounts to a non-trivial constraint on $G(s)$. This implies the main result of Willems-Hesselink [26].

Corollary 6.4: If G(s) is a non-degenerate 2×2 transfer function with McMillan degree 4, then the set of closed loop polynomials which cannot be achieved by output feedback is a non-empty open subset of $\mathbb{R}^4$.

This set is open since the image of χ: $\mathrm{Grass}_{\mathbb{R}}(2,4) \to \mathbb{P}^4_{\mathbb{R}}$ is compact and, by virtue of the second part of Theorem 5.1, has a closed intersection with $\mathbb{R}^4 \subset \mathbb{P}^4_{\mathbb{R}}$, the subspace of polynomials having finite roots. Notice that if $\det G(s) \equiv 0$, then

$$\det(I + KG(s)) = 1 + \mathrm{tr}(KG(s))$$

is linear in K and so to say this correspondence is not surjective is to say there exists $K \neq 0$ such that $\mathrm{tr}(KG(s)) = 0$. That is, thinking of $\mathrm{tr}(K\cdot)$ as a linear functional on the space of 2×2 matrices, this implies that $G(s) \subset V \subsetneq \mathbb{R}^4$, for V some subspace of $\mathbb{R}^4$. This constraint fails to hold for the generic G(s) satisfying $\det G(s) = 0$, indeed

$$G(s) = \begin{bmatrix} s+1 & 0 \\ s+2 & 0 \end{bmatrix} \begin{bmatrix} (s-1)(s-2) & (s-1)(s-2) \\ 0 & (s-3)(s-4) \end{bmatrix}^{-1}$$

satisfies $\det G(s) \equiv 0$ but there exists no proper $V \supset G(s)$, for all s. In particular, one can place poles over $\mathbb{R}$ for the generic G(s) which satisfies $\det G(s) \equiv 0$. By Theorem 6.2, these conditions are necessary for pole-placement as well.

Corollary 6.5 Let G(s) be a real 2×2 transfer function of McMillan degree 4. One can place poles arbitrarily by output feedback if, and only if, $\det G(s) \equiv 0$, and no real linear combination of the g_{ij} vanishes.

Remark: If one imbeds Grass(2,4) $\subset \mathbb{P}^5$ by the Plucker imbedding, (6.7) gives the equation for pole-placement in the so-called Plucker coordinates. (6.7)' represents the fact that in the Plucker coordinates these equations amount to 4 linear equations - representing the constraint that the hyperplane in $\mathbb{P}^5$ corresponding to K pass through the 4 points $G(\lambda_1),\ldots,G(\lambda_4)$ - and the single quadratic constraint which defines (the dual) G(2,4) as a quadric in (the dual) $\mathbb{P}^5$. This is quite general, the Plucker equations define Grass(p,m+p) $\subset \mathbb{P}^N$ as the intersection of quadrics - for example, if $m = 2$, $p = 3$ and $n = 6$, the pole placement equations become 6 linear equations and 3 quadratic equations. In this case, however, they are always solvable over $\mathbb{R}$ since, in this case, the remarkable formula (6.4) gives $d(2,3) = 5$. Thus, for non-degenerate 2 by 3 transfer functions of McMillan degree 6, we can always place poles arbitrarily over $\mathbb{R}$.

In general if $mp = n$ then the Schubert calculus tells us that there are, generically, a certain number of feedback gains $K_1,K_2,\ldots,K_d$ which satisfy

$$\det(G(\lambda_i)K_j+I) = 0 \qquad i=1,2,\ldots n$$

and hence place the poles at the locations $\lambda_1,\lambda_3,\ldots\lambda_n$. Suppose that G(s) is real for s real and suppose that the $G_i = G(\lambda_i)$ appear in complex conjugate pairs. Then if K_j is a solution $\bar{K}_j$ is also; the complex solutions occur in complex conjugate pairs. If the total number of solutions is odd then, of course, one solution must be real. We know that for typical values of G_i in $\mathbb{R}^{m\times p}$ there are solutions. However the set of self-conjugate G_i is not open in $\mathbb{C}^{m+p} \times \ldots \times \mathbb{C}^{m+p}$ (n factors) and so we must reason with some care in order to show that for typical self-conjugate sets of $\{G_i\}$ we have d(m,p) roots.

Theorem 6.6: Suppose G(s) is nondegenerate. If mp = n and d(m,p) is odd then there exists an open dense set of self-conjugate G_i such that

$$\det(G_i K+I) = 0 \quad i = 1,2,\ldots n \tag{6.9}$$

has a real solution. Moreover d(m,p) is odd if and only if min(p,m) = 1 or min(p,m) = 2 and max(p,m) = 2^k-1 for some k = 2,3,...

Proof: Let $J(K_i)$ denote the Jacobian of the pole placement map evaluated at K_i. Suppose that $\{G_i\}_{i=1}^n$ is given and suppose that the pole placement map has d(m,p) inverse images $K_1, K_2, \ldots K_d$, then

$$\psi = \det J(K_1) \det J(K_2) \ldots \det J(K_d)$$

is a function of $G_1, G_2, \ldots G_n$ but not a function of K. We know that ψ is nonzero generically. It is also an analytic function of the entries of G_i. Thus if it vanishes identically for, say, an open subset of real G_i's then it vanishes identically which is a contradiction.

I. Berstein [2] proves that d(m,p) is odd if and only if the given conditions are satisfied. Q.E.D.

Corollary 6.7 Suppose G(s) is a 3×2, or a 2×3, nondegenerate transfer function. Then, the McMillan degree of G is less than or equal to 6, and the poles of G may be placed arbitrarily over $\mathbb{R}$ if, and only if, the degree equals G.

As a final remark, the well known formula for det(A + B) when B is of rank one provides some insight into the pole placement problem. If b and c are vectors and K = bc', then

$$\begin{aligned}\det(G(s)K+I) &= \det(G(s)bc'+I)\\ &= c'G(s)b + 1\end{aligned}$$

Regarding the space of monic polynomials $\chi(s) = s^n+p_{n-1}s^{n-1}+\ldots p_o$ as an n-dimensional vector space and writing

$$c'G(s)b = q(s)/p(s)$$

we see that by inserting a scale factor in front of b the whole pencil of polynomials

$$\chi(s) = p(s)+kq(s)$$

may be achieved. This line of thinking leads easily to the following strengthening of Kimura's result [13]. (It is stronger than Kimura's results because we show that the pole placement map is onto and not just almost onto.)

Theorem 6.7: Given G(s) of McMillan degree n the image of the pole placement map is the whole space if for any given polynomial q of degree n-1 or less there exists vectors c and b such that

$$c'G(s)b = q(s)/p(s)$$

Moreover, if $G(s) = C(Is-A)^{-1}B$ with A, B, C chosen generically this condition will be satisfied provided $m+p-n \geq 1$.

Proof: Consider the set of all transfer functions c'G(s)b with $||c|| = 1$ and $||b|| = 1$. Under the hypothesis this set intersects every line passing through zero in the real vector space of all polynomials of degree n-1 or less. Using the above argument we see that this means that the pole placement map is onto.

To see that this condition is generically satisfied, we note that it is equivalent to asking that n polynomials in the vectors G(s)b and G(s)c should be independent over $\mathbb{R}$. Clearly this is generically true.

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ON ROOT LOCI IN SEVERAL VARIABLES, CONTINUITY IN THE HIGH GAIN LIMIT*

by

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Abstract

In this note we discuss the continuity of the closed loop poles of a linear multivariable system with respect to a multidimensional polynomial family of direct output gains $K(\lambda_1,\ldots\lambda_r)$. This is based on, and contains an exposition of, the geometric formulation for including infinite gains which was developed in the lectures [2] and extended and applied in [1] to the study of output feedback systems. This has been a basic tool in recent work on the classical problem of pole-placement by output feedback and in [1] the lack of continuity of the root-loci, in certain situations, was discussed with special emphasis on the complex case. Here, after presenting two somewhat surprising counterexamples to this continuity, we give in Theorem 1 and the ensuing discussion necessary and sufficient conditions for continuity of the root-loci at a real infinite gain. This should have significant impact on the problem of constructing graphical tests for the stability of systems subject to 2-dimensional variations in the gain parameter.

1. In the past decade, in one sense a fairly conclusive theory for sketching the root-loci of a (square) m×m, strictly proper rational transfer function G(s), as a function of the 1-parameter direct output gain $K(\lambda) = \lambda I$, has been developed (see, e.g. [5], [6]). The root-loci, or closed-loop poles, consist of n algebraic functions $s_1(\lambda),\ldots,s_n(\lambda)$ which evolve (as $\lambda \to \infty$) on the m sheets of a Riemann surface X, branched over the Riemann sphere. Explicitly, we first take $X_o \subset \mathbb{C} \times \mathbb{C}$ to be the locus defined by the return-difference determinant

$$0 = F(\lambda,s) = \det(I+K(\lambda)G(s)) \quad OLCP(s) \tag{1.1}$$

where OLCP(s) denotes the open-loop characteristic polynomial of (a minimal realization of) G(s). Since we are interested in the behavior near $\lambda = \infty$ we adjoin such points, $S^2 = \mathbb{C} \cup \{\infty\}$, and consider the Riemann surface $X = \overline{X}_o \subset S^2 \times S^2$ together with the 2 natural projections $p_1(\lambda,s) = \lambda$ and $p_2(\lambda,s) = s$ which each exhibit X as a branched cover

$$p_i : X \to S^2 \qquad i = 1,2$$

of the Riemann sphere with n (respectively, m) sheets. Since $F(\lambda,s) = CLCP(\lambda,s)$, for fixed λ_o the n points $s_1(\lambda_o),\ldots,s_n(\lambda_o)$ coincide with the closed-loop poles determined by $K(\lambda_o)$.

Just as in the classical case (i.e., m = 1), the method for sketching the root-loci consists of determining:

(i) the initial values $s_1(0),\ldots,s_n(0)$ which are of course the open-loop poles;

(ii) the final values $s_1(\infty),\ldots,s_n(\infty)$ which in this case are the open loop zeroes provided $\det G(s) \not\equiv 0$; and

(iii) the angles of departure ($\lambda << \infty$) and angles of approach ($0 << \lambda$) on the appropriate sheets of X.

We shall return to a discussion of (ii) presently, but remark that the determination of (iii) is a classical problem in algebraic function theory, solved by Puiseux in 1850. Explicitly, suppose we are interested in the behavior of those $s_i(\lambda)$ which tend to ∞ with λ. These can occur in several groups, or cycles, corresponding to the infinite branches of X over the point $\lambda = \infty$. Choosing one such branch, we can obtain a Puiseux expansion for $\frac{1}{s}$ as a function of $\frac{1}{\lambda}$:

$$\frac{1}{s_j} = a_{j1}\left(\frac{1}{\lambda}\right)^{1/q_j} + a_{j2}\left(\frac{1}{\lambda}\right)^{2/q_j} + \ldots \tag{1.2}$$

which converges absolutely in a neighborhood of $\frac{1}{\lambda} = 0$. Here, j indexes the algebraic function s_j in this particular cycle. Inverting (1.2) one obtains the asymptotic expansion

$$s_j(\lambda) = \alpha_{j1}\sqrt[q_j]{\lambda} + \alpha_{j2}\sqrt[q_j]{\lambda^2} + \ldots \tag{1.3}$$

which determines the angle of approach of $s_j(\lambda)$ as $\lambda \to \infty$. For example, if $\alpha_{j1} \neq 0$ then $s_j(\lambda)$ tends to infinity asymptotically as a q_j-th root of unity. In other words, this cycle of root-loci tends to ∞ in a Butterworth pattern and thus the root-loci tend to ∞ in superimposed Butterworth patterns. It is a happy fact that the leading fractional powers appearing in (1.2) can be read off, in a glance, from the Newton diagram of X - that is, from the Newton diagram of the polynomial $F(\lambda,s)$. Moreover, the leading coefficients in (1.2) can be obtained from substituting (1.3) into (1.1) and equating coefficients.

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2. Quite clearly, the analysis above extends to the important case of 1-parameter families $K(\lambda)$ of gains which vary polynomially with λ. The extension to nonsquare $G(s)$ involves considerably more thought, especially vis-a-vis (ii), requiring a careful development of the concept of infinite gain and leading naturally to questions concerning the robustness of the asymptotic expansions (1.3) and of the root loci themselves. In this sense, the root-locus theory is far from complete and, since robustness is likely to play an important role in the analysis of 2-parameter families of gains, we present here a theory of robustness based on the notions of infinite gain presented in [1] and [2]. We shall now give 2 rather surprising examples of this lack of robustness, the first is adapted from the single variable discussion in [1].

Example 1 Consider the transfer function and the 2-parameter family of gains, respectively, defined by

$$G(s) = \begin{bmatrix} 1/(s^2-1) & 0 \\ 0 & 1/(s^2-4) \end{bmatrix} , \quad K(\lambda,\mu) = \begin{bmatrix} 1-\mu\lambda & \lambda \\ \mu^2\lambda-\mu & \mu\lambda-1 \end{bmatrix}$$

We shall compute the root-loci, as a function of (λ,μ), along the 2 asymptotic curves, γ_1: $\mu = 0$ and γ_2 : $\mu\lambda = 1$, in the direction of increasing λ. Along γ_1, we see that the root-loci is constant and coincides with the roots of $s^2(s^2-5)$, yet along γ_2 the root-loci is constant, given by the roots of $(s^2-1)(s^2-4)$. Furthermore, along γ_3 : $\mu^2\lambda = 1$ which is also asymptotic to γ_1, the root-loci depend on λ and tend to the roots of s^4-5s^2-1. This example shows that the final values of the root-locus are not continuous with respect to the gain! Furthermore, it can occur that along one curve a system appears stable in the high gain limit but along a slightly perturbed curve, the system is unstable for high gains. We note that this pathology is not due to:

(1) inequalities $m > p$ or $p < m$, or to

(2) the absence of diagonal dominance in $G(s)$, or to

(3) the vanishing of $\det G(s)$.

The second example is deceptively trivial.

<u>Example 2</u>. Given $G(s)$ and $K(\lambda,\mu)$ defined by

$$G(s) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Big/ s, \quad K(\lambda,\mu) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda-2\lambda^2\mu \end{bmatrix}$$

Along γ_1 we have the root locus given by $s(\lambda) = -\lambda$, yet along γ_2 one computes $s(\lambda) \equiv 0$.

N.B. The condition $\det G(s) \equiv 0$ is generic (indeed, always satisfied) for 2×2 transfer functions of McMillan degree 1.

These examples can be interpreted in the following context. A polynomial family of $(m\times p)$ gains $K(\lambda_1,\ldots,\lambda_r)$ is said to have dimension r just in case the Jacobian of the function $K : \mathbb{R}^r \to \mathbb{R}^{mp}$ is not identically zero. Thus, $\dim K \leq mp$. Here, and in the following sections, $G(s)$ is a $p\times m$ real transfer function having McMillan degree n and the variety $\Sigma^n_{m,p}$ of all such functions is coordinatized by the Hankel parameters of $G(s)$ as in [3]. In particular, a generic set of G is a complement of a proper subvariety in the corresponding space of Hankel matrices.

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Theorem 1. If dim $K \leq n$, then for the generic transfer function G and any curve γ in $\mathbb{R}^r$ along which K tends to infinity, the root-loci are continuous. That is, for γ_1 asymptotic to γ the root-loci agree in the high gain limit. If dim K > n, then there exists some curve γ in $\mathbb{C}^r$ along which the root loci are discontinuous. Moreover, there exists some real family K for which γ may be taken to be real.

Note that Example 1 illustrates the necessity of the generic hypothesis on G in the first statement, while Example 2 illustrates the remaining assertions.

3. Returning to Example 1, we investigate the asymptotic behavior of the gain $K(\lambda,\mu)$ itself along γ_1. Here, $u = K(\lambda,\mu)y$ is the linear relation

$$u_1 = y_1+\lambda y_2 \quad , \qquad u_2 = -y_2 \tag{3.1}$$

or, equivalently, for $\lambda \neq 0$

$$u_1/\lambda = y_1/\lambda + y_2, \qquad u_2 = -y_2 \tag{3.1$'$}$$

As $\lambda \to \infty$, this linear relation approaches the linear relation

$$0 = y_2, \quad u_2 = -y_2 \quad , \tag{3.2}$$

that is, the linear relation (also of rank 2), $u_2 = y_2 = 0$. It is easily seen that this relation is also the limit of $K(\lambda,\mu)$ along γ_2 and γ_3. The system-theoretic interpretation of the discontinuity at this infinite gain is simply that for $\lambda >> 0$ each frequency $s \in \mathbb{C}$ behaves like an approximate pole, so that "in the limit" each $s \in \mathbb{C}$ deserves to be called a pole. Explicitly, given any $s_o \in \mathbb{C}$, any $\varepsilon > 0$ and any $c > 0$ there exists an input u with $||\hat{u}(s_o)|| < \varepsilon$ and an $N(c,\varepsilon) >> 0$, such that the resulting output $y(s_o)$ of the closed-loop system corresponding to $K(\lambda)$ defined by (3.1) satisfies

$$||\hat{y}(s_o)|| > c \quad \text{for} \quad \lambda > N(c,\varepsilon) \quad .$$

For this condition to be satisfied, one must choose $u_2(s_o) \neq 0$ which will then amplify $\hat{u}_1(s_o)$ and hence $\hat{y}_1(s_o)$ in the closed loop equations. In this light, the infinite gain constraints (3.2) reflect the fact that, only under the condition $u_2 = 0$ (and therefore $y_2 = 0$), will the limiting "transfer function" yield a finite output from a finite input.

4. We can make this precise in the language of classical algebraic geometry. Consider, for $K : \mathbb{R}^p \to \mathbb{R}^m$, the graph of K as a dimensional subspace

$$gr(K) = \{(y,Ky) : y \in \mathbb{R}^p\} \subset \mathbb{R}^p \oplus \mathbb{R}^m$$

We may also consider K as a linear map, $K_{\mathbb{C}} : \mathbb{C}^p \to \mathbb{C}^m$ and therefore define $gr(K_{\mathbb{C}})$ as a subspace of $\mathbb{C}^p \oplus \mathbb{C}^m$. Of course $gr(K_{\mathbb{C}})$ determines $K_{\mathbb{C}}$. Now consider the set Grass(p,m+p) of all p-planes in $\mathbb{C}^p \oplus \mathbb{C}^m$. This set may be naturally regarded as a compact manifold, indeed a variety (see [4]), the Grassmannian variety. The generic p-plane is complementary to $\mathbb{C}^m \subset \mathbb{C}^p \oplus \mathbb{C}^m$ and is therefore the graph of some linear function $K : \mathbb{C}^p \to \mathbb{C}^m$. In this way, Grass(p,m+p) is a compactification of the space of (finite) gains and therefore has dimension mp. For example, the 2-plane in $\mathbb{C}^4$ defined by (3.2) is an (infinite) point of Grass(2,4) which is not the graph of a linear gain Ky = u, but which is the limit of the 2-planes $gr(K(\lambda)_{\mathbb{C}})$ where $K(\lambda)$ is defined by (3.1). Thus, Grass(p,m+p) is a model for the space of finite and infinite gains, with p-planes $V \subset \mathbb{C}^p \oplus \mathbb{C}^m$ falling into these 2 classes according to whether V is complementary to $\mathbb{C}^m$ or not. We shall denote this latter subvariety by $\sigma(\mathbb{C}^m)$.

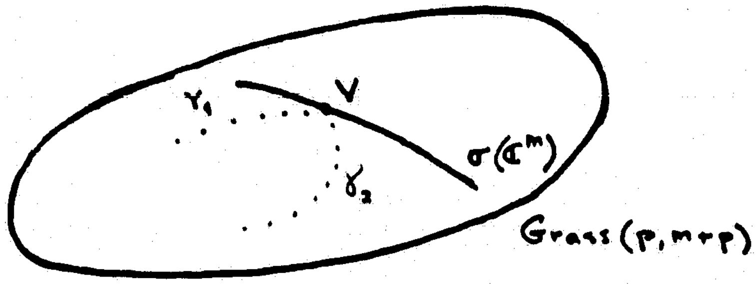

Figure 1

In Figure 1, we have depicted an infinite gain $V \in \sigma(\mathbb{C}^m)$ together with 2 asymptotic sequences γ_1, γ_2 of finite gains approaching V in the (high gain) limit.

5. In this model for including high gain limits, we can ask whether the root-loci are continuous at infinite gains. Referring to Figure 1 and examples 1 and 2, it is not hard to see the difficulty involved. In the standard context of this problem, the methods by which the root-loci are defined at V is by choosing some sequence, say γ, approaching V and defining the root-loci as the corresponding limiting n-tuple on the Riemann sphere (which exists, by Bolzano-Weierstrass). The analytical

question which remains is whether this limit is independent of γ. This was determined in [1] and [2] by defining the root-loci at V intrinsically - in the case when the limits will agree.

Lemma ([1]) Let K be a finite gain, defined over $\mathbb{R}$ or $\mathbb{C}$, and let $\Phi = \{\phi_1,\ldots,\phi_m\}$ be a set of linear functionals which define $gr(K_{\mathbb{C}}) \subset \mathbb{C}^p \oplus \mathbb{C}^m$. Then the poles of the closed-loop system $G(s)(I-KG(s))^{-1}$ are the zeroes of the rational function

$$F_{\Phi}(s) = \det(\phi_i(g_j(s))) \tag{5.1}$$

where $g_j(s)^t$ is the j-th row of $[G(s),I_m]$.

N.B. (5.1) is proportional to the return-difference determinant $\det(I-KG(s))$, in particular its zeroes are independent of the choice of Φ- a fact which can also be checked directly.

Turning to infinite gains V, choose $\Phi = \{\phi_1,\ldots,\phi_m\}$ so that $V = \bigcap_{i=1}^{m} \ker \phi_i$, and form $F(s)$ exactly as above. Provided $F_{\Phi}(s)$ does not vanish identically, we may define the infinite root-locus $\chi(V)$ as the n zeroes, finite or infinite, of $F_{\Phi}(s)$. Moreover, continuity of the root-loci at V follows from the continuity of the roots of a (non-zero) polynomial on its coefficients. However, when $F_{\Phi}(s) \equiv 0$ this is nonsense.

Theorem 2 ([1], [2]) The root-loci are continuous at an infinite gain V if, and only if, $F_{\Phi}(s) \not\equiv 0$.

6. Example 1'. Consider the infinite gain V defined by (3.2), thus $V = \bigcap_{i=1}^{2} \ker \phi_i$ where, for example,

$$\phi_1(y_1,y_2,u_1,u_2) = y_2, \quad \phi_2(y_1,y_2,u_1,u_2) = u_2 \tag{6.1}$$

while

$$g_1(s)^t = (1/(s^2-1),0,1,0) \text{ and } g_2(s)^t = (0,1/(s^2-4),0,1) \tag{6.2}$$

Thus,

$$F_\phi(s) = \det \begin{bmatrix} 0 & 1/(s^2-4) \\ 0 & 1 \end{bmatrix} \equiv 0 \quad ,$$

as claimed.

Example 2'. Along γ_1, $K(\lambda,\mu) = \lambda I_2$ so that $u = K(\lambda)y$ is simply

$$u_i = \lambda u_i, \quad \text{or } u_i/\lambda = y_i \quad \text{for } \lambda \neq 0 \tag{6.3}$$

In the limit $\lambda \to \infty$, we obtain the infinite gain $V = \mathbb{C}^p$ defined by $y_i = 0$, for $i = 1,2$ or equivalently by the linear functionals

$$\phi_i(y_1,y_2,u_1,u_2) = y_i \quad . \tag{6.3)'$$

Thus

$$F_\phi(s) = \det G(s) \equiv 0 \, ! \tag{6.4}$$

Example 3. For any square system $G(s)$, the root loci corresponding to $K(\lambda) = \lambda I_m$ will be continuous at $V = \lim_{\lambda\to\infty} K(\lambda)$, defined as in (6.3)', if and only if the well-known condition $\det G(s) \not\equiv 0$ is satisfied. Alternatively, the technical condition that open-loop zeroes should exist coincides with the condition that these particular final values of the root locus should be independent of the high gain limit

$$V = \lim_{\lambda\to\infty} \lambda K \, , \qquad \det K \neq 0 \quad .$$

This gives additional insight into the condition (6.4) as well as Theorem 2. As a special case (m=1), note that root loci are always continuous in the scalar case for nonzero g(s).

7. Now, to derive Theorem 1 from Theorem 2 we note that ([2], Lemma 4.A) $F_\Phi(s) \equiv 0$ if, and only if, $F_\Phi(s_i) = 0$ for $i=1,\ldots,n+1$ and where s_i are distinct points in $\mathbb{C}$. Furthermore, for any fixed s_o and $V \in \mathrm{Grass}(p,m+p)$ arbitrary, the equation $F_\Phi(s_o) = 0$ defines a hypersurface on Grass(p,m+p) - indeed, a Schubert hypersurface (see [1], [2]). For generic G these may be chosen in general position for distinct s_i and therefore ([7], p. 57) for any subvariety $X \subset \mathrm{Grass}(p,m+p)$ the intersection

$$\bigcap_{i=1}^{n+1} \{V: F_\Phi(s_i) = 0\} \cap X$$

is empty if, and only if, $\dim X \leqslant n$. Finally, the condition $\dim K \leqslant n$ on a family of gains $K(\lambda_1,\ldots,\lambda_r)$ is the condition that the algebraic dimension over $\mathbb{C}$ of $\overline{K(\mathbb{C}^r)} \subset \mathrm{Grass}(p,m+p)$ is less than or equal to n, but over $\mathbb{C}$ this coincides with the geometric dimension of $X = \overline{K(\mathbb{C}^r)}$. This shows that if $r > n$, there exists some point V of discontinuity $\overline{K(\mathbb{C}^r)} \subset \mathrm{Grass}(p,m+p)$. However, $\overline{V}$ must also be a point of discontinuity since we may choose the n+1 points s_i to be real, and in that case

$$G(s_i) \cap V \neq (0) \text{ iff } G(s_i) \cap V \neq (0) \quad .$$

Indeed,

$$G(s_i) \cap V \cap \overline{V} \neq (0) \quad \text{for } i = 1,\ldots,n+1.$$

Choosing a real p-plane w, such that $V \cap \overline{V} \subset w$, one has

$$G(s_i) \cap w \neq (0) \quad \text{for } i = 1,\ldots,n+1$$

and hence for all s. Such a ω is therefore a point of discontinuity for the root-locus map and also lies in the closure of the family

$$K : \mathbb{R}^{mp} \to \mathbb{R}^{mp}$$

defined by $K(x) = x$.

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"OUTPUT FEEDBACK AND GENERIC STABILIZABILITY"*

by

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ABSTRACT

We consider questions of pole placement and stabilization for generic linear systems with prescribed state, input and output dimensions, where the controller must be implemented by linear memoryless output feedback. We present a criterion, in terms of a special pole-placement property, for generic stabilizability and apply this to describe constraints on the dimensions which are consistent with generic stabilizability. We also discuss the rationality and solvability by radicals of stabilizing or pole positioning gains, and we describe how decision algebra can theoretically handle existence questions for generic systems.

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1. INTRODUCTION

In this paper, we are concerned with questions of pole-assignability and stabilizability for real linear input-output systems

$$\frac{dx}{dt} = Fx + Gu$$

$$y = Hx \qquad (1.1)$$

or

$$x(t+1) = Fx(t) + Gu(t)$$

$$y(t) = Hx(t) \qquad (1.1)'$$

where we allow constant gains $u = Ky$ as feedback. The equations of pole-assignability are real polynomials and it is natural to attempt to solve these equations by eliminating the unknown variable K. Similar remarks apply to the equations of stabilizability which include, however, algebraic inequalities arising for example from the Routh-Hurwitz criteria. In what follows, we shall use various results from classical algebraic geometry, including elimination theory and the Schubert calculus of enumerative geometry, which apply to the equations of pole-placement.

Put geometrically, elimination theory consists in the study of a projection

$$p_1 : X \times Y \to X \qquad (1.2)$$

restricted to an algebraic, or semialgebraic, set $Z \subset X \times Y$, where X and Y can be taken to real or complex vector spaces, e.g. $X = \mathbb{R}^N$, $Y = \mathbb{R}$. The main problem in elimination theory consists in finding a description of the set

$$p_1(Z) = \{x : \exists y \text{ such that } (x,y) \in Z\}$$

in terms of Z. A basic example is given by

$$Z = \{(x, y) : x = y^2\} \tag{1.3}$$

which is algebraic but for which $p_1(Z)$ is only semialgebraic if we take real coefficients*.

In relation to the pole assignability question for a prescribed F, G, H, we can identify the entries of K with the space Y and the coefficients of the closed loop characteristic polynomial, call them $p_1, \ldots, p_n$, with X. Then

$$Z = \{(p_1, \ldots, p_n, K) : \det(sI - F - GKH) = s^n + \sum_{i=1}^{n} p_i s^{n-i}\}$$

and pole assignability of a generic closed-loop polynomial holds if and only if $p_1(Z)$ coincides with all of R^n save a proper subvariety.

Among the results we obtain using classical algebraic geometry are: the condition $mp \leq n$ is necessary for the stabilizability of the generic (F, G, H). This condition is well known to be necessary for pole-assignability of the generic (F, G, H), and our result raises the question as to whether or not, in terms of the values, m, n, p, these two questions might not be equivalent. As unlikely as this may be, at the time we write there is no counterexample (although there is evidence in this direction for $m = 2$, $n = 9$, and $p = 6$, see [5]). We also show that if a stabilizing gain exists, then such a gain can be found by a rational procedure. On the other hand, we show that if

*Semialgebraic sets are defined in Section 3, equation (3.4).

$mp = n$ a rational procedure for finding a gain K which assigns a given characteristic polynomial (assuming such a K exists) does not exist unless $\min(m,p) = 1$, in which case a linear formula can be found. Moreover using square roots as well as rational operations only helps if $\min(m,p) = \max(m,p) = 2$. This is of course in contrast with pole assignment by state feedback, and answers in the negative a question raised in [1].

We also argue that one can in principle determine by rational calculations whether, given m, n, p, generic F, G, H are pole assignable, generically pole assignable, or stabilizable. We say "in principle" since the number of calculations required is enormous.

We use several tools to prove the results. One of the theorems is due to Tarski-Seidenberg, which asserts that if Z is semialgebraic, then $p_1(Z)$ is semialgebraic. This theorem can be used iteratively to reduce the question of the existence of a solution $x \in \mathbb{R}^n$ to a set of semialgebraic equations to the question of existence of a solution to another set of semialgebraic equations in, for example, the unknown $x_1 \in \mathbb{R}$. Such existence can be decided by a rational procedure in the coefficients of the resulting semialgebraic equations. The Tarski-Seidenberg theorem is extremely qualitative, and "worst-case" analysis ([7]) shows that such a decision procedure takes at least 2^{k^n} steps, where $k > 0$ is a constant and n is the length of the input formula.

We also use a classical form of elimination theory, over $\mathbb{C}$: if $Z \subset \mathbb{C}^N \times \mathbb{C}^M$ is defined by equations which are homogeneous in y, then $p_1(Z) \subset \mathbb{C}^N$ is definable by polynomial equations. In particular, $p_1(Z)$ is closed.

A topological form of this elimination theorem also holds over $\mathbb{R}$, and is crucial in showing that (for $mp \leq n$) the image of the pole-placement map is Euclidean closed in $\mathbb{R}^n$ for the generic system ([4]). Our proof of Theorem 1 relies on this result.

We must also use rather explicit elimination arguments which have appeared in the literature. Among these are the works by Willems-Hesselink [23] and, more recently, Morse-Wolovich-Anderson [19] which treat the case $m = p = 2$, and $m = 2$, $p = 3$. These authors, after considerable calculation, obtain a single explicit equation in a single unknown and it is possible to obtain some quantitative and qualitative results from the form of the equations. Finally, we use the results of Brockett-Byrnes [3] who determined the degree of this equation, for general m, p, using methods of the Schubert calculus. This calculus was developed in the 19th century in order to deduce the degree of the final equation one would obtain in certain problems of enumerative geometry, without going through the elimination theory first. It is a fortunate fact that the return difference equation corresponds to a classical equation of enumerative geometry, enabling one to determine this degree as a function of m and p.

2. STATEMENTS OF THE MAIN RESULTS

Let us suppose that (F, G, H) is a triple of matrices which correspond to either a discrete or a continuous time system having m inputs, n states, and p outputs. We consider the questions, for m, n, p fixed:

Question 1: Is it true that for all (F, G, H), except perhaps those contained in a proper algebraic set, one can arbitrarily assign the (closed-loop) eigenvalues of $F+GKH$ by suitable choice of output feedback K?

Question 2: Is it true that one can stabilize all (F, G, H), except perhaps those contained in a proper algebraic set, by some output feedback K?

Concerning Question 1, it is known ([13], [23]) that $mp \geq n$ is a necessary condition on the parameters m, n, p. In Section 3 we derive a stabilizability criterion as a limiting form of the equivalence of generic stabilizability for continuous and for discrete time systems. This can be thought of as an equivalence between generic stabilizability and the generic existence to an output feedback deadbeat control problem for nondegenerate systems (in the sense of [3], [4]):

Theorem 1: If $mp \leq n$, the following statements are equivalent:

i) m,n,p are such that the generic (F,G,H) is stabilizable

ii) m,n,p are such that for any nondegenerate (F,G,H) there exists a gain K such that the closed loop polynomial is s^n.

This result holds for $mp > n$ as well, with nondegenerate replaced by the weaker term generic. Since we do not need the general result here, we shall only prove it in the case $mp \leq n$. From Theorem 1 we obtain

Theorem 2: $mp \geq n$ is necessary for generic stability.

This result of course implies that $mp \geq n$ is necessary for Question 1 as well, but also raises the question as to whether the answers to questions 1 and 2 might not agree, as functions of the parameters m, n, and p. On the one hand, if $\max(m,p) \geq n$ then generically either G or H is of rank n so that one is in the state feedback situation where the answer to Question 1, and therefore to Question 2, is well known to be in the affirmative under the generic hypothesis of reachability. On the other extreme, Theorem 2 shows that for $mp < n$ the answer to both questions is in the negative, so that explicit calculations for $mp \sim n$ are therefore quite interesting. However, aside from a few special cases, our knowledge is incomplete.

Example 1 $(m = p = 2)$: If $n = 4$, it has been shown by Willems-Hesselink ([23]) that pole placement does not hold for an open subset of (F, G, H). In [3] it is shown that pole placement does not hold unless the transfer function $T(s) = H(sI - F)^{-1}G$ has rank 1. In particular, pole placement does not hold for (F,G,H) in an open, dense set. In [19], necessary and sufficient conditions for generic pole placement, for a particular system of this dimension are derived.

Thus, by Kimura's Theorem [16] and the Willems-Hesselink counter-example, the answer to Question 1 is yes if, and only if, $n \leq 3$.

In [23] it is asserted that a modification by P. Molander of the techniques in [23] shows that the answer to Question 2 is in the negative if $n = 4$. Thus, the answers to Questions 1 and 2, if $m = p = 2$, are yes if, and only if, $n \leq 3$. Since this result is unpublished, in section 4 we present a verification of Molander's conclusion as a corollary to our generic stabilizability criterion. This of course gives another proof of the Willems-Hesselink theorem.

<u>Theorem 3 (Molander)</u>: There is a nonempty open set of (nondegenerate) 2 x 2 systems of degree 4 which are not stabilizable by constant output gain feedback.

<u>Example 2</u> ($m = 2$, $p = 2^k-1$): It is known in this case that the answer to Question 1, and therefore to Question 2, is in the affirmative ([3]) provided $mp \geq n$. By Theorem 2, the answer to both questions, for these values of m,p, is therefore yes if, and only if $mp \geq n$.

<u>Example 3</u>. ($m = 2$, $p = 4$) At present, one is able to deduce from the results proved in [3] and more refined topological methods that the answer to Question 1, and therefore to Question 2, is in the affirmative whenever $n \leq 7$. Theorem 2 then asserts that the only case which remains to be analyzed is $n = 8$, where it has been conjectured ([6]) that the answer to Question 1 is in the negative.

We should mention, however, that there are cases (e.g. $m = 2$, $p = 6$, $n = 9$) where generic stabilizability is known to hold, but where Question 1 remains unanswered ([5]).

Until now, we have only discussed the existence of solutions to the problems of pole positioning and stabilization. Equally important is the consideration of what kind of algorithm might exist for finding a gain K which places the poles, or stabilizes the system, provided such a gain exists. In Sections 5 and 6 we analyze each of these questions and prove

<u>Theorem 4</u>: Suppose there exists a gain which stabilizes the system (F, G, H). Then, one can find such a K by an algorithm which is rational in the coefficients of (F, G, H).

In [1] the question was raised as to whether rational formulae exist for a gain K which places the closed loop characteristic polynomial at $p(s) = s^n + p_1 s^{n-1} + \ldots + p_n$. That is, provided such a gain K exists, can one find K as a rational function of $(F, G, H, p_1, \ldots, p_n)$? This holds for the case of state feedback and, in particular, where $\min(m,p) = 1$ and $\max(m,p) \geq n$. In this case, a linear formula for K follows from consideration of the phase-variable canonical form. However, as the equation obtained by Willems-Hesselink (see also [3], [19]) shows for the case $m = p = 2$, $n = 4$, there exist precisely 2 gains (possibly a complex conjugate pair) counted with multiplicity which place a given real monic polynomial

$$s^4 + p_1 s^3 + \ldots + p_4.$$

Moreover, the coefficients of such a 2×2 gain K are given by the solution formula for a quadratic equation. Thus, in general, a rational formula does not exist. If $mp = n$, we can give a more precise answer to the question raised in [1]:

Theorem 5: If $mp = n$, the following statements are equivalent for the generic (F,G,H) and monic polynomial $p(s)$:

(a) there exists a rational formula, in the coefficients of $p(s)$ and entries of (F,G,H), for some K which places the closed loop polynomial at $p(s)$;

(b) there exists a linear formula, in the coefficients of $p(s)$ and entries of (F,G,H), for such a K;

(c) $\min(m,p) = 1$ and $\max(m,p) = n$.

Theorem 6: If $mp = n$, the following statements are equivalent for the generic (F,G,H) and monic polynomial $p(s)$:

(a) There exists a formula, involving rational expressions and square roots, for some K which places the closed loop polynomial at $p(s)$;

(b) either $\min(m,p) = 1$ or $\min(m,p) = \max(m,p) = 2$.

Indeed, if $mp = n$ we conjecture that the only cases for which there exists formulae for K involving rational operations and radicals are

i) $\min(m,p) = 1$ and $\max(m,p) = n$; or

ii) $\min(m,p) = \max(m,p) = 2$.

This conjecture appears natural in the light of our techniques (Section 6), which are an application of Galois theory and of the methods used in [3] enabling one to express the number $d_{m,p}$ of (perhaps complex) gains K which place the poles of a given generic (= nondegenerate) system at a given monic polynomial if $mp = n$. In fact

$$d_{m,p} = \frac{1! \ldots (p-1)!(mp)!}{m! \ldots (m+p-1)!}$$

This agrees with the Willems-Hesselink calculation ([20]), that $d_{2,2} = 2$ and with the recent calculation made by Morse-Wolovich-Anderson ([19]) that $d_{2,3} = d_{3,2} = 5$*.

Our methods for proving Theorem 4 rely quite heavily on the Tarski-Seidenberg Theorem (Prop. 3.2). In the course of the proof we need several other results from "decision algebra". With these results in hand, it only requires modest additional effort to show that the question raised in this paper, i.e. whether or not Questions 1 and 2 are equivalent for any fixed m,n,p triple, can in fact be answered by decision algebra. This is shown in the Appendix.

The actual application of a decision algebra based checking procedure is of course extremely impractical to implement but we should emphasize that, at present, this is the only method which is even in principle capable of answering this equivalence question for arbitrary m,n,p. For this reason, we feel it is worthwhile to give a proof of this statement.

*Based on our techniques and those in [12], the authors of [6] have confirmed our conjecture in the case $m = 2$, $p = 3$ by showing that the Galois group of the output feedback problem is the full symmetric group, S_5.

3. PROOF OF THEOREMS 1 and 2:

We shall begin by proving that, for m, n, p fixed, stabilizability for the generic $(F, G, H) \in \mathbb{R}^{n^2+n(m+p)}$ is equivalent to the property that $(s-\rho)^n$, $\rho \in \mathbb{R}$, may be assigned as the closed loop characteristic polynomial for the generic $(F, G, H) \in \mathbb{R}^{n^2+n(m+p)}$. It is intuitively clear that Question 2 should not distinguish between continuous time and discrete time stabilizability. This follows from the first lemma where $\varepsilon = 1$ and $\rho = 0$.

Lemma 3.1: The following statements are equivalent:

i) m,n,p are such that for all (F, G, H) - except perhaps those contained in a proper algebraic set - there exists a stabilizing gain K.

ii) m,n,p are such that for all (F, G, H) - except perhaps those contained in a proper algebraic set- for all real ρ and all $\varepsilon > 0$, there exists a gain K such that the eigenvalues of $F + GKH$ are contained in an ε-disc centered about ρ.

Proof: We first note that to say (1.1) is stabilizable is to say the system

$$\dot{x} = Fx + Gu, \quad y = Hx + Ju, \tag{3.1}$$

with J arbitrary but fixed is stabilizable. For, if K is a stabilizing gain for (1.1), and $I - KJ$ is nonsingular, then the gain $u = \bar{K}y$, where

$$\bar{K} = (I - KJ)^{-1}K,$$

stabilizes (3.1). If $I - KJ$ is singular, we may choose $\tilde{K}$ sufficiently close to K so that $\tilde{K}$ is a well-defined stabilizing gain for (3.1).

Now consider the conformal transformation

$$\phi(z) = [(z - \rho)/\varepsilon + 1]\,[(z - \rho)/\varepsilon - 1]^{-1}$$

and define the rational matrix valued function

$$V(z) = W(\phi(z)) = \tilde{H}(zI - \tilde{F})^{-1}\tilde{G} + \tilde{J} \qquad (3.2)$$

where $W(z)$ is the open loop transfer function,

$$W(z) = H(zI - F)^{-1}G + J \qquad (3.3)$$

Now let $\bar{K}$ be a gain such that the closed-loop poles of

$$W(z)(I + KW(z))^{-1}$$

are at $z_1, \ldots, z_n$. Then, generically, the poles $\phi(z_1), \ldots, \phi(z_n)$ of

$$V(z)(I + KV(z))^{-1}$$

will be finite. Since

$$Re[z] < 0 \quad \text{if, and only if,} \quad |\phi(z) - \rho| < \varepsilon$$

$\bar{K}$ stabilizes $W(z)$ with respect to $Re[z] < 0$ if, and only if, it stabilizes $V(z)$ with respect to the ε-disc centred about ρ.

We claim that, consequently, a generic (F,G,H,J) is stabilizable with respect to $\mathrm{Re}[z]<0$ if, and only if, a generic $(\tilde{F},\tilde{G},\tilde{H},\tilde{J})$ is stabilizable with respect to the ε-disc $B(\rho;\varepsilon)$. Assuming the claim, by our first observation the "direct part" J may be omitted, and the Lemma is proved.*

To verify the claim, we first develop W(z) in a Laurent series

$$W(z) = J + \sum_{i=1}^{\infty} L_i z^{-i}$$

and form the $n\times n$, $p\times m$-block Hankel matrix

$$h_W = [L_{i+j-1}]$$

Then W(z) determines, and is determined by, a point in the set

$$\mathscr{H}^n_{m,p} = \{(J,L_1,\ldots,L_{2n}) : \mathrm{rank}\ h_W = n\}$$

$\mathscr{H}^n_{m,p}$ is, by definition, an open subset of an algebraic set of matrices. Moreover, $\mathscr{H}^n_{m,p}$ is the image of the rational map

$$\Pi : \mathscr{M} \subset \mathbb{R}^{n^2+n(m+p)+mp} \to \mathscr{H}^n_{m,p}$$

defined on the open dense set $\mathscr{M}$ of minimal systems by

$$\Pi(F,G,H,J) = (J,L_1,\ldots,L_{2n})$$

where of course

$$H(sI-F)^{-1}G + J = J + \sum_{i=1}^{\infty} L_i z^{-i}$$

*Argument along these lines has been developed independently by J.C. Willems.

Therefore, $\mathscr{H}^n_{m,p}$ is irreducible, as the image of an irreducible algebraic set ([21]). In this language, we have:

(i) ϕ induces, via (3.2), a rational map

$$\Phi : \mathscr{H}^n_{m,p} \to \mathscr{H}^n_{m,p}$$

with singularities on the algebraic set $\{h_W : W \text{ has a pole at } 1\}$, since $V(z) = \Phi(W(z)) = W(\phi(z))$ is proper if, and only if, $W(1)$ is finite.

(ii) image $\Phi = \mathscr{H}^n_{m,p} - \{h_V : V \text{ has a pole at } \dot{\varepsilon} + \rho\}$ for similar reasons as in (i).

Furthermore, since stability of minimal systems is an input-output property, if $\mathscr{D}$ is a self-conjugate subset of $\mathbb{C}$, then

(iii) the set

$$U = \{\sigma = (F,G,H,J) : \sigma \text{ is stabilizable with respect to } \mathscr{D}\}$$

is open and dense in $\mathscr{M}$, if and only if,

$$\Pi(U) \subset \mathscr{H}^n_{m,p}$$

is open and dense in $\mathscr{H}^n_{m,p}$.

The claim then follows from (i), (ii), and (iii).

Q.E.D.

Remark: A similar, perhaps well-known, result is that for fixed m,n,p stabilizability is generic if, and only if, for generic (F,G,H) there exists a gain K such that the closed-loop spectrum lies in $Re[s] < \sigma$ or $Re[s] > \sigma$, with $\sigma \in \mathbb{R}$ arbitrary.

The next proof relies on the following result which is stated in the notation of (1.2). For f, g polynomials, set:

$$U\{f_i\} = \{x \in \mathbb{R}^n: \ f_i(x) > 0, \quad \forall i\},$$

$$V\{g_i\} = \{x \in \mathbb{R}^n: \ g_i(x) \geq 0, \quad \forall i\}. \tag{3.4}$$

A subset $Z \subset \mathbb{R}^n$ is called semialgebraic if it is finite union of finite intersections of sets of the form (3.4). For example, the algebraic set

$$Z = \{x \in \mathbb{R}^n: \ g(x) = 0\}$$

is semialgebraic. A subset of the form $U\{f_i\}$ is called a basic open semialgebraic set, and those of the form $V\{g_i\}$ are called basic closed semialgebraic sets.

Proposition 3.2: If $Z \subset X \times Y$ is a semialgebraic set, then $p_1(Z) \subset X$ is a semialgebraic set. Thus, the existence of Y such that

$$p_1(x_0, y) = x_0$$

can be checked by a finite number of rational operations in x_0.

This theorem is of course a version of the Tarski-Seidenberg Theorem. It is worth noting that a recent improvement on this result has been made ([8], [9]), viz. if it is known that $p_1(Z)$ is Euclidean closed (or open), then $p_1(Z)$ is a finite union of basic closed (or open) semialgebraic sets. Of course, $p_1(Z)$ is not necessarily closed, even if Z is closed.

Lemma 3.3: If $mp \leq n$, then the following statements are equivalent

i) m,n,p are such that the generic (F,G,H) is stabilizable.

ii) m,n,p are such that for all real ρ and for the generic (F,G,H), there exists a gain K such that the closed loop characteristic polynomial is $(s-\rho)^n$.

Proof: Statement (ii) obviously implies (i). For the converse, consider the function, for $\sigma = (F,G,H)$,

$$\chi_\sigma : \mathbb{R}^{mp} \to \mathbb{R}^n, \text{ defined via}$$

$$\chi_\sigma(K) = (p_1,\ldots,p_n) \tag{3.5}$$

where

$$s^n + p_1 s^{n-1} + \ldots + p_m = \det(sI - F - GKH)$$

If statement (i) holds, then for each r there exists an open dense subset $U_r \subset \mathbb{R}^{n^2} \times \mathbb{R}^{nm} \times \mathbb{R}^{np} = \mathbb{R}^N$ such that for $(F,G,H) \in U_r$,

$$(p_1,\ldots,p_n) \in \text{image } (\chi_\sigma)$$

where the roots of $s^n + p_1 s^{n-1} + \ldots + p_n$ lie in a $1/r$-disc centered about ρ. By the Baire category theorem,

$$U = \bigcap_{r=1}^{\infty} U_r$$

is a dense subset of $\mathbb{R}^N$ such that for $(F,G,H) \in U$,

$$(\bar{p}_1,\ldots,\bar{p}_n) \in \overline{\text{image } (\chi_\sigma)}$$

where

$$s_n + \bar{p}_1 s^{n-1} + \ldots + \bar{p}_n = (s-\rho)^n.$$

Now, according to Theorem, Section 4, of [4] provided $mp \leq n$ there exists an open dense subset $W \subset \mathbb{R}^N$ - the set of nondegenerate systems - such that image (χ) is Euclidean closed for $(F,G,H) \in W$. Thus, if

$$(F, G, H) \in U = U' \cap W$$

then

$$(\bar{p}_1, \ldots \bar{p}_n) \in \text{image}\,(\chi_\sigma)$$

Now, any real gain K may be regarded as a point in $\mathbb{R}^{mp}$ and we may consider the real algebraic set

$$V^\rho = \{(F,G,H,K) : \det(sI-F-GKH) = (s-\rho)^n\} \subset \mathbb{R}^N \times \mathbb{R}^{mp}. \qquad (3.6)$$

By the Tarski-Seidenberg Theorem (Proposition 3.2)

$$p_1(V^\rho) \subset \mathbb{R}^N,$$

the projection onto the first factor, is a semialgebraic set in $\mathbb{R}^N$; i.e., $p_1(V^\rho)$ is defined by a finite set of equations and inequations as in (3.4).

Since

$$U \subset p_1(V^\rho) = \mathbb{R}^N$$

is dense, it follows that $p_1(V^\rho)$ may be defined by algebraic conditions (perhaps disjunctive).

$$f_1(F,G,H) > 0,\ldots,f_r(F,G,H) > 0$$

from which it follows that $p_1(V^\rho)$ is open and dense. Since $(F,G,H) \in p_1(V^\rho)$ if, and only if, there exists a K such that the closed loop characteristic polynomial is $(s-\rho)^n$, the lemma is proved.

Q.E.D.

For the more precise assertion in part (ii) of Theorem 1, we need the following.

Lemma 3.4: For any $p = (p_1,\ldots,p_n) \in \mathbb{R}^n$, the subset

$$V_p = \{\sigma = (F,G,H) \in W : \chi_\sigma(K) = p \text{ for some } K\}$$

is closed in W.

Remark: The corresponding assertion for (F,G,H) minimal can be false. This is quite analogous to the fact that the set

$$\{x \in \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } xy = 1\}$$

is not closed in $\mathbb{R}$, while the set

$$\{x \in \mathbb{R} - \{0\} : \exists y \in \mathbb{R} \text{ such that } xy = 1\}$$

is closed in the open dense subset $W = \mathbb{R} - \{0\} \subset \mathbb{R}$.

Proof: As in [4], we may think of $K \in \mathbb{R}^{mp}$ as a point in Grass(p,m+p) - the set of p-plane in $\mathbb{R}^{m+p}$ - via the assignment

$$K \longmapsto \text{graph}(K) = \{(y,Ky)\} \subset \mathbb{R}^p \oplus \mathbb{R}^m.$$

It is known (see e.g. [4] and references cited therein) that Grass(p,m+p) may be regarded as a compact manifold of dimension mp. Moreover,

$$\text{Grass}(p,m+p) = \mathbb{R}^{mp} \cup \sigma(\infty)$$

where $\sigma(\infty)$ is the closed subset defined by

$$\sigma(\infty)=\{\Pi \in \text{Grass}(p,m+p) : \dim(\Pi \cap \mathbb{R}^m) \geq 1\}$$

That is, $\Pi \in \sigma(\infty)$ if, and only if, Π is not complementary to U. Thus, $\Pi \notin \sigma(\infty)$ if, and only if,

$$\Pi = \text{graph}(K), \text{ for some linear } K : \mathbb{R}^p \to \mathbb{R}^m$$

On the other hand, one may regard the monic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_n$$

as a point $(p_1,\ldots,p_n) \in \mathbb{R}^n$ and therefore ([4]) as a point, via the homogeneous coordinates

$$[p_1,\ldots,p_n,1] \in \mathbb{RP}^n,$$

in real projective n-space. Of course, $\mathbb{RP}^n = \text{Grass}(1,n+1)$ by definition. According to ([4], Remarks, p. 103), for nondegenerate σ the map χ_σ extends continuously to a map

$$\chi_\sigma : \text{Grass}(p,m+p) \to \mathbb{RP}^n,$$

satisfying:

$$\chi_\sigma(\Pi) = [p_1,\ldots,p_n,0] \qquad (3.7a)$$

if, and only if,

$$\Pi \in \sigma(\infty) \tag{3.7b}$$

Matters being so, consider the continuous function

$$\chi : W \times \mathrm{Grass}(p,m+p) \to \mathbb{RP}^n$$

defined via

$$\chi(F,G,H,\Pi) = \chi(\sigma,\Pi) = \chi_\sigma(\Pi)$$

Therefore, if $\overline{p} = [1,0,\ldots,0,1]$ corresponds to $\overline{p}(s) = s^n$,

$$Z = \chi^{-1}(\overline{p}) \subset W \times \mathrm{Grass}(p,m+p)$$

is a closed subset. Since Grass(p,m+p) is compact,

$$p_1(Z) \subset W$$

is closed and, by virtue of (3.6),

$$p_1(Z) = \{\sigma = (F,G,H) : \chi_\sigma(K) = \overline{p}, \text{ for some } K\} = V_{\overline{p}}$$

Q.E.D.

On the other hand, $U \cap W \subset V_{\overline{p}}$ is dense in W by the Baire Category Theorem, and therefore

$$V_{\overline{p}} = W$$

from which (ii), and Theorem 1, follows.

Q.E.D.

We now turn to a proof of Theorem 2. Clearly, it suffices to consider the case $mp \leq n$; thus, the preceding lemmata and Theorem 1 are applicable.

Consider, then, the algebraic set of nilpotent $n \times n$ real matrices

$$\mathcal{N} = \{N : N^k = 0, \text{ for some } k\}$$

and the algebraic set $V = V^0$ obtained by setting $\rho = 0$ in (3.6). We define the polynomial mapping

$$\Phi : \mathcal{N} \times \mathbb{R}^{nm} \times \mathbb{R}^{np} \times \mathbb{R}^{mp} \to \mathbb{R}^{n^2} \times \mathbb{R}^{nm} \times \mathbb{R}^{np} \qquad (3.8)$$

via

$$\Phi(N,G,H,K) = (N - GKH, G, H)$$

From Theorem 1, we have:

Lemma 3.5: If $mp \leq n$ and if the generic system is stabilizable, the image of Φ contains an open, dense set.

Denote by $\mathcal{N}_{\mathbb{C}}$ the algebraic set of $n \times n$ complex matrices. It is known (see e.g. [17], [20]) that $\mathcal{N}_{\mathbb{C}}$ is an irreducible algebraic set. Therefore there exists an open dense subset U of $\mathcal{N}_{\mathbb{C}}$ which is itself a complex manifold and therefore has a dimension. Indeed ([17], [20]),

$$\dim_{\mathbb{C}}(U) = n^2 - n$$

The points of U are called simple, and one of the thorny points in real algebraic geometry ([18]) is that in general an irreducible real algebraic set $V_{\mathbb{R}}$ may contain none of the simple points of $V_{\mathbb{C}}$.

This, for example, is the reason for the failure of the Hilbert Nullstellensatz over $\mathbb{R}$, and the most well known example of this phenomenon is

$$W_{\mathbb{R}} = \{(x,y) : x^2 + y^2 = 0\}$$

If $V_{\mathbb{R}}$ contains a simple point of $V_{\mathbb{C}}$, then for example $\dim_{\mathbb{R}}(V_{\mathbb{R}})$ is defined as above and

$$\dim_{\mathbb{R}}(V_{\mathbb{R}}) = \dim_{\mathbb{C}}(V_{\mathbb{C}}) \tag{3.9}$$

It is an elementary computation to check that the real matrix

$$N = \begin{bmatrix} 0 & 1 & & & \\ & 0 & \cdot & & 0 \\ & & \cdot & \cdot & \\ 0 & & & \cdot & 1 \\ & & & & 0 \end{bmatrix}$$

is a simple point of $\mathcal{N}_{\mathbb{C}}$. Thus, $\dim_{\mathbb{R}} N$ exists. We will now give a self-contained proof of

<u>Lemma 3.6</u>: $\dim_{\mathbb{R}}(\mathcal{N}) = n^2 - n$.

<u>Proof</u>: Since the matrix N consists of a single Jordan block, the dimension of the centralizer

$$Z(N) = \{T \in GL(n,\mathbb{R}) : TN = NT\}$$

is n, according to the Frobenius dimension formula ([15] Vol. II, Thm. 19, p. 111). Now consider the orbit of N under $GL(n,\mathbb{R})$

$$\mathcal{O}(N) = \{TNT^{-1} : T \in GL(n,\mathbb{R})\} \simeq GL(n,\mathbb{R}) / Z(N)$$

In particular,

$$\dim_{\mathbb{R}} \mathcal{O}(N) = \dim GL(n,\mathbb{R}) - \dim Z(N) = n^2 - n$$

We claim $\overline{\mathcal{O}(N)} = \mathcal{N}$ from which follows:

(i) $\mathcal{N}$ is irreducible, since $\mathcal{O}(N)$ is irreducible; and

(ii) $\dim_{\mathbb{R}} \mathcal{O}(N) = \dim_{\mathbb{R}}(\mathcal{N})$, by the closed orbit lemma and (3.9).

Following [20], note that if N_i is any nilpotent Jordan canonical form, then clearly there is a 1-parameter diagonal subgroup $T_\lambda \in GL(n,\mathbb{R})$ such that

$$\lim_{\lambda\to\infty} T_\lambda N T_\lambda^{-1} = N_i$$

Therefore, $\overline{\mathcal{O}(N)} = \mathcal{N}$.

Q.E.D.

Now suppose that m, n, and p are such that the generic system is stabilizable, and $mp \leq n$. By Lemma 3.5 and ([21], Thm. 7, p. 60) one has

$$\dim \mathcal{N}_n + n(m+p) + mp \geq n^2 + n(m+p) \tag{3.10}$$

In the light of Lemma 3.6 and (3.10),

$$n^2 - n + mp \geq n^2$$

yielding

$$mp \geq n \ .$$

In conclusion, if $mp \leq n$ then $mp = n$ is necessary for generic stabilizability, whence Theorem 2.

Q.E.D.

4. Proof of Theorem 3.

In the proof of Theorem 1 (cf. Lemma 3.2) we made use of certain facts concerning $p \times m$ systems of degree n which also allow us to show, together with Theorem 1, that for $n = 4$, $m = p = 2$, generic stabilizability is not possible. Specifically:

(i) if $mp \leq n$, then the class W of nondegenerate systems is open and dense in $\mathbb{R}^{n^2} \times \mathbb{R}^{mn} \times \mathbb{R}^{np}$; and

(ii) for any monic polymonial $p(s)$ of degree n, the set

$$V_p = \{(F,G,H) \in W: \det(sI-F-GKH) = p(s) \text{ for some } K\}$$

is closed in W (Lemma 3.4).

In light of Theorem 1, if $\bar{p}(s) = s^n$ then generic stabilizability implies that $V_{\bar{p}}$ is dense and closed in W, hence coincides with W. Therefore, to find one nondegenerate system for which $\bar{p}(s)$ is not assignable as a closed loop polynomial is to prove that stabilizability is not generic.

We shall now give a "frequency domain" criterion [3] (which can be taken as a definition, compare [4]) for nondegeneracy. If $T(s)$ is the transfer function

$$T(s) = H(sI-F)^{-1}G \tag{4.1}$$

of the system, denote by $t_i(s)$ the i-th column of the $(p+m) \times m$ matrix.

$$\mathcal{T}(s) = \begin{bmatrix} T(s) \\ I \end{bmatrix}$$

If $\phi(y,u)$ is a complex linear functional on $\mathbb{C}^p \oplus \mathbb{C}^m$, then we can form the scalar rational function

$$\phi(t_i(s)) \qquad \text{for } i = 1,\ldots,m \; .$$

Now suppose $\Phi = \{\phi_1,\ldots,\phi_p\}$ is any linearly independent set of linear functionals on (m+p)-space, and form the determinant

$$\Phi(s) = \det\left[\phi_i(t_j(s))\right] \tag{4.2}$$

(F,G,H) is said to be nondegenerate provided

$$\Phi(s) \not\equiv 0 \quad \text{in } s \tag{4.3}$$

for any choice of Φ.

Remark 1. If (F,G,H) is scalar, then (F,G,H) is nondegenerate since (4.2-(4.3) reduces, for $\phi(u,y) = au + by$, to

$$ag(s) + b \not\equiv 0 \quad \text{in } s \; .$$

2. The zeroes of the set $\Phi = \{\phi_1,\ldots,\phi_m\}$ defines a p-plane in (u,y)-space which is the graph either of a linear function $u = Ky$, i.e. a finite constant gain, or of a linear relation between u and y, i.e. an infinite constant gain. The zeroes of (4.2) are then, modulo pole-zero cancellation, the closed loop poles at this gain and (4.3) just asks that these zeroes be finite in number, i.e. that the root-locus map χ be defined and continuous at this gain.

Example 4. Suppose $m = p = 2$, $n = 4$, and consider

$$G = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} I & 0 \end{bmatrix}$$

and

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

We claim (F,G,H) is nondegenerate, to this end we compute (clearing denominators)

$$\mathscr{T}(s) = \begin{bmatrix} s^3 - 1 & - s \\ s & s^3 \\ s^4+s-1 & 0 \\ 0 & s^4+s-1 \end{bmatrix}$$

and consider 2 linear functionals

$$\phi_1(y,u) = a_1y_1+a_2y_2+a_3u_1+a_4u_2, \quad \phi_2(u,y)=b_1y_1+b_2y_2+b_3u_1+b_4u_2 \quad .$$

Thus,

$$\Phi(s) = \det [\phi_i(t_j(s))] = \det\begin{bmatrix} \alpha_{11}(s) & \alpha_{12}(s) \\ \alpha_{21}(s) & \alpha_{22}(s) \end{bmatrix} \tag{4.4}$$

where

$$\begin{aligned}
\alpha_{11}(s) &= a_3 s^4 + a_1 s^3 + (a_2+a_3)s - a_3 - a_1, \\
\alpha_{12}(s) &= a_4 s^4 + a_2 s^3 + (a_4-a_1)s + a_4 , \\
\alpha_{21}(s) &= b_3 s^4 + b_1 s^3 + (b_2-b_3)s + b_3 - b_1, \\
\alpha_{22}(s) &= b_4 s^4 + b_2 s^3 + (b_4-b_1)s + b_4
\end{aligned} \tag{4.5}$$

Now, (4.4) vanishes just in case there exists c_s - a priori depending on s - such that

$$\begin{aligned}
c_s \alpha_{11}(s) &= \alpha_{21}(s) \\
c_s \alpha_{12}(s) &= \alpha_{22}(s)
\end{aligned} \tag{4.6}$$

for all but finitely $s \in \mathbb{C}$. Comparing coefficients shows that c_s is constant for all but finitely many and hence all, s and therefore an inspection of (4.5)-(4.6) shows that

$$c\,\phi_1 = \phi_2$$

contradicting linear independence of the functionals ϕ_i.

Recall, in the proof of Lemma 3.3 the fact that image (χ) is closed for all nondegenerate (F,G,H) was used rather crucially. If

$$K = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} ,$$

it is readily verified that

$$-\det(sI-F-GKH) = s^4 + (-x_1-x_4)s^3 + (x_1x_4-x_2x_3)s^2 + (1-x_2+x_3)s + (1+x_1) \qquad (4.7)$$

By the quadratic formula, it is easily verified that image(χ) is a closed semialgebraic set. Furthermore, if (4.7) is to be s^4, we require

$$x_1 + x_4 = x_1x_4 - x_2x_3 = 1 - x_2 + x_3 = 1 + x_1 = 0 \qquad (4.7)'$$

whence

$$x_2x_3 = -1 \quad , \quad x_2 = 1 + x_3 \ ,$$

whence

$$x_3^2 + x_3 + 1 = 0 \qquad (4.8)$$

This equation (4.8) cannot be satisfied by any real x_3, i.e. there is no real gain producing closed loop poles at $s = 0$. Since (F,G,H) is nondegenerate our previous remarks imply Theorem 3, thereby verifying Molander's conclusion.

5. PROOF OF THEOREM 4:

In addition to the Tarski-Seidenberg Theorem (Proposition 3.2), we shall also need a somewhat different result from decision algebra, which deals with the question of describing the set of x_o for which $p_1^{-1}(x_o) \cap Z = \{x_o\} \times Y$; i.e. for which $(x_o, y) \in Z$ for all y. In the course of deriving this result we also will state the Tarski-Seidenberg theorem in what is perhaps a more familiar form ([15] Vol. III, [22]).

Notational conventions are as follows: x, y, z denote collections of indeterminates, with each of x, y, z considered to be shorthand for a number of indeterminates $x_1, \ldots, x_n$ etc. Particular real values taken by these quantities will be denoted by $\hat{x}, \hat{y}, \hat{z}$; p, q, r, s, perhaps with subscripts will denote polynomials in x, y, z with real coefficients. We shall regard $p(x, y) = 0$ or $q(x, y) \geq 0$ as examples of equations or inequations, (i.e. descriptions of problems for which solutions are sought, should they exist), and we shall regard $p(\hat{x}, \hat{y}) = 0$ or $q(\hat{x}, \hat{y}) \geq 0$ as examples of equalities or inequalities (i.e. statements of fact that can be verified by arithmetic, and which show that $\hat{x}, \hat{y}$ are solutions of $p(x, y) = 0$ or $q(x, y) \geq 0$).

We shall reserve script letters $\mathcal{S}$, $\mathcal{T}$, etc. to denote collections of a finite number of equations and inequations or equalities and inequalities of the following type. $\mathcal{S}(x)$ is an abbreviation for:

either $\{p_{i1}(x) = 0$ and $q_{j1}(x) > 0$ and $r_{k1}(x) \neq 0$ and $s_{\ell 1}(x) \geq 0\}$

or $\{p_{i2}(x) = 0$ and $q_{j2}(x) > 0$ and $r_{k2}(x) \neq 0$ and $s_{\ell 2}(x) \geq 0\}$

or

⋮

or $\{p_{it}(x) = 0$ and $q_{jt}(x) > 0$ and $r_{kt}(x) \neq 0$ and $s_{\ell t}(x) \geq 0\}$

where it is understood that $p_{i\alpha}(x) = 0$ is shorthand for $p_{1\alpha}(x) = 0$ and $p_{2\alpha}(x) = 0$ and ... and $p_{i_\alpha \alpha}(x) = 0$, and similarly for $q_{j\alpha}$ etc. Naturally, $S(\hat{x})$ is an abbreviation for the associated set of equalities and inequalities. We can talk of the problem of solving $S(x)$ and of $S(\hat{x})$ holding, or of $\hat{x}$ being a solution of $S(x)$.

The above type of $S(x)$ is more or less standard in decision algebra. However, we shall sometimes use a simple modification. Each $s_{\alpha\beta} \geq 0$ is a disjunction: $s_{\alpha\beta} > 0$ or $s_{\alpha\beta} = 0$. This means that any $S(x)$ and thus any $S(\hat{x})$ can be rewritten to exclude inequations or inequalities of the $\geq$ type.

Lemma 5.1: The statement $S(\hat{x})$ does not hold is equivalent to a statement $\bar{S}(\hat{x})$ holds where $\bar{S}(x)$, termed the negator of S, is itself a collection of equations and inequations of the standard form.

Proof: "$S(\hat{x})$ holds" is a disjunction ("or" statement) of conjunctions ("and" statements) of formulas of the type $p(\hat{x}) = 0$, $q(\hat{x}) > 0$, $r(\hat{x}) \neq 0$ and $s(\hat{x}) \geq 0$. Hence "$S(\hat{x})$ does not hold" is a conjunction of disjunctions of negations of these formulas, i.e. of $p(\hat{x}) \neq 0$, $-q(\hat{x}) \geq 0$, $r(\hat{x}) = 0$ and $-s(\hat{x}) > 0$. Any conjunction of disjunctions can be rearranged as a disjunction of conjunctions, and in this way, $\bar{S}(x)$ is defined.

Obviously, $\bar{\bar{S}} = S$.

Next, we recall the main result of decision algebra, the Tarski-Seidenberg theorem. We break it into two parts.

Proposition 5.2: A) Consider an equation/inequation set $S(x, y)$. Then one can determine by a finite number of rational calculations a second such set $T(y)$ such that $T(\hat{y})$ holds if and only if there exists at least one $\hat{x}$ such that $S(\hat{x}, \hat{y})$ holds.

B) The solvability of any equation/inequation set $T(y)$ is determinable by a finite number of rational calculations.

We remark that the set $T(y)$ in part A may be empty: this would imply that there are no pairs $\hat{x}, \hat{y}$ for which $S(\hat{x}, \hat{y})$ holds.

Proposition 5.3: Consider an equation/inequation set $S(x, y)$. Then the set of values $\hat{y}$ of y such that for all $\hat{x}$, $S(\hat{x}, \hat{y})$ holds, is definable by an equation/inequation set $T(y)$.

Proof: Let $\bar{S}(x, y)$ be the negator of $S(x, y)$, existing by Lemma 5.1. By Proposition 5.2A, we can find $\bar{T}(y)$ such that $\bar{T}(\hat{y})$ holds if and only if there exists at least one $\hat{x}$ such that $\bar{S}(\hat{x}, \hat{y})$ holds. Let T be the negator of $\bar{T}$. Then $T(\hat{y})$ holds if and only if there exists no $\hat{x}$ such that $\bar{S}(\hat{x}, \hat{y})$ holds, i.e. if and only if for all $\hat{x}$, $S(\hat{x}, \hat{y})$ holds.

The following algorithm, in conjunction with Propositions 5.2 and 5.3, gives a proof of Theorem 4. We find it convenient to break this into two parts.

Part I: Find a cube containing a stabilizing gain K.

I.1 Choose $N > 0$, and consider the semialgebraic set

$$Z \subset \mathbb{R}^{mp-1} \times \mathbb{R}$$

defined by $Z = Z_1 \cap Z_2$, with

$$Z_1 = \{K: \det(sI - F - GKH) \text{ is Hurwitz}\},$$

$$Z_2 = \{K: \Sigma(k_{ij})^2 < N\}.$$

From the Routh-Hurwitz criterion, it follows that Z_1 is a (basic open) semialgebraic set and it is then clear that Z is semialgebraic. Using Proposition 5.2 inductively, we can decide by rational operations whether there exists a gain $K \in Z$. If $Z \neq \phi$, go to Step II.1. Otherwise, go to step I.2.

I.2 Replace N by $2N$ and go to Step I.1. Since a stabilizing gain exists by hypothesis, we will eventually move to Part II.

Part II: Find a cube contained in the set of stabilizing gains K.

II.1 We suppose there is a stabilizing gain K in the cube $||K|| < N$. Using Proposition 5.3 inductively, we can decide by rational operations whether all such K are stabilizing. If so, choose any K such that $||K|| < N$. If not, go to step II.2.

II.2 Divide the cube into 2^{mp} cubes with sides of length N. Return to step I.1 with this list of cubes.

This algorithm will stop at some stage, since the set of stabilizing gains is open and therefore contains a cube of sufficiently small size.

Q.E.D.

Example 5. One might ask whether one can bound the number of steps in this program simply in terms of m, n, p. The answer is no, as we now illustrate. Consider the open loop system with transfer function

$$w(s) = \frac{1}{s^3 + as^2 + bs}$$

where $a,b > 0$. For negative feedback with gain k, the closed loop characteristic polynomial is $s^3 + as^2 + bs + k$ and therefore is Hurwitz if, and only if, $k \in (0, ab)$. It follows that the size of a cube (here, an interval) contained in the open set of stabilizing gains can be made arbitrarily small by suitable choice of ab. In turn, the number of steps in Part II of the algorithm can be made arbitrarily large, though for fixed (a, b) it is of course finite.

6. PROOF OF THEOREMS 5 AND 6:

Since we have already demonstrated the existence of linear formulae for the appropriate values of m, n, p, it is enough to show that these are the only values for which such formulae can exist. Moreover, it suffices to prove this last assertion over $\mathbb{C} = \mathbb{R}(\sqrt{-1})$. Consider the closed loop characteristic coefficient map χ, defined in (3.4), extended to gains with complex coefficients

$$\chi_{\mathbb{C}}: \mathbb{C}^{mp} \to \mathbb{C}^{n} \tag{6.1}$$

where (F, G, H) is understood to be a generic, but fixed, system with $n = mp$. We first analyze the question as to whether there exists a formula for $(k_{ij}) \in \chi^{-1}(p)$ which is rational in the coordinates of $p = (p_{\ell}) \in \mathbb{C}^n$. Thus, we consider the field K_1 of all rational expressions (or functions) in the p_{ℓ}, and the field K_2 of all rational functions in the (k_{ij}):

$$K_1 = \mathbb{C}(p_{\ell}), \qquad K_2 = \mathbb{C}(k_{ij}) \tag{6.2}$$

Since $\chi_{\mathbb{C}}$ is polynomial, if $f \in K_1$ then $f \circ \chi_{\mathbb{C}} \in K_2$. For generic (F, G, H), image $\chi_{\mathbb{C}}$ contains an open set ([13]) so that

$$f \circ \chi_{\mathbb{C}} = 0 \Longrightarrow f = 0. \tag{6.3}$$

By virtue of (6.3), we can think of K_1 as a subfield of K_2, i.e.

$$K_1 \simeq \chi_{\mathbb{C}}^{*} K_1 \subset K_2 \tag{6.4}$$

where $\chi_{\mathbb{C}}^{*} f = f \circ \chi_{\mathbb{C}}$, and an easy dimension argument shows that (6.4) is a finite field extension. That is K_2, as a vector space over the field of scalars K_1, is finite dimensional. For example, to say rational formulae for $(k_{ij}) \in \chi_{\mathbb{C}}^{-1}(p_\ell)$ exist is to say the dimension of this vector space

$$\delta = [K_2 : K_1] = \dim_{K_1}(K_2) \tag{6.5}$$

is equal to 1, i.e. $K_1 = K_2$. We shall now give a formula for δ, in terms of m, p. In [4] it was shown that $\chi_{\mathbb{C}}$ is proper and it follows from the proof in [4] that

$$R_1 \simeq \chi_{\mathbb{C}}^{*} R_1 \subset R_2$$

is an integral ring extension, where

$$R_1 = \mathbb{C}[p_\ell], \qquad R_2 = \mathbb{C}[k_{ij}].$$

In this case (since the field $\mathbb{C}$ has characteristic zero), δ is given by the number d of solutions, counted with multiplicity, to the equation

$$\chi_{\mathbb{C}}(K) = p$$

([18]pp.116-117). On the other hand, d has been computed using methods of the Schubert calculus in [3] to be

$$d = \frac{1! \ldots (p-1)!(mp)!}{m! \ldots (m+p-1)!} \tag{6.6}$$

Thus, Theorem 5 follows from the following elementary observation.

Lemma 6.1: In (6.6), $d = 1 \iff \min(m,p) = 1$.

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As for Theorem 6, from the explicit form of the solution to the pole-placement equations, derived via elimination methods by Willems-Hesselink [2 3], it is clear that (over $\mathbb{R}$ or $\mathbb{C}$) quadratic formulae and rational expressions are sufficient to express K as a function of $(p_1,\ldots,p_n)$ for generic (F,G,H) when $m=p=2$, and $n = 4$. We shall now prove that, except for the linear cases $\min(m,p) = 1$, this is the only case when formula - involving square roots and rational operations - for K in terms of $(p_1,\ldots,p_n)$ exist.

To this end, we consider a Galois extension

$$K_1 \subset K, \tag{6.9}$$

that is, a minimal normal extension of $K_1 = \mathbb{C}(p\ )$ which contains all of the roots to the equation

$$\chi_{\mathbb{C}}(K) = (p\ ). \tag{6.10}$$

If a solution expressible by square roots and rational operations alone exists, then

$$\delta' = [K : K_1]$$

is a power of 2 ([2]). On the other hand, by Artin's Theorem of the Primitive Element [2], we may regard $K_2 \subset K$ and therefore

$$\delta = [K_2 : K_1] \quad \text{divides} \quad [K : K_1] \quad ,$$

from which it follows that

$$\delta = d_{m,p} = 2^r\ , \quad \text{for some} \quad r.$$

Theorem 6 therefore follows from the following result:

Lemma 6.2: If $\min(m,p) \geq 2$ and $m+p \geq 5$, then $d_{m,p}$ is divisible by an odd prime.

Remark: It is known [14] that any prime $q \leq \frac{\min(m,p)+1}{2}$ divides $d_{m,p}$, so that only the cases $\min(m,p) = 2$, 3 or 4 remain. The proof we present here, however, is valid for all m,p and is based on an application of the strong form of Bertrand's postulate ([11], p. 373) shown to us by W.H. Gustafson.

Proof: By the strong form of Bertrand's postulate, there is a prime q satisfying

$$m+p-1 < q < 2(m+p)-4, \tag{6.11}$$

under the hypothesis $m+p \geq 5$. Clearly, q does not divide the denominator of $d_{m,p}$. On the other hand, if $\min(m,p) \geq 2$, then

$$mp > q$$

so that q divides the numerator of $d_{m,p}$. Hence, $q \mid d_{m,p}$.

Q.E.D.

APPENDIX: "IN PRINCIPLE" ANSWERS TO QUESTIONS 1 AND 2 BY DECISION ALGEBRA

In [1], indications of the applicability of decision algebra to problems of systems theory were given. In particular, it was shown that one can determine, at least in principle, by rational operations whether a given system (F, G, H) can be stabilized. We shall extend these results to show that one can answer Questions 1 and 2 by rational operations using decision theoretic techniques, but we emphasize that such results are very qualitative. In fact, a "worst-case" analysis ([7]) shows that any decision procedure takes at least 2^{k^n} steps, where $k > 0$ is a constant and n is the length of the input formula.

However, in the absence of any other technique which allows one for example, even in principle, to distinguish between Questions 1 and 2, we thought it worthwhile to point out that this is a question which can be answered by the Tarski-Seidenberg theory. An interesting special case is whether or not we can place poles for generic 2×4 systems with McMillan degree 8. One does know that there exist 14 complex solutions to the pole-placement equations, but at present one does not know whether any of these are real.

The new ingredient here is the consideration of the generic system (F, G, H) rather than a particular choice of system (F_0, G_0, H_0), and we shall need to present some further results from decision algebra. The notation is as in Section 5.

<u>Lemma A.1</u>: Consider an equation/inequation set $\mathcal{S}(x, y, z)$. Then there exists a set $\mathcal{T}(y)$ such that $\mathcal{T}(\hat{y})$ holds if and only if for all $\hat{z}$, there exists $\hat{x}$ depending on $\hat{y}, \hat{z}$ with $\mathcal{S}(\hat{x}, \hat{y}, \hat{z})$ holding.

Proof: By Proposition 5.1′, there exists $R(y, z)$ such that $R(\hat{y}, \hat{z})$ holds if and only if $S(x, \hat{y}, \hat{z})$ is solvable, i.e. if and only if there exists at least one $\hat{x}$, depending on $\hat{y}$ and $\hat{z}$, such that $S(\hat{x}, \hat{y}, \hat{z})$ holds. By Proposition 5.3, there exists $T(y)$ such that $T(\hat{y})$ holds if and only if $R(\hat{y}, \hat{z})$ holds for all $\hat{z}$. Then clearly, $T(\hat{y})$ holds if and only if for all $\hat{z}$, there exists $\hat{x}$ such that $S(\hat{x}, \hat{y}, \hat{z})$ holds.

In Proposition 5.3 and Lemma A.1, the set $T(y)$ may be empty. The following Lemma replaces the "all $\hat{x}$" in Proposition 5.3 by "almost all", and in this sense may enable one to get a practical result when the $T(y)$ of this proposition is empty.

<u>Lemma A.2</u>: Consider an equation/inequation set $S(x, y)$. Then there exists an equation/inequation set $T(y)$ such that $T(\hat{y})$ holds if and only if $S(\hat{x}, \hat{y})$ holds for all $\hat{x}$ save a set contained in a proper variety depending on $\hat{y}$.

Proof: Given a polynomial $p(x, y)$, it is clear that there exists a possibly empty $P(y)$ such that $P(\hat{y})$ holds if and only if $p(x, \hat{y})$ is the zero polynomial, i.e. $p(\hat{x}, \hat{y}) = 0$ for all $\hat{x}$. Further, if $p(\hat{x}, \hat{y}) = 0$ for all $\hat{x}$ save those lying in a proper variety, $p(\hat{x}, \hat{y}) = 0$ for all $\hat{x}$.

Given a polynomial $r(x, y)$, it is clear that there exists $R(y)$ such that $R(\hat{y})$ holds if and only if $r(x, \hat{y}) \neq 0$ is solved by all x save those on a proper variety depending on $\hat{y}$.

Given a polynomial $s(x, y)$, it is clear that there exists $\bar{S}(y)$ such that $\bar{S}(y)$ holds if and only if $s(\hat{x}, \hat{y}) < 0$ for some $\hat{x}$. Hence $S(\hat{y})$ holds if and only if $s(\hat{x}, \hat{y}) \geq 0$ for all $\hat{x}$. Further, if $s(\hat{x}, \hat{y}) \geq 0$ for all $\hat{x}$ save those in a proper variety, $s(\hat{x}, \hat{y}) \geq 0$ for all $\hat{x}$.

Given a polynomial $q(x, y)$, it is clear that there exists $Q_1(\hat{y})$ such that $q(\hat{x}, \hat{y}) \geq 0$ for all $\hat{x}$ and $Q_2(\hat{y})$ such that $q(\hat{x}, \hat{y}) \neq 0$ for all $\hat{x}$ save those in a proper variety. Let $Q(y)$ denote the conjunction of $Q_1(y)$ and $Q_2(y)$. Then $Q(\hat{y})$ holds if and only if $q(\hat{x}, \hat{y}) > 0$ for all $\hat{x}$ save those in a proper variety depending on $\hat{y}$.

Suppose now that $S(x, y)$ is the disjunction of equation/inequation sets $S_i(x, y)$ where each $S_i(x, y)$ is a conjunction of

$$p_{\alpha i}(x,y) = 0 \quad q_{\beta i}(x,y) > 0 \quad r_{\gamma i}(x,y) \neq 0 \quad s_{\delta i}(x,y) \geq 0$$

By the discussion above, it is clear that there exists $T_i(y)$ such that $T_i(\hat{y})$ holds if and only if $S_i(\hat{x}, \hat{y})$ holds for all $\hat{x}$ save those in a proper variety depending on $\hat{y}$. $T(y)$ is obtained as the disjunction of the $T_i(y)$.

Now consider the system (1.1), subject to output feedback $u = Ky$. The coefficients of the closed loop characteristic polynomial, as a function of K, give rise to the polynomial mapping (3.4)

$$\chi: \mathbb{R}^{mp} \to \mathbb{R}^n$$

and we write $\chi_{(F,G,H)}$ to emphasize the dependence on the open loop system (1.1). Then, Question 1 asks whether $\chi_{(F,G,H)}$ is surjective for the generic (F,G,H) and we claim that this question can be answered within the scope of decision algebra. To this end, let $X = \mathbb{R}^{mp}$ $Y = \mathbb{R}^{n^2+nm+np}$ and $Z = \mathbb{R}^n$, so that $(K, (F,G,H), (p_\ell)) \in X \times Y \times Z$, and consider the algebraic subset $W \subset X \times Y \times Z$ defined by the equations

$$S(x, y, z): \quad \chi_{(F,G,H)}(K) = (p_\ell) \tag{A.1}$$

By Lemma A.1, there exists an equation/inequation set T in $y = \{F, G, H\}$ such that $T(\hat{y})$ holds if and only if for all $\hat{z}$, i.e. for all p_ℓ, there exists $\hat{x}$, i.e. a value of K, such that $S(\hat{x}, \hat{y}, \hat{z})$ holds, i.e. such that (A.1) holds.

Let $\bar{T}(y)$ denote the negator of T, and write $\bar{T}(y)$ as a disjunction of conjunctions $\bar{T}_i$. As observed in Section 4, we can assume without loss of generality that each $\bar{T}_i$ contains equations $p_{\alpha i}(y) = 0$, and inequations $q_{\beta i}(y) > 0$ and $r_{\gamma i}(y) \neq 0$, <u>without</u> inequations of the type $s_{\delta i}(y) \geq 0$. We can determine, see Proposition 5.2B, whether any $\bar{T}_i$ defines an empty set of solutions; if so, we discard it.

Now with $\bar{T}_i$ of the form just noted, and with each possessing a solution, we can readily answer Question 1.

If $\bar{T}_i(\hat{F}, \hat{G}, \hat{H})$ holds for any i, pole positionability for all α_i via choice of K is not possible, and conversely. It follows that if each $\bar{T}_i$ includes one or more equalities, then the set of $\hat{F}, \hat{G}, \hat{H}$ for which pole-positionability is not possible lies within a proper variety, and that for almost all $\hat{F}, \hat{G}, \hat{H}$, pole positionability for all p_ℓ can be achieved.

On the other hand, if $\bar{T}_i$ contains no equalities then it is clear that there exists a neighbourhood of any one solution of $\bar{T}_i(y)$ which consists entirely of solutions. (The fact that $\bar{T}_i$ contains no inequations of the type $s_{\delta i}(y) \geq 0$ is crucial). In this case, it cannot be true that for almost all $\hat{F}$, $\hat{G}$, $\hat{H}$, pole positionability can be achieved for all p_ℓ.

This analysis of the $\bar{T}_i$ answers Question 1.

Now one can also ask whether image χ is almost all of $\mathbb{R}^n$, for almost all (F, G, H). Let us identify K with x and F, G, H and the p_ℓ with y. Equations (A.1) yield a collection $S(x, y)$ of polynomial equations. By Proposition 5.2A, there exists $T(y) = T(F, G, H, p_\ell)$ such that $T(\hat{y})$ holds if and only if $S(x, \hat{y})$ is solvable. Using arguments like those above, it is easy to check whether or not the set of $\hat{y}$ for which $\bar{T}(\hat{y})$ is true is contained in a proper variety. If it is, then and only then will it be true that for almost all F, G, H, the map χ is almost onto R^n.

We shall now turn to an analysis of Question 2.

If the closed loop characteristic polynomial has all roots in the half plane $Re[s] < 0$, certain polynomial inequalities in the p_i obtainable from the Hurwitz determinants, see [4], must hold, and conversely. Accordingly, we have

$$
\begin{aligned}
&p_i(F, G, H, K) = p_i && i = 1, \ldots, n \\
&q_j(p_i) > 0 && j = 1, \ldots, n
\end{aligned}
\tag{A.2}
$$

Identify K and p with x and F, G, H, with y. Regard (A.2) as an equation/inequation set $S(x, y)$. By Tarski-Seidenberg - A, there exists $T(y)$ such that $T(\hat{y}) = T(\hat{F}, \hat{G}, \hat{H})$ holds if and only if (A.2) can be satisfied by some K, p_i. If the set of $\hat{y}$ such that $\bar{T}(\hat{y})$ holds is contained in a proper variety, then and only then Question 2 has an affirmative answer. The discussion of Question 1 described how one could check whether the set of $\hat{y}$ such that $\bar{T}(\hat{y})$ holds is or is not contained in a proper variety.

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THE MCMILLAN AND NEWTON POLYGONS OF A FEEDBACK SYSTEM AND THE CONSTRUCTION OF ROOT LOCI*

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Abstract

In this paper the local behaviour of Root Loci around zeros and poles is investigated. This is done by relating the Newton diagrams which arise in the local analysis to the McMillan structure of the open-loop system by means of what we shall call the McMillan polygon. This geometric construct serves to clarify the precise relationship between the McMillan structure, the principal structure, and the branching patterns of the root loci. In addition, several rules are obtained which are useful in the construction of the root loci of multivariable control systems.

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1. Introduction.

The well known root locus method for SISO feedback-systems, as first proposed by Evans [4], gives a set of simple rules which permit the graphical estimation of the loci of the closed-loop poles as a function of the feedback gain.

When trying to generalise this method several problems arise. First of all, the orders of the Butterworth patterns, which determine the branching pattern of the root loci at the open-loop poles and zeros, do not always correspond to the McMillan orders of the system. Secondly, Butterworth patterns of noninteger order have been observed in [13,21] and this has led to several open questions concerning the nature of the actual branching patterns.

As far as the authors are aware, only when a simple null structure condition is put on the system, has a satisfactory analysis been presented and this has primarily been concerned with the problem of determining the asymptotic behaviour of the root loci [7,8,11,17]. In this paper we show that the aforementioned condition is also necessary and we analyse what happens in absence of the condition. The principal tool in this analysis is the Newton diagram, which has been pioneered in this context by Postlethwaite [25], Postlethwaite and MacFarlane [15] in the case of square systems and scalar gain $K(\lambda) = \lambda I$ (the case of nonsquare systems and polynomial gain $K(\lambda)$ is treated in [2]), and which has also been used to a large extent in the recent book by Hahn [23], to which the referee was kind enough to draw our attention. It seems remarkable that Newton polygons, which were introduced three centuries ago by Newton [10] as a graphical tool to compute the exponents of the leading terms in fractional power series expansions, have received so little attention as a tool for developing asymptotic expansions. Newton first derived the binomial series for fractional powers of $(1-x)$ by regarding such a function as a root of the algebraic equation

$$y^n - f(x) = 0 \ .$$

In general, he gave an algorithm - based on the Newton polygon - for finding the branches of any algebraic relation

$$f(x,y) = 0 \ .$$

According to Abhyankar [1], this method was apparently forgotten until it was revived by Puiseux [16] in 1850.

At any rate, the method of the Newton polygon is a simple and efficient algorithm for determining the branching patterns of root loci, and it requires only rational operations. For a feedback system, the relationship between the Newton diagrams which arise for both finite and infinite branches of the root loci, and the McMillan orders of the open loop system seems to be best expressed geometrically. Accordingly, we introduce the McMillan polygon of a system and relate this to the corresponding Newton polygon. This gives rise to a geometric explanation of several points which have often seemed to require a very sophisticated and detailed analysis. We shall also derive a number of rules which we believe will be helpful when estimating the behaviour of multivariable root loci.

For simplicity, we have considered throughout this paper the case of a square transfer function $G(s)$ "acted" on by the proportional gain $K(\lambda) = \lambda I$, subject to the condition $\det G(s) \not\equiv 0$. The main results do hold, mutatis mutandis, for rectangular $G(s)$ and polynomial gain $K(\lambda)$ - subject, however, to an important constraint. Explicitly, if

$$K(\lambda) = K_o + K_1\lambda + \ldots + K_r\lambda^r, \qquad r \geq 1$$

then one would expect the asymptotic limit of the closed-loop poles to coincide with the limit of the closed-loop poles obtained by using the highest order term

$$\tilde{K}(\lambda) = K_r\lambda^r$$

provided $K_rG(s)$ has maximal rank. Of course, the asymptotic rates of the root-loci will change due to λ^r factor, but this change is easily accounted for and does not affect the limiting values of the root loci, viz. the McMillan zeroes of $K_rG(s)$. If $mp \leq n$, then for generic $G(s)$ the highest order term $K_r\lambda^r$ does determine the limiting values of the root-loci. Perhaps surprisingly, if $mp > n$ this is not the case: for every $G(s)$ there exist maximal rank K_r for which this is not the case, even if $r = 1$. Indeed several examples of this discontinuity are given in [2] and [3]. For any given K_r and $G(s)$, there exists an explicit algebraic criterion for this degeneracy to occur ([2], [3]). In the case of square $G(s)$ and maximal rank K_r, this turns out to be the familiar constraint

$$\det G(s) \equiv 0 .$$

In general, the condition for the results given below to extend to the rectangular (and polynomial) cases is that the root-locus map be non-degenerate for λK_r. Again in the cases treated below, this amounts to: $\det G(s) \not\equiv 0$.

2. Statement of the problem and main results.

Consider the feedback system of Figure 1, with

$u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$

$G_o(s) \in \mathbb{R}(s)^{m\times m}$, strictly proper, with $\det G_o(s) \not\equiv 0$

$K \in \mathbb{R}^{m\times m}$, $\lambda \in \mathbb{R}^+$

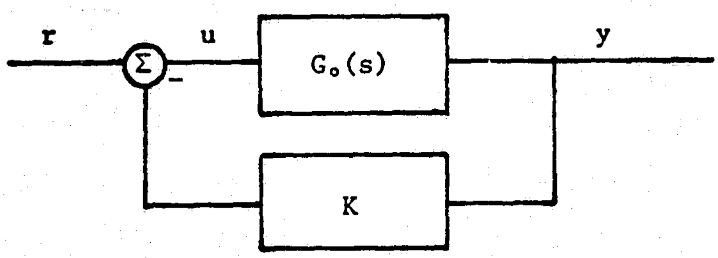

Figure 1.

The open loop system $G_o(s)$ is assumed to be linear, time invariant and finite dimensional. If $\{N(s),D(s)\}$ is a left coprime factorisation of $G_o(s)$, the the open, respectively closed loop characteristic polynomials are given by

$$OLCP(s) = \det [D(s)] \quad (1)$$

$$CLCP(s,\lambda) = \det [D(s) + \lambda N(s) K] \quad , \quad (2)$$

and both are related by the return-difference determinant

$$r(s,\lambda) = \det [I_m + \lambda G(s) K] \quad (3)$$

$$= \frac{CLCP(s,\lambda)}{OLCP(s)} \quad (4).$$

As the feedback gain λ varies, the closed loop poles are given by the algebraic functions $s_i(\lambda)$, defined as the zeros of the closed loop characteristic polynomial. The root locus is the locus of these solutions as λ runs through the positive real line [0,∞].

A graphical estimation of the loci, very simply stated, should allow one to answer the natural questions : (i) where do the loci start ; (ii) where do the loci end ; (iii) how do they behave in between. In the present setting ($K(\lambda)=\lambda I$), the answer to the first two questions has been known for some time and is the same as in the SISO case : the closed loop poles start at the open loop poles and move to the open loop zeros. However, only in answering the third question does the precise meaning of this statement become clear. The behaviour of the root loci in between the initial and final points is estimated by means of a number of rules and the most important rule predicts the local behaviour at the poles and zeros. More precisely, it predicts the angles of approach and departure at the finite points and also predicts the asymptotic directions.

As stated earlier the main tool used in analysing the branching behaviour is the Newton polygon. Consider the closed loop characteristic polynomial, where λ is substituted for $1/g$:

$$CLCP(s,\lambda) = \det [D(s) + \lambda N(s) K]$$

$$CLCP(s,g) = \det [g D(s) + N(s) K] = \Sigma c_{ij} g^i s^j \quad (5),$$

$$\text{with } CLCP(s,g) = 1/\lambda^m \; CLCP(s,\lambda) \quad (6).$$

Instead of the expression (5), it is also possible to express the CLCP at any point of interest s_o as :

$$CLCP(s,g) = \Sigma c^o_{ij} g^i (s-s_o)^j \quad (7).$$

Definition 2.1.

The Newton polygon of the closed loop characteristic polynomial at the point s_o is the polygon obtained as the convex hull of the points (i,j) where i and j are the exponents of the nonzero terms $c^o_{ij} g^i (s-s_o)^j$ in (7).□

Definition 2.2.

The Newton boundary is the lower boundary of the Newton polygon.□

As an example consider the polynomial

$$r(s,g) = s^3 + s^4 g^3 + s g + s^2 g^2 + s g^4 + g^3 + g^2 \quad (8)$$

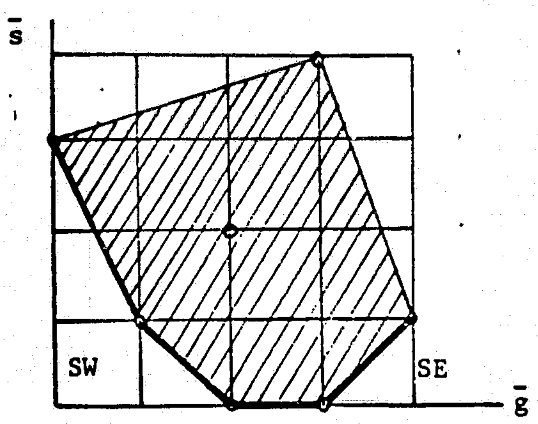

Figure 2.

Denoting the exponents of s and g by $\bar{s}$ and $\bar{g}$ respectively, we get at s=0, the Newton polygon and boundary (thick line) of Figure 2.

The Newton boundary determines the leading exponents of the Puiseux expansions : the SW boundary corresponds to expansions of (s,g) around (0,0), the SE boundary to expansions around $(0,\infty)$. More precisely, the negative reciprocals of the slopes of the line segments of the SW and SE Newton boundaries yield the branching patterns of the arriving and departing branches respectively, at the point $s_\circ$. In the example above there is one second order and one first order pattern for the arriving branches : $s_{1,2} \sim \pm \alpha \sqrt{g}$, $s_3 \sim \beta g$. There is one first order pattern for the departing branches : $s_4 \sim \gamma 1/g \sim \gamma \lambda$.

Definition 2.3.

The McMillan polygon of a transfer matrix G(s), at a pole-zero location $s_\circ$, is obtained as convex hull of the points $(m-i, \delta_p^\circ - c_i^\circ)$, where m is the size of G(s), δ_p° the polar degree of $s_\circ$ and c_i° the maximum content at $s_\circ$ of the [ixi] minors of G(s). □

For the definition of the content of a rational matrix and its relationship with the McMillan structure we refer to [19,20,22]. Recall that the following simple relationships hold : let σ_i° denote the McMillan orders of $s_\circ$, arranged in decreasing order (for a pole σ is positive, for a zero σ is negative), Thus the polar degree and the zero degree are given by

$$\delta_p^\circ = \sum_{i \text{ s.t. } \sigma_i^\circ > 0} \sigma_i^\circ \qquad (9)$$

$$\delta_z^\circ = \sum_{i \text{ s.t. } \sigma_i^\circ < 0} |\sigma_i^\circ| \qquad (10).$$

The i-minor content satisfies

$$c_i^\circ = \sum_{j=1}^{i} \sigma_j^\circ \qquad (11)$$

whence

$$\delta_z^\circ = \delta_p^\circ - c_m^\circ \qquad (12).$$

Definition 2.4.

The McMillan boundary is the lower boundary of the McMillan polygon. □
As an example consider the 4×4 transfer matrix, whose McMillan form at $s_\circ$ is given by

$$G(s) \sim_{s_\circ} \begin{bmatrix} 1/(s-s_\circ) & & & \\ & 1 & & \\ & & (s-s_\circ) & \\ & & & (s-s_\circ)^2 \end{bmatrix} \quad (13).$$

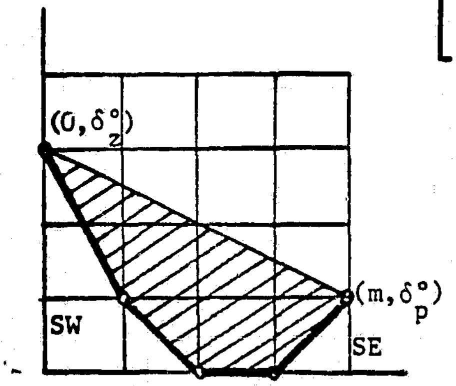

Figure 3.

The polar degree is 1, the zero degree is 3 and the contents are : $c_\circ^\circ = 0$, $c_1^\circ = 1$, $c_2^\circ = 1$, $c_3^\circ = 0$, $c_4^\circ = -2$. The McMillan polygon and boundary (thick line) are shown in figure 3.

Before stating the first result, we recall the simple null structure assumptions that were used in several papers [7,11,17], in the course of deriving computational methods for the evaluation of the exponents and coefficients of the leading term in the Puiseux expansion. Around the point $s = \infty$ it is shown in [7,11,17] that, as far as the asymptotic expansions of the root loci is concerned, the system is equivalent to a block diagonal matrix :

$$G_\circ(s)\, K \sim_\infty \begin{bmatrix} Q_1\ 1/s^{i_1} & & & \\ & Q_2\ 1/s^{i_2} & & \\ & & \ddots & \\ & & & Q_l\ 1/s^{i_l} \end{bmatrix} \quad (14),$$

provided the matrices Q_i have simple null structure. It is easy to see (cfr. section III) that the above equivalence also holds at finite pole-zero locations :

$$G_*(s)\,K \sim_{s_*} \begin{bmatrix} Q_1\, 1/(s-s_*)^{i_1} & & \\ & \ddots & \\ & & Q_l (s-s_*)^{i_l} \end{bmatrix} \qquad (15).$$

Theorem 2.1.

The branch pattern of the root loci at a pole or zero (including infinity) consists of a superposition of Butterworth patterns, with orders equal to the McMillan orders at that point if, and only if, the matrices Q_i which arise when block-diagonalizing the transfer matrix as in (15) have simple null structure. Furthermore this condition is generically satisfied.

proof : cfr. section III. □

The above theorem determines those conditions under which the orders of the Butterworth patterns formed by the branches at a pole-zero location correspond to the McMillan orders. Explicitly, it shows that the simple null structure conditions not only are sufficient, but necessary as well. In view of the definitions above, Theorem 2.1 also can be interpreted as giving necessary and sufficient conditions for the Newton and McMillan boundaries to coincide.

As an immediate consequence of the semicontinuity of the Newton boundary, the following important result holds.

Theorem 2.2.

The Newton boundary of an invertible system, subject to a full rank feedback matrix, is contained within the McMillan polygon of the system. □

Theorem 2.2 thus gives a priori bounds on the possible branching patterns which can arise when the conditions of Theorem 2.1 are not satisfied. For the example of Figure 3, all additional Newton boundaries are given in

Figure 4. Indeed, for a given McMillan structure it can now be investigated which of the possible Newton boundaries actually correspond to some feedback system. A conjecture in this spirit has been made by Owens [11] for the asymptotic branches of the root loci : " The m×m linear, time-invariant, invertible system G(s), having McMillan orders $N = \{n_1,\ldots,n_m\}$ at ∞, can only have zeros of orders equal to arithmetic means of subsets of N."
This conjecture, however, turns out to be too strong. The following example shows that different orders are possible:

$$G(s) \sim \begin{bmatrix} z^2 & z \\ z^4 & 0 \end{bmatrix}$$

, where z formally represents $s-s_o$ in expansions at s_o, or $1/s$ at ∞. (16)

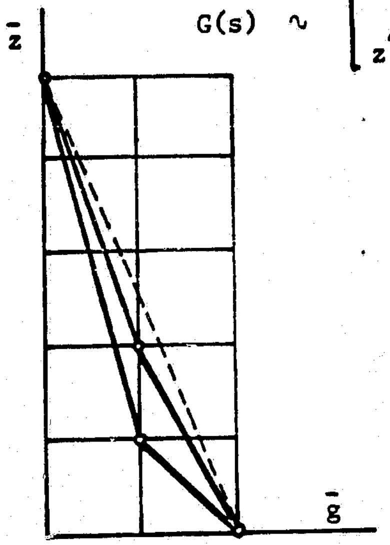

Figure 5.

The McMillan polygon and the Newton boundary for the scalar feedback matrix λI_2 are shown in figure 5. Clearly, the pattern (2,3) is not obtained as an arithmetic mean of (1,4). Indeed, in general the following is true :

Corollary 2.3.

For a given McMillan structure, every possible Newton boundary, in the sense of Theorem 2.2, corresponds to some invertible linear system G(s) subject to a scalar feedback λI_m . □

Corollary 2.3 shows that the class of possible Newton boundaries is much wider than the one suggested by Owens. Returning to the example of Figure 4, Corollary 2.3 guarantees the existence of systems that have the Newton boundaries of the diagrams 1 through 9 of figure 4. For example :

$$G_1 = \begin{bmatrix} z^{-1} & z^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & z & 0 & z \\ 0 & 0 & 0 & z^2 \end{bmatrix}, \; G_9 = \begin{bmatrix} 1 & z^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z \\ 0 & z & 0 & 0 \end{bmatrix} \quad (17).$$

For the general construction of these examples we refer to the proof of the corollary in section III. At this point it is worth remarking that the existence of so-called "noninteger order" branch patterns is in harmony with this analysis. In particular, in this case the end points of a line segment with noninteger slop will have integer coordinates for which there will exist a number, equal to an integer multiple of q, of p/qth order patterns, where p and q are coprime (cfr[2]).

Instead of fixing only the McMillan structure and varying the transfer matrix, the following more practical problem can be considered: for a given transfer matrix, which Newton boundaries correspond to some choice of the feedback matrix? A first result in this sense was stated by Kouvaritakis and Shaked [7]. In the present terminology, this may be stated as:

Corollary 2.4.

For any system, the McMillan boundary is attainable as the Newton boundary through an appropriate choice of the feedback matrix. □

In section III, it will be shown how this result easily follows from the proof of Theorem 2.1, and that it holds for both finite and infinite branches of root loci.

The next result formalises the intuition behind the conjecture made by Owens [13]:

Corollary 2.5.

When the system is diagonalisable by a suitable choice of the feedback matrix and by constant similarity transformations, then branch patterns corresponding to arithmetic means of the McMillan orders are attainable through an appropriate choice of the feedback matrix. □

Some hypotheses on G(s) are necessary in order that the above result holds, as the example (16) shows. Clearly the transfer matrix (16) is not diagonalisable by constant transformations. A straightforward calculation shows that only the patterns (1,4) and (2,3) are attainable, e.g. the straight line of figure 3, corresponding to the arithmetic mean (5/2,5/2) is not attainable. On the other hand, the conditions of Corollary 2.5 are not necessary as the following example shows. Consider

$$G(z) = \begin{bmatrix} z & z^2 \\ 0 & z^3 \end{bmatrix} \qquad (18).$$

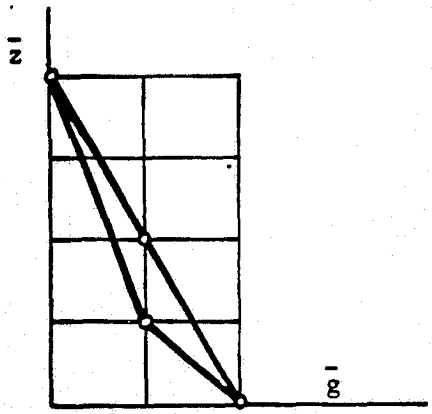

Figure 6.

The McMillan polygon is shown in figure 6. Although the system is not diagonalisable, both the patterns (1,3) and (2,2) are attainable.

To conclude this section we will show how the above results simplify in the case of scalar systems. Due to the linearity of the CLCP with respect to the feedback gain, the McMillan polygon reduces to one line segment. As a result the Newton boundary always must coincide with the McMillan

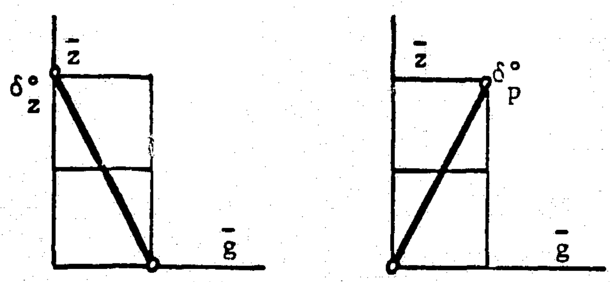

Figure 7.

boundary and the root loci always branch according to the McMillan orders. Furthermore, poles and zeros never coincide, which is also reflected in the McMillan polygons of figure 7 : the boundary is either downsloping for a zero or upwardsloping for a pole.

This simplicity of the McMillan polygon for scalar systems ultimately accounts for the existence of necessary and sufficient conditions for rules such as [18] :

1. the center of gravity rule : $\sum_i s_i(\lambda) = \text{constant}$,

2. the product rule : $\prod_i s_i(\lambda) \sim \lambda$.

to hold. However, one can easily generalise the sufficiency of these rules using the McMillan polygon.

Proposition 2.6.

The center of gravity of the root locus remains fixed if the system has m zeros at infinity, each of order at least equal to 2. □

Proposition 2.7.

In order for the product of the closed loop poles to vary proportionally to λ^m it is sufficient that there are m poles at the origin, each of McMillan order at least equal to 1. □

In appendix A, it will be shown how some further insight can be gained from the McMillan polygons. As a specific application, propositions 2.6 and 2.7 will be proved in appendix B. Finally appendix C will introduce a further notion, namely the principal boundary, and investigate its relationship with the Newton boundary.

3. Proofs of the main results.

Theorem 2.1.

The branch pattern of the root loci at a pole or a zero (including infinity) consists of a superposition of Butterworth patterns, with orders equal to the McMillan orders at that point if, and only if, the matrices Q_i which arise when block-diagonalizing the transfer matrix as in (15), have simple null structure. Furthermore this condition is generically satisfied.

Proof

(i) Sufficiency.

The closed loop characteristic polynomial satisfies

$$CLCP(s,\lambda) = r(s,\lambda) \cdot OLCP(s) \qquad (21),$$

or after a change of variables $\lambda = 1/g$

$$CLCP(s,g) = \det[\, g\, I_m + G(s)] \cdot OLCP(s)$$

$$= r(s,g) \cdot OLCP(s) \qquad (22).$$

In (22) we set, for notational convenience,

$$G(s) = G_o(s)\, K \qquad (23).$$

Of course the McMillan structures of $G(s)$ and $G_o(s)$ are the same. Expanding the first factor in the righthand side of (22) :

$$CLCP(s,g) = OLCP(s) \cdot g^m + OLCP(s) \cdot tr[G(s)] \cdot g^{m-1} + \ldots$$
$$+ OLCP(s) \cdot \det[G(s)] \qquad (24).$$

At the point of interest s_o, the OLCP can be written as

$$OLCP(s) = (s - s_o)^{\delta_p^o} \cdot p(s) \quad , \text{with } p(s_o) \neq 0 \qquad (25).$$

Recall also that the zero-polynomial satisfies

$$z_G(s) = OLCP(s) \cdot \det[G(s)] \qquad (26),$$

Again, at the point of interest $s_\circ$, this can be written as

$$z_G(s) = (s - s_\circ)^{\delta_z^\circ} \cdot q(s) \quad , \text{with } q(s_\circ) \neq 0 \qquad (27).$$

In (25) and (27), δ_p° and δ_z° represent the polar and zero degree of $s_\circ$, respectively. Substituting (25) and (27) in (24) gives

$$CLCP(s,g) = (s - s_\circ)^{\delta_p^\circ} \cdot p(s) \cdot g^m + \ldots + (s - s_\circ)^{\delta_z^\circ} \cdot q(s) \qquad (28).$$

From $\quad p(s_\circ) \neq 0 \ , \quad q(s_\circ) \neq 0$

it follows that the Taylor expansions

$$p(s) = \Sigma_{i=0}^{\infty} p_i (s - s_\circ)^i \quad , \quad q(s) = \Sigma_{j=0}^{\infty} q_j (s - s_\circ)^j \qquad (29),$$

have non zero leading coefficients

$$p_\circ \neq 0 \quad , \quad q_\circ \neq 0 \qquad (30).$$

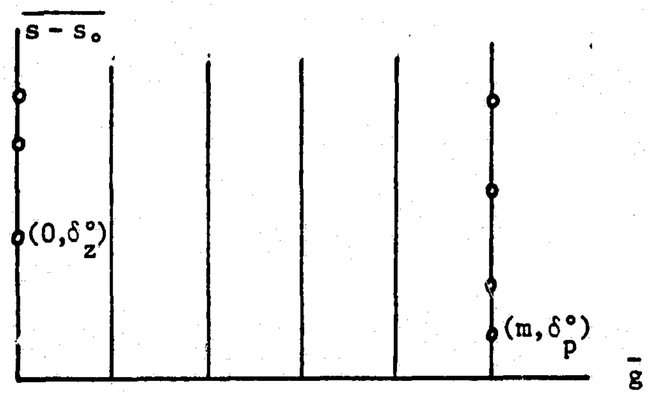

Figure 8.

Part of the Newton diagram of equation (28) hence looks as in figure 8, where $\overline{(.)}$ stands for the exponent associated with the variable (.) . Since we are investigating the local behaviour around $s=s_\circ$, $g=0,\infty$ only the lowest points in the diagram are of interest. In view of (30), the lowest points on the lines $\bar{g}=0$ and $\bar{g}=m$ are δ_z° and δ_p°, respectively. This fixes the initial and final points of the Newton boundary.

Next we will compute the lowest exponents in $(s - s_\circ)$ along the intermediary lines $\bar{g} = 1,\ldots,m-1$. Consider the return-difference

$$r(s,g) = g^m + tr[G(s)]\ g^{m-1} + \ldots + det[G(s)] \qquad (31).$$

From (21) or (24) it follows that in lieu of the CLCP, we may also use the return-difference (31) to compute the lowest exponents; i.e. the difference of degree in $(s - s_\circ)$ between (24) and (31) is fixed and equal to δ_p°. Now, consider the Laurent expansion of (23) at $s_\circ$

$$G(s) = G_1(s - s_\circ)^{-\sigma_1} + G_2(s - s_\circ)^{-\sigma_2} + \ldots \qquad (32),$$

with $-\sigma_1 \leq -\sigma_2 \leq \ldots$.

Our determination of the lowest degree terms in the characteristic coefficients $r_i[G(s)]$, $i=1,\ldots,m$ (here $r_i(T)$ is the coefficient of ξ^{n-i} in $\det[\xi I - T]$, so that $r_i[G(s)]$ is the sum of the principal i-minors of G(s)) is identical to Owens method of dynamic transformations for computing the asymptotic directions [11]. Owens showed that the local expansions of G(s) are the same as the expansions for

$$L(s - s_\circ)\ T^{-1}\ G(s)\ T \qquad (33),$$

where $L(s - s_\circ)$ is a unimodular "left dynamic" transformation which preserves the structure of G(s) at $(s - s_\circ)$, or equivalently has no poles at $s_\circ$ (for asymptotic directions $s-s_\circ$ is replaced by $^1/s$). T is a constant similarity transformation. Briefly this result was derived in the following way : consider

$$\det[gI_m + G(s)] = \det[\ gI_m + L(s - s_\circ)\ T^{-1}\ G(s)\ T\ L^{-1}(s - s_\circ) \quad (34).$$

By Schur's lemma, the local expansions of the root loci of G(s) and $L(s - s_\circ)T^{-1}G(s)T$, at $s_\circ$ coincide (cfr [11],formulas (38)...(45); it is however our belief that at most one dynamic transformation is needed. That is, in the equation (38) of [11], the effect of the left transformation cancels and in fact only the effect of the right transformation is actually used). Recall, the essence of such dynamic transformations is that by suitable choice of T and $L(s - s_\circ)$, the Laurent coefficients

$$\{ G_1, G_2, \dots \}$$

may be put into upper triangular form

$$\left\{ \begin{bmatrix} Q_1 & 0 & 0 \dots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} Q_2 & x & x \dots \\ 0 & Q_3 & 0 \\ 0 & 0 & 0. \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} Q_4 & x & x \dots \\ 0 & Q_5 & x \\ 0 & 0 & Q_6 \\ & & & \ddots \end{bmatrix}, \dots \right\} \tag{35}$$

In order for the transformations to exist, however, the blocks Q_1, Q_3, Q_6,..., are assumed to have simple null structure. From (35), it is now clear how to construct the lower boundary of the Newton polygon. Let

$$\text{rank } [Q_1] = \rho_1 \;, \quad \text{rank } [Q_3] = \rho_2 \;, \quad \text{etc.} \tag{36}$$

Along the following vertical lines we get as minimal exponents :

$$\bar{g} = m - \rho_1 \longrightarrow \overline{s - s_\circ} = - \sigma_1 \rho_1 + \delta^\circ_p$$

$$\bar{g} = m - \rho_1 - \rho_2 \longrightarrow \overline{s - s_\circ} = - \sigma_1 \rho_1 - \sigma_2 \rho_2 + \delta^\circ_p \tag{37}$$

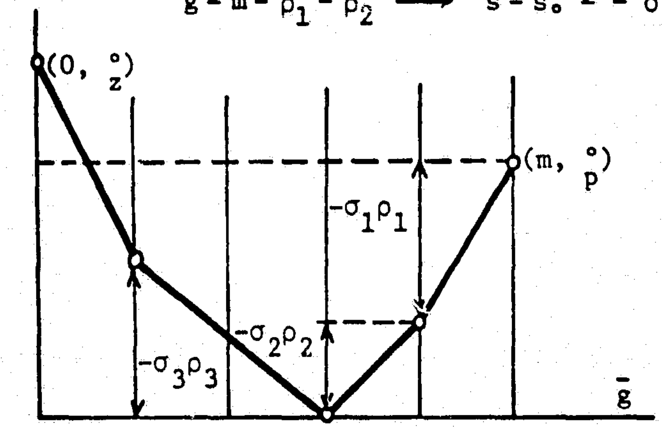

Figure 9.

The corresponding Newton diagram is shown in figure 9. At this point the relationship between the Newton and the McMillan boundary also becomes clear. As is shown by Vandooren et al [19], the rank increments $\bar{\rho}_i$ of the following Toeplitz matrices

$$\begin{bmatrix} G_1 & G_2 & G_3 & & G_l \\ & G_1 & G_2 & & \\ & & G_1 & \ddots & \\ & & & & G_l \end{bmatrix} \qquad l = 1,2,\dots$$

determine the McMillan structure of the system (23) at $(s - s_\circ)$. Explicitly, the McMillan structure is given by

$$\text{diag}\left\{\underbrace{(s-s_\circ)^{-\sigma_1},\ldots,(s-s_\circ)^{-\sigma_1}}_{\rho_1\text{-times}},\ \underbrace{(s-s_\circ)^{-\sigma_2},\ldots,(s-s_\circ)^{-\sigma_2}}_{\rho_2\text{-times}},\ldots\right\} \tag{39}$$

Under simple null structure assumption the rank increments $\bar{\rho}_i$, are, however, equal to the ranks ρ_i introduced for the Newton boundary, since the Toeplitz Matrices (38) take on the special form

$$\begin{bmatrix} \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} Q_2 & x & x \\ 0 & Q_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} Q_4 & x & x \\ 0 & Q_5 & 0 \\ 0 & 0 & Q_6 \end{bmatrix} & \cdots \\ & \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} Q_2 & x & x \\ 0 & Q_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \cdots \\ & & \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \cdots \\ & & & \ddots \end{bmatrix} \tag{40}$$

In terms of the contents of the rational matrix G(s), it also follows ([17]) that

$$\begin{aligned} c_\circ(G^{\bar{\rho}_1}) &= -\sigma_1\bar{\rho}_1 = -\sigma_1\rho_1 \\ c_\circ(G^{\bar{\rho}_1+\bar{\rho}_2}) &= -\sigma_1\bar{\rho}_1 - \sigma_2\bar{\rho}_2 = -\sigma_1\rho_1 - \sigma_2\rho_2 \end{aligned} \tag{41}$$

Along the vertical lines corresponding to the abscisses $m - \rho_i$, the Newton and the McMillan boundaries coincide. Since these points are the vertices of the boundaries, they have to coincide entirely. Note that, indeed, only the vertices of the linear segments of the McMillan boundary were considered. Whether or not an intermediate integer point (i,j) corresponding to some non zero coefficient c_{ij} actually occurs on such a line segment cannot be inferred from (35). However, as far as the orders are concerned, those intermediary points are irrelevant.

Finally, we should also remark that due to the property

$$\delta_p^\circ = \Sigma_{-\sigma_i < 0} |\sigma_i \rho_i| \qquad (42),$$

it follows that the McMillan boundary has to touch the $\bar{g}$-axis.

(ii) Necessity.

Recall that Owens' method of triangularizing the system matrix inductively considers higher order terms in the Laurent expansion. Suppose now that at some point in this algorithm, say at the ℓ-th Laurent term, the submatrix Q_l does not have simple null structure. The transformed transfer matrix at this point looks like

$$\text{diag}\{Q_1, Q_2, \ldots, Q_l\}$$

where Q_l can be put into Jordan form :

$$\begin{bmatrix} \bar{Q}_l & & \\ & J_1 & \\ & & \ddots & \\ & & & J_{i_l} \end{bmatrix}, \text{ where } \bar{Q}_l \text{ has full rank and } J_i = \begin{bmatrix} 0 & 1 & & \\ & \cdot & 1 & \\ & & \cdot\cdot & \\ & & & \cdot 1 \\ & & & 0 \end{bmatrix}$$

Clearly, the rank of Q_l satisfies

$$\rho_l = d_l - i_l \text{ , where } d_l \text{ stands for the size of } Q_l \qquad (43).$$

From the results of Vandooren et al. , it follows that the vertex of the McMillan boundary will occur at the abscis corresponding to ρ_l. However, no nonzero principal minors with the McMillan content exist. By convexity of the Newton polygon the Newton boundary hence must lie above the McMillan boundary as is shown in figure 10.

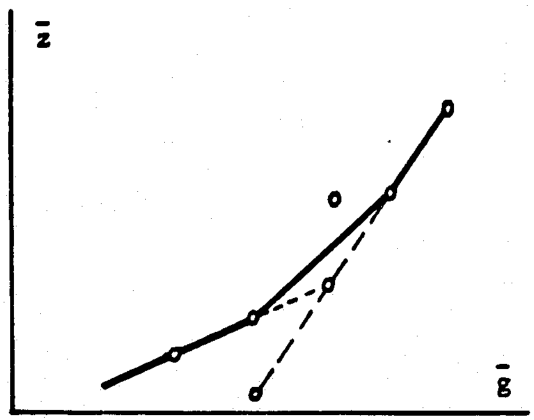

Figure 10.

(iii) Genericity

First we shall make precise what is meant by generic. Intuitively, this means that almost all systems have this property. Algebraically, genericity is defined in the following way. Let $\Sigma^n_{m,p}$ denote the set of all systems with m inputs, p ouputs and McMillan degree n. By passing to the Markov, or Hankel, parameters

$$G(s) = \Sigma^{\infty}_{i=1} H_i s^{-i} \tag{44}$$

we get a new parameterization of $\Sigma^n_{m,p}$, viz. we think of G(s) as being determined by the string

$$H_G = (H_1, H_2, \dots , H_{2n}) \tag{45}$$

of p×m matrices. A generic property is then a property P for which the set S_P of systems which do not have P is defined by polynomial equations,i.e.

$$H_G \in S_P \quad \Longleftrightarrow \quad f_k(H_G) = 0 \tag{46}$$

for some set of polynomials f_k. For example, the property that the (truncated)

Hankel matrix

$$H_G' = [\, H_{i+j-1} \,]_{i,j=1}^{n}$$

has rank n is a generic property. And, the simple null structure hypothesis is also generic in exactly this sense. Since the zero set of a set of polynomials has empty interior, a generic property holds for an open, dense set of systems. □

Theorem 2.2.

The Newton boundary of an invertible system, subject to a full rank feedback matrix, is contained within the McMillan polygon of the system.

Proof

The proof of the second part of Theorem 2.1 shows that the McMillan boundary forms a lower boundary for the Newton polygon. Since the Newton polygon is convex and since the initial and final points $(0,\delta_z^\circ)$;(m,δ_p°) resp. are the same for every Newton polygon the result follows immediately. □

Corollary 2.3.

For a given McMillan structure, every possible Newton boundary, in the sense of Theorem 2.2, corresponds to some invertible linear system G(s), subject to a scalar feedback λI_m.

Proof

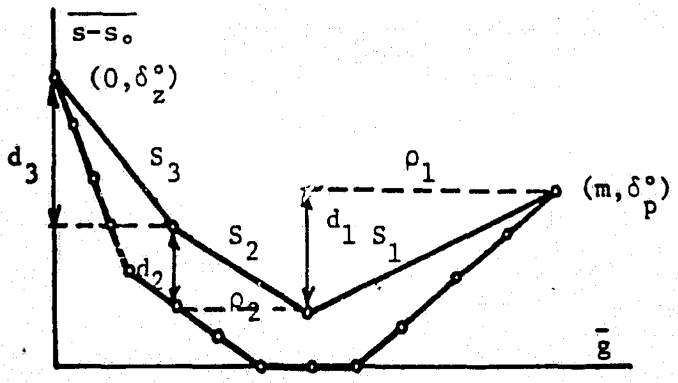

Figure 11.

Consider a McMillan polygon as in Figure 11, and suppose we want to find a system whose Newton boundary is given by the segments S_i of figure 11. Formally the segments S_i are defined as those subsegments of the Newton boundary that link points with integer coordinates. Each segment is characterised by a dimension ρ_i and by a degree d_i. Consider the transfer matrix

$$\left[\begin{array}{ccc|ccc|ccc}
0 & z^{-\sigma_1} & & & & & & & \\
 & & z^{-\sigma_2} & & & & & & \\
 & & \ddots & & & & & & \\
z^{n_1} & & 0 & z^{-\sigma_{\rho_1}} & & & & & \\
\hline
 & & & 0 & z^{-\sigma_{\rho_1+1}} & & & & \\
 & & & & & \ddots & & & \\
 & & & z^{n_2} & & 0 & z^{-\sigma_{\rho_1+\rho_2}} & & \\
\hline
 & & & & & & 0 & \ddots & \\
 & & & & & & & \ddots & z^{-\sigma_{m-1}} \\
 & & & & & & z^{n_l} & & 0
\end{array}\right] \tag{47}$$

where

$$n_i = d_i + \sum_{j=\rho_i+1}^{\rho_i+\rho_{i+1}-1} \sigma_j \tag{48}.$$

In words : we take a matrix with McMillan terms just off the main diagonal. Per segment we introduce a term to adjust the degree of the corresponding block, such as to agree with the change in ordinate along the segment.

It can be checked that :

1. The matrix has McMillan structure $(\sigma_1, \sigma_2, \ldots, \sigma_m)$.
2. The matrix (47) has the Newton boundary corresponding to the segments S_i. □

Corollary 2.4.

Through an appropriate choice of the feedback matrix, the McMillan boundary always is attainable for every system.

As in the proof of Theorem 2.1, we will again apply the algorithm of Owens, namely triangularise the system by dynamic transformations

$$G(s) \longrightarrow L(s-s_o)\, T^{-1}\, G(s)\, T \qquad (49).$$

If the matrices Q_i occuring on the block diagonal (cfr.(14)), have simple null structure, then the Newton and McMillan boundaries will coincide. So suppose that for some Laurent coefficient G_l, the submatrix Q_l does not have simple null structure. Explicitly, the dynamic transformations and the transformed system matrix take on the forms

$$L = \begin{bmatrix} L_1 & 0 \\ L_2 & I_{m-r} \end{bmatrix} \begin{matrix} \}\, r \\ \}\, m-r \end{matrix}, \quad r = \rho_1 + \ldots \rho_{l-1} \qquad (50)$$

$$L\, T^{-1}\, G\, T = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \qquad (51),$$

where the entries of the first r rows, below the diagonal, have a degree in $(s-s_o)$ that is sufficiently large to be irrelevant in the local analysis around the point s_o. The lowest degree term in H_4 is by construction

$$Q_l\, (s-s_o)^l \qquad (52)$$

Multiplying (51) on the right by a suitable permutation matrix

$$K = \begin{bmatrix} I_r & C \\ 0 & \bar{K} \end{bmatrix} \qquad (53),$$

will render $Q_l \bar{K}$ simple null. Furthermore, since

$$L\, T^{-1} G\, T\, K = \begin{bmatrix} H_1 & H_2\bar{K} \\ H_3 & H_4\bar{K} \end{bmatrix}$$

it follows that the properties of the first r columns are not changed, therefore the inductive hypothesis of the algorithm is maintained.

It remains to check that the permutation matrix (53) corresponds to a choice of the feedback matrix. This, however, follows from the constancy of the entries of the similarity transformation T :

$$L(s-s_o)\, T^{-1} G(s)\, T\, K \;=\; L(s-s_o)\, T^{-1} G(s)\, \tilde{K}\, T \qquad (54),$$

where

$$\tilde{K} = T K T^{-1}$$

□

Corollary 2.5.

When the system is diagonalizable by a suitable choice of the feedback matrix and by constant similarity transformations, the branch patterns corresponding to arithmetic means of the McMillan orders are attainable through an appropriate choice of the feedback matrix.

Proof

By assumption

$$T^{-1}G(s)\,T \;\sim_{s_o}\; \begin{bmatrix} (s-s_o)^{-\sigma_1} & & \\ & \ddots & \\ & & (s-s_o)^{-\sigma_m} \end{bmatrix} \qquad (55).$$

$$-\sigma_1 \le \;\cdots\cdots\; \le -\sigma_m$$

First consider the case where Butterworth patterns do correspond to the McMillan orders, except for two patterns of orders σ_l and σ_k, that are to be replaced by two patterns of $(\sigma_l+\sigma_k)/2$ -th order. This is achieved by permuting the corresponding entries in (55), which becomes after reordering

$$\begin{bmatrix} (s-s_o)^{-\sigma_1} & & & & \\ & \ddots & & & \\ & & 0 & (s-s_o)^{-\sigma_l} & \\ & & (s-s_o)^{-\sigma_k} & 0 & \\ & & & & \ddots \\ & & & & & (s-s_o)^{-\sigma_m} \end{bmatrix} \qquad (56),$$

It is easily checked that the matrix (56) has the desired Newton boundary. The permutation matrix P corresponds to a feedback matrix

$$K = T\,P\,T^{-1} \tag{57}$$

In general the transfer matrix (55) can be put in the form

$$\mathrm{diag}\,\{Q_1, Q_2, \ldots, Q_q\} \tag{58}$$

where

$$Q_i = \begin{bmatrix} 0 & (s-s_\circ)^{-\sigma_1^i} & & \\ & & \ddots & \\ & & & (s-s_\circ)^{-\sigma^i_{q_i-1}} \\ (s-s_\circ)^{-\sigma^i_{q_i}} & & & 0 \end{bmatrix}$$

by some permutation matrix, which corresponds to a feedback matrix according to equation (57). The Newton boundary of the system (58) has segments that correspond to branches of orders equal to the arithmetic mean of the McMillan orders that occur in the blocks Q_i. □

Remark

For notational simplicity the system was assumed to be diagonalisable. It can be checked, however, that it is sufficient that the off diagonal terms have a degree in $(s-s_\circ)$ which is high enough, such as to be of no importance in the local analysis. From the results so far it can be seen that, for instance, $\max\{\delta_z^\circ, \delta_p^\circ\}$, certainly is an upper bound.

APPENDIX A : Some further properties of root loci.

1. The asymptotic behaviour.

The results of section II can be specialised to analyse the asymptotic behaviour of a strictly proper, invertible system. The McMillan structure at infinity is defined as the McMillan structure at the origin after performing the substitution $s = 1/z$ ([20],[22]). As in ([11]), the Laurent expansion reduces to a Taylor expansion in the variable z

$$G(z) = G_1 z + G_2 z^2 + \ldots\ldots \qquad (59),$$

where the G_i are the usual Markov parameters. Because of the properness of the sytem, there are no poles at infinity, i.e. $\delta_p^\infty = 0$, which is reflected in the absence of negative powers in (59). In terms of the McMillan polygon this implies that there is no SE boundary. A typical McMillan polygon is shown in Figure 12. From this diagram it is clear δ^∞ poles must go to infinity, for all full rank feedback matrices.

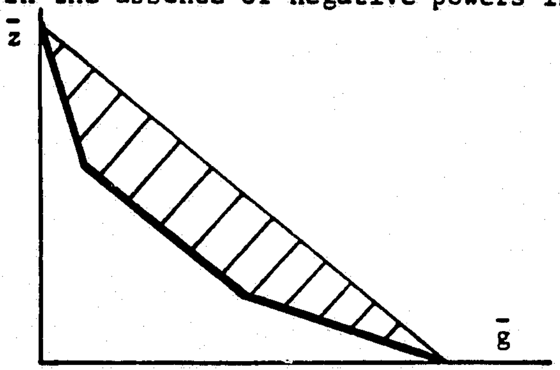

Figure 12.

Uniform rank systems are defined by the generic condition that the first term in the Laurent expansion must have full rank. As a result the McMillan polygon collapses to a straight line. The Newton boundary hence coincides, for all full rank feedback matrices, with the McMillan boundary -

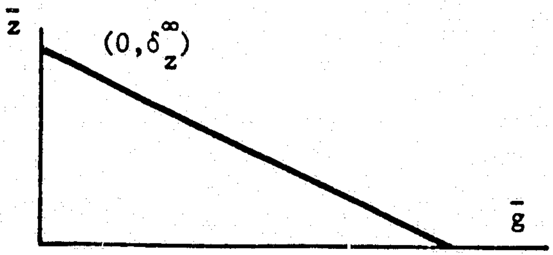

Figure 13.

illustrating the robustness of uniform rank systems.

2. Branching behaviour at finite poles and zeros.

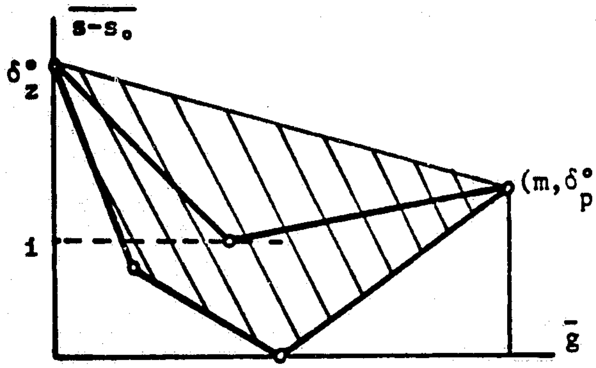

Figure 14.

A typical McMillan polygon at a finite pole-zero location is shown in figure 14. Non-coincidence of the Newton and McMillan boundaries not only implies that the branching behaviour is different from the one predicted by the McMillan orders, but also implies that the total number of branches can be less than $\delta^\circ_p + \delta^\circ_z$. Indeed, a factor $(s - s_o)^i$, with i equal to the distance of the Newton boundary to the $\bar{g}$-axis, is common to all coefficients and hence

$$CLCP(s,g) = (s - s_o)^i \, p'(s,g) \qquad (60).$$

As a result there is a fixed point of multiplicity i : i arriving and i departing branches disappear. In Appendix C we shall return to this phenomenon.

APPENDIX B : Proof of propositions 2.6 and 2.7

Before proceding to the proof of the propositions, recall that in section II two Newton diagrams, nl. one for g and one for 1/g, were combined. In the same way, we can combine the Newton diagrams for s and 1/s. For strictly proper systems, this gives rise to a Newton polygon as in figure 15.

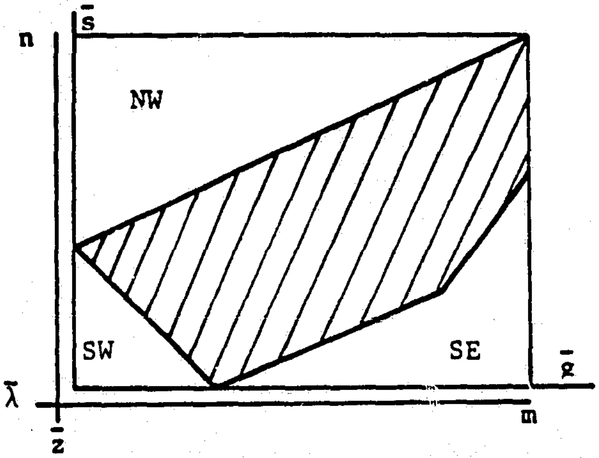

Figure 15.

The NW-boundary corresponds to the asymptotic behaviour (zeros at infinity). The SW boundary corresponds to the arrival of the root loci at the zero s=0, and the SE boundary to the departure at the pole s=0.

Proof (Center of gravity and product rules)

Because the open loop system is assumed to be strictly proper, the CLCP will be monic in the variable s. As a result a necessary and sufficient condition for the center of gravity rule to hold, is that the coefficient of s^{n-1} does not depend on λ. This is depicted in Figure 16 : the upper shaded area should not intersect the Newton polygon. A sufficient condition for this to be true, clearly is that the NW boundary of the

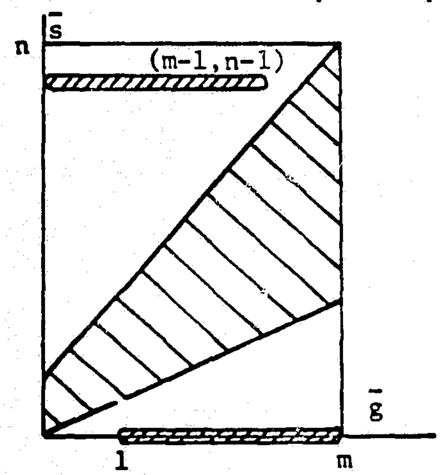

Figure 16.

McMillan polygon , which contains the Newton polygon, has a slope at least equal to 2.

A necessary and sufficient condition for the product rule to hold, is that the coefficient of s° only depends on λ^{m}. This condition again can be displayed graphically : the lower shaded area of figure 16 should not intersect the McMillan polygon. A sufficient condition for this to be true clearly is that the SW boundary is absent and that the SE boundary has a slope at least equal to 1. □

Remark

The conditions of propositions 2.6 and 2.7 are only sufficient. In view of the results of section II, these propositions could be rephrased as : "A necessary and sufficient condition for the center of gravity (product) rule to hold, for all full rank feedback matrices K, is that the open loop system has m zeros (poles) at infinity (s=0), each having an order not less than 2 (1). □

APPENDIX C : The principal polygon.

Consider again the return difference :

$$\det(g I_m + G(s)) = g^m + \operatorname{tr} G(s)\ g^{m-1} + \ldots + \det G(s) .$$

Each coefficient constists of a sum of all principal minors of a certain dimension. The relationship between the McMillan and Newton boundaries hence can also be explained in the following way : if the i-minor content of G(s) at s_o is $c_o(G^i)$, then generically some principal i-minors will have a content $c_o(G^i)$. Furthermore, generically no cancelations will occur when summing these principal minors. As a result both boundaries generically coincide and the two phenomenon that can cause a difference between the boundaries, are :(1) the content of every principal minor is smaller than the i-minor content,(2) cancelations occur when adding the principal minors.

Definition C.1 ([20,22])

The content of a rational function g(s) at s=q, is : +k, if g(s) has a pole of order k at s=q; -k, if g(s) has a zero of order k at s=q. □

Definition C.2.

The principal i-minor content of a rational matrix G(s) at q, denoted by $pc_q(G^i)$, is defined to be the maximum of the contents at q of the principal i-minors of G(s). □

Definition C.3.

The principal polygon of a transfer matrix is defined in the same way as the McMillan polygon, but replacing contents by principal contents. □

Definition C.4.

The principal boundary is the lower boundary of the principal polygon.□

Proposition C.1.

The Newton boundary is contained within the principal polygon.

proof

Follows immediately from the introductary discussion. □

Remark

Unlike the McMillan boundary, the points generating the principal boundary need not in general lie on the principal boundary. □

The importance of the principal polygon is that, for low dimensional systems, it allows one to estimate the Newton polygon by hand: one only has to scan through the principal minors. We emphasize, however that in accordance with (2) quoted above, the actual Newton boundary might be different from the principal boundary. As a result the principal structure cannot predict the number of fixed points of the root loci, as one might have thought (see also [24], esp. pp. 26 and 56). The following example illustrates this: consider

$$G(s) = \begin{bmatrix} s^2/(s+s_o)^3 & 1/s & 0 \\ 1/s & 0 & -1/s \\ 0 & 1/s & 0 \end{bmatrix}$$

with

$$\det[gI + G(s)] = g^3 + g^2 \frac{s^2}{(s+s_o)^3} + g(1/s^2 - 1/s^2) + 1/(s+s_o)^3$$

$$= \frac{1}{(s+s_o)^3} [(s+s_o)^3 g^3 + g^2 s^2 + 1]$$

and

$$\text{CLCP}(s,g) = s^2 [(s+s_o)^3 g^3 + (3s_o + 1)s^2 g^2 + 1] .$$

In Figure 17 the Newton and the principal boundaries are shown. From inspection of the Newton boundary it follows that there is a fixed point, $s^2 = 0$. This fixed point is, however, not predicted by the principal boundary.

To summarize, the principal polygon can be helpful to estimate the Newton polygon and this estimate will be better than the McMillan polygon. However, the principal structure is not invariant under multiplication by the feedback matrix, nor need the estimate actually coincide with the Newton boundary.

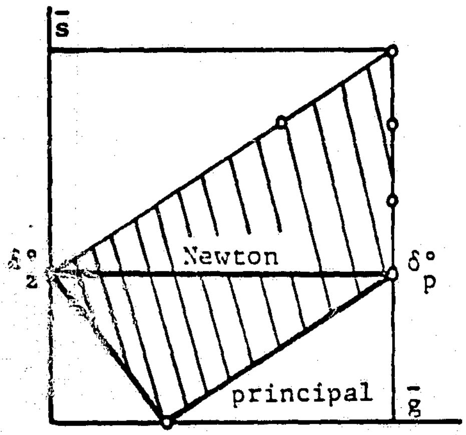

Figure 17.

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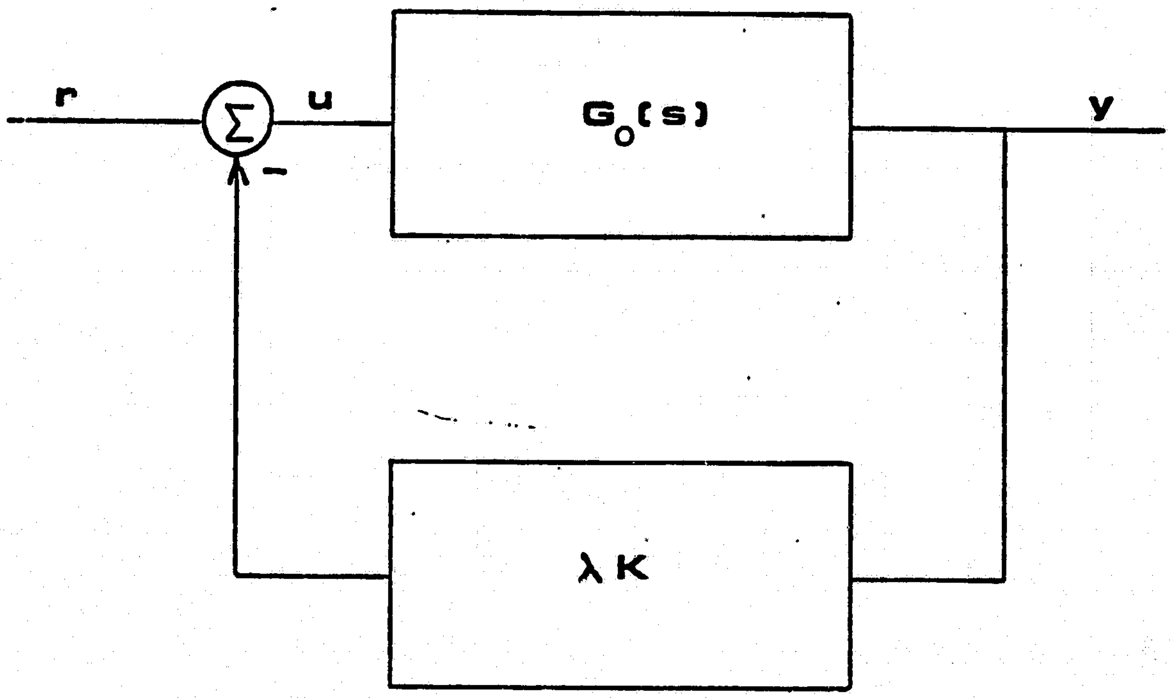
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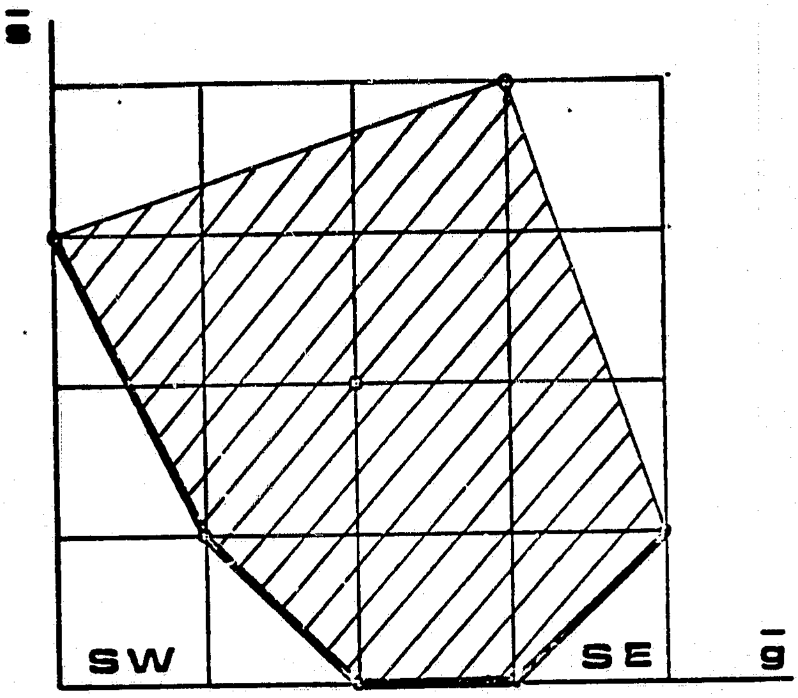
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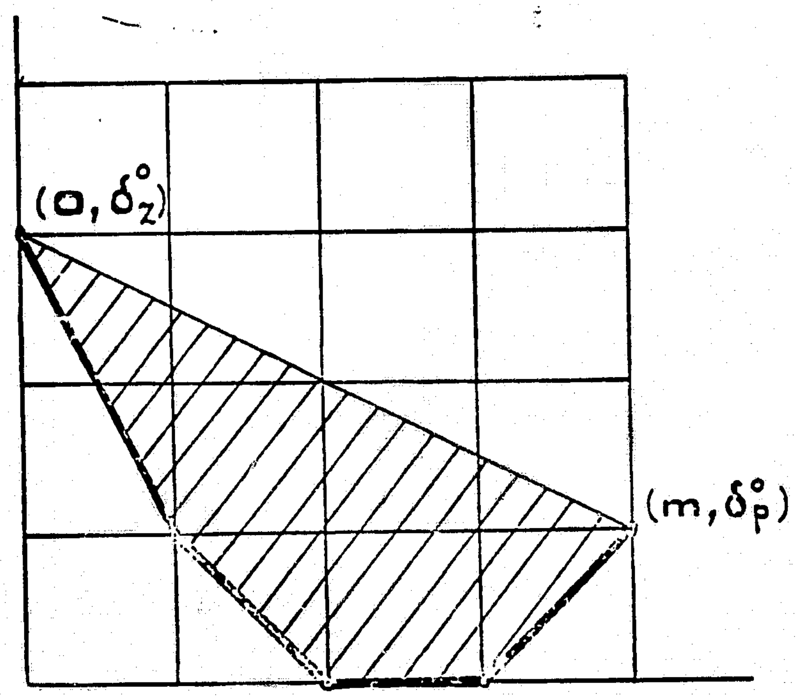
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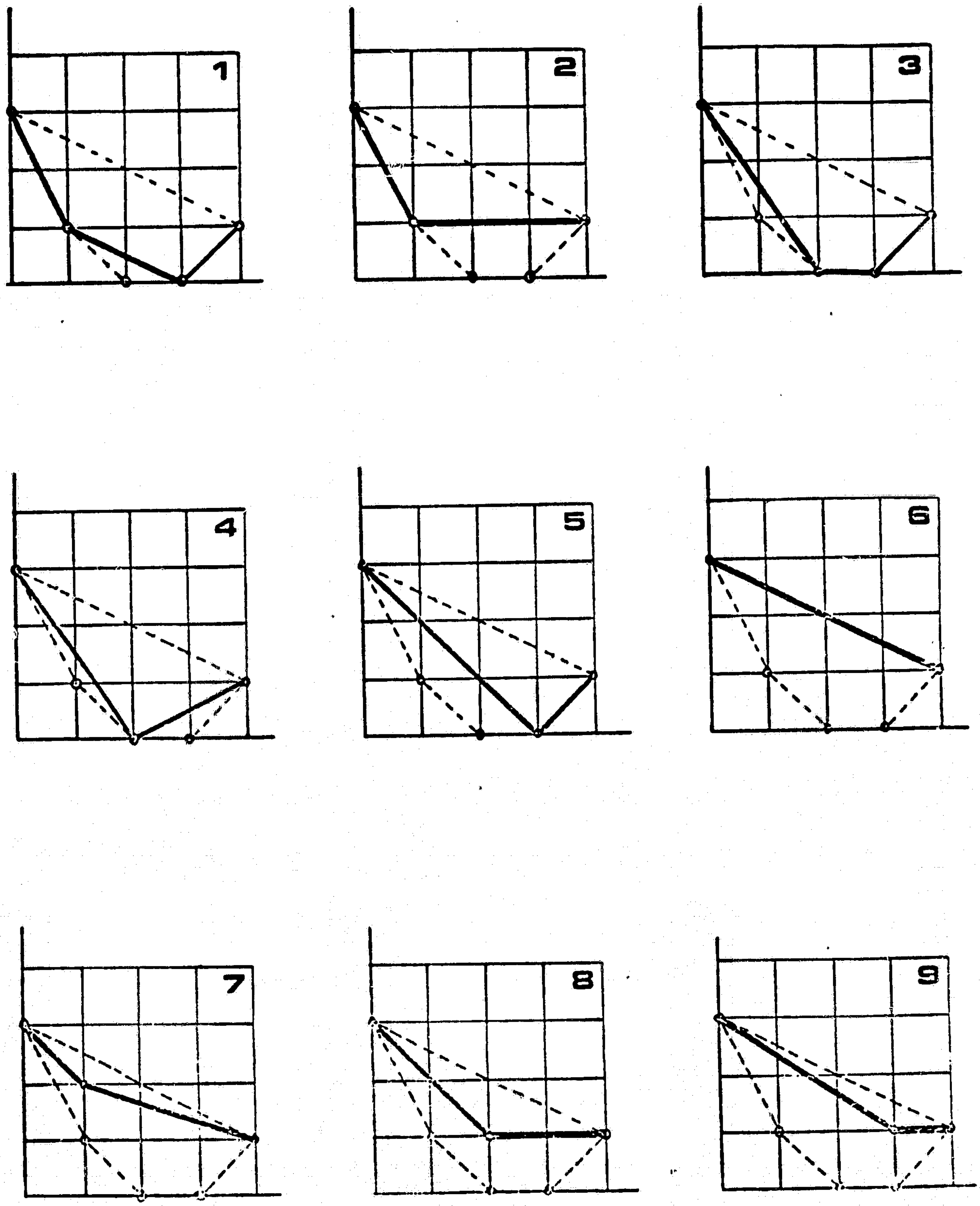

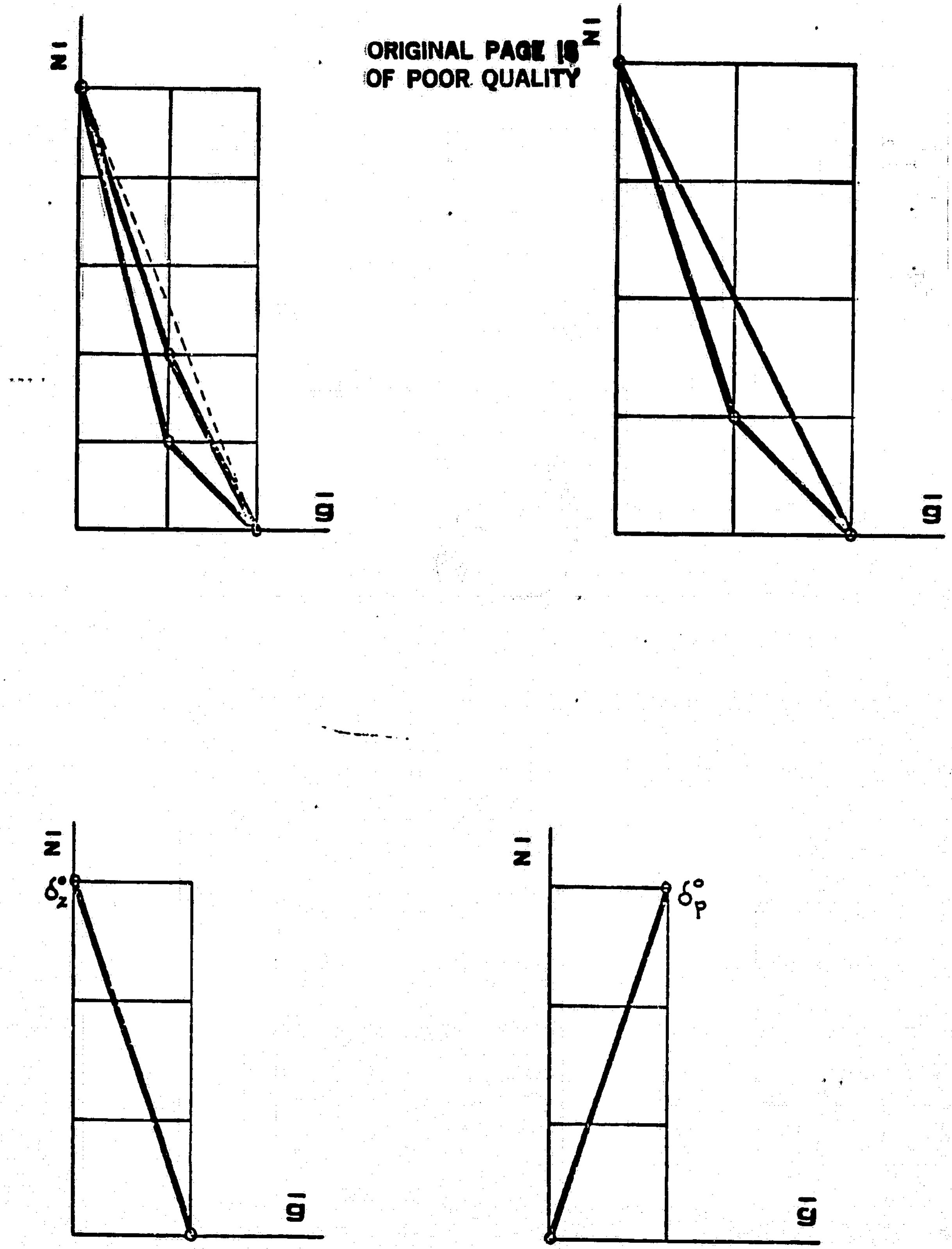
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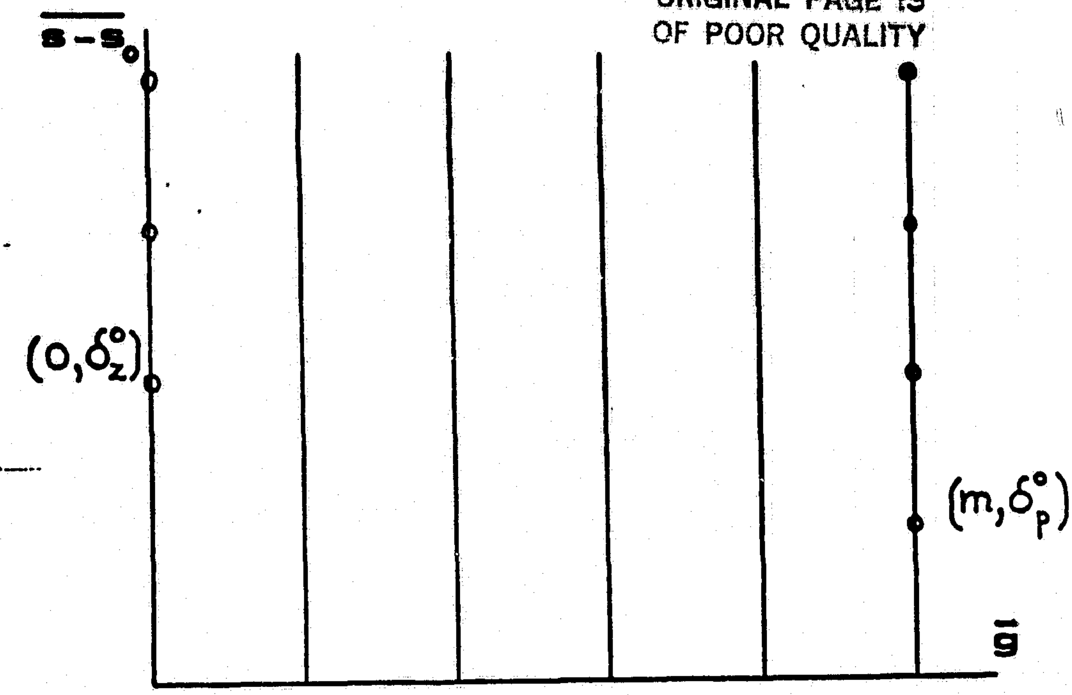
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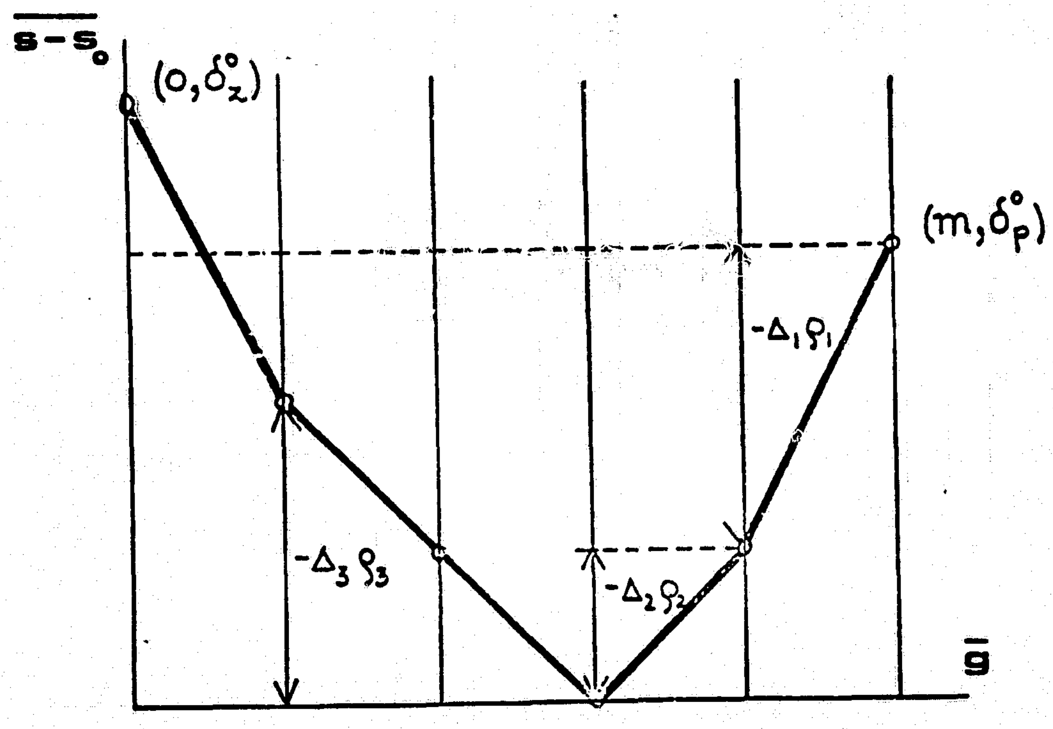
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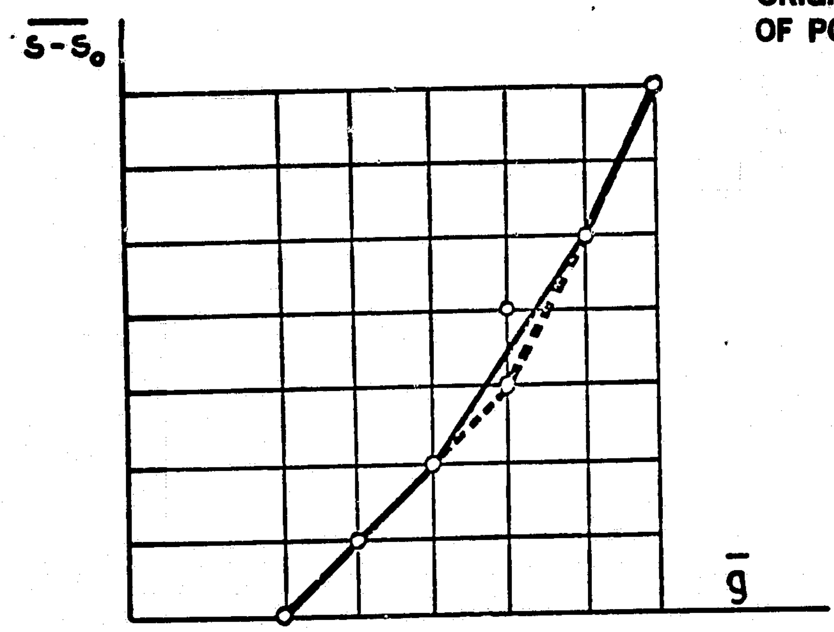
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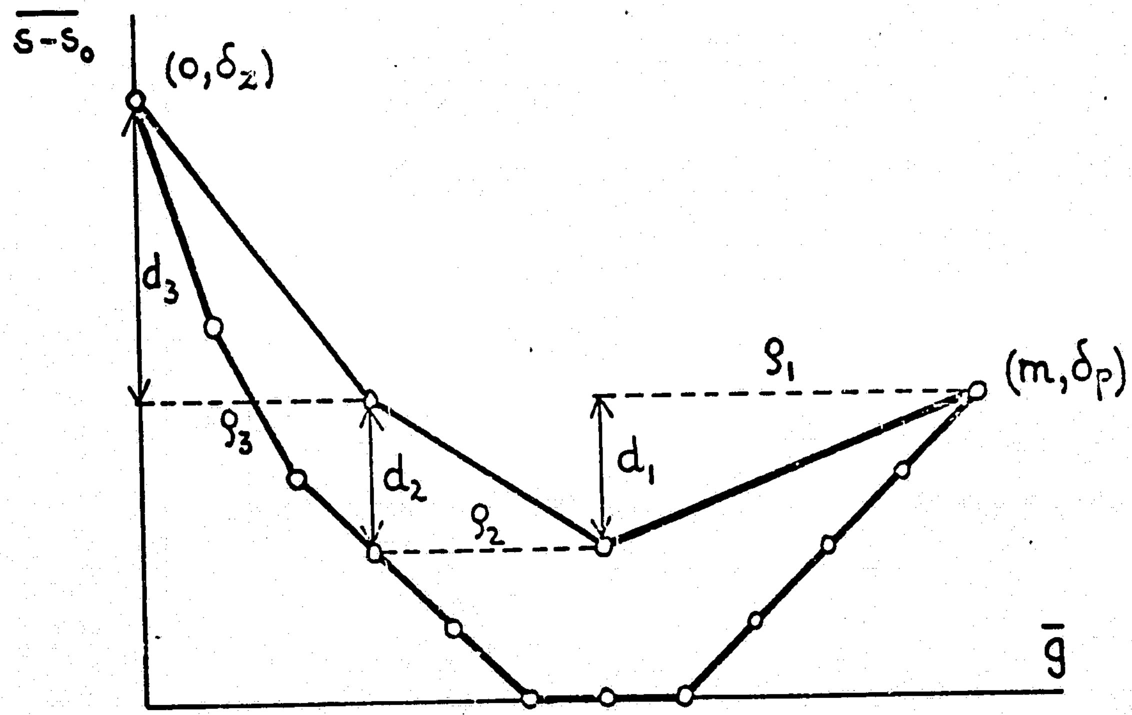
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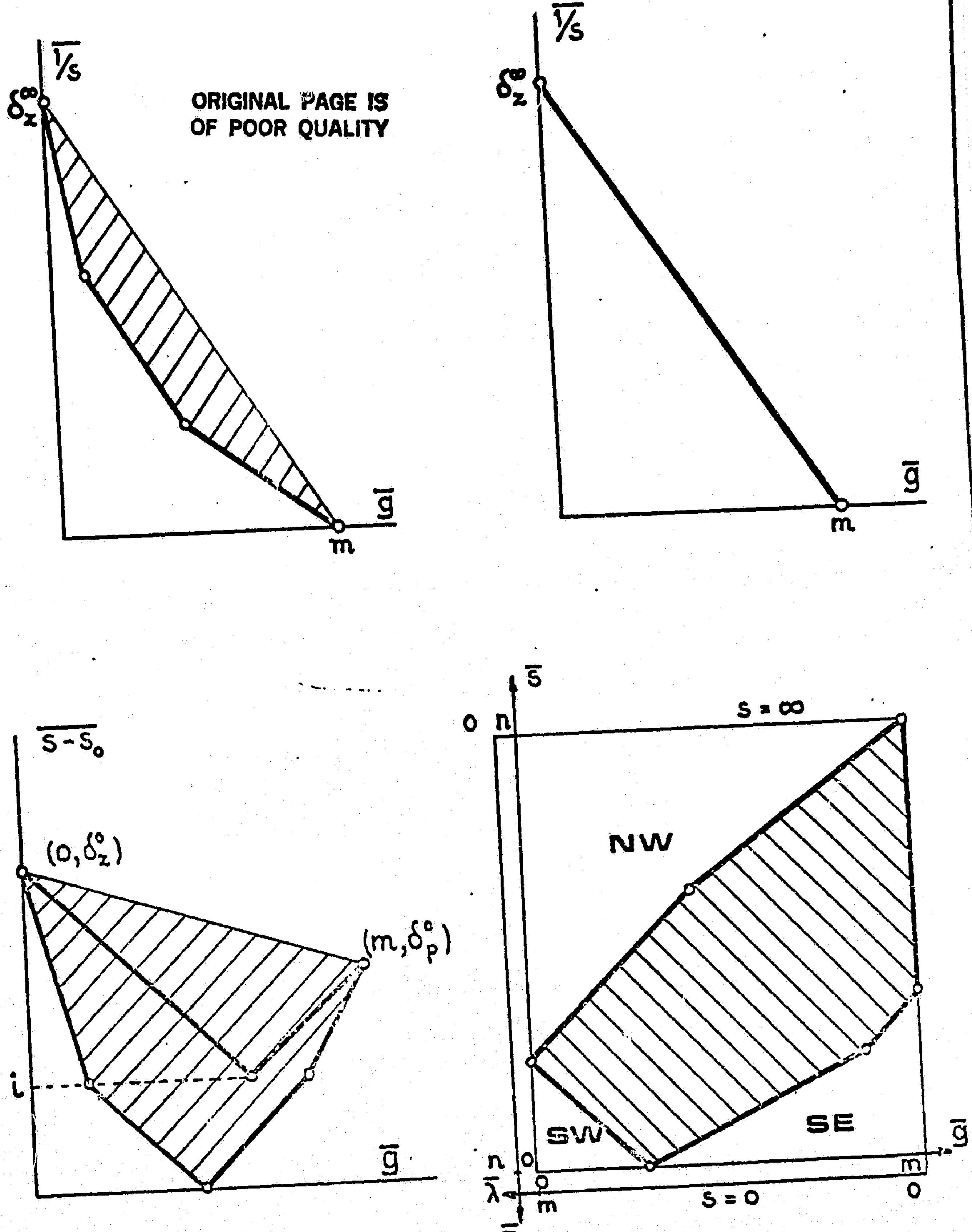
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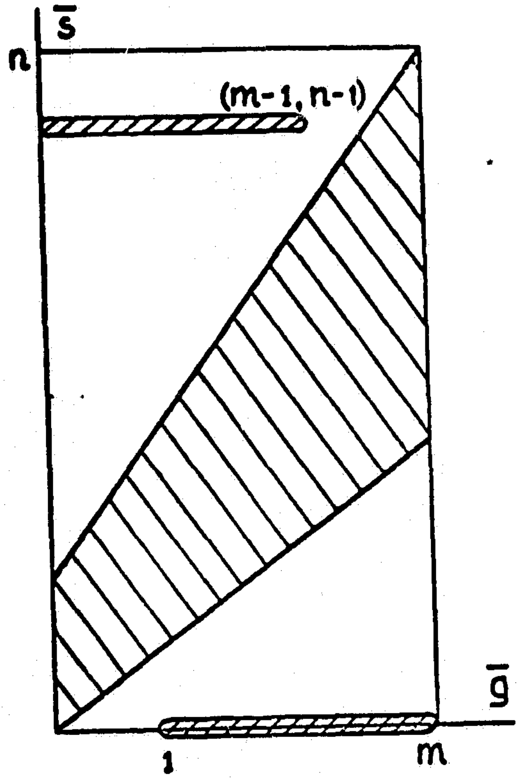

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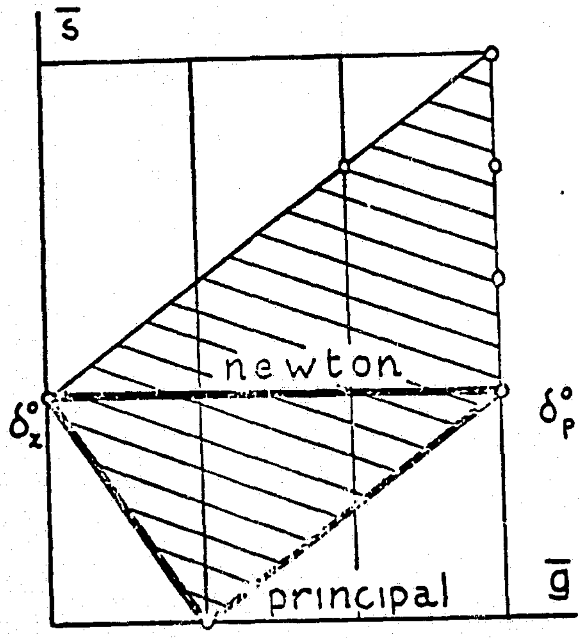

SIMULTANEOUS STABILIZATION AND ITS CONNECTION WITH THE PROBLEM OF INTERPOLATION BY RATIONAL FUNCTIONS.

by

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ABSTRACT :

A necessary and sufficient condition for simultaneous stabilization has been obtained for an r-tuple of m input p output plants under the restriction $r \leq m+p$, $\min(m,p) = 1$. In particular if $r < m+p$, a generic r-tuple is stabilizable and if $r = m+p$, only a semialgebraic set [2] of plants is stabilizable. The general case $\min(m,p) \geq 1$ can be vectored down to the above case so that in general a sufficient condition " $r \leq \max(m,p)$ " may be written down for generic simultaneous stabilization. This generalization supports special cases due to Saeks et al [8] for $m=p=1$, $r=2$ and Vidyasagar et al [12] for $\max(m,p) > 1$, $r=2$.

1. INTRODUCTION

We consider a set of r real,linear,time-invariant,proper dynamical systems,each of a given fixed Mcmillan degree n_i,i=1,r with m inputs and p outputs and ask the following question:

" When does there exist a non-switching, m input, p output real,linear,time invariant,proper compensator of arbitrary large but fixed Mcmillan degree q, which stabilizes each of the r given plants ? "

In this paper we give an answer to the above question for min(m,p)=1 and $r \leq m + p$. In fact an r-tuple of min(m,p)$\geq$ 1 plants can be vectored down to the min(m,p)=1 case (as shown by Stevens [10]) so that in this way one obtains a sufficient condition for generic,simultaneous stabilization given by ' $r \leq \max(m,p)$ '. This inequality however has also been derived by Ghosh and Byrnes [5]. Note that one can use this inequality to prove corollary 4.3 due to Vidyasagar and Viswanadham [12] on a result about the generic stabilizability of a pair of multi input-output plants.

Coming back to the case min(m,p)=1, we topologize the set 'S' of r-tuples of plants in the topology of section-2. If q is apriorily fixed we know that (see [4]) the set $\tilde{S}$ of r-tuples of plants which admit a stabilizing compensator is an open semialgebraic [2] subset of S. A semialgebraic set is a finite union and intersection of sets defined by algebraic equations and inequations, and it is a classical result by Tarski [11] and Seidenberg [9], that the property of being semialgebraic is preserved by a rational map. Indeed it was a pioneering idea of Anderson et al [1] to apply these concepts in system theory and show that the set of plants which can be stabilized by some non-dynamic compensator is an open semi-algebraic subset in the space of all plants.

In this paper we allow q to be arbitrarily large but finite and show that for min(m,p)=1, $r \leq m+p$, $\tilde{S}$ is an open semi-algebraic subset of S. Moreover for $r < m+p$, we show that $\tilde{S}$ is dense in S, so that simultaneous stabilizability is generic for the case min(m,p)=1 , $r < m + p$.

Without any loss of generality, we assume that $m \geq p$. Consider an r-tuple of $1 \times m$ plants to be stabilized by an $m \times 1$ compensator. In section 4, the solvability of the generic simultaneous stabilizability problem has been shown equivalent to a problem of interpolation by rational functions. These problems have been described in section 4 as PR1,PR2,PR3. Indeed we have the following general theorems :

Theorem 4.1

Under generic hypothesis (3.5), an r-tuple ($r < m+1$) of $1 \times m$ plants is generically simultaneously stabilizable.

Theorem 4.5

Under generic hypothesis (3.5), an r-tuple ($r = m+1$) of $1 \times m$ simultaneously stabilizable plants form a semialgebraic set.

Theorem 4.7

Under generic hypothesis (3.5), an r-tuple ($r > m+1$) of $1 \times m$ plants is simultaneously stabilizable iff the problem PR3 (see section 4,case III) has a solution.

The above theorems have been proved by using an interpolation result due to Youla et al [14,corollary 2,pp165] referred to in this paper as "Youla's lemma". Notice the important distinction between $r < m+1$ and $r \geq m+1$. This might be expected from the necessary and sufficient condition derived in [5]. As an illustration of the results obtained in section 4, we consider in Example 4.6 the case r=3,m=2,p=1 and obtain the semialgebraic set of stabilizable plants.

In section 5, we analyse the case $m = p = 1$. For $r = 1$, ofcourse the problem always has a solution. For $r = 2$ the statement of corollary 5.3 can be used to describe the semialgebraic set of plants that may be stabilized by some compensator. This reproves results due to Saeks et al [8], Vidyasagar et al [12]. For $r \geq 3$, the problem of simultaneous stabilization reduces to the problem of existence of a stable,minimum phase compensator as described by corollary 5.4 . Thus a necessary condition for simultaneous stabilizability of three single input single output plants may be obtained (see theorem 4.8) and we have the folklore example 5.5 of a triplet of simultaneously unstabilizable single input single output plants every pair of which, however, may be simultaneously stabilized. (see [15])

2. BASIC SET UP AND NOTATIONS

For details about the basic mathematical set up, the reader is referred to Vidyasagar et al [12],[13], Saeks et al [7],[8] and Desoer et al [3]. The following notations are used :

- H : Ring of all stable rational functions with real coeffici
- J : Set of multiplicative units in H.
- F : Quotients field of H [6, pp 88-90]
- C^+ : Closed right half complex plane.
- R^+ : Non-negative real line including infinity.

As per the above notation, H is an integral domain and the class of single input single output unstable systems considered are the elements of the quotient field F of H.

Thus every single input single output plant can be written as n/d where n,d $\in$ H. An r-tuple of m input 1 output plant of Mcmillan degre n can be written as

$$[\ n^{(1)}_{p_j} / d_{p_j}\ ,\ n^{(2)}_{p_j} / d_{p_j}\ ,\ \ldots\ldots\ ,\ n^{(m)}_{p_j} / d_{p_j}\ ] \qquad (2.1)$$

where $n^{(i)}_{p_j}$, d_{p_j} $\in$ H , and Mcmillan degree of $n^{(i)}_{p_j}$, $d_{p_j} \leq$ n ,i=1,2,..,m j=1,2, .. , r. Hence an r tuple of plants may be topologized in $\mathbb{R}^N$ for N = r [(n+1) (m+1) - 1]. (see [5] for details.)

A 1 input m output compensator of Mcmillan degree q can be written as

$$[\ n^{(1)}_{c_j} / d_{c_j}\ ,\ n^{(2)}_{c_j} / d_{c_j}\ ,\ \ldots\ldots\ ,\ n^{(m)}_{c_j} / d_{c_j}\ ]^T \qquad (2.2)$$

where $n_{c_j}^{(i)}, d_{c_j} \in H$ and Mcmillan degree of $n_{c_j}^{(i)}, d_{c_j} \leq q$
$i = 1,2, \ldots, m$; $j = 1,2,\ldots,r$. with the restriction

$$n_{c_j}^{(i)} / d_{c_j} = n_{c_k}^{(i)} / d_{c_k} \quad j,k=1,2,\ldots,r \tag{2.3}$$

For notational simplicity, we define the following quantities

Fix integers r,m and s, $1 \leq s \leq \min(r, m+2)$

Let $\eta_{1j}^{(i)} = n_{p_j}^{(i)}$, $\eta_{1j}^{(m+1)} = d_{p_j}$; $i=1,2,\ldots,m$;
$j=1,2,\ldots,(r-s+1)$. Define recursively the following :

$$\eta_{s_1,j}^{(i)} \triangleq \eta_{s_1-1,1}^{(1)} \eta_{s_1-1,j+1}^{(i+1)} - \eta_{s_1-1,j+1}^{(1)} \eta_{s_1-1,1}^{(i+1)}$$

$$\eta_{s_1,j}^{(m+1)} \triangleq \eta_{s_1-1,j+1}^{(1)} \xi_{s_1-1}$$

where, $$\xi_{s_1} \triangleq \eta_{s_1-1,1}^{(1)} \xi_{s_1-1} , \quad \xi_1 = 1 \tag{2.4}$$

for all, $i = 1,2,\ldots,m$; $s_1 = 1,2,\ldots,s$; $j = 1,2,\ldots, r-s+1$.

3. THE SIMULTANEOUS STABILIZATION PROBLEM

Following Vidyasagar et al [13], a necessary and sufficient condition for simultaneous stabilization of r plants given by (2.1) by a compensator (2.2),(2.3) is the solvability of the equations

$$\sum_{i=1}^{m} n_{p_j}^{(i)} n_{c_j}^{(i)} + d_{p_j} d_{c_j} = \Delta_j \tag{3.1}$$

for $d_{c_j}, n_{c_j}^{(i)} \in H$ and $\Delta_j \in J$; for all $i = 1,2,\ldots,m$;
$j = 1,2,\ldots,r$. with the restriction (2.3)

In the notation of (2.4), the equation (3.1) may be written as

$$\sum_{i=1}^{r_1} \gamma_{s,j}^{(i)} \eta_{s,j}^{(i)} + \sum_{i=r_1+1}^{m+1} \Delta_{s,j}^{(i)} \eta_{s,j}^{(i)} + \Delta_j \xi_s = 0 \qquad (3.2)$$

where $r_1 = m-s+2$; $j = 1,2, \ldots\ldots, r-s+1$

and the condition (2.3) may be written as the following :

" $\gamma_{s,j_1}^{(i_1)} / \gamma_{s,j_2}^{(i_1)}$, $i_1 = 1,2,\ldots,r_1$ and $\Delta_{s,j_1}^{(i_2)} / \Delta_{s,j_2}^{(i_2)}$, $i_2 = r_1+1,\ldots,m+1$-

are the same for i_1 , i_2 in the respective domains and for a fixed j_1 , j_2 ($1 \leq j_1,j_2 \leq r-s+1$) (3.3)

By (3.1) we have the following

SIMULTANEOUS STABILIZATION $\Leftrightarrow \exists\, \gamma_{s,j}^{(i)} \in H$; $\Delta_{s,j}^{(i)} \in J$ $\Delta_j \in J$ satisfying (3.2),(3.3) for $s=1$ (3.4)

The purpose of this section is to show that (3.4) is true for all $s = 1,2, \ldots, \min(r,m+2)$ under the following generic assumption :

" For a fixed s, such that $1 \leq s \leq \min(r,m+2)$; $\eta_{s,j_1}^{(1)}$ and $\eta_{s,j_2}^{(1)}$ have no root in common in C^+ for every $j_1,j_2 \in \{1,2,\ldots, r-s+1\}$ $j_1 \neq j_2$ (3.5)

Note that in particular for $r \leq m+1$, $s = r$; (3.2) is just a single equation as opposed to r equations in (3.1) . We have the following Lemma :

LEMMA 3.1

Under hypothesis (3.5), the set of equations (3.2),(3.3) has a solution for j = 1,2 for $\gamma_{s,j}^{(i)}$, $\Delta_{s,j}^{(i)}$ iff the equation (3.6) given below have a solution for some Δ , $\Delta_{s,1}^{(i)} \in J$, $\gamma_{s,1}^{(i)} \in H$.

$$\xi_s \eta_{s,1}^{(1)} \Delta = \xi_s \eta_{s,2}^{(1)} + \sum_{i=2}^{r_1} \gamma_{s,1}^{(i)} [\eta_{s,1}^{(1)} \eta_{s,2}^{(i)} - \eta_{s,2}^{(1)} \eta_{s,1}^{(i)}] + \sum_{i=r_1+1}^{m+1} \Delta_{s,1}^{(i)} [\eta_{s,1}^{(1)} \eta_{s,2}^{(i)} - \eta_{s,2}^{(1)} \eta_{s,1}^{(i)}] \qquad (3.6)$$

Proof : See Appendix I

THEOREM 3.1

Under the generic hypothesis (3.5), a set of r, $1 \times m$ plants is simultaneously stabilizable by a non-switching compensator iff (3.2), (3.3) have a solution for $\gamma^{(i)}_{s,j}$ $i = 1,2, \ldots, r_1$ and $\Delta^{(i)}_{s,j}$ $i = r_1+1, \ldots, m+1$ $\forall s = 1,2,\ldots,\min(r,m+2)$.

Proof :

We prove this theorem by induction over s. Note that the theorem is true for s = 1 by (3.1). Assume that the theorem is true for some s. To prove the theorem for s+1, the strtegy is as follows :

Consider the pair of equations defined by (3.2) for j=1 and $j = j_1$ $1 < j_1 \leq r-s+1$. There would be 'r-s' pairs of equations and by lemma 3.1, each pair can be reduced to a single equation (of the type (3.6)). Thus under the generic hypothesis (3.5) the set of equations (3.2) is equivalent to a set of equations obtained by replacing s by s+1 in (3.2) upto multiplication by a stable, minimum phase rational function.

4. THE INTERPOLATION PROBLEM

Our result on simultaneous stabilization (Theorem 3.1) can be posed as an interpolation problem. We pose the interpolation problem for each of the following cases seperately :

Case I ($r < m + 1$)

Choose the maximum value r of s. From (3.2) $r_1 = m-r+2$, j=1 so that (3.2) may be written as

$$\gamma_{s,1}^{(1)} \eta_{s,1}^{(1)} = - [\sum_{i=2}^{r_1} \gamma_{s,1}^{(i)} \eta_{s,1}^{(i)} + \sum_{i=r_1+1}^{m+1} \Delta_{s,1}^{(i)} \eta_{s,1}^{(i)} + \Delta_1 \xi_s] \quad (4.1)$$

Since $\gamma_{s,1}^{(1)} \in H$, we have the following interpolation problem :

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PR 1 " Find $\gamma_{s,1}^{(2)} \in H$ which intersects

$$- [\sum_{i=3}^{r_1} \gamma_{s,1}^{(i)} \eta_{s,1}^{(i)} + \sum_{i=r_1+1}^{m+1} \Delta_{s,1}^{(i)} \eta_{s,1}^{(i)} + \Delta_1 \xi_s] / \eta_{s,1}^{(2)}$$

at those points $s_0 \in C^+$ where $\eta_{s,1}^{(1)}$ vanishes. "

Proof of Theorem 4.1 (Solution of PR 1)

Assume, $\Delta_1 = 1$, $\Delta_{s,1}^{(i)} = 1 \quad i = r_1+1, \ldots, m+1$,
$\gamma_{s,1}^{(i)} = 1 \quad i = 3,4, \ldots r_1$.

The problem thus reduces to obtaining $\gamma_{s,1}^{(2)} \in H$ which interpolates a symmetric set of complex tuples. It is therefore sufficient to choose a real polynomial $\gamma_{s,1}^{(2)}$ of sufficiently large degree.

COROLLARY 4.2

A sufficient condition for simultaneous generic stabilization for $\min(m,p) \geq 1$ is given by $r \leq \max(m,p)$

Proof :

Given a set of r, $p \times m$ ($m \geq p$, say) transfer functions, with distinct simple poles, $G_1, G_2, \ldots, G_r$. A plant G_i has the decomposition

$$G_i = \sum_{j=1}^{n} \frac{T_j^i}{s - \lambda_j^i}$$

where n is the Mcmillan degree of G_i, T_j^i is a rank one matrix of order $p \times m$ and $\lambda_1^i, \lambda_2^i \ldots \lambda_n^i$ are the poles of G_i. Now consider an arbitrary non-zero $1 \times p$ vector v such that $v.T_j^i \neq \underline{0} \; \forall \; i=1,..,r; \; j=1,..,n$, and consider the mapping $G_i \longmapsto G_i' = v.G_i \; \forall \; i=1,\ldots r$

Since G_i' constitute a set of r, $1 \times m$ plants they can be generically simultaneously stabilizable by some compensator if $r' < \max(m,p) + 1$ (By Theorem 4.1). Moreover the generic condition in G_i' pulls back to that in G_i. Finally (see [5]) any r tuple of plants have a constant gain output feedback K such that the closed loop systems have distinct simple poles.

COROLLARY 4.3 (Vidyasagar, Viswanadham [12])

A pair of max(m,p) > 1 plants is generically, simultaneously stabilizable.

Proof : Immediate from corollary 4.2 .

Case II ($r = m+1$)

Choose the maximum value r of s. Hence $s = r = m+1$, $r_1 = 1$, $j = 1$ and (3.2) may be written as

$$\gamma^{(1)}_{m+1,1}\, \eta^{(1)}_{m+1,1} + \sum_{i=2}^{m+1} \Delta^{(i)}_{m+1,1}\, \eta^{(i)}_{m+1,1}\, \Delta_1\, \xi_{m+1} = 0 \qquad (4.2)$$

Since $\gamma^{(1)}_{m+1,1} \in H$, we have the following interpolation problem .

PR 2 " Find $\Delta^{(2)}_{m+1,1} \in J$, which intersects

$$-\left[\sum_{i=3}^{m+1} \Delta^{(i)}_{m+1,1}\, \eta^{(i)}_{m+1,1} + \Delta_1\, \xi_{m+1}\right] / \eta^{(2)}_{m+1,1}$$

at those points $s_0 \in C^+$ where $\eta^{(1)}_{m+1,1}$ vanishes . "

Solution of PR 2

We want conditions on the plant parameters for which PR 2 has a solution. First of all we consider the following lemma :

LEMMA 4.4

Let $\xi_1, \xi_2, \ldots, \xi_{t_1} \in H$ be given and let $s_1, s_2, \ldots, s_t \in C^+$ be a symmetric set of complex numbers. Then $\exists\, \Delta_i$, $i = 1, \ldots t_1$ such that

$$\sum_{i=1}^{t_1} \Delta_i\, \xi_i \Big|_{s=s_j} = 0 \ ; \quad j=1,\ldots,t \qquad (4.3)$$

if and only if, $\exists$ non-zero real numbers $r_1^{j_1}, r_2^{j_1}, \ldots\ldots, r_{t_1}^{j_1}$ such that

$$\sum_{i=1}^{t_1} r_i^{j_1} \xi_i \Big|_{s=s_{j_1}} = 0 \qquad (4.4)$$

where $s_{j_1} \in \{ s_1, s_2, \ldots, s_t \} \cap R^+$, such that $r_i^{j_1}$ has the same sign for fixed i and $\forall j_1$.

<u>Proof</u> :

Writing (4.3) as

$$\Delta_1 \Big|_{s=s_j} = - \sum_{i=2}^{t_1} \Delta_i \xi_i / \xi_1 \Big|_{s=s_j} \qquad (4.5)$$

it may be concluded that (4.5) holds if and only if $\exists \Delta_1 \in J$, which interpolates the points

$$\left(s_j , - \sum_{i=2}^{t_1} \Delta_i \xi_i / \xi_1 \Big|_{s=s_j} \right) \qquad (4.6)$$

By Youla's lemma [14] a necessary and sufficient condition is given by the following :

" $\exists \Delta_i$ $i=2,\ldots,t_1$ such that $r_1^{j_1} = - \sum_{i=2}^{t_1} \Delta_i \xi_i / \xi_1 \Big|_{s=s_{j_1}}$

have the same sign, $\forall j_1$ such that $s_{j_1} \in \{s_1, s_2, \ldots\ldots$

$\ldots, s_t \} \cap R^+$ " (4.7)

From the condition (4.7) we have

$$\Delta_2 \Big|_{s=s_{j_1}} = \left[- r_1^{j_1} \xi_1 - \sum_{i=3}^{t_1} \Delta_i \xi_i \right] / \xi_2 \Big|_{s=s_{j_1}} \qquad (4.8)$$

We apply Youla's lemma repeatedly and the proof follows.

<u>PROOF OF THEOREM 4.5</u>

We now obtain the solution of the interpolation problem PR 2.

By Lemma 4.4, it is clear that a necessary and sufficient condition is the existence of non-zero real numbers $r_i^{j_1}$ such that

$$\left. \sum_{i=2}^{m+1} r_i^{j_1} \eta_{m+1,1}^{(i)} + r_{m+2}^{j_1} \xi_{m+1} \right|_{s=s_{j_1}} = 0 \qquad (4.9)$$

$\forall$ j_1 such that $s_{j_1} \in \{ s_1, s_2, \ldots, s_t \} \cap R^+$; where $s_i \in C^+$

such that, $\eta_{m+1,1}^{(1)}(s_i) = 0$ (4.10)

and where for a fixed i , $r_i^{j_1}$ has the same sign for all j_1 .

Let us define

$$V(s) \triangleq [(-1)^{e_1} \eta_{m+1,1}^{(2)}, (-1)^{e_2} \eta_{m+1,1}^{(3)}, \ldots, (-1)^{e_m} \eta_{m+1,1}^{(m+1)}, (-1)^{e_{m+1}} \xi_{m+1}]$$

It is now straightforward to show the following (see [4] for details)

" SIMULTANEOUS STABILIZATION $\iff$ $\exists$ integers $e_1, e_2, \ldots, e_{m+1}$ such that all the vectors $V(s_{j_1})$ (s_{j_1} as defined in 4.10) have at least one negative component. (4.11)

The above condition clearly defines a semialgebraic condition.

EXAMPLE 4.6

For m=2,p=1,r=3 consider the triplet of plants

$$[n_{p_1}^1/d_{p_1} \quad n_{p_1}^2/d_{p_1}],[n_{p_2}^1/d_{p_2} \quad n_{p_2}^2/d_{p_2}],[n_{p_3}^1/d_{p_3} \quad n_{p_3}^2/d_{p_3}]$$

$$[n_{c_1}^1/d_{c_1} \quad n_{c_1}^2/d_{c_1}]=[n_{c_2}^1/d_{c_2} \quad n_{c_2}^2/d_{c_2}]=[n_{c_3}^1/d_{c_3} \quad n_{c_3}^2/d_{c_3}]$$

i.e. $n_{c_i}^s / d_{c_i} = n_{c_j}^s / d_{c_j}$ $\quad i \neq j \quad s=1,2;\ i,j \in \{1,2,3\}$

For simultaneous stabilization we need to solve the equation (see (4.2)

$$\bar{n}_{21} + \Delta_1 \bar{n}_{13} + \Delta_2 \bar{n}_{23} = \delta \cdot \kappa$$

for $\Delta_1, \Delta_2 \in J$; $\delta \in H$

where $\bar{n}_{ij} = n_{p_i}^1 n_{p_j}^2 - n_{p_j}^1 n_{p_i}^2$

and $\kappa = d_{p_1} \bar{n}_{23} + d_{p_2} \bar{n}_{31} + d_{p_3} \bar{n}_{12}$

Let $s_1, s_2, s_3, \ldots\ldots, s_t$ be the set of points in R^+ where κ vanishes. By (4.11) the set of simultaneously unstabilizable plants is given by the following condition

" For every triplet $e_1, e_2, e_3 \in \{0,1\}$

$$(-1)^{e_1} \bar{\eta}_{21}(s_{t_1}) > 0 \,,\; (-1)^{e_2} \bar{\eta}_{13}(s_{t_1}) > 0 \,,\; (-1)^{e_3} \bar{\eta}_{32}(s_{t_1}) > 0$$

for some $t_1 \in \{1,2,\ldots\ldots\ldots,t\}$ "

<u>Case III ($r > m+1$)</u>

Choose the maximum value $m+2$ of S. Then (3.2) reduces to

$$\sum_{i=1}^{m+1} \Delta_{m+2}^{(i)} \eta_{m+2,j}^{(i)} + \Delta_j \xi_{m+2} = 0 \qquad (4.12)$$

where we have normalized $\Delta_{m+2,k}^{(i)}$ to satisfy

$$\Delta_{m+2,j}^{(i)} = \Delta_{m+2,k}^{(i)} \quad \forall\; i = 1,2,\ldots\ldots,m+1 \text{ and } j,k \in \{1,2,\ldots,r-m-1\}$$

From (4.12) we may now state the interpolation problem as follows :

<u>PR 3</u> " Find $\Delta_{m+2}^{(2)} \in J$ which intersects

$$\left[-\Delta_j \xi_{m+2} - \sum_{i=3}^{m+1} \Delta_{m+2}^{(i)} \eta_{m+2,j}^{(i)} \right] / \eta_{m+2,j}^{(2)} \qquad (4.13)$$

at those points $s_0^j \in C^+$ and only those in the closed right half plane where $\eta_{m+2,j}^{(1)}$ vanishes $\forall\; j = 1,2,\ldots\ldots,r-m-1$ "

<u>Proof of Theorem 4.7</u>

Immediate from the argument given above.

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Note :

A solution of PR 3 would involve finding a stable, minimum phase rational function, $\Delta^{(2)}_{m+2}$, which intersects the graphs of r-m-1 rational functions, given by (4.14) as specified in the statement of PR3 In particular, since $\Delta^{(2)}_{m+2}$ is stable we have the following :

Theorem 4.8 (Necessary condition)

A necessary condition for simultaneous stabilization of r plants [$r > m+p$, $\min(m,p) = 1$] is given by the existence of non-zero real numbers $r_i^{j_1}$ such that

$$\left[\sum_{i=2}^{m+1} r^{j_1}_{i,j} \, \eta^{(i)}_{m+2,j} + r^{j_1}_{m+2,j} \, \xi_{m+2} \right] \Bigg|_{s=s^{(j)}_{j_1}} = 0$$

$\forall\, j = 1,2, \ldots , r-m-1$

$\forall\, j_1$ such that $s^{(j)}_{j_1} \in R^+$ and $\eta^{(1)}_{m+2,j}(s^{(j)}_{j_1}) = 0$

and where for a fixed 'i' , $r^{j_1}_{i,j}$ has the same sign for all j, j_1 defined above and for a fixed j, $r^{j_1}_{m+2,j}$ has the same sign for all j_1 as defined above.

Proof :

Straightforward and follows from the proof of Theorem 4.5 (see [4] for details)

5. THE SINGLE INPUT SINGLE OUTPUT CASE

The case m=p=1 has been studied extensively by Saeks et al [7],[8] and also by Vidyasagar et al [12]. For our purposes we restate Theorem 3.1 for this special case and reprove as corollaries, some of the results known. Refer back to the plants and compensators in the notation of (2.1) , (2.2) and (2.3).

Theorem 5.1

For $m = p = 1$, SIMULTANEOUS STABILIZATION <=> " the following set of equations in H, has a solution for $\Delta_i \in J$ $i = 1,2,\ldots,r$ and $n_{c_1}^{(1)}, d_{c_1} \in H$ "

$$(r > 2) \quad \Delta_1 \eta_{1i} + \Delta_2 \eta_{2i} = \Delta_i \eta_{12}, \quad i=1,2,\ldots,r-2 \tag{5.1}$$

$$(r = 2) \quad n_{c_1}^{(1)} \eta_{21} + d_{p_2} = d_{p_1} \Delta_1 \tag{5.2}$$

$$(r = 1) \quad n_{c_1}^{(1)} n_{p_1}^{(1)} + d_{c_1} d_{p_1} = \Delta_1 \tag{5.3}$$

$$\text{where} \quad \eta_{ij} = n_{p_i}^{(1)} d_{p_j} - n_{p_j}^{(1)} d_{p_i} \quad i \neq j \tag{5.4}$$

Proof :

The proof is immediate from that of Theorem 3.1 (see [4] for details)

Corollary 5.2 (Saeks et al [8])

A necessary and sufficient condition for a pair of single input single output plants to be simultaneously stabilizable is given by the following condition :

$$\text{" } d_{p_2} / d_{p_1} \Big|_{s = s_0} \text{ has the same sign for all } s_0 \in R^+$$

where η_{21} vanishes "

Proof :

Writing (5.2) as $n_{c_1}^{(1)} = (d_{p_1} \Delta_1 - d_{p_2}) / \eta_{21}$ and using the argument of Theorem 4.5 the result follows.

Corollary 5.3 (Assume $r > 2$)

Let p_i, $i=1,2,\ldots,r$ be generic plants (satisfying (3.5)) with coprime representation $n_{p_i}^{(1)}/d_{p_i}$. There exists a compensator which simultaneously stabilizes the r plants $p_i, i=1,\ldots,r$ iff there exists a stable, minimum phase dynamic compensator Δ_2 / Δ_1 which places the poles of η_{2i} / η_{1i} at those points in C^+ where η_{12} vanishes and places the rest of the poles in the left half plane. $\forall$ $i = 1,2,\ldots,r-2$.

Proof :

Immediate from (5.1) and section 4 case III.

Note : Even for r=3 the existence criterion of a stable, minimum phase compensator satisfying the above condition is not known.

Example 5.5

In this example we construct a triplet of plants which are stabilizable in pair but unstabilizable simultaneously.

Let

$$P_1 = \frac{s + 1}{s - 2}, \quad P_2 = \frac{s + 3}{s - 1}, \quad P_3 = \frac{s + A}{s + B}$$

From Corollary 5.2 and Theorem 4.8 the required algebraic conditions which A,B need to satisfy may be constructed. The choice A=-1.1,B=-4.5 satisfies these conditions. The details have been omitted.

6. CONCLUSION

This paper addresses the question of simultaneous stabilization under the restriction min(m,p)=1, $r \leq m+p$. A sufficient condition for simultaneous stabilization has been obtained by using these techniques, for the general case $\min(m,p) \geq 1$.The case min(m,p)=1, $r > m+p$ still remains open. Future research might be in the direction of finding an appropriate necessary condition for $\min(m,p) \geq 1$.

7. ACKNOWLEDGEMENT

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7. APPENDIX I

Proof of Lemma 3.1

Scaling (3.2) for j=1,2, with $\Delta_j = -1$ and by a slightly tedious but straightforward algebraic manipulation the following set of equations have been obtained :

$$\gamma^{(1)}_{s,1} = [\ \xi_s - \sum_{i=2}^{r_1} \gamma^{(i)}_{s,1}\ \eta^{(i)}_{s,1} - \sum_{i=r_1+1}^{m+1} \Delta^{(i)}_{s,1}\ \eta^{(i)}_{s,1}\ ]\ /\ \eta^{(1)}_{s,1} \qquad (I1)$$

$$\gamma^{(i)}_{s,2}\ W(s) = \gamma^{(i)}_{s,1}\ \eta^{(1)}_{s,1}\ \xi_s \qquad i = 1,\ldots\ldots,r_1 \qquad (I2)$$

$$\Delta^{(i)}_{s,2}\ W(s) = \Delta^{(i)}_{s,1}\ \eta^{(1)}_{s,1}\ \xi_s \qquad i = r_1+1,\ldots\ldots,m+1 \qquad (I3)$$

where,

$$W(s) = \xi_s\ \eta^{(1)}_{s,2} + \sum_{i=2}^{r_1} \gamma^{(i)}_{s,1}\ [\ \eta^{(1)}_{s,1}\ \eta^{(i)}_{s,2} - \eta^{(1)}_{s,2}\ \eta^{(i)}_{s,1}\ ]$$
$$+ \sum_{i=r_1+1}^{m+1} \Delta^{(i)}_{s,1}\ [\ \eta^{(1)}_{s,1}\ \eta^{(i)}_{s,2} - \eta^{(1)}_{s,2}\ \eta^{(i)}_{s,1}\ ] \qquad (I4)$$

Sufficiency :

By assumption $\exists\ \gamma^{(i)}_{s,1} \in H\ \ i=2,3,\ldots,r_1;\ \Delta^{(i)}_{s,1} \in H\ \ i=r_1+1,\ldots,m+1$ which satisfies (3.6)

From (I2),(I3) and (3.6) we have

$$\gamma^{(i)}_{s,2} = \gamma^{(i)}_{s,1}\ \Delta^{-1} \in H\ ;\ i=1,\ldots,r_1 \qquad (I5)$$

$$\Delta^{(i)}_{s,2} = \Delta^{(i)}_{s,1}\ \Delta^{-1} \in H\ ;\ i=r_1+1,\ldots\ldots,m+1 \qquad (I6)$$

provided $\gamma^{(1)}_{s,1} \in H$

To show that $\gamma^{(1)}_{s,1} \in H$, let $s_0 \in C^+$ be such that $\eta^{(1)}_{s,1}(s_0) = 0$. From (I4) and (3.6) $W(s_0) = 0$ and

$$W(s_0) = \eta^{(1)}_{s,2}\ [\ \xi_s - \sum_{i=2}^{r_1} \gamma^{(i)}_{s,1}\ \eta^{(i)}_{s,1} - \sum_{i=r_1+1}^{m+1} \Delta^{(i)}_{s,1}\ \eta^{(i)}_{s,1}] \qquad (I7)$$

By generic assumption (3.5), $\eta^{(1)}_{s,2} \neq 0$ so that

$$\left[\xi_s - \sum_{i=2}^{r_1} \gamma_{s,1}^{(i)} \eta_{s,1}^{(i)} - \sum_{i=r_1+1}^{m+1} \Delta_{s,1}^{(i)} \eta_{s,1}^{(i)} \right] \Bigg|_{s=s_0} = 0 \quad \text{(I7)}$$

By (I1),(I7) $\gamma_{s,1}^{(1)} \in H$

Necessity :

Assume that (3.2) has a solution. Define Δ by (3.6). We want to show that $\Delta \in J$

Let $s_0 \in C^+$ be such that

$$\eta_{s,1}^{(1)} (s_0) = 0 \quad \text{(I8)}$$

Since by assumption $\gamma_{s,1}^{(1)} \in H$, we have the equation I7 from I1. By (I8), (I7), and (3.6) $W(s_0) = 0$. Let $s_1 \in C^+$ be such that $\xi_s(s_1) = 0$. If $s > 1$, (I3) $\Rightarrow$ either $W(s_1) = 0$ or $\Delta_{s,2}^{(i)} (s_1) = 0$. However since $\Delta_{s,2}^{(i)} \in J$ we have $W(s_1) = 0$. If s=1, on the other hand $\xi_s(s_1)$ cannot vanish by definition. Thus we conclude that $\Delta \in H$

To show that $\Delta^{-1} \in H$ we proceed as follows :
Let $s_2 \in C^+$ be such that $W(s_2) = 0$. Then either $\xi_s(s_2) = 0$ or $\eta_{s,1}^{(1)} (s_2) = 0$ for if not by (I2),(I3)

$$\gamma_{s,1}^{(i)}(s_2) = 0 \quad i = 1,\ldots,r_1 \quad \text{(I9)}$$

$$\Delta_{s,1}^{(i)}(s_2) = 0 \quad i = r_1+1,\ldots,m+1 \quad \text{(I10)}$$

However by (3.2),(I9),(I10) we have $\Delta_j(s_2) = 0$ which is absurd since $\Delta_j \in J$. Hence $\Delta^{-1} \in H$

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SIMULTANEOUS STABILIZATION AND SIMULTANEOUS POLE-PLACEMENT BY NONSWITCHING DYNAMIC COMPENSATION

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The "simultaneous stabilization problem" - in either discrete or continuous time - consists in answering the following question:

Given an r-tuple $G_1(s),\ldots,G_r(s)$ of $p\times m$ proper transfer functions, does there exist a compensator $K(s)$ such that the closed-loop systems $G_1(s)(I+K(s)G_1(s))^{-1},\ldots,G_r(s)(I+K(s)G_r(s))^{-1}$ are (internally) stable?

As pointed out in [13], this question arises in reliability theory, where $G_2(s),\ldots,G_r(s)$ represents a plant $G_1(s)$ operating in various modes of failure and $K(s)$ is a nonswitching stabilizing compensator. Of course, for the same reason, it is important in the stability analysis and design of a plant which can be switched into various operating modes. The simultaneous stabilization problem can also apply to the stabilization of a nonlinear system which has been linearized at several equilibria. Finally, it has been shown [14], [20] that to solve the case $r=2$ is to solve the well-known problem considered by Youla et al in [21]: When can a single plant be stabilized by a stable compensator? This correspondence also serves to give some measure of the relative depth of this problem.

In order to describe the results obtained via this correspondence, we need some notation. First, set n_i = McMillan degree of $G_i(s)$. In the scalar input-output setting $(m=p=1)$, we regard each $G_i(s)$ as a point in $\mathbb{R}^{2n_i+1}$, viz. if

$$G_i(s)=p_i(s)/q_i(s),\ \text{where}$$

$$p_i(s)=a_{oi}+\ldots+a_{n_i i}s^{n_i},\ \text{and}\ \ q_i(s)=b_{1i}+\ldots+b_{ni}s^{n_i-1}+s^{n_i}$$

then $G_i(s)$ corresponds to the vector $(a_{oi},\ldots,a_{n_i i},b_{1i},\ldots,b_{ni})$ $\mathbb{R}^{2n_i+1}$.

Moreover, since p_i and q_i are relatively prime, this vector lies in the open dense set $\mathrm{Rat}(n_i) \subset \mathbb{R}^{2n_i+1}$ (see [3] for the strictly proper case). In [14], Saeks and Murray used the techniques of fractional representations [8] and the correspondence mentioned above to give explicit inequalities defining the open set

$$U \subset \mathrm{Rat}(n_1) \times \mathrm{Rat}(n_2)$$

of pairs $(G_1(s),G_2(s))$ which are simultaneously stabilizable. In [20] Vidyasagar and Viswanadham showed, using similar techniques, that provided $\max(m,p) > 1$ the open set U of pairs $(G_1(s),G_2(s))$ which can be stabilized is in fact dense.

This can be made precise by topologizing a point $G_i(s)$ in the set

$$\Sigma^n_{m,p} = \{p \times m \; G_i(s) \; ; \; \text{degree } G_i(s) = n_i\}$$

as a vector in $\mathbb{R}^{(n_i+1)(mp)}$ via its Hankel parameters: If

$$G_i(s) = \sum_{j=0}^{\infty} H_{ij} s^{-j}$$

then $G_i(s)$ corresponds to the $n+1$ $p \times m$ block matrices $\{H_{i0},\ldots,H_{i,n+1}\}$ which determines $G(s)$. It is known that $\Sigma^n_{m,p}$ is an $(n(m+p)+mp)$-manifold (see [7], [12], [5]), although this is not important here. What is important is that $\Sigma^n_{m,p}$ is a topological space.

One of our main results concerns the generic stabilizability problem; that is,

<u>Question 1.1</u>. Fix m,p,r, and n_i. Is the set U of r-tuples $G_1(s),\ldots,G_r(s)$ which can be simultaneously stabilized open and dense in $\Sigma^{n_1}_{m,p} \times \ldots \times \Sigma^{n_r}_{m,p}$?

It is also important to ask, for reasons of global robustness of algorithms finding such a compensator, for compensators with a fixed degree of complexity.

Question 1.2. Fix m,p,r, and n_i. What is the minimal value of q (if one exists) for which the set W_q of r-tuples which can be simultaneously stabilized, by a compensator of degree $\leq q$, is open and dense in $\Sigma_{m,p}^{n_1} \times \dots \times \Sigma_{m,p}^{n_r}$?

It should be noted that, in the case $r=1$, Question 1.2 is an outstanding, unsolved, classical problem. In this paper, we prove:

Theorem 1.1. In either discrete or continuous time, a sufficient condition for generic simultaneous stabilizability is

$$\max(m,p) \geq r \tag{1.1}$$

Indeed, if (1.1) holds, then the generic r-tuple can be stabilized by a compensator of degree less than or equal to q, where q satisfies:

$$q[\max(m,p)+1-r] \geq \sum_{i=1}^{r} n_i - \max(m,p) \tag{1.2}$$

In the case $r=1$, it is unknown whether generic stabilizability implies generic pole-assignability; that is, whether or not these properties of m,n, and p are really different (see [4]). Perhaps not surprisingly then, Theorem 1.1 follows from:

Theorem 1.2. A sufficient condition for generic simultaneous pole-assignability is (1.1), where the compensator $K(s)$ can be taken to be of degree q satisfying (1.2).

Here, simultaneous pole-assignability means the assignability of r sets of self-conjugate sets of numbers $\{s_{1i},\ldots,s_{n_i+q,i}\} \subset \mathbb{C}$. In fact sharper bounds on q can be obtained (see [18], [11]). Our proof relies on the recent pole-placement techniques derived for $r=1$ by P.K. Stevens in his thesis [18], which contains an improvement on existing results in the literature, see also [9], [17]. We shall prove Theorem 1.2 only in the strictly proper case; the proper case involves more technical arguments from algebraic geometry which can be found in [11]. We shall, however, give an independent proof of Theorem 1.1 in the nonstrictly proper case, based on the equivalence of generic stabilizability and existence of a solution to a generic "deadbeat control" problem, which we can solve if (1.1) is satisfied. This argument extends the argument given in [4] for the case $r=1$ and $q=0$.

Note that if $r=1$, then (1.1) is always satisfied in which case (1.2) implies

Corollary 1.3. (Brasch-Pearson [2]). The generic $p \times m$ plant $G(s)$ of degree n can be stabilized by a compensator of order q, where q satisfies

$$(q+1)\max(m,p) \geq n \tag{1.3}$$

If $r=2$ and $\max(m,p) > 1$, then (1.1) is again satisfied, so we obtain rather easily:

Corollary 1.4. (Vidyasagar-Viswanadham [20]). If $r = 2$ and $\max(m,p) > 1$, then the generic pair $(G_1(s), G_2(s))$ is simultaneously stabilizable.

Moreover, in this case we know an upper bound on the order of the required compensator. For example, if $m = p = 2$, $r = 2$, then q can be taken to satisfy

$$q \geqslant n_1 + n_2 - 2$$

On the other hand, in [20] the explicit conditions defining the closed set

$$\sum\nolimits_{m,p}^{n_1} \times \sum\nolimits_{m,p}^{n_2} - U$$

of pairs not simultaneously stabilizable were derived. Such conditions can be derived from our proof, but instead we refer to [10], where Theorem 1.1 (excepting (1.2)) is proved by interpolation methods also yielding a set of explicit conditions in the range $r \leqslant \max(m,p)$.

Finally, we prove that the condition (1.1) is sharp in the following sense.

Theorem 1.5. If $\min(m,p) = 1$, then for fixed m, p, r and n_i the following statements are equivalent for proper plants:

(i) $q \in \mathbb{N}$ satisfies $q(\max(m,p) + 1 - r) + \geqslant \sum_{i=1}^{r} n_i$;

(ii) the generic r-tuple $G_1(s), \ldots, G_r)$ is simultaneously stabilizable in discrete or continuous time by a compensator of degree $\leqslant q$;

(iii) the generic r-tuple $G_1(s), \ldots, G_r(s)$ is simultaneously stabilizable in discrete or continuous time.

In the strictly proper case it follows that (i)-(iii) is also equivalent to generic simultaneous pole assignability. This holds in the proper case as well, but requires a separate argument [11].

Corollary 1.6. If $\min(m,p) = 1$ and $r \leq \max(m,p)$ then the generic r-tuple is simultaneously stabilizable by a compensator of order precisely given by the least integer q satisfying (1.2).

As a further corollary, we obtain one of the results obtained by Saeks and Murray in [ ], see also [15]:

Corollary 1.7. (Saeks-Murray). Suppose $m = p = 1$ and $r = 2$. Simultaneous stabilizability is not a generic property.

We remark that these results hold also over the field $\mathbb{C}$ of complex numbers - in particular, the complex analogue of Corollary 1.7 dispels a folklore conjecture concerning simultaneous stabilization using compensators with complex coefficients.

Finally, over any field, the method of proof of Theorem 1.2 gives linear equations for a compensator simultaneously placing $r(n+q)$ poles when the generic hypothesis is satisfied.

2. POLE PLACEMENT AND THE GENERALIZED SYLVESTOR MATRIX: A PROOF OF THEOREM 1.2

In this section we proceed to prove Theorem 1.2. Note that Theorem 1.1 and Corollaries 1.3 and 1.4 follow immediately in the strictly proper case from this theorem. Without any loss of generality we can assume that $m \geq p$, for, if $K(s)$ stabilizes $G_i^t(s)$ then $K^t(s)$

stabilizes $G_i(s)$.

Suppose, first of all, that $p = 1$, so that we are given a set of r, m input 1 output plants of McMillan degree $\leq n$ represented as

$$\left[\frac{\sum_{i=0}^{n} p_{1i}^{k} s^{i}}{\sum_{i=0}^{n} p_{m+pi}^{k} s^{i}}, \frac{\sum_{i=0}^{n} p_{2i}^{k} s^{i}}{\sum_{i=0}^{n} p_{m+pi}^{k} s^{i}}, \ldots\ldots\ldots, \frac{\sum_{i=0}^{n} p_{m+p-1,i}^{k} s^{i}}{\sum_{i=0}^{n} p_{m+p,i}^{k} s^{i}}\right] \tag{2.1}$$

for $k = 1,2,\ldots,4$. A 1 input, m output, compensator of McMillan degree $\leq q$ is represented as

$$\left[\frac{\sum_{i=0}^{q} a_{1i} s^{i}}{\sum_{i=0}^{q} a_{m+pi} s^{i}}, \frac{\sum_{i=0}^{q} a_{2i} s^{i}}{\sum_{i=0}^{q} a_{m+pi} s^{i}}, \ldots\ldots\ldots, \frac{\sum_{i=0}^{q} a_{m+p-1,i} s^{i}}{\sum_{i=0}^{q} a_{m+p_i} s^{i}}\right] \tag{2.2}$$

Note that in (2.1) and (2.2) the coefficients $p_{ji}^{k}\ \forall k$ and a_{ji} has been defined up to a nonzero scale factor. Moreover, for a strictly proper plant or compensator, $p_{jn_i}^{k} = 0$, $a_{jq} = 0\ \forall\ j = 1,\ldots,m+p-1;\ k = 1,\ldots,r$.

The associated return difference equation, $\det(I + K(s)G_k(s) = 0$ is given by

$$\Pi_k(s) = \sum_{j=1}^{m+p} \left[\sum_{i=0}^{n} p_{ji}^{k} s^{i}\right]\left[\sum_{i=0}^{q} a_{ji} s^{i}\right] \tag{2.3}$$

$$\forall\ k = 1,2,\ldots,r$$

A generic r-tuple of plants define a mapping χ, via equation (2.3), between the plant parameters and the coefficient of the return difference polynomials given by

$$\chi : \mathbb{R}^{(q+1)(m+p)} \rightarrow \mathbb{R}^{r(n+q+1)} \tag{2.4}$$

where

$$\chi\,(A_0,\ldots,A_q) = (A_0,\ldots,A_q)\begin{bmatrix} p_0 & \cdots & p_n & & & \\ & p_0 & \ddots & p_n & & \\ & & \ddots & & \ddots & \\ & & & p_0 & \cdots & p_n \end{bmatrix} \tag{2.5}$$

where

$$A_i = (a_{1i}\ ,\ a_{2i}\ ,\ \ldots\ ,\ a_{m+pi}) \tag{2.6}$$

$$p_i = \begin{bmatrix} p^1_{1i} & p^1_{1i} & \cdots & p^r_{1i} \\ p^1_{2i} & p^2_{2i} & \cdots & p^r_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p^1_{m+pi} & p^2_{m+pi} & \cdots & p^r_{m+pi} \end{bmatrix} \tag{2.7}$$

The matrix in the right hand side of (2.5) is classically known as the generalized Sylvestor matrix and is of order $(q+1)\,(m+p)\times r(n+q+1)$. For $r=1$ its rank has been analyzed by Bitmead, Kailath, Kung in [1]. In particular, for a generic plant, it is known to have full rank. For $r \geq 1$, we have the following:

<u>Lemma 2.1</u>. The generalized Sylvestor matrix is of full rank for a generic r-tuple.

<u>Proof</u>: See Appendix I.

Lemma 2.2. Assume $\min(m,p) = 1$. A sufficient condition for generic pole assignment, for an r tuple of strictly proper plants by a proper compensator is given by

$$(q+1)(m+p-r) \geq \sum_{i=1}^{r} n_i - r + 1 \tag{2.8}$$

Proof: We prove this Lemma assuming for notational convariance that $n_i = n \ \forall \ i = 1,\ldots,r$ and analyze the mapping χ as defined by (2.4), (2.5). Assume

$$a_{m+p,q} = -1 \quad , \quad p^k_{m+p,n} = 1 \ \forall \ k = 1,\ldots,r$$

and that the coefficient of s^{n+q} in all the r return difference polynomials (2.3) has been normalized to 1.

Thus a sufficient condition for generic pole assignment is that χ' is onto. Here the mapping

$$\chi' : \mathbb{R}^{(q+1)(m+p)-1} \to \mathbb{R}^{(n+q)} \tag{2.9}$$

is given by

$$\chi(A_0,\ldots,A_{q-1}A'_q) = (A_0,\ldots,A_{q-1}A'_q) \begin{bmatrix} p_0 & \cdots & p_n & & & \\ & p_0 & \cdots & p_n & & \\ & & \ddots & & & \\ & & & p_0 & \cdots & p_n \\ & & & p'_0 & \cdots & p'_n \end{bmatrix} \tag{2.10}$$

where

$$A'_q = (a_{1q}a_{2q},\ldots,a_{m+p-1q})$$

and p'_i is obtained from p_i by deleting its $(m+p)^{th}$ row.

By Lemma 2.1 the matrix in the right-hand side of (2.10) is of full rank for a generic r-tuple of plants, and has the order $(q+1)(m+p-1) \times r(n+q)$. Therefore, a sufficient condition for generic pole placement is given by

$$(q+1)(m+p)-1 \geq r(n+q) \tag{2.11}$$

which is same as (2.8) for $n_i = n \ \forall \ i=1,\ldots,r$.

The proof of Theorem 1.2 now proceeds by a reduction to the case $\min(m,p)=1$, which has been treated in Lemmas 2.1-2.2. This procedure, which is called "vectoring down", is adopted from the case $r=1$, studied in P.K. Stevens thesis [18].

<u>Lemma 2.3</u>. Given an r-tuple of $p \times m$ plants $G_i(s)$ of degrees n_i, each with distinct simple poles, there is an open dense set of $1 \times p$ vectors $v \in \mathbb{R}^p$ such that $vG_i(s)$ has degree n_i.

<u>Proof</u>: If $r=1$, then we may expand $G(s)$

$$G(s) = \sum_{i=1}^{n} \frac{R_i}{s-\lambda_i}$$

in a partial fraction expansion, where $\lambda_i \in \mathbb{C}$ and each R_i has rank 1. Now, the set U_1 of real vectors v such that $vR_1 \neq 0$ is clearly open and dense in $\mathbb{R}^p$. Defining $U_2,\ldots,U_n$ similarly, set $V = \bigcap_{i=1}^{n} U_i$. Thus, V is an open dense set of vectors with the required property.

If $r>1$, one obtains, as above, sets $V_1,\ldots,V_r$ in $\mathbb{R}^p$ having an open dense intersection $\bigcap_{i=1}^{n} V_i$. Q.E.D.

Lemma 2.4. Given an r-tuple of $p \times m$ plants $G_i(s)$ there exists a constant gain output feedback k such that the closed loop systems $G_i(s)(I+kG_i(s))^{-1}$ have distinct simple poles.

Proof: For $r=1$, the set W_1 of K such that the closed loop system has simple poles is the complement in $\mathbb{R}^{mp}$ of an algebraic set. It is well known [2] that this set is nonempty; therefore, W_1 is open and dense. Taking any K in the open dense set $\bigcap_{i=1}^{r} W_i$ gives the desired conclusion. Q.E.D.

Thus, choosing any $(v,K) \in \mathbb{R}^p \times \mathbb{R}^{mp}$ we have a mapping from an an open dense set

$$\Phi_{(v,k)} : \Sigma^n_{m,p} \times \cdots \times \Sigma^n_{m,p} \to \Sigma^n_{m,1} \times \cdots \times \Sigma^n_{m,1}$$

$$\Phi_{(v,k)}(G_i(s))^r_{i=1} = (vG_i(s)(\Phi + KG_i(s))^{-1})^r_{i=1}$$

which is rational in the Hankel parameters (H_{ij}) of (G_i). Applying Lemmas 2.1-2.2 to the case $\min(m,p)=1$, i.e. $\Sigma^n_{m,1} \times \cdots \times \Sigma^n_{m,1}$, gives - via composition with Φ - an open dense set of $\Sigma^n_{m,p} \times \cdots \times \Sigma^n_{m,p}$ which can be simultaneously pole-assigned. Q.E.D.

3. GENERIC STABILIZABILITY CONDITION OF AN r-TUPLE OF PROPER PLANTS

In this section we proceed to prove Theorem 1.1 independent of Theorem 1.2. We first show that the following three statements are equivalent.

I. A generic r-tuple of proper plants is stabilizable with respect to the open left half plane by a proper compensator of degree $\leq q$.

II. A generic r-tuple of proper plants is stabilizable with respect to the interior of the unit disc, by a proper compensator of degree $\leq q$.

III. A generic r-tuple of proper plants is pole assignable at the origin by a proper compensator of degree $\leq q$.

Lemma 3.1. I <=> II

Proof: Consider the conformal transformation

$$\phi(s) = (s+1)/(s-1) \tag{3.1}$$

which maps the r-tuple of proper plants $g_1, g_2, \ldots, g_r$ onto the r-tuple of proper plants $g_1', \ldots, g_r'$ where $g_i'(s) = g_i(\phi(s))$ except for the algebraic set of plants satisfying - "$g_i(s)$ has a pole at $s=1$ for some $i=1,\ldots,r$". The proof now follows from the two facts.

1. $\phi(s)$ maps the open left half plane onto the interior of the unit disc.
2. The mapping

$$(g_1, \ldots, g_r) \longmapsto (g_1', \ldots, g_r')$$

and its inverse, map the generic r-tuple of proper plants to the generic r-tuple of proper plants.

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Lemma 3.2. II <=> III

Proof: Sufficiency is clear and follows by an analagous argument of Lemma 3.1 with $\phi(s) = s + a$, $a > 0$, $a \in \mathbb{R}$.

To prove necessity, we have the following: For each $r = 1,2,\ldots$ (shown easily by assuming statement II and considering $\phi(s) = as$, $a > 0$, $a \in \mathbb{R}$). $\exists$ an open dense set of U_r of r-tuple of plants for which there exist a compensator of degree $\leq q$ which places the poles in the interior of the disc D_r of radius $1/r$ centered at the origin. Consider the set

$$U = \bigcap_{r=1}^{\infty} U_r$$

Clearly, U is a dense set by the Baire Category Theorem [13]. Since the mapping χ given by (2.4 is linear, it has a closed image. Moreover, every r-tuple of plants in U admits a sequence of compensators which places the poles arbitrary close to the origin. Since image of χ is closed, U is contained in a set V of all r-tuple of plants for which there exists a compensator which places the poles at the origin. By the Tarski [19] - Seidenberg [16] theory of elimination over $\mathbb{R}$, V is indeed defined by union and/or intersection of sets given by polynomial equations or inequations $f_\alpha > 0$, $f_\beta = 0$. Finally, since U is dense in V, $f_\beta(U) = 0 \Rightarrow f_\beta \equiv 0$ so that V is defined by strict polynomial inequalities. Hence V is open. Moreover, since U is dense, V is also dense.

Lemma 3.3. For a generic r-tuple $(r \leq m+p)$ of $\min(m,p) = 1$ plants

$$\text{III} \Leftrightarrow (q+1(m+p-r) \geq r(n-1)+1$$

<u>Proof</u>: The only nontrivial part is to prove sufficiency for the case

$$r(n+q) < (q+1)(m+p) < r(n+q+1)$$

(The other cases follow easily from the fact that the associated Sylvestor's matrix is of full rank for a generic r-tuple.)

To prove sufficiency, for the above case we want to show that the vector

$$(0,0,\ldots\ldots,0\ ,\ s_1,s_2,\ldots\ldots s_r)$$
$$\longleftarrow r(n+q) \longrightarrow \qquad \longleftarrow\ r\ \longrightarrow$$

indeed belongs to the image of χ (defined by (2.5)) for some $s_i \neq 0,\ i=1,\ldots,r$.

Partition the Sylvestor's matrix in (2.5 as $[S_1 S_2]$ where S_1 is of order $(q+1)(m+p) \times r(n+q)$. Clearly we are solving the pair of equations

$$[A_0,\ldots,A_q]S_1 = [0,\ldots,0] \tag{3.2}$$

$$[A_0,\ldots,A_q]S_2 = [s_1,\ldots,s_r] \tag{3.3}$$

We claim that for a generic r-tuple of plants (3.2) has a solution for a nonzero vector A_q for otherwise if $A_q = 0$ we have

$$[A_0,\ldots,A_{q-1}]S_1' = (0,\ldots,0) \tag{3.4}$$

where S_1' is of order $q(m+P) \times r(n+q)$ obtained by deleting the last $m+p$ rows of S_1. From (3.4) $(A_0,\ldots,A_{q-1}) = \underline{0}$ since S_1' is of full rank generically. Thus the only solution of (3.2) is the zero vector which is a contradiction since the kernel of S_1 is at least of dimension 1. On the other hand, for $A_q \neq 0$, for a generic r-tuple of plants the right-hand side of (3.3) is a vector none of whose entries are zero.

Theorem 1.1 then follows from Lemma 3.1, 3.2, 3.3 and the vectoring down technique used in the proof of Theorem 1.2 in Section 2.

4. PROOF OF THEOREM 1.5

To say there exists $q \in \mathbb{N}$ satisfying (1.2) is to $\max(m,p) \geq$ Thus, (ii) follows from (i) by Theorem 1.1.

(ii) => (iii) since (iii) is weaker than (ii).

By Lemma 3.1, in order to prove (iii) => (i) it suffices to assume that $G_1(s),\ldots,G_r(s)$ are simultaneously stabilizable in continuous-time.

Proposition 4.1. The generic $(m+1)$-tuple of $1 \times m$ proper continuous time plants of degree n is not simultaneously stabilizable by a proper compensator of finite (but not a priori bounded) degree.

Proof: Consider the domain of (simultaneous) stability

$$\mathcal{D} = \{(c_{i1},\ldots,c_{1n},\ldots,c_{r,n_r}) : \sum_{j=0}^{n_i+q} c_{i,j}s^j \text{ has all roots in } D_1\}$$

and its convex hull $\Omega(\mathcal{D}) \subset \mathbb{R}^{n_1+q} \times \ldots \times \mathbb{R}^{n_r+q}$. Clearly, a necessary condition for generic simultaneous stabilizability is

$$\text{image}(\chi_\eta) \cap \Omega(\mathcal{D}) \neq \emptyset,$$

for an open dense set of η. Since

$$\Omega(\mathcal{D}) \subset \{(c_{ij}) : c_{ij} > 0\}$$

it will suffice to prove:

Lemma 4.2. If $r = m + p$, then there exists an open set of r-tuples η such that image (χ_η) contains no vector with only positive entries.

We fix the value of q and construct the associated Sylvestor matrix S. We claim that the open set E of plants defined by

$$E \triangleq \{(P_0, P_1, \ldots, P_n) | P_0^{-1} P_j \ \forall\ j = 1, \ldots, n \text{ has all the entries negative}\}$$

cannot be stabilized by a proper compensator of degree $\leq q$.

Suppose the above is not true, then there exist $\eta \in E$, such that

$$\text{image}(\chi_\eta) \cap \Omega(\mathcal{D}) \neq \emptyset$$

or in other words $\exists \underline{\alpha} | \alpha_i > 0\ \forall\ i = 1, r(n+q+1)$ and

$$\underline{a}\, S = \underline{\alpha} \tag{4.1}$$

has a solution. Writing S as

$$S \triangleq [S' \quad S'']$$

where

$$S' = \begin{bmatrix} P_0 & P_1 & \cdots\cdots & P_q \\ 0 & P_0 & \cdots\cdots & P_{q-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & P_0 \end{bmatrix} \tag{4.2}$$

and $P_j = 0$ for all $j > n$

Equation (4.1) can be written as

$$\underline{a}'\ [I \mid S'^{-1} S''] = \underline{\alpha} \tag{4.3}$$

where S'^{-1} is given as follows

$$S'^{-1} = \begin{bmatrix} X_0 & X_1 & \cdots\cdots & X_q \\ & X_0 & \cdots\cdots & X_{q-1} \\ \cdots & \cdots & \cdots & \cdots \\ & & & X_0 \end{bmatrix}$$

where $X_0 = P_0^{-1}$

$$-(P_1, P_2, \ldots, P_{r+1}) \begin{bmatrix} X_r \\ X_{r-1} \\ \cdot \\ \cdot \\ X_1 \end{bmatrix} = X_{r+1}\ \forall\ r = 0, \ldots, q-1$$

$$P_j = P_0^{-1} P_j \quad ; \quad j = 1, \ldots, q$$

The identity matrix of order $(q+1)(m+p)$ in (4.3) forces a' to have all the entries positive. Moreover, since $\eta \in E, S'^{-1} S''$ has all its entries negative so that $\underline{a}'(S'^{-1} S'')$ has all the entries negative which is a contradiction since $\underline{\alpha}$ is a positive vector.

Finally it is shown that E is not an empty set. For a fixed $P_0 = P_0^*$ choose the vector δ to be so that $P_0^{*-1}\delta$ has all its entries negative. Let

$$P_j^* = \underbrace{(\delta, \delta, \ldots, \delta)}_{\longleftarrow m+p \longrightarrow} \qquad j = 1, \ldots, n$$

so that

$$(P_0^*, P_1^*, \ldots, P_n^*) \in E$$

Q.E.D.

<u>Remark</u>: If image(χ_η) is affine hyperplane, then the necessary condition

$$\text{image}(\chi_\eta) \cap \Omega(\mathcal{D}) \neq \emptyset$$

of course is sufficient, i.e. implies

$$\text{image}(\chi_\eta) \cap \mathcal{D} \neq \emptyset$$

This fact was used by Chen, together with

<u>Lemma 4.3</u>. (Chen [6]) If $r = 1$, $\Omega(\mathcal{D}) = \{(c_1, \ldots, c_n) : c_i > 0\}$

to give precise conditions for stabilizability in the case $r = 1$, $q = 0$, $\min(m,p) = 1$, and $\max(m,p) = n - 1$. This technique can be adapted in the cases $r \geq 1$ to give explicit conditions - in certain cases - defining the open set of simultaneously stabilizable plants when $r > \max(m,p)$, see [11].

Note that Corollary 1.6 now follows from our previous results on the generic rank of the generalized Sylvestor matrix, while Corollary 1.7 follows either from Theorem 1.5 or Proposition 4.1.

APPENDIX I: PROOF OF LEMMA 2.1

The generalized Sylvestor matrix is co-ordinatized by $r(n+1)(m+p)$ parameters, and it is sufficient to show the existence of one principal minor with nonvanishing determinant.

By reordering the rows and columns, the generalized Sylvestor matrix can be written as

$$\bar{S} = [Q_1, Q_2, \ldots, Q_{m+p}]^T \tag{1}$$

where

$$Q_i = [P_{i1}, P_{i2}, \ldots, P_{ir}] \tag{2}$$

$$p_{jk} = \underbrace{\begin{bmatrix} p_{j0}^k & p_{j1}^k & \cdots\cdots & p_{jn}^k & & 0 \\ & \ddots & \ddots & & \ddots & \\ & & \ddots & \ddots & & \ddots \\ 0 & & & p_{j0}^k & p_{j1}^k & \cdots\cdots\; p_{jn}^k \end{bmatrix}}_{(n+q+1)} \Bigg\updownarrow (q+1) \tag{3}$$

in the notation of (2.7). Moreover, each p_{jk} is referred to as a 'block' of S.

Define a set M of matrices as follows: "m belongs to M provided m is obtainable from one of the matrices p_{jk} in (3) either by deleting the first α_1 columns or the last α_2 rows $\alpha_1, \alpha_2 \geq 0$."

<u>Proposition A.1</u>. Every element m of M has the property that there exists a principal minor $m_p \in M$ of m, a coordinate p_m^* and an integer j_m^* such that $p_m^{*j_m^*}$ is a summand in $\det m_p$ where j_m is the order of the minor.

Proof: Clear from the structure of p_{jk}.

The following is an algorithm to construct a principal minor with nonidentically-vanishing determinant.

Algorithm:

Set $S = \overline{S}$, Initialize $\xi = 0$

1. Set $\xi = \xi + 1$.
2. Look at P_{11}. Obtain the principal minor m of P_{11}, satisfying Proposition 1. If there is more than one possible choice, choose the one containing the first column. Define $\alpha_\xi = p_m^*$ and $j_\xi = j_m^*$.
3. Delete the rows and columns corresponding to the coordinate p_m^* from S. Renumber the blocks of the resulting matrix and call it S. (Every block of S is to be identified as a minor of the corresponding block in $\overline{S}$ obtained by row or column deletion.)
4. Do the same "delete" operation as in step 3 in $\overline{S}$.
5. If S is empty, terminate. Otherwise go to 6.
6. Set $k = \xi$.
   Construct the principal minor $\overline{m}_p$ of $\overline{S}$ by choosing those elements of $\overline{S}$ whose corresponding row and column has been deleted in Step 6.

Proposition A.2. During the execution of the above algorithm, S can always be decomposed into blocks belonging to M.

Proof: Clearly $\overline{S}$ satisfies the above proposition, since each block p_{jk} belongs to M. Each iteration of the algorithm deletes either the first α_1 columns of the first block column of S or the last α_2 rows of the first block row of S. The proposition thus follows from the definition of M.

<u>Proposition A.3</u>. $\bar{m}_p$ constructed in Step 6 of the algorithm has a nonidentically-vanishing determinant.

<u>Proof</u>: We prove the proposition by showing that det $\bar{m}_p$ has a summand given by $\prod_{i=1}^{k} \alpha_i^{j_i}$, in the notation of the algorithm. This is clear, however, by observing that in the ξ^{th} iteration the matrix S has a principal minor, the determinant of which has the summand $\prod_{i=\xi}^{k} \alpha_i^{j_i}$, where k is defined in Step 6 of the algorithm.

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CONTROL THEORY, INVERSE SPECTRAL PROBLEMS, AND REAL ALGEBRAIC GEOMETRY*

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0. Tuning Natural Frequencies by Feedback

Consider a linear control system

$$\frac{dx}{dt} = Ax(t) + Bu(t) \quad , \quad y = Cx \tag{0.1}$$

defined for $x \in \mathbb{R}^n$, with control $u(t) \in \mathbb{R}^m$ for each t, and output or observation vector $y \in \mathbb{R}^p$. A, B, and C are real matrices of the appropriate sizes. The oldest problem in mathematical control theory ([1], [2], [21]) is to understand the extent to which linear feedback, i.e. a linear function

$$u = -Ky, \tag{0.2}$$

can alter the dynamical characteristics of (0.1); specifically, the location of the eigenvalues of the perturbed system

$$\frac{dx}{dt} = (A - BKC)x(t) \tag{0.1$'$}$$

For example, a very important problem arising in applications is whether or not a real matrix K can be found which stabilizes (0.3). This condition is necessary and sufficient for the existence of asymptotically constant output solutions to the "closed-loop" system

$$\frac{dx}{dt} = (A - BKC)x(t) + Bu(t) \quad , \quad y = Cx \tag{0.1$''$}$$

and is for this reason part of the analysis and design of engineering systems which generate constant motion [21]. It is also an important problem to produce, via feedback, periodic motions of prescribed frequency or to eliminate such motions ([1], [2]).

These considerations, as well as many others, motivate the following additive inverse spectral problem:

Question 0.1 Given (A,B,C) can one find, for any self-conjugate set $\{s_1,\ldots,s_n\} \subset \mathbb{C}$, a real $m \times p$ matrix such that

$$\mathrm{spec}(A - BKC) = \{s_1,\ldots,s_n\}$$

Since the eigenvalues of A arise as the poles of the function

$$G(s) = C(sI - A)^{-1}B \tag{0.3}$$

for an open, dense set of (A,B,C), see [7], this problem is often referred to as "pole-placement". It corresponds to the physical problem of tuning the natural frequencies of the system (0.1) by feedback (0.2).

Evidently, for A,B,C fixed,

$$\det(sI - A + BKC) = s^n + c_1(K)s^{n-1} + \ldots + c_n(K)$$

is a system of real algebraic (in fact, polynomial) equations

$$c_1(K) = c_1,\ldots,c_n(K) = c_n$$

in K, and Question 0.1 asks if these can be solved for all c. Alternatively, define the function (for A,B,C fixed)

$$\chi : \mathbb{R}^{mp} \to \mathbb{R}^n \tag{0.4a}$$

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via

$$\chi(K) = (c_1(K),\ldots,c_n(K)) \tag{0.4b}$$

Question 0.1 then asks whether, for fixed (A,B,C), χ is a surjection. In this paper, I will present some new results in real algebraic geometry as well as their application to this problem. These results extend many of the existing results on this problem, some of which I shall now review.

First, note that $mp \geq n$ is clearly necessary. Using an elementary argument, viz. the dominant morphism theorem, R. Hermann and C.F. Martin proved

<u>Proposition 0.1</u> [15] If $mp \geq n$, then for generic (A,B,C) the complexified map

$$\chi_{\mathbb{C}} : \mathbb{C}^{mp} \to \mathbb{C}^n$$

has an open dense image.

Using the "high-gain" techniques introduced in [10], one can improve this result to

<u>Proposition 0.2</u> [12] If $mp \geq n$, the complexified map $\chi_{\mathbb{C}}$ is surjective.

As it turns out, over $\mathbb{C}$ it is sufficient to prove Proposition 0.2 in the case $mp = n$ and in this case it is known [10] that $\chi_{\mathbb{C}}$ is proper for generic A,B,C. Indeed, Brockett and Byrnes showed that its degree is given by a formula well known in several areas of mathematics. The Cauchy-Riemann equations imply that $\deg\chi_{\mathbb{C}}$ actually counts the honest number (with multiplicity) of solutions to (0.4) over $\mathbb{C}$. Thus, this formula has the advantage of giving sufficient conditions for $\chi_{\mathbb{R}}$ to be surjective, viz. whenever $\deg\chi_{\mathbb{C}}$ is odd (see Corollary 0.5).

<u>Theorem 0.3</u> [8] If $mp = n$, then for generic A,B,C (explicitly, for nondegenerate A,B,C in the sense of [8], [10]) one has

$$\deg\chi_{\mathbb{C}} = \frac{1!\ldots(p-1)!(mp)!}{m!\ldots(m+p-1)!} \tag{0.5}$$

In general, the real difficulties, so to speak, emerge when one asks that K be real. In Section 2, I present sufficient conditions for a system of real algebraic equations to have a solution. In the case at hand, this criterion produces constants $c_{m,p}$ - as well as effectively computable lower bounds $c'_{m,p}$ - yielding for generic A,B,C:

Theorem 0.4 $c'_{m,p} \geq 2\left[\frac{n-1}{2}\right]+1$ implies χ is surjective.

Here we define $k \in \mathbb{N}$ by $2^k < m+p \leq 2^{k+1}$ and then set

$$c'_{m,p} = \begin{cases} 2^{k+1}-1 & \text{if } \min(m,p) = 2,\ \max(m,p) \neq 2^k-1 \\ 2^{k+1}-2 & \text{if } \min(m,p) = 2,\ \max(m,p) = 2^k-1 \\ 2^{k+1}-1 & \text{if } \min(m,p) = 3,\ m+p = 2^k+1 \\ 2^{k+1} & \text{otherwise} \end{cases} \tag{0.6}$$

This theorem has several corollaries. For example, over the complex field the analogous inequalities assert that $mp \geq n$ is necessary and sufficient that $\chi_{\mathbb{C}}$ be surjective for generic (A,B,C). Over the real field, the crude lower bounds $c'_{m,p}$ yield

Corollary 0.5 (Brockett-Byrnes [8]) If $mp = n$, then the conditions

$$\min(m,p) = 1 \quad \text{or} \quad \min(m,p) = 2 \quad \text{and} \quad \max(m,p) = 2^r-1 \tag{0.7}$$

are sufficient that χ be surjective for generic (A,B,C).

Remark: This, however, is only one of the results obtained in [8] on stabilizability and pole-assignment. For example, an explicit characterization of the open dense set of (A,B,C) for which Corollary 0.5 is valid is given as well.

I also obtain a stronger version (viz. surjectivity) of

Corollary 0.6 (Kimura [19]) If $m+p-1 \geq n$, then image χ contains an open dense set in $\mathbb{R}^n$, for generic (A,B,C).

1. Systems of Real Algebraic Equations

Our interest is in the following basic problem. Consider the system of equations

$$f_i(x) = y_i \qquad x \in \mathbb{R}^N, \quad i = 1,\ldots,n \tag{1.1}$$

which is to be solved for all $y \in \mathbb{R}^n$ subject to the constraint $x \in X \subset \mathbb{R}^N$, where X is the real algebraic set defined (not necessarily as a complete intersection) by equations

$$g_i(x) = 0 \qquad i = 1,\dots,r \tag{1.2}$$

That is, we ask whether $f : X \to \mathbb{R}^n$ is surjective. In what follows we shall assume

$$d_i = \deg(f_i) \quad \text{is odd, for } i = 1,\dots,n \tag{H1}$$

We note that (H1) is not a restriction on the class of problems considered, only on the form of the equations. For, by the introduction of "slack variables", we can render any set of equations in a form satisfying (H1).

Example 1.1 To solve $y = p(x)$, $x \in \mathbb{R}$, is to solve the "slack equations"

$$f(x_1,x_2) = y \quad , \quad g(x_1,x_2) = 0 \tag{1.3}$$

where

$$f(x_1,x_2) = x_2 \quad , \quad g(x_1,x_2) = x_2 - p(x_1) \tag{1.3'}$$

Thus, we ask for surjectivity of

$$f : X \to \mathbb{R}$$

where $X \subset \mathbb{R}^2$ is a curve, viz. the graph of p.

We shall also need another hypothesis. One can express any polynomial F on $\mathbb{R}^N$ as

$$F = F^h + F^r$$

where F^h is homogeneous, and $\deg(F^r) < \deg(F^h)$. Consider the algebraic sets

$$Z_f^h = \bigcap_{i=1}^{n} (f_i^h)^{-1}(0) \quad , \quad X^h = \bigcap_{i=1}^{n} (g_i^h)^{-1}(0) \tag{1.4}$$

We ask that the "base locus" condition

$$\dim (Z_f^h \cap X^h) \leq \dim X - n \qquad \text{(H2)}$$

be satisfied, where throughout this note we mean geometric dimension.

Remark: (H2) implies the (obvious) necessary condition for surjectivity of f:

$$\dim X \geq n \qquad (1.5)$$

On the other hand, if (1.5) is satisfied, then the generic f (with d_i fixed) satisfies (H2). For example, f in (1.3)' always satisfies (H2).

Example 1.2 (no constraints) If $X = \mathbb{R}^N$, then Bezout's Theorem on $\mathbb{C}P^N$ implies the existence of real solutions (possibly at infinity) to (1.1) for any f satisfying (H1). If $N = n$, then (H2) is the condition that the base locus of the rational map f be empty and therefore, for finite y, a finite solution always exists.

Example 1.3 (compact constraints) If X is a compact real algebraic set, no f can be surjective.

Example 1.3 of course cannot occur over $\mathbb{C}$, since complex varieties admit unbounded holomorphic functions. The main theme which we suggest is that the topology of the real algebraic set (1.2) influences quite strongly the solubility of equations (1.1) defined on these sets. And, the topology of complex algebraic varieties is so remarkably well-behaved that this issue does not arise over $\mathbb{C}$.

2. The Main Theorems on Real Algebraic Geometry

The key to distinguishing, for example, the real algebraic sets arising in Example 1.2 and 1.3 is to study their behaviour at infinity. To this end, we consider the inclusion of the closure

$$i : \overline{X} \hookrightarrow \mathbb{R}\mathbb{P}^N \qquad (2.1)$$

of X in $\mathbb{R}\mathbb{P}^N$, where $\mathbb{R}^N \subset \mathbb{R}\mathbb{P}^N$ via the standard construction

$$(x_1, \ldots, x_N) \to [x_1, \ldots, x_N, 1]$$

Berstein and Ganea [6] introduced a homotopy invariant of maps, the category of a map, defined in our setting as

$$\operatorname{cat}(i) = \min.\operatorname{card}.(U_\alpha): (U_\alpha) \text{ an open cover of } \overline{X} \text{ such that } i|_{U_\alpha} : U_\alpha \to \mathbb{R}\mathbb{P}^N \text{ is null homotopic}$$

Note that in Examples 2 and 3, cat(i) is $N+1$ and 1 respectively. We can now state our basic existence theorem:

<u>Theorem 2.1</u> If $\operatorname{cat}(i) > n$, then (1.1)-(1.2) is solvable for any f satisfying (H1) and (H2). Indeed, for any $y \in R^n$

$$\dim f^{-1}(y) \geq \operatorname{cat}(i) - n - 1 \tag{2.2}$$

If $\dim X = n$, then f is in fact proper in light of (H2), and (2.2) asserts that f is a finite-to-one surjection. If X is smooth, then f has a well-defined degree, $\deg_{\mathbb{R}} f$. Using characteristic classes, one sees that (2.2) works at least as well as mod(2) methods:

<u>Proposition 2.2</u> If $\deg_{\mathbb{R}} f$ is odd and $\overline{X}$ is smooth, $\operatorname{cat}(i) = n+1$. In particular, if $\deg_{\mathbb{R}} f$ is odd, $\pi_1(\overline{X})$ contains a subgroup of index 2 (and therefore $\overline{X}$ is not simply-connected) and the mod(2) Betti numbers $\beta_i(\overline{X})$ are nonzero for $i = 0,\ldots,n$.

<u>Remark</u>: This last topological conclusion is of course reminiscent of the Kähler conditions. We denote by $X_{\mathbb{C}}$, $i_{\mathbb{C}}$, etc. the objects one obtains by complexifying. If $\overline{X}_{\mathbb{C}}$ is smooth, then the Kähler conditions together with a theorem of Eilenberg [13] imply

$$\operatorname{cat}(i_{\mathbb{C}}) \geq \dim_{\mathbb{C}} \overline{X}_{\mathbb{C}} + 1$$

with equality if $\overline{X}_{\mathbb{C}}$ is simply connected [18], [25]. In particular, $\operatorname{cat}(i_{\mathbb{C}}) > n$ is implied by (H2) and is thus superfluous over $\mathbb{C}$, illustrating our philosophy. In this sense, $\operatorname{cat}(i_{\mathbb{C}})$ seems to play the role of $\dim_{\mathbb{C}}$ for real algebraic sets. Moreover, $\operatorname{cat}(i_{\mathbb{C}}) > n$ is implied by the condition

$$\operatorname{rank} Jf(x_o) = n \quad , \quad \text{for some } x_o \in X$$

In this case, the dominant morphism theorem asserts that (1.1) is solvable for almost all $y \in \mathbb{C}^n$. (H2) is a stronger hypothesis, but strengthens this theorem. Thus, Theorem 2.1 may be thought of as a "dominant morphism theorem" over $\mathbb{R}$.

Further connections between $\deg_{\mathbb{R}}(f)$ and the topology of $\overline{X}$ can be derived in several cases.

Theorem 2.3 Suppose $\overline{X}$ is a smooth orientable n-manifold.

(i) If n is odd, then

$$\deg_{\mathbb{R}}(f) \text{ is odd} \iff \operatorname{cat}(i) = n+1$$

(ii) If n is even, $\deg_{\mathbb{R}}(f) = 0$.

Remark: In Example 1.1, we have 2 cases. If deg(p) is odd, then $\deg_R f = \pm 1$, $\operatorname{cat}(i) = 2$ and $\ker(i_*) = 2\mathbb{Z} \subset \mathbb{Z} \simeq \Pi_1(\overline{X})$. If deg(p) is even, $\deg(f) = 0$, $\operatorname{cat}(i) = 1$ and of course $\Pi_1(X) = \{0\}$. Nontrivial applications will be indicated in Sections 3 and 4.

Assertion (ii) has a corollary which seems of independent interest.

Corollary 2.4 Suppose $\overline{X}_{\mathbb{C}}$ is smooth and has odd degree in $\mathbb{C}\mathbb{P}^N$. If $\dim_{\mathbb{C}}(\overline{X}_{\mathbb{C}})$ is even, then $\overline{X}_{\mathbb{R}}$ is not orientable.

I now consider the simplest case compatible with the conclusion of Proposition 2.2. The following result can be obtained from Theorem 2.1 using an analogue of the Hopf Degree Theorem, for maps to $\mathbb{R}\mathbb{P}^n$ (see [24] and also [5], [25]).

Proposition 2.5 Suppose $\overline{X}$ is smooth, nonorientable, dimX is even, and $\Pi_1(\overline{X}) \simeq \mathbb{Z}_2$. Then

$$\deg_{\mathbb{R}}(f) \text{ is odd} \iff \operatorname{cat}(i) = n+1$$

Theorem 2.1 applies however in the non-equidimensional cases, and even when $\overline{X}$ has singularities - in particular it applies in the absence of mod(2) orientability of $\overline{X}$.

In fact, in Section 4 I give an example, with $\overline{X}$ singular, where Theorem 2.1 gives a better result than the mod(2) theory. This example arises in an analysis of Question 0.1.

3. Applications to Inverse Eigenvalue Problems

As has been indicated above, any system of real algebraic equations can be put in the form (1.1)-(1.2), where f satisfies (H1), but major technical problems remain in the application of Theorem 2.1 - especially the calculation of cat(i) or even explicit knowledge of the embedding $\overline{X} \subset \mathbb{R}\mathbb{P}^N$. One class of problems for which there is a quite natural transformation of the basic equations into the desired form arise in the study of inverse spectral problems.

If A_o is a fixed $n \times n$ real matrix, consider the effect on $\mathrm{spec}(A_o)$ of an additive perturbation $A_o + A$, where $A \in X$ - an algebraic set of matrices, such as the diagonal or the rank one matrices. The inverse spectral problem asks, in part, whether the resulting map

$$\chi_{A_o} : A \to \mathbb{R}^n \quad , \quad \chi_{A_o}(A) = \text{characteristic coeff's of } A_o + A \tag{3.1}$$

is surjective. Quantitatively, one has the Weinstein-Aronszajn formula

$$\det(I + A(sI - A_o)^{-1}) = \Phi(s) \tag{3.2}$$

where $\Phi(s)$ is rational, vanishing on $\mathrm{spec}(A + A_o)$, having poles on $\mathrm{spec}(A_o)$, and satisfying $\Phi(\infty) = 1$. In particular, $\Phi(s_o) = 0$ whenever $s_o \notin \mathrm{spec}(A_o)$ and $s_o \in \mathrm{spec}(A + A_o)$. The vanishing of (3.2) also has a geometric interpretation in $\mathrm{Grass}_{\mathbb{R}}(n,2n)$, where we think of A as a point (and X as a subset) via the correspondence

$$A \to \mathrm{graph}(A) \tag{3.3}$$

For s_o fixed, we can consider dually the hypersurface $\sigma(s_o) \subset \mathrm{Grass}_R(n,2n)$ defined by

$$\sigma(s_o) = \{W : \dim(W \cap \mathrm{graph}(s_o I - A_o)^{-1}) \geq 1\} \tag{3.4}$$

Then, vanishing of (3.2) is the equation of incidence

$$\mathrm{graph}(A) \in \sigma(s) \tag{3.5}$$

and, since $\sigma(s_o)$ is a hyperplane section for the Plücker imbedding

$$\mathscr{P}_{n,n} : \mathrm{Grass}_{\mathbb{R}}(n,2n) \quad \mathbb{R}\mathbb{P}^N \tag{3.6}$$

the inverse eigenvalue equations

$$\Phi(s_1) = 0, \ldots, \Phi(s_n) = 0 \tag{3.6'}$$

become linear in the Plücker coordinates of graph(A), which are to be solved in $\mathbb{R}\mathbb{P}^N$ subject to the constraints

$$\mathrm{graph}(A_o) \in \overline{X} \subset \mathrm{Grass}_{\mathbb{R}}(n,2n)$$

<u>Theorem 3.1</u> The mapping (3.1) is the restriction of a central projection

$$\Pi : \mathbb{R}P^N - B_\Pi \to \mathbb{R}P^n$$

to $X \subset \overline{X} \subset \mathrm{Grass}(n,2n)$. Thus, on an affine open $\mathbb{R}^N$ containing X the equations (3.1) take the form (1.1)-(1.2) and the inclusion (2.1) is the composition of $\overline{X} \subset \mathrm{Grass}(n,2n)$ with the Plücker imbedding $\mathscr{P}_{n,n}$.

<u>Corollary 3.2</u> If $\dim X = n$ is even and if the base locus condition is satisfied, then $\deg_{\mathbb{R}}(\chi) = 0$ if $\overline{X}$ is orientable.

In the next two examples, Theorem 3.1 is illustrated in well-known inverse eigenvalue problems. Although less sophisticated arguments suffice in each case, these are given in the way of illustrations of a unified viewpoint and also as a preliminary to Section 4.

<u>Example 3.3</u> (rank 1 perturbations) Let A_o be a 2×2 matrix and consider the algebraic set

$$X = \{A : \mathrm{rank}\ A \leq 1\} \subset \mathscr{M}_2(\mathbb{R})$$

As above, $\overline{X} \subset \mathrm{Grass}_{\mathbb{R}}(2,4)$ via the correspondence

$$A \longmapsto \mathrm{graph}(A) \subset \mathbb{R}^2 \oplus \mathbb{R}^2$$

Indeed, if V denotes 2-plane in $\mathbb{R}^2 \oplus \mathbb{R}^2$ then

$$\overline{X} = \{V : \dim V \cap (\mathbb{R}^2 \oplus \{0\}) \geq 1\} \tag{3.7}$$

since to say A has rank ≤ 1 is to say

$$\ker A = \text{graph}(A) \cap \mathbb{R}^2 \oplus \{0\}$$

has dimension ≥ 1. In particular, $\overline{X}$ is a singular Schubert hypersurface in $\text{Grass}_{\mathbb{R}}(2,4)$. In order to compute cat(i) note that $\text{Grass}_{\mathbb{R}}(2,4) - \overline{X}$ is a chart on $\text{Grass}_{\mathbb{R}}(2,4)$; that is,

$$\text{Grass}_{\mathbb{R}}(2,4) - \overline{X} \simeq \mathbb{R}^4 \tag{3.8}$$

By Lefschetz Duality, the inclusion

$$\overline{X} \hookrightarrow \text{Grass}_{\mathbb{R}}(2,4)$$

induces an isomorphism

$$H^i(\text{Grass}_{\mathbb{R}}(2,4);\mathbb{Z}_2) \to H^i(X;\mathbb{Z}_2) \quad , \quad i = 0,\ldots,3$$

In particular, the mod(2) Betti numbers of $\overline{X}$ are

$$\beta_o = 1 \quad , \quad \beta_1 = 1 \quad , \quad \beta_2 = 2 \quad , \quad \text{and} \quad \beta_3 = 2$$

By Eilenberg [13], $\text{cat}(\mathcal{P}_{2,2} \circ i)$ is bounded below by the height $\text{ht}(w_1)$, in the ring $H^*(\overline{X};\mathbb{Z}_2)$, of the nonzero element w_1 of $H^1(\overline{X};\mathbb{Z}_2)$. From the Schubert calculus ([20], [22]) one knows that $w_1^2 \neq 0$ and therefore

$$\text{cat}(\mathcal{P}_{2,2} \circ i) = 3$$

Finally, from (3.2) one sees that, for generic A_o, the base locus condition is satisfied.

A similar calculation for arbitrary n gives a proof of the well-known

<u>Corollary 3.3</u> For generic real A_o and any self-conjugate subset $\{s_1,\ldots,s_n\} \subset \mathbb{C}$ there exists an A of rank ≤ 1 such that

$$\text{spec}\{A_o + A\} = \{s_1,\ldots,s_n\}$$

Example 3.4 (diagonal perturbations) Let A_o be a 2×2 real matrix and consider the algebraic set of diagonal matrices

$$X = \{A : A = \mathrm{diag}(a_1, a_2) \quad , \quad a_i \in \mathbb{R}\}$$

Again, the correspondence $A \longmapsto \mathrm{graph}(A)$ induces an inclusion $X \subset \mathrm{Grass}(2,4)$. Moreover, $\overline{X}$ is the intersection of 2 Schubert hypersurfaces

$$\sigma_1 = \{V : \dim(V \cap \mathrm{span}\{e_1, e_3\}) \geq 1\} \quad , \text{ and}$$

$$\sigma_2 = \{V : \dim(V \cap \mathrm{span}\{e_2, e_4\}) \geq 1\}$$

where e_1, e_2, e_3, e_4 are the standard basis vectors in $\mathbb{R}^4$. Elementary geometry shows

$$\overline{X} = \sigma_1 \cap \sigma_2 \simeq \mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$$

so that $\overline{X}$ is a 2-torus. The base locus condition is satisfied for all A_o and therefore

$$\deg_{\mathbb{R}}(\chi_{A_o}) = 0$$

for all A_o, according to Corollary 3.2. This is in harmony with the fact that, e.g., χ_o fails to be surjective.

More generally, for any n $\overline{X}$ is an n-torus. Over $\mathbb{C}$, $\overline{X}_{\mathbb{C}} \simeq \mathbb{C}\mathbb{P}^1 \times \ldots \times \mathbb{C}\mathbb{P}^1$ and one has

$$\mathrm{cat}(i_{\mathbb{C}}) = n + 1$$

from which one deduces the well-known result:

Corollary 3.4 ([14], [3]) For an arbitrary $n \times n$ real or complex matrix A_o and an arbitrary subset $\{s_1, \ldots, s_n\} \subset \mathbb{C}$, there exists a diagonal matrix $A = \mathrm{diag}(a_1, \ldots, a_n)$, with $a_i \in \mathbb{C}$, such that

$$\mathrm{spec}(A_o + A) = \{s_1, \ldots, s_n\}$$

4. Pole Placement by Output Feedback

I now turn to the problem of arbitrarily tuning the natural frequencies of a control system (A_o,B_o,C_o) by use of output feedback F (Section 0). In this setting X is given as

$$X = \{B_oKC_o : B_o \ n \times m,\ C_o \ p \times n \ \text{ are fixed}\}$$

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For generic B_o, C_o, $\overline{X}$ is itself a Grassmannian

$$\overline{X} = \text{Grass}_{\mathbb{R}}(p,m+p) \subset \text{Grass}_{\mathbb{R}}(n,2n)$$

i is the Plücker imbedding, and the base locus condition is satisfied for generic (A_o,B_o,C_o). (See [11].)

In this setting, Theorems 2.1-3.1 assert that $c_{m,p} = \text{cat}(\mathcal{P}_{m,p}) > n$ implies arbitrary eigenvalue placement (Theorem 0.4). Eilenberg's Theorem asserts that $\text{cat}(\mathcal{P}_{m,p})$ is bounded from below by the height of the first Stiefel-Whitney class

$$\text{cat}(\mathcal{P}_{m,p}) > \text{ht}(w_1) \text{ in } H^*(\text{Grass}(p,m+p)\ ;\ \mathbb{Z}_2) \tag{4.1}$$

This height has recently been calculated by Hiller [17] and by Stong [26], but the sufficient conditions which these estimates yield also follow, by Poincaré duality, from mod(2) intersection theory. Indeed, starting with the interpretation of the vanishing of (3.2) as an equation in the Schubert calculus, we can obtain these same results by constraining the perturbation variety $A \in X$ to be a Schubert variety $Z \subset \text{Grass}(p,m+p)$ and applying Pieri's formula [9].

These calculations can be refined using Lefschetz Duality and Theorem 2.1 as in Example 3.3. In the above notation, the inclusion of the Schubert hypersurface (for s real)

$$\sigma(s) \hookrightarrow \text{Grass}(p,p+m)$$

induces an isomorphism in cohomology except of course for degree pm. Although Poincaré duality fails to hold for $\sigma(s)$, Theorem 2.1 applies to χ restricted to $\sigma(s)$ and one can therefore improve the estimate in (3.7) by one, in all cases except $\min(m,p) = 2$, $\max(m,p) = 2^r - 1$ where $\text{ht}(w_1) = mp$ [5], by first "placing a pole at s" and then considering the remaining $n-1$ constraints. Combining the calculations made in [17], [26] with this observation leads to the definition of $c'_{m,p}$ in (0.6). Combining this computation with Theorem 2.1, we obtain a proof of Theorem 0.4.

The first case not treated by Theorem 0.4 (or by Corollary 0.5) is the case $m=p=2$, $n=4$. This had already been studied by Willems and Hesselink in [27], where they showed that for generic (A,B,C), image (χ) misses an open set of infinite Lebesque measure in $\mathbb{R}^4$. This has since been checked in various ways [8], [23] but it is interesting to note that one can see this result, within the present framework, by either part (ii) of Theorem 2.3 or by the real algebraic methods presented in [4]. Explicitly, take s_1, s_2, s_3 and $s_4 \in \mathbb{C}$ so that $s_i = \overline{s}_{i+2} \notin \mathbb{R}$ and consider the submanifold of real points

$$\mathrm{Grass}_{\mathbb{R}}(2,4) \subset \mathrm{Grass}_{\mathbb{C}}(2,4) \tag{4.2}$$

Following the technique in [4], note first that

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$$0 \neq [\mathrm{Grass}_{\mathbb{R}}(2,4)] \in H_4(\mathrm{Grass}_{\mathbb{C}}(2,4);\ \mathbb{Z})$$

This can be seen from the fact that multiplication by $\sqrt{-1}$ maps the tangent bundle $T(\mathrm{Grass}_{\mathbb{R}}(2,4))$ to the normal bundle of (4.2). Since $\mathrm{Grass}_{\mathbb{R}}(2,4)$ is orientable, the self-intersection number of $[\mathrm{Grass}_{\mathbb{R}}(2,4)]$ in $H_*(\mathrm{Grass}_{\mathbb{C}}(2,4);\ \mathbb{Z})$ can be calculated as the Euler characteristic of $\mathrm{Grass}_{\mathbb{R}}(2,4)$; i.e. as 2. For generic (A,B,C) there exists a real K placing the eigenvalues $s_1,\ldots,s_4$ if, and only if, there is a point in

$$\sigma(s_1) \cap \sigma(s_2) \cap \mathrm{Grass}_{\mathbb{R}}(2,4) \subset \mathrm{Grass}_{\mathbb{C}}(2,4)$$

Moreover, we have the formula

$$\deg_{\mathbb{R}}(\chi) = \#(\sigma(s_1) \cap \sigma(s_2) \cap \mathrm{Grass}_{\mathbb{R}}(2,4)) \in \mathbb{Z}$$

for $\deg_{\mathbb{R}}(\chi)$ <u>in the integers</u>. An elementary calculation in $H_*(\mathrm{Grass}_{\mathbb{C}}(2,4);\ \mathbb{Z})$ shows that

$$\deg_{\mathbb{R}}(\chi) = 0$$

Remark: From the classification of smooth functions in this dimension range, one can then see that χ is not surjective, but this requires more elaborate argument.

This technique will apply whenever $\mathrm{Grass}_{\mathbb{R}}(p,m+p)$ is orientable and has a nonzero Euler characteristic, viz. whenever m and p are even, reducing the calculation of $\deg_{\mathbb{R}}(\chi)$ to a problem in the Schubert calculus. In the present setting, this calculation may be avoided by appealing to Theorem 2.3-part (ii); i.e.

Corollary 4.1 If $m,p \in 2\mathbb{Z}$ and $mp = n$, then for generic (A,B,C),

$$\deg_{\mathbb{R}}(\chi) = 0$$

5. Acknowledgements

In an innovative paper [16] Hermann and Martin introduced Grassmannians into "frequency domain" control theory, interpreting the transfer function (0.3) as a holomorphic curve

$$G : \mathbb{CP}^1 \to \mathrm{Grass}_{\mathbb{C}}(m,m+p)$$

defined via the correspondence $s \longmapsto$ graph G(s). This had an influence on some earlier work on the main problem considered here, see [8] and [10], although subsequently it was realized that the Weinstein-Aronszejn formula - which was known and discovered independently by electrical engineers under the name "return-difference determinant" - provided a much easier starting point. Also, the dual point of view, viz. that G(s) induces a hypersurface in the dual Grassmannian $\mathrm{Grass}_{\mathbb{C}}(p,m+p)$, and the introduction of nonzero feedback laws as points in $\mathrm{Grass}_{\mathbb{C}}(p,m+p)$ enable one to go further than [16] in the study of feedback systems by incorporating output rather than state feedback.

This dual point of view was introduced in [10] and developed much further in [8] where the Schubert calculus was used as an essential tool in studying feedback systems. In the time since the lectures [10] have appeared, I have profited from conversation and correspondence with many people - particularly M.F. Atiyah, I. Berstein, R.W. Brockett, J. Harris and D. Mumford whom I gratefully acknowledge.

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POLE PLACEMENT BY STATIC AND DYNAMIC OUTPUT FEEDBACK

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Abstract

In this paper, we give new results concerning pole-assignability by static and dynamic output feedback, based on the interpretation of transfer functions, feedback laws, poles and zeroes ([3], [5], [12], [19]) in terms of the incidence geometry of m-planes and p-planes in (m+p)-space. As an illustration of the most basic ideas, we give a short proof of the Brasch-Pearson Theorem. A more careful analysis of this proof yields a significant extension of this theorem, which we then considerably sharpen in the case of pole-assignment by constant gain output feedback. As a final application we introduce a root-locus design technique for non-square systems: zeroplacement by pre- or post-compensation. This zeroplacement problem is then analyzed by methods similar to those developed for pole placement by output feedback.

1. Exact Pole-Assignability: "Vectoring Down"

The first problem we consider is: Given a real $p \times m$ transfer function $G(s)$ with McMillan degree $\delta(G) = n$, what is the minimal q such that for any self-conjugate subset $\{s_1, \ldots, s_{n+q}\} \subset \mathbb{C}$, there exist a real compensator $K(s)$ of order q which places the poles of the closed loop system at $s_1, \ldots, s_{n+q}$? We consider also those poles which have been cancelled in the closed loop transfer function $G(s)(I+K(s)G(s))^{-1}$ by expressing the closed-loop system in state space form. We shall illustrate our techniques by giving a new, elementary proof of the well-known Brasch-Pearson Theorem [2], before giving more delicate improvements on this theorem and on the existing results on pole placement by constant gain output feedback.

Given $v \in \mathbb{R}^m$, as in [19], we can "vector down" $G(s)$ by passing the input channels through v, i.e. we can form the new $p \times 1$ transfer function $G(s)v$. A partial fraction decomposition shows that the poles of $G(s)v$ are among the poles of $G(s)$; moreover,

Lemma 1.1: For fixed $G(s)$, there is an open dense set $V \subset \mathbb{R}^m$ of v such that $\delta(G) = \delta(G \cdot v) = n$.

Remark: This follows from our results in section 2, we see Lemma 2.2 below. If $G(s) = (sI-A)^{-1}B$, with (A,B) controllable, then Lemma 1.1 is a well-known consequence of Heymann's Lemma, viz. that (A,Bv) is controllable for almost any input channel Bv.

Now suppose a self-conjugate subset $\{s_1, \ldots, s_{n+q}\} \subset \mathbb{C}$ is given. Choosing v as in Lemma 1.1, we seek a $1 \times p$ compensator $K(s)$ such that if

$$G(s)v = N(s)D(s)^{-1} \quad \text{and} \quad K(s) = Q(s)^{-1}P(s)$$

are coprime factorizations, then

$$Q(s)D(s) + P(s)N(s) = 0 \Leftrightarrow s \in \{s_1, \ldots, s_{n+q}\} \quad (1.1)$$

By equating coefficients on the left-hand side of (1.1), we obtain a linear map - the generalized Sylvestor resultant [1]:

$$S_q : \mathbb{R}^{(q+1)(p+1)} \to \mathbb{R}^{(n+q+1)} \quad (1.2)$$

Pole-assignability by a compensator of the form $vK(s)$ is therefore equivalent to surjectivity of S_q, whose rank is given in a simple, beautiful formula [1]:

$$\text{rank } S_q = (p+1)(q+1) - \sum_{\nu_i < q+1} (q+1-\nu_i) \quad (1.3)$$

where (ν_i) are the observability indices of $G(s)v$ - or, what is the same, of $G(s)$ for v as in Lemma 1.1. We then easily have:

Theorem 1.2: Suppose $G(s)$ has observability indices (ν_i). Then $G(s)$ can be arbitrarily pole-assigned with a compensator of order q where q satisfies

$$(q+1)p - \sum_{\nu_i < q+1} (q+1-\nu_i) \geq n \quad (1.4)$$

By duality, the same result holds, mutatis mutandis, for controllability indices (κ_i)

Choosing, for example, $q = \nu_{max} - 1$ we obtain

$$\sum_{\nu_i < q+1} (q+1-\nu_i) = \sum_{\nu_i \leq q+1} (q+1-\nu_i) = p(q+1) - n \quad (1.5)$$

Combining (1.4)-(1.5), we have

Corollary 1.3 (Brasch-Pearson [2]): Choose $q = \min(\kappa_{max}, \nu_{max}) - 1$. Then $G(s)$ can be arbitrarily pole-assigned using a compensator of order q.

Since the left-hand side of (1.4) is an increasing function of q, achieving its maximum at $q = \nu_{max} - 1$,

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Theorem 1.2 is in fact equivalent to Corollary 1.3.

2. Generic Pole Assignability by Output Compensation

In this section we examine the "vectoring down" process more closely, investigating the effect of preprocessing by a vector v which is not in general position in the sense of Lemma 1.1. We illustrate this in the "generic" case. Our main result is then a strengthening of the Brasch-Pearson result for generic (A,B,C). More precisely, a property P of triples (A,B,C) is generic provided it is satisfied by all (A,B,C) except perhaps those which lie in a proper algebraic subset of $\mathbb{R}^{n^2+n(m+p)}$; that is, a subset X defined by real polynomial equations

$$X = \{(A,B,C) : p_r(a_{ij}, b_{k\ell}, c_{mn}) = 0 \ , \ r = 0,1,\ldots\}$$

Our main theorem ([19] and also [18]) is then:

Theorem 2.1: The generic triple (A,B,C) is output pole-assignability at the generic set of poles (e.g., distinct) by a compensator of order q satisfying

$$(q+1)(\max(m,p)) + \min(m,p) - 1 \geq 2\left[\frac{n-1}{2}\right] + 1 \qquad (2.1)$$

Here, as is customary [16], $\left[\frac{a}{b}\right]$ for $a,b \in \mathbb{Z}$ denotes the greatest integer less than or equal to a/b.

The proof reposes on a more careful analysis of the effect of "vectoring down" on the poles of G(s). Suppose first that G(s) has distinct simple poles, and consider the partial fraction expansion

$$G(s) = \sum_{i=1}^{n} \frac{R_i}{s-\lambda_i} \qquad (2.2)$$

Then, G(s)v (or $w^tG(s)$) will have a pole at λ_i if, and only if, R_iv (or w^tR_i) does not vanish. More generally, suppose, without loss of generality, that $p \leq m$ and consider the coprime factorization

$$G(s) = N(s)D(s)^{-1} \qquad (2.3)$$

leading to the matrix

$$\tilde{G}(s) = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} \qquad (2.3)'$$

If G(s) has a simple pole at s_1, as in (2.1), then $w^tR_1 = 0$ if, and only if, $w^t \in \mathbb{R}^p \subset \mathbb{C}^p$ is orthogonal to the row span of R_1 in $\mathbb{C}^p$. Equivalently, thinking of column span $\underset{\sim}{G}(s)$ as an m-dimensional subspace of $\mathbb{C}^p + \mathbb{C}^m$, we have

$$w^tR_1 = 0 \iff w \perp \text{column span } \underset{\sim}{G}(s) \cap \mathbb{C}^p \qquad (2.4)$$

Such incidence conditions are familiar from the earlier work of Kimura [14] on pole placement, and the algebraic geometric results of Hermann-Martin [12], and have come to play a sizable role in the geometric theory of pole-assignability ([3], [4], [5], [19]). Note, in particular, that

$$\text{col.span } \underset{\sim}{G}(s_1) \cap \mathbb{C}^p \neq \{0\} \iff s_1 \text{ is a pole of } G(s) \quad (2.5)$$

compare [12] and also [9]. The right-hand side generalizes in the case of non-simple poles to:

Lemma 2.2 ([19]): $\delta(G(s)) = \delta(w^tG(s))$ if, and only if, w is not orthogonal to col.span $\underset{\sim}{G}(s) \cap \mathbb{C}^p$, for a pole of G(s).

Since G(s) can have only finitely many poles, Lemma 1.1 follows by induction. Moreover, it follows easily from Lemma 1.2 (for example) that given a $p \times m$ transfer function G(s), with $p \leq m$, one can place $2\left[\frac{p-1}{2}\right]+1$ self-conjugate poles $\{s_i\}$ using a compensator of order 0, i.e. a constant gain output feedback law K. The proof of Theorem 2.1 now proceeds as follows:

(i) given $s_1,\ldots,s_{n+q}$ place the self-conjugate subset (after reordering) $s_1,\ldots,s_{2\left[\frac{p-1}{2}\right]+1}$ by output feedback.

(ii) choose $w \in \mathbb{R}^p$ orthogonal to the planes col.span $\underset{\sim}{G}(s_1),\ldots,$ col.span $\underset{\sim}{G}(s_{2\left[\frac{p-1}{2}\right]+1})$.

Thus, $w^tG(s)$ is of McMillan degree $n - 2\left[\frac{p-1}{2}\right] - 1$. By Lemma 1.2, if $w^tG(s)$ has the generic set of controllability indices, we can place the remaining poles by a compensator of order q, where q satisfies

$$m(q+1) \geq n - 2\left[\frac{p-1}{2}\right] - 1 \qquad (2.6)$$

Recalling that $p = \min(m,p)$ and $m = \max(m,p)$, (2.6) implies Theorem 2.1.

Q.E.D.

3. Generic Pole Assignability by Constant Gain Output Feedback

If $m = p = 2$, $n = 4$ then Theorem 2.1 asserts that the generic (A,B,C) can be (generically) pole-assigned by a compensator of order $q = 1$. By the main result of Willems-Hesselink [22] (see also [3], [15]), we know that one cannot generically assign poles by a compensator of order 0. Thus, the bound in (2.1) – or in the Brasch-Pearson Theorem – is the best bound possible on the necessary order of complexity of the pole-assigning compensator required for a generic system. Nonetheless, since the proof of Theorem 2.1, as well as the original proof of the Brasch-Pearson Theorem [14], employed a q-th order compensator of a very special form (viz. a "vectored-up" compensator) it is rather likely that the bound in Theorem 2.1 is not sharp in all cases. Indeed, if n is even then the well-known result of Kimura [14] asserts that

$$m + p - 1 \geq n \qquad (3.1)$$

is sufficient for generic pole-assignability of the generic system, while (2.1) only guarantees that $m+p \geq n$ is sufficient. As it turns out, a more delicate analysis of the geometric interpretation (2.5) of poles, allows one to sharpen (3.1) and many other existing results on pole assignability. Among the new results one can prove by these methods is the following.

Define $k \in \mathbb{N}$ by $2^k < m + p \leq 2^{k+1}$ and set

$$c_{m,p} = \begin{cases} 2^{k+1}-1 \text{ if } \min(m,p)=2,\ \max(m,p)\neq 2^k-1 \\ 2^{k+1}-2 \text{ if } \min(m,p)=2,\ \max(m,p)=2^k-1 \\ 2^{k+1}-1 \text{ if } \min(m,p)=3,\ m+p=2^k+1 \\ 2^{k+1} \quad \text{otherwise} \end{cases}$$

Theorem 3.1: $c_{m,p} \geq 2\left[\frac{n-1}{2}\right]+1$ implies arbitrary pole assignability for the generic triple (A,B,C).

Note that, in any case,

$$m+p-1 \leq c_{m,p} \leq mp \qquad (3.1)$$

The left-hand inequality implies a strengthened form of:

Corollary 3.2 (Kimura): $m+p-1 \geq n$ implies generic pole-assignability for the generic system (A,B,C)

The right-hand inequality reflects the necessary condition [22] $mp \geq n$ for pole-assignability, and one can ask when $c_{m,p} = mp$. Of course if $\min(m,p) = 1$, then $mp \geq n$ is sufficient for pole-assignability of the generic system. On the other hand, $mp \geq n$ is not sufficient if $\min(m,p) = \max(m,p) = 2$. The case $c_{m,p} = mp$ occurs precisely in the cases discovered in [3]:

Corollary 3.3 (Brockett-Byrnes): The generic $p \times m$ system of degree n is pole-assignable provided $mp \geq n$ and

$$\min(m,p) = 1 \text{ or } \min(m,p) = 2 \text{ and } \max(m,p) = 2^r - 1$$

In [3] one can find an explicit characterization of the generic property alluded to above; see also [15] in the cases $\min(m,p) = 2$, $\max(m,p) = 2,3$.

For example, consider a 3×3 system G(s). The technique of "vectoring down" yields, as does Corollary 3.2, that generic pole-assignability holds provided the McMillan degree of G(s) does not exceed 5. On the other hand, if M is a 2×2 matrix chosen generically, MG(s) is a 3×2 transfer function satisfying

$$\delta(G(s)) = \delta(MG(s))$$

Applying Corollary 3.3, we see that arbitrary pole-assignability is possible provided $\delta(G(s)) \leq 6$. Theorem 3.1 asserts, however, that arbitrary pole-assignability holds provided $\delta(G(s)) \leq 7$. This claim can also be deduced from Corollary 3.3 in the same way that Theorem 2.1 follows from Lemma 1.2. Indeed, this example illustrates the spirit of the proof of Theorem 3.1, which we shall now sketch.

First, note that to say s_1 is a pole of the closed-loop system $G(s)(I+KG(s))^{-1}$ is to say

$$\det(D(s_1)+KN(s_1)) = 0 \qquad (3.2)$$

or, equivalently [3],

$$\det \begin{pmatrix} I & N(s_1) \\ -K & D(s_1) \end{pmatrix} = 0 \qquad (3.2)'$$

Since the matrices

$$\begin{pmatrix} I \\ -K \end{pmatrix}, \quad \begin{pmatrix} N(s) \\ D(s) \end{pmatrix}$$

are of full rank (at each $s \in \mathbb{C}$), (3.2)' implies that

$$\dim \ \text{col.span} \begin{pmatrix} I \\ -K \end{pmatrix} \cap \text{col.span} \begin{pmatrix} N(s_1) \\ D(s_1) \end{pmatrix} \geq 1 \qquad (3.3)$$

Thus, to say K places the poles of G(s) at $s_1, \ldots, s_n$ is to say that the p-plane

$$\text{col.span} \begin{pmatrix} I \\ -K \end{pmatrix} \subset \mathbb{C}^p \oplus \mathbb{C}^m \qquad (3.4)$$

intersects each of the n m-planes

$$\text{col.span} \begin{pmatrix} N(s_i) \\ D(s_i) \end{pmatrix} \subset \mathbb{C}^p \oplus \mathbb{C}^m, \ i = 1, \ldots, n \qquad (3.5)$$

nontrivially. This line of reasoning was the basis of the original proof of Corollary 3.3, noting especially that if $mp = n$, then the number of complex p-planes (3.4) satisfying (3.3) for $i = 1, \ldots, n$ is finite. In fact this number,

$$d_{m,p} = \frac{1!\ldots(p-1)!(mp)!}{m!\ldots(m+p-1)!} \qquad (3.6)$$

was derived by Schubert in his study of the enumerative geometry of planes ([3]). The Brockett-Byrnes Theorem follows from determining when (3.6) is odd.

The second ingredient in the proof of Theorem 3.1 is a new development [13], [20] in the Schubert calculus, enabling us to relax the condition $mp = n$ while still retaining quantitative analogues of (3.6) - see also [4] for an independent derivation using the classical methods of enumerative geometry. Those results yield sufficient conditions involving either $c_{m,p}$ or $c_{m,p} - 1$.

Finally, the condition $c_{m,p} \geq n$ can be derived by developing a modified enumerative geometry for p-planes which satisfy (3.3) for a single, fixed real pole (see [6]).

4. Results

There is a geometric interpretation of multivariable zeroes [19] which is quite analogous to the geometric interpretation (2.5) of poles. The literature on multivariable zeroes is too extensive to be surveyed here; following Verghese [21], we will think, intuitively, of a zero as an "absorbed motion." Suppose $p \leq m$; then a pair $(s,u) \in \mathbb{C}^* \times (\mathbb{C}^m - \{0\})$ is a zero provided there exists a p-plane $V \subset \mathbb{C}^m$ such that

$$u \in \text{col.span} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} \cap V \qquad (4.1)$$

Zeroes, then, are also characterized by an incidence relation. As an application, suppose G(s) is a $p \times m$ non-square transfer function. In order to analyze the feedback properties of G(s), it is often useful to "square-down" G(s) by either pre- or post-multiplication, obtaining a square system MG(s), or G(s)M.

In this setting, one may employ multivariable root-locus methods - such as [17] - to analyze the stability of resulting closed-loop systems. Since, in the square case, the root-loci move from the open-loop poles to the open-loop zeroes (under full rank feedback, provided $\det(MG(s)) \neq 0$ or $\det(G(s)M) \neq 0$, it is of considerable interest to choose M so that the "squared-down" system has as many zeroes as possible in the left half-plane. Thus, we consider the zero-placement problem:

Can one place $c(G(s))$ zeroes of $G(s)$ arbitrarily by output feedback?

Here, $c(G(s))$ denotes the "content" of $G(s)$, which equals the total number of poles of $G(s)$ minus the total number of zeroes ([21]). Due to the geometric characterization (4.1) of zeroplacement as a problem in enumerative geometry, we can apply the previous results with appropriate changes [19]:

Theorem 4.1: If c_{ij} is defined as in section 3,

$$c_{\max(m,p)-\min(m,p),\min(m,p)} \geq c(G(s))$$

implies zeroplacement for the generic $G(s)$.

We remark that if $n - mp$ is large, then pole-placement is impossible, while in this range zero-placement can be effectively incorporated into a root-locus design technique.

Remark: Similar techniques apparently can be applied to quite general feedback problems. For example, the results of section 2 have recently been applied to the problems of simultaneous pole-assignability and simultaneous stabilizability of a set of r $p \times m$ plants $G_1(s),\ldots,G_r(s)$ yielding, for example, the result that the generic r-tuple of plants may be simultaneously stabilized provided $r \leq \max(m,p)$ (see [10], [11]).

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HIGH GAIN FEEDBACK AND THE STABILIZABILITY OF MULTIVARIABLE SYSTEMS

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ABSTRACT

The problems of determining the minimal order of a stabilizing compensator for a fixed linear, multivariable system and for the generic $p \times m$ system of fixed degree are considered. An elementary geometric argument gives sufficient conditions for the generic stabilizability by a compensator of order $\leq q$. A more delicate geometric argument, involving pole-placement in the high gain limit, is then used to derive necessary conditions, obtained jointly with B.D.O. Anderson, for the lower bound $q \geq 1$. Taken together, these results determine the minimal order in certain low dimensional cases. The general upper bound, however, is not always tight and in many cases can be improved upon by more powerful techniques. For example, based on a geometric model for finite and infinite gains, sufficient conditions for $q = 0$ are derived in this paper in terms of a topological invariant (of the "gain space") introduced by Ljusternick and Šnirel'mann in the calculus of variations. Using the Schubert calculus, an estimate of the Ljusternick-Šnirel'mann category is obtained, yielding a stabilizability criterion which, to my knowledge, contains the previous results in the literature on stabilizability by constant gain output feedback, as special cases.

0. INTRODUCTION

The purposes of feedback in system theory are manifold, including (for example) stabilization, decoupling, optimization, and increased insensitivity to perturbations. Indeed, the study of the possible effects of feedbacks on the dynamical characteristics of a control system engaged the interests of the earliest quantitative research efforts in mathematical control theory ([1], [2], [31]). Recently, the study of "high gain feedback" has been formalized in several ways leading to a robust extension ([36], [37]) of the elegant (A,B)-invariant subspace theory, which is capable of answering questions such as "almost disturbance decoupling", and to new results in the classical problem of pole-

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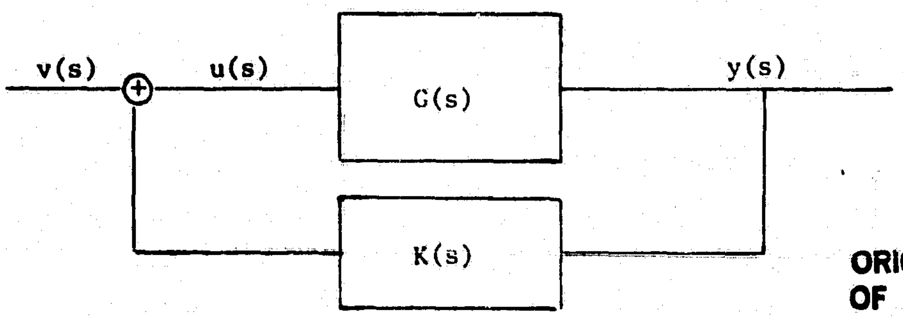
assignability by output feedback ([7], [8]). In this paper, I would like to illustrate the algebraic-geometric aspects of the analysis of high-gain feedback in the less understood context of stabilization by static and dynamic output feedback. More precisely, I would like to begin by focusing on a specific problem, which is representative of a genre of classical linear system theory.

<u>Question 0.1</u>. Given a $p \times m$ rational transfer function $G(s)$, strictly proper with McMillan degree

$$\delta(G) = n,$$

what is the minimum degree, $q = \delta(K)$, of a proper compensator $K(s)$ which (internally, in the sense of [38]) stabilizes $G(s)$ in closed-loop:



Equivalently, we ask that the (n+q)-poles (i.e., including the cancelled poles) of $G(s)(I-K(s)G(s))^{-1}$ lie in the left-half plane.

There are several reasons to ask for an upper bound for q, not the least of which is the desire to stabilize G(s) with a compensator having at most a certain degree of complexity. Second, the set of $m \times p$ compensators of order $\leq q$ is naturally an algebraic set of finite dimension, viz. $q(m+p)+mp$. Indeed, the set $\Sigma(n,p,m)^*$ of compensators of order q can be parameterized as a smooth finite-dimensional manifold ([13], [25]). Thus, the techniques of calculus on finite-dimensional manifolds can be used on $\Sigma^*(n,m,p)$ in developing algorithms for finding a stabilizing compensator.

I will also consider the question of whether a given G(s) can be arbitrarily closely approximated (say, uniformly in $s \in \mathbb{CP}^1$) by a transfer function of the same degree which is stabilizable by a compensator of degree q. Since stabilizable systems form an open set, this is then equivalent to the question:

<u>Question 0.2</u>. Is the set U_q of $p \times m$ systems G(s) of degree n, which are (internally) stabilizable by a compensator of degree q, open and dense in the space $\Sigma(n,m,p)$ of all $p \times m$ systems of degree n?

To make this precise, one need only know how to regard $\Sigma^*(n,m,p)$ as a topological space. Develop G(s) in its Laurent expansion $G(s) = \sum_{i=1}^{\infty} L_i s^{-i}$,

L_i $p \times m$ real matrices. Since $\delta(G) = n$, $G(s)$ determines and is determined by the entries of

$$h_G = (L_1, \ldots, L_{2n}) \in \mathbb{R}^{2nmp} \tag{0.1}$$

where h_G must satisfy the constraint

$$\text{rank} \begin{bmatrix} L_1 & L_2 & \cdots & L_n \\ L_2 & \cdots & \cdots & \cdots \\ \vdots & & & \\ L_n & & & L_{2n-1} \end{bmatrix} = n \tag{0.2}$$

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Thus, $\Sigma(n,m,p)$ is in bijection, via (0.1), with the subset $\text{Hank}(n,m,p) \subset \mathbb{R}^{2nmp}$ of points satisfying (0.2). In this way, $\Sigma(n,m,p)$ is regarded as a subspace of $\mathbb{R}^{2nmp}$, so that

$$G_i(s) \to G(s)$$

if, and only if, Hankel (or Markov) parameters

$$(L_1^{(i)}, \ldots, L_{2n}^{(i)}) \to (L_1, \ldots, L_{2n})$$

converge. Thus, the meaning of the question, is $U_q \subset \Sigma(n,m,p)$ open and dense, is clear.

I will refer to Questions 0.1 and 0.2 as stabilizability and generic stabilizability, respectively, by a compensator of degree q. I should remark that the question of the simultaneous stabilizability of an r-tuple of plants, which arises in problems of reliability and fault tolerance, has recently been quite successfully studied by B.K. Ghosh using extensions of these methods, see ([16], [18]).

It is a pleasure to acknowledge the influence of my friends and coauthors Brian Anderson, Roger Brockett, Bijoy Ghosh, and Peter Stevens on my thinking about this problem. Indeed a great deal of this paper (cf. references) is based on or surveys joint work with these authors. In addition, I would also like to acknowledge interesting conversations and correspondence on this topic with Ted Djaferis, Sanjoy Mitter, Steve Morse, and Jans Willems.

1. STABILIZABILITY WITH DYNAMIC COMPENSATION

Let $G(s)$ be a $p \times m$ transfer function of degree, $\delta(G)$, n and consider $v \in \mathbb{R}^m$ as an input channel, leading to the new $p \times 1$ transfer function $G(s)v$. According to Brasch-Pearson [6], there exists v such that

$$\delta(G(s)v) = \delta(G(s)) \tag{1.1}$$

Actually ([12], [34]), the set of input channels v such that (1.1) holds is open and dense in $\mathbb{R}^m$, with the same statement holding for output channels $w^t \in (\mathbb{R}^p)^*$.

This is easily seen in the case where G(s) has simple poles, for then G(s) admits a partial fraction decomposition

$$G(s) = \sum_{i=1}^{n} \frac{R_i}{s-\lambda_i} \quad , \quad \text{rank } R_i = 1 \tag{1.2}$$

Then, G(s)v (or $w^t G(s)$) will have a pole at λ_i if, and only if, $R_i v$ (or $w^t R_i$) does not vanish. Since the poles of G(s)v are among those of G(s), and since G(s) has finite degree, the set of such v (or w^t) is open and dense.

More generally, consider a coprime factorization $G(s) = N(s)D(s)^{-1}$ leading to the martix

$$\tilde{G}(s) = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} \tag{1.3}$$

If G(s) has a simple pole of s_1, then $w^t R_1 = 0$, if, and only if, $w \in \mathbb{R}^p \subset \mathbb{C}^p$ is orthogonal to the column span of R_1 in $(\mathbb{C}^p)^*$. Alternatively, regarding column span $\tilde{G}(s)$ as an m-dimensional subspace of $\mathbb{C}^p \oplus \mathbb{C}^m$ and $\mathbb{C}^p$ as a p-dimensional subspace,

$$w^t R_1 = \{0\} \iff w \perp (\text{column span } \tilde{G}(s) \cap \mathbb{C}^p) \tag{1.4}$$

Thus [34], if G(s) has poles at $s_1, \ldots, s_r$ and if $w \in \mathbb{R}^p$ is chosen so that w is not orthogonal to the subspaces col.sp.$\tilde{G}(s_1) \cap \mathbb{C}^p, \ldots,$ col.sp.$\tilde{G}(s_r) \cap \mathbb{C}^p$ of $\mathbb{C}^p \subset \mathbb{C}^p \oplus \mathbb{C}^m$, then

$$\delta(w^t G(s)) = \delta(G(s)) \tag{1.5}$$

<u>Lemma 1.1</u>. ([34]) $\delta(G(s)) = \delta(w^t G(s))$ if, and only if, w is not orthogonal to (col.sp.$\tilde{G}(s) \cap \mathbb{C}^p$), for s a pole of $\mathbb{C}^p$. Thus, the set of w satisfying (1.5) is open and dense in $\mathbb{R}^p$.

The same result of course holds for $v \in \mathbb{R}^m$, mutatis mutandis. Incidence conditions such as (1.4) are familiar from the earlier work of Hautus [22] and Kimura [26] on pole-placement and from the seminal algebraic geometric interpretation of transfer functions due to Hermann-Martin [30], and have come to play a sizable role in the geometric theory of pole-assignability ([7]-[10], [12], [34]). Note especially that

$$\text{col.sp.}\tilde{G}(s_i) \cap \mathbb{C}^p \neq \{0\} \iff s_i \text{ is a pole of } G(s) \tag{1.6}$$

Compare [26] and [30].

These concepts can be illustrated in the following:

Theorem 1.2. The generic $G(s)$ can be stabilized by a compensator of order q satisfying

$$(q+1)\max(m,p) + \min(m,p) - 1 \geq n. \tag{1.7}$$

(1.7) improves, by $\min(m,p)-1$, the generic stabilizability result which one obtains from the generic form of the Brasch-Pearson Theorem (see [6], [14]). If $q=0$, i.e. if one asks for stabilization by constant gain output feedback, (1.7) agrees with the condition one obtains from Kimura's Theorem [26].

Proof. Let $v \in \mathbb{R}^m$. If $K(s)$ is $1 \times p$ compensator, consider coprime factorizations

$$G(s)v = N(s)D(s)^{-1} \quad \text{and} \quad K(s) = Q(s)^{-1}P(s)$$

Then, the return-difference determinent, as a function of (P,Q), is a linear function

$$S_q : \mathbb{R}^{(q+1)(p+1)} \to \mathbb{R}^{n+q+m+2} \tag{1.8}$$

$$S_q(P,Q) = Q(s)D(s) + P(s)N(s)$$

in the coefficients of $P(s)$, $Q(s)$, and $QD+PN$. According to [5], the rank of "the generalized Sylvestor resultant" is given by the beautiful formula

$$\text{rank } S_q = (q+1)(p+1) - \sum_{\nu_i < q+1} (q+1-\nu_i) \tag{1.9}$$

where the ν_i are the observability indices of $G(s)v$. Therefore, for generic $G(s)$, S_q is surjective provided

$$q(p+1) + p \geq \delta(G(s)v) + q \tag{1.10}$$

The proof now proceeds as follows, we assume without loss of generality that $p \leq m$:

(i) Choose $s_1,\ldots,s_{m-1} \in \mathbb{R}^-$ and an $m \times p$ K_0 placing the poles s_i;

(ii) Choose w^t, as in Lemma 1.1, orthogonal to $\tilde{G}(s_i) \cap \mathbb{C}^p$;

(iii) Since $(w^t G(s)) = n-m+1$, provided q satisfies (1.7) for generic $G(s)$ one can find $K(s)$ with $\delta(K) = q$ placing any self-conjugate set $\{s_m,\ldots,s_{n+q}\}$ of poles in $\mathbb{R}^-$.

Remark 1.3. (Concerning Question 0.1) One can also obtain results on minimum order compensation for a fixed, not necessarily generic, $p \times m$ $G(s)$ by using the formula (1.9) for the rank of the generalized Sylvestor resultant.

Remark 1.4. (Concerning the Brasch-Pearson Theorem) A more elementary argument [12] gives a proof of the Brasch-Pearson Theorem [6]. Explicitly, simply choose $v \in \mathbb{R}^m$ to satisfy (1.1) and use the identity (1.9) to obtain the criterion

$$(q+1)p - \sum_{\nu_i < q+1} (q+1-\nu_i) \geq n \tag{1.11}$$

for arbitrary pole-assignability, with a dual criterion in terms of the controllability indices (K_i). Choosing $q = \max(\nu_i) - 1$ in (1.11), one has the assertion that if $G(s)$ is a $p \times m$ transfer function having controllability indices (ν_i), then $G(s)$ can be arbitrarily pole-assigned using a compensator of order q. The proof of Theorem 1.2 is based on the argument given by Stevens ([12], [34]) which proves, by choosing v more carefully as above, that for generic $G(s)$ q may be taken to satisfy

$$(q+1)\max(m,p) + 2\left[\frac{\min(m,p)-1}{2}\right] \geq n \tag{1.12}$$

A form of (1.12) seems to be implicit in the algorithm described by Seraji [33].

Remark 1.5. (Concerning Related Work) I should also comment on the interesting results obtained by Hammer (esp. [20]), based on an algebraic study of the interplay between feedback and precompensation (see also [19], [21]) which also has application to the stability of systems. Using this theory one can prove, for example, that if

$$G(s) = \frac{(s+1)}{(s+2)(s+3)}$$

and $p(s) = s^2 + as + b$, with $a, b, > 0$, then there exists a compensator $K(s)$ such that the (uncancelled) poles of $G(s)(I+K(s)G(s))^{-1}$ are the roots of $p(s)$, while the cancelled poles are all stable. Although this seems to provide a better result than the Brasch-Pearson Theorem, which would yield the assertion that an arbitrary cubic can be assigned using a compensator of degree 1, these two results cannot be compared since they give solutions to different problems. For example, a dimension count shows that, also in Hammer's result, $\delta(K) \geq 1$ for an open dense set of such quadratic $p(s)$. Moreover, and more crucial for the solution of Questions 0.1 and 0.2, no upper bound on $\delta(K)$ is given in [20].

2. NECESSARY CONDITIONS FOR STABILIZABILITY

In this section, I will sketch a proof of a theorem, obtained jointly with B.D.O. Anderson [11], asserting that $mp \geq n$ is a necessary condition for generic stabilizability by constant gain feedback. Together with Theorem 1.2 this yields, for example, that the minimum order of a stabilizing compensator for the generic $2 \times p$ system of degree $2p+1$ is 1. Before proceeding to the theorem, I will give some low-dimensional examples illustrating the tightness of the estimate (1.7).

For example, it follows from [18] that (1.7) gives the minimum order of stabilizing compensator if $\min(m,p) = 1$.

Example 2.1. Suppose $\min(m,p) = 2$. By the above remarks, (1.7) provides the generic minimum order of a stabilizing compensator provided either $m+p-1 \geq n$ or $mp+1 = n$ holds. The case $mp = n$ is rather interesting. If $m = p = 2$, one deduces that the minimum order compensator satisfying $\delta(K) = q \leq 1$ from (1.7). On the other hand, in [11] a proof is given of an unpublished result, attributed to P. Molander, which is equivalent to $\delta(K) = q \geq 1$. Thus, in this case (1.7) is tight provided $n \leq 5$. If $m = 2$, $p = 2^r - 1$, then (1.7) again yields $\delta(K) \leq 1$, while the pole-placement results obtained in [7] implies $\delta(K) = 0$ whenever $\min(m,p) = 2$, $\max(m,p) = 2^r - 1$.

Theorem 2.2. ([11]) $mp \geq n$ is necessary for generic stabilizability by constant gain output feedback.

Proof. All compensators are assumed to have degree 0. First of all, it is intuitive - from the algebraic system theoretic perspective - that generic stabilizability in continuous time is equivalent to generic stabilizability. Indeed [11], the generic system is stabilizable with respect to $\mathrm{Re}[z] < 0$ if, and only if, the generic $p \times m$ system of degree n is stabilizable with respect to the disc $\mathbb{D}(0;\rho) = \{|z| < \rho\}$.

Assuming that stabilizability is generic, the set

$$U_r = \{(A,B,C) : C(sI-A)^{-1}B \text{ is stabilizable with respect to } \mathbb{D}(0;1/r)\}$$

is open and dense in $\mathbb{R}^N$, $N = n^2 + n(m+p)$. By the Baire Category Theorem, $U = \bigcap_{r=1}^{\infty} U_r$ is dense in $\mathbb{R}^N$. Now consider the algebraic subset of $\mathbb{R}^N \times \mathbb{R}^{mp}$,

$$\tilde{V} = \{(A,B,C;K) : \mathrm{spec}(A-BKC) = \{0\}\} \tag{2.2}$$

If $p_1 : \mathbb{R}^N \times \mathbb{R}^{mp} \to \mathbb{R}^N$ is the projection, $p_1(x,y) = x$, on the first factor, then $V = p_1(\tilde{V})$ is a semialgebraic set by the Tarski-Seidenberg Theorem. That is, V is described by the conditions,

$$f_i(A,B,C) = 0 \quad , \quad g_j(A,B,C) > 0 \tag{2.3}$$

for f_i, g_j polynomials in the entries of (A,B,C). I claim that $U \subset V$. Explicitly, this follows from

Lemma 2.3. ([8]) If $mp \leq n$ then the polynomial function for $\sigma = (A,B,C)$

$$\chi_\sigma : \mathbb{R}^{mp} \to \mathbb{R}^n, \text{ defined by}$$

$$\chi_\sigma(K) = \text{characteristic coeff's of } (A-BKC)$$

has a closed image for an open dense subset of $\sigma \in \mathbb{R}^N$.

Indeed, $\sigma \in U$ if, and only if, for every $\varepsilon > 0$ $\exists K_\varepsilon$ such that $\text{spec}(A-BK_\varepsilon C) \subset \mathbb{D}(0;\varepsilon)$. Taking $\varepsilon = 1/k$, for $\sigma \in U$ there exists $K_k \in \mathbb{R}^{mp}$ such that $\chi_\sigma(K_k) \to$ coeff's of $\{s^n\}$. By Lemma 2.2, there exists $K \in \mathbb{R}^{mp}$ such that $\chi_\sigma(K) =$ coeff's of s^n, for generic $\sigma \in \mathbb{R}^N$.

<u>Nota Bene 2.4</u>. If $m = p = 1$, then image χ_σ is a straight line in $\mathbb{R}^n$, so that Lemma 2.2 is valid for all $\sigma \in \mathbb{R}^N$. Kimura [26] contains an example, pp. 514-515: Example 3, of a 2×2 system σ of degree 3 for which image χ_σ is not a closed set.

Since it suffices to prove the theorem if $mp \leq n$ we can assume, without loss of generality, that $U \subset V$. In particular, any f_i in (2.2) must vanish identically, since U is dense. Thus, V is open and dense in $\mathbb{R}^N$.

Now consider the algebraic subset of $n \times n$ real matrices

$$\mathcal{N}_n = \{N : N \text{ is nilpotent}\}$$

It is known [28] that $\mathcal{N}_n$ is an irreducible algebraic subset of dimension $n^2 - n$. Matters being so, generic stabilizability (for $mp \leq n$) implies that the function

$$\Phi : \mathcal{N}_n \times \mathbb{R}^{nm} \times \mathbb{R}^{np} \times \mathbb{R}^{mp} \to \mathbb{R}^{n^2} \times \mathbb{R}^{nm} \times \mathbb{R}^{np}$$

defined via

$$\Phi(N,B,C,K) = (N + BKC, B, C)$$

has an image containing an open, dense subset. In this case, then

$$(n^2-n) + nm + np + mp \geq n^2 + nm + np$$

Equivalently,

$$mp \geq n$$ Q.E.D.

<u>Corollary 2.5</u>. If $\min(m,p) = 2$ and $n = mp + 1$, then the minimal order of a stabilizing compensator for the generic system is $q = 1$.

3. A GEOMETRIC MODEL FOR HIGH GAIN FEEDBACK

As has been remarked (Nota Bene 2.4), Lemma 2.3 fails to hold if the genericity hypothesis is violated. The major point involved here is the dichotomy: Suppose for $\sigma = (A,B,C)$ fixed one has gain K_r such that the roots of $A - BKC$ lie in $\mathbb{D}(0;1/r)$. As $r \to \infty$, either

(i) $K_r \to K$ as $r \to \infty$, in which case $\chi_\sigma(K_r) \to \chi_\sigma(K)$; or

(ii) $K_r \to \infty$.

Only in the latter case can s^n fail to lie in image χ_σ. To analyze this case, I would like to make the statement, $K_r \to \infty$, explicit in terms of feedback. Now feedback, $u = Ky$, is just a bilinear relation between inputs u and outputs y, with a special property, viz, that u is a function, Ky, of y. If K_λ is a 1-parameter family of feedback laws, say

$$\begin{aligned} u_1 &= \lambda y_1 \\ u_2 &= \lambda y_2 \end{aligned} \tag{3.1}$$

then passing to the limit, $\lambda \to \infty$, also defines a bilinear relation between inputs and outputs, viz.

$$\begin{aligned} 0 &= y_1 \\ 0 &= y_2 \end{aligned} \tag{3.1$'$}$$

Note that the equations (3.1) and (3.1)' both define 2-dimensional subspaces of $\mathbb{R}^4 = Y \oplus U$: (3.1) corresponds to the subspace graph(K_λ), where $u = K_\lambda y = \lambda y$, while (3.1)' corresponds to the graph of a bilinear relation which is not a function $u = Ky$ for any $K : Y \to U$. In this spirit, I shall consider a feedback law, including "high gain limits", as the graph of bilinear relation R on $Y \times U$ of rank p, i.e. as a p-plane, viz. graph(R), in $Y \oplus U$.

Of course, not every p-plane V is of the form graph(K), for such a V must be complementary to the subspace $\mathbb{R}^m \subset \mathbb{R}^p \oplus \mathbb{R}^m$. In this sense, the space of all p-planes in $\mathbb{R}^{m+p}$ Grass(p,m+p) contains the space of feedback laws K, qua graph(K), as an open dense subspace and one can interpret those p-planes V such that

$$\dim\,(V \cap \mathbb{R}^m) \geq 1 \tag{3.2}$$

as infinite gains or as high gain limits.

Following [7], [8], I shall describe how one might assign a set of "closed-loop" poles to the p-plane graph(R)

Modulo-zero cancellations, the poles of $G^K(s)$ are given by the return difference equation

$$0 = \det(I - KG(s) \iff \det \begin{bmatrix} I & G(s) \\ K & I \end{bmatrix} = 0 \tag{3.3}$$

Thus, to say s is a pole of $G^K(s)$ is to say

$$\dim \left(\text{col.span} \begin{bmatrix} I \\ K \end{bmatrix} \cap \text{col.span} \begin{bmatrix} G(s) \\ I \end{bmatrix} \right) \geq 1 \tag{3.3$'$}$$

where

$$\text{col.span}\begin{bmatrix} I \\ K \end{bmatrix} = \text{graph}(K) \subset \mathbb{R}^p + \mathbb{R}^m$$

is a p-plane in $\mathbb{R}^{m+p}$. Note, if $K = 0$ then (3.3)' reduces to the Hermann-Martin identity (1.6).

By definition p-planes V satisfying (3.2) are called infinite gains, those not satisfying (3.2) are finite gains, in the ordinary sense. In this language, Lemma 2.3 follows from the complex analogue of

3.1. The High Gain Lemma. For generic G(s), if $s_1,\ldots,s_n \in \mathbb{R}$ are such that

$$\bigcap_{i=1}^{n} \sigma(s_i) \neq \emptyset \quad \text{in Grass}(p,m+p)$$

then this intersection contains a finite gain.

Thus, the High Gain Lemma asserts intuitively that if $s_1,\ldots,s_n$ can be placed in the high gain limit, then $s_1,\ldots,s_n$ can be placed by a finite gain. If the root-locus map χ_σ were continuous at infinity, stabilizability in the high gain limit would imply stabilizability by finite gain. However, if $mp > n$, χ_σ is never continuous at ∞ [9] and therefore, cf. Theorem 2.2, in most cases of interest one requires a more subtle argument - such as 3.1. Details will appear in a future paper.

4. STABILIZABILITY BY STATIS OUTPUT FEEDBACK

Using (3.3) one can interpret the vanishing of the return-difference determinant geometrically; in terms of the compact manifold Grass(p,m+p). There is a classical topological invariant of any space X, discovered by Ljusternick and Šnirel'mann [29] in the calculus of variations, which will play a sizable role in the present analysis. Explicitly, consider any covering (U_α) of X by open sets U_α which are contractible in X and define L-S cat(X) to be the minimum cardinality of such a cover. Set

$$k(m,p) = \text{L-S cat}(\text{Grass}(p,m+p)) - 1 \tag{4.1}$$

Theorem 4.1. $k(m,p) \geq n$ implies generic stabilizability.

Pf. If one defines $\sigma(s_i) \subset \text{Grass}(p,m+p)$, for $s_i \in \mathbb{R} \cup \{\infty\}$ and for G(s) fixed, via

$$\sigma(s_i) = \{V : \dim(V \cap \text{graph}(G(s_i))) \geq 1\}$$

then $\sigma(s_i)$ is a hypersurface in Grass(p,m+p). Clearly

Lemma 4.2. $\mathrm{Grass}(p,m+p) - \sigma(s_i) \simeq \mathbb{R}^{mp}$.

Now, to say K places the poles of $G^K(s)$ at the distinct real numbers $s_1,\dots,s_n$ is to say (3.3), or equivalently (3.3)', holds for each $s = s_i$. That is, $\mathrm{graph}(K) \in \bigcap_{i=1}^{n} \sigma(s_i) \subset \mathrm{Grass}(p,m+p)$, and in particular,

$$\bigcap_{i=1}^{n} \sigma(s_i) \neq \emptyset \qquad (4.2)$$

Lemma 4.3. Suppose $s_1,\dots,s_n \in \mathbb{R}$. Then $k(m,p) \geq n$ implies (4.2) for any $G(s)$.

Proof. If $\bigcap_{i=1}^{n} \sigma(s_i) \neq \emptyset$, then $(U_i)_{i=1}^{n}$ covers $\mathrm{Grass}(p,m+p)$ where $U_i = \mathrm{Grass}(p,m+p) - \sigma(s_i)$. Since $U_i \simeq \mathbb{R}^{mp}$, one has L-S cat$(\mathrm{Grass}(p,m+p)) \leq n$ and therefore, by definition (4.1), one obtains the contradiction $k(m,p) < n$.

This tautology does not imply, by choosing $s_i < 0$, stabilizability by finite gains, for none of the points V of $\bigcap_{i=1}^{n} \sigma(s_i)$ might be of the form graph(K).

For generic $G(s)$, however, there exists a finite gain by the High Gain Lemma. In the next section, I will give some applications of Theorem 4.1.

5. APPLICATIONS TO GENERIC STABILIZABILITY BY CONSTANT GAIN FEEDBACK

First, define the integer s by

$$2^s < m+p \leq 2^{s+1} \qquad (5.1)$$

Corollary 6.1. If $\min(m,p) = 2$, then

$$\max(m,p) + 2^s - 1 \geq n$$

implies generic stabilizability.

For $\max(m,p) \leq 5$, the bound in Corollary 5.1 coincides with the pole-placement bounds which one can derive, in various cases, from the literature ([7], [10], [26], [27]). However, for $\max(m,p) = 6$, Corollary 5.1 asserts that generic stabilizability holds provided $n \leq 9$ in contrast to the best known value, viz. 8, for pole-placement [27].

Corollary 5.2. If $\min(m,p) = 3$, then the following equalities imply generic stabilizability:

(i) $2^{s+2} - 3(2^{r-1}) - 4 \geq n$, if $m+p = 2^{s+1} - 2^r + 1$;

(ii) $2^{s+2} - 3(2^{r-1}) - 2 + t \geq n$, if $m+p = 2^{s+1} - 2^r + 2 + t$, $0 \leq t \leq 2^{r-1} - 2$;

(iii) $2^{s+2} - 5 \geq n$, if $m + p = 2^{s+1}$.

Corollary 5.3. If $\min(m,p) = 4$, then the following inequalities imply generic stabilizability:

(i) $2^{s+1} + 2^{s} - 7 \geq n$, if $m + p = 2^{s} + 1$

(ii) $2^{s+1} + 2^{s} + 2^{r+1} + j - 7$, if $m + p = 2^{s} + 2^{r} + j + 1$, where $s > r \geq 0$ and $0 \leq j \leq 2^{r} - 1$.

In fact, one can always assert that

$$m + p - 1 \leq k(m,p) \leq mp \tag{5.2}$$

The left-hand side of (5.2) implies that Theorem 4.1 will do at least as well as any stabilizability result derived from Kimura's Theorem [26] while the right-hand side apparently reflects Theorem 2.2.

Proofs. Eilenberg's Theorem [15] asserts, in the case at hand, that

$$k(m,p) \geq \text{nil}\ \ (H^*(\text{Grass}(p,m+p);\mathbb{Z}_2)) \tag{5.3}$$

The cohomology ring $H^*(\text{Grass}(p,m+p);\mathbb{Z}_2)$ is given in terms of generators and relations as

$$R = \mathbb{Z}_2 [w_1,\dots,w_m,v_1,\dots,v_m]/I \quad , \quad I = \left(\sum_{i+j=r} w_i v_j\right) \tag{5.4}$$

and nil (R) is the maximum number of nontrivial terms in a nonzero produce in R. It follows from the Schubert calculus ([4], p. 130) that one can always find a nontrivial product of $m + p - 1$ Schubert generators in (5.4), thereby proving the left-hand side of (5.2). The right-hand side follows from the general fact [24] $\text{cat}(X) \leq \dim X + 1$ for any path connected, paracompact space X.

The corollaries now follow from calculations [3], [23], [35] of the order of nilpotency for the rings in (5.4), in the range $2 \leq \min(m,p) \leq 4$.

It should be remarked that the calculation of the order of nilpotency of the rings (5.4) is entirely algorithmic for fixed m and p and in this way a table, giving values of n - as a function of m and p - for which generic stabilizability will hold, can be constructed. Taken together, Corollaries 5.1-5.3 yield such a table for $m + p \leq 9$.

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