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RELATIVISTIC PERTURBATIONS ON THE MOTION AND TRACKING OF THE LAGEOS SATELLITE

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SECTION 1.0
INTRODUCTION

Laser tracking of the LAGEOS satellite is presently being performed with noise levels at the few centimeter level and with systematic errors (biases, refraction, timing, etc.) considered to be at or reducible to the sub-centimeter level. To take full advantage of such accuracies, the measurement and ephemeris modeling in the data reduction program must have comparable accuracies. The most widely used computer program for laser data processing is the GSFC Geodyn program. In its original formulation, this program integrates satellite equations of motion based on Newtonian mechanics, and allows the estimation of orbital and geodetic parameters so that a best fit is obtained, in a weighted least squares sense, to an input data set consisting of various data types. This report describes the modifications to Geodyn to substitute the Einstein gravitational theory for Newtonian gravitation. This results in modifications to both the satellite equations of motion and to the modeling of satellite tracking measurements.

Since the Newtonian theory is a very close approximation to the Einstein theory (or general theory of relativity), observable deviations from Newtonian theory are small, and the applicable equations of motion or observation equations can be formulated so that they differ only by small terms which we will denote as relativistic perturbations. In the implementation of the Einstein theory in an orbital data reduction program, one still deals with station coordinates and satellite coordinates which resemble the normal Cartesian coordinates, one has a time system defined for the coordinate system, and one still measures round trip transit times from ground stations to satellites. However, the satellite equations of motion are altered though the addition of complicated perturbation terms, the clocks at tracking stations do not measure the same time used in the equations of motion, and one does not obtain a range measurement by simply multiplying the transit time by a constant. And one must continually be aware that parameters considered constant in the Newtonian theory may no longer be considered so in the relativistic theory.

In general, no attempt will be made in this report to derive the perturbation equations which should be implemented to transform Geodyn (or comparable computer program developed for estimating orbits for earth satellites) into a relativistic program. Most of the desired relations are well documented in the literature and, in some cases, have a very complex derivation. Likewise, no extensive discussion will be given of general relativity theory itself, for which textbooks may be consulted [e.g., Misner, Thorne and Wheeler, 1973]. The basic theory deviates from classical mechanics in that the "interval" between points is the integral $\int ds$, where

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \qquad (1)$$

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Here

$g_{\mu\nu}$ is the metric tensor

$\mu, \nu \quad = \quad 1, 2, 3, 4$

and summation of μ, ν from 1 to 4 in (1) is implied. The coordinates x^{μ} include, of course, three spatial coordinates and the time coordinate. The path followed by any unconstrained body is such that the integral $\int ds$ is a maximum. In Newtonian theory, a coordinate system can be chosen in which $g_{\mu\nu} = \pm \delta_{\mu\nu}$ and one has Cartesian coordinates. In the Einstein theory, the $g_{\mu\nu}$ depend upon the distribution of mass bodies, but a coordinate system can be chosen for the motion of planetary bodies such that $g_{\mu\nu}$ is almost the Minkowski metric (diagonal -1,-1,-1,+1). In fact, the only available solutions for the motion of planetary masses are in such a system.

In the implementation of relativistic effects, two important points must be emphasized:

1. The coordinate system for integration and measurement computation becomes a solar system barycenter system. The equations of motion for the satellite are derived in this system, and it has already been implicitly assumed in the gravitational third body perturbations in orbit programs such as Geodyn. In practice, Geodyn normally integrates the satellite acceleration relative to the center of the earth in order to

maintain higher precision. Care thus needs to be exercised in the observation computation to insure that, equivalently, coordinates referenced to the solar system barycenter, or differences of such coordinates, are being used.

2. A number of changes are required to implement Einstein gravitation into a program such as Geodyn. The net effects of all changes on observables (such as baselines between stations) are expected to be small (e.g., below the decimeter level). This does not necessarily mean that the effects of individual changes are small. The relativistic theory is different from Newtonian theory, but must be accepted only as a complete theory.

This report includes consideration of both of these points. The effects of using an explicit earth-centered coordinate system will be examined with regard to the computation of the laser ranging observable. And simulation results will be presented to demonstrate the effects of the individual modifications required to implement the Einstein gravitation theory in an orbital data reduction program designed for earth orbiting satellites.

SECTION 2.0
RELATIVISTIC SOLUTIONS FOR n-BODY MOTION

Although quite elegant in tensor formulation, the Einstein equations for determining the ten metric tensor components are quite difficult to solve for a general set of physical bodies. For satellite motion, the solution needed is one which can account for the motion of the satellite in the gravitational field of the earth, sun, and moon. The influence of the other solar system bodies should be considered, but relativistic effects from even Jupiter and Saturn would not be expected to be significant in comparison with those of the sun and earth.

The solution of the Einstein field equations for a single point mass was first obtained by Schwarzschild [1916] within a year of the publication of the general relativity theory. Shortly thereafter, approximate solutions for the field of n mass points [Droste, 1916] and for the motion of n heavy bodies [de Sitter, 1916; Eddington and Clark, 1938] were also obtained. The approximate n-body solution should be perfectly adequate for the application to earth satellite motion. The most appealing approach to the motion problem has been that taken in the so-called Einstein-Infeld-Hoffmann (EIH) approximation technique, since the equations of motion were obtained directly from the field equations. These equations of motion give the accelerations for a spherically symmetric body moving in the field of n-1 other spherically symmetric bodies, in a non-rotating coordinate system centered at the center-of-mass of the n bodies. To be consistent with the solution derivation, these accelerations must be effectively computed in the solar system barycenter system for both the satellite and for the earth. Station coordinates must also be in the same system.

The coordinate system generally adopted for the n-body problem has been a non-rotating system whose origin is at the center of mass of the n-bodies. The coordinates in this system are nearly Cartesian and the equations of motion for the bodies are similar to those for Newtonian n-body motion but with small additional terms. In the Geodyn implementation, these additional terms are treated as the relativistic perturbations, with the integration otherwise proceeding as for integration of the Newtonian equations of motion. Absolute time in the integration is replaced by coordinate time.

The acceleration of one of the n-bodies in this center of mass system may be obtained by carrying out the coordinate time differentiation in the EIH solution [Infeld and Plebański, p. 112]. In the notation of Moyer [1971, Eq. 35], the result after deleting the Newtonian term is:

$$\delta\ddot{\mathbf{r}}_i = \sum_{j\neq i} \frac{\mu_j(\mathbf{r}_j-\mathbf{r}_i)}{r_{ij}^3}\Bigg\{-\frac{4}{c^2}\sum_{\ell\neq i}\frac{\mu_\ell}{r_{i\ell}} - \frac{1}{c^2}\sum_{k\neq j}\frac{\mu_k}{r_{jk}}$$

$$+\left(\frac{\dot{s}_i}{c}\right)^2 + 2\left(\frac{\dot{s}_j}{c}\right)^2 - \frac{4\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j}{c^2} - \frac{3}{2c^2}\left[\frac{(\mathbf{r}_i-\mathbf{r}_j)\cdot\dot{\mathbf{r}}_j}{r_{ij}}\right]^2$$

$$+\frac{1}{2c^2}(\mathbf{r}_j-\mathbf{r}_i)\cdot\ddot{\mathbf{r}}_j\Bigg\} + \frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\left[(\mathbf{r}_i-\mathbf{r}_j)\cdot(4\dot{\mathbf{r}}_i-3\dot{\mathbf{r}}_j)\right](\dot{\mathbf{r}}_i-\dot{\mathbf{r}}_j)$$

$$+\frac{7}{2c^2}\sum_{j\neq i}\frac{\mu_j\,\ddot{\mathbf{r}}_j}{r_{ij}} \qquad (2)$$

where

μ_j = gravitational constant for body j.

x, y, z
$\dot{x}, \dot{y}, \dot{z}$
$\ddot{x}, \ddot{y}, \ddot{z}$ = rectangular components of position, velocity, and acceleration relative to a non-rotating frame of reference centered at the barycenter of the system of n bodies.

$$r_{ij} = [(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2]^{1/2}$$

$$\dot{s}_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2$$

All dots denote differentiation with respect to coordinate time.

This acceleration is the relativistic perturbation of a body, and should be valid for both the acceleration of the LAGEOS satellite in the solar system barycenter system, and the acceleration of the earth in the same system. The relativistically computed acceleration of the difference between the satellite and earth coordinates is thus obtainable by differencing the satellite acceleration from (2) and the earth acceleration from (2), or,

$$\delta\ddot{r})_{sat\text{-}earth} = \delta\ddot{r})_{satellite} - \delta\ddot{r})_{earth} \qquad (3)$$

We call this difference the acceleration of the satellite relative to the earth. The satellite-earth coordinate differences may, using these perturbations, be integrated as an alternative to separate integration of satellite coordinates and earth coordinates. A considerable increase in precision is thereby achieved. In practice for the LAGEOS satellite, only the sun and earth constitute significant perturbations in Eqn. (2) or (3).

Associated with the n-body equations of motion is a metric tensor and line element. Again in the notation of Moyer [1971, Eqns. 22-30], the n-body line element can be written

$$
\begin{aligned}
ds^2 = &-\left(1 + \frac{2}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\right)\left[(dx_i)^2 + (dy_i)^2 + (dz_i)^2\right] \\
&+ \frac{8}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\,\dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_i\,(dt)^2 \\
&+ \Bigg\{1 - \frac{2}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}} + \frac{2}{c^4}\left[\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\right]^2 - \frac{3}{c^4}\sum_{j\neq i}\frac{\mu_j\,\dot{s}_j^2}{r_{ij}} \\
&+ \frac{2}{c^4}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\sum_{k\neq j}\frac{\mu_k}{r_{jk}} \\
&- \frac{1}{c^4}\sum_{j\neq i}\mu_j\left[\frac{(\mathbf{r}_j-\mathbf{r}_i)\cdot\ddot{\mathbf{r}}_j}{r_{ij}} + \frac{\dot{s}_j^2}{r_{ij}} - \frac{[(\mathbf{r}_j-\mathbf{r}_i)\cdot\dot{\mathbf{r}}_j]^2}{r_{ij}^3}\right]\Bigg\}c^2(dt)^2
\end{aligned}
\tag{4}
$$

This line element will be needed in the derivation of the relativistic range calculations. It could also be used in developing the equations of motion of the i'th particle by finding the trajectory which extremizes $\int ds$. The result is Eqn. (2) (plus the Newtonian effects).

SECTION 3.0
RANGING MEASUREMENT CALCULATIONS

The basic measurement made by a satellite ranging station is the time that elapses on the station clock between the transmission and reception of a signal. For the laser tracking case, the signal is simply reflected, so the delay at the spacecraft* is relatively easily calculated, at least for a spherical satellite such as LAGEOS. The problem is then to relate the (corrected) time interval measurement to the coordinates of the tracking station (at transmission and reception) and the satellite.

This problem has been addressed a number of times for the most significant perturbing body, with the result [e.g., Moyer, 1971, p. 17]

$$t_j - t_i = \frac{r_{ij}}{c} + \frac{2\mu_S}{c^3} \ln\left[\frac{r_i + r_j + r_{ij}}{r_i + r_j - r_{ij}}\right] \qquad (5)$$

with the signal traveling from point i at coordinate time t_i to point j at coordinate time t_j. r_i and r_j are the magnitudes of the position vectors from points i and j to the sun and μ_S is the gravitational constant of the sun.

*Delay here means the correction necessary to the measurement to make it effectively a measurement to the spacecraft center of mass.

This result can also be obtained using the line element from Eqn. (4). If we retain only first order perturbation terms (for an earth orbiting satellite), drop the subscript i, and consider that the path described is a null geodesic, we have

$$0 = -\left(1 + \frac{2}{c^2}\sum_j \frac{\mu_j}{r_j}\right)\left[(dx)^2 + (dy)^2 + (dz)^2\right]$$

$$+ \left(1 - \frac{2}{c^2}\sum_j \frac{\mu_j}{r_j}\right) c^2\, dt^2 \,, \qquad (6)$$

where

$$r_j = \sqrt{(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2}$$

and j ranges over all perturbing bodies. If all body motion is considered to be slow, or to be negligible compared to the total transit time, then we can set

$$x = X_1 + \alpha\,(X_2 - X_1)$$

$$y = Y_1 + \alpha\,(Y_2 - Y_1) \qquad (7)$$

$$z = Z_2 + \alpha\,(Z_2 - Z_1)$$

where

$$(X_1,\ Y_1,\ Z_1),\quad (X_2,\ Y_2,\ Z_2)$$

are the coordinates of the transmitter and receiver, respectively. The parameter α has the range 0-1. Substituting into Eqn. (6), transposing the first term to the left, and taking the square root of both sides, we obtain, approximately

$$\left(1 + \frac{1}{c^2} \sum_j \frac{\mu_j}{r_j}\right) r_{12}\, d\alpha = \left(1 - \frac{1}{c^2} \sum_j \frac{\mu_j}{r_j}\right) c\, dt$$

or

$$c\, dt = \left(1 + \frac{2}{c^2} \sum_j \frac{\mu_j}{r_j}\right) r_{12}\, d\alpha \tag{8}$$

Integrating both sides,

$$c\, \Delta t = r_{12} + \frac{2\, r_{12}}{c^2} \sum_j \mu_j \int_0^1 \frac{d\alpha}{r_j} \tag{9}$$

To perform the last integration, we write

$$r_j = \sqrt{(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2}$$

$$= \sqrt{[X_1+\alpha(X_2-X_1)-x_j]^2+[Y_1+\alpha(Y_2-Y_1)-y_j]^2+[Z_1+\alpha(Z_2-Z_1)-z_j]^2}$$

$$= \sqrt{r_{j1}^2 + 2\ \alpha\ \mathbf{R}_{12} \cdot \mathbf{R}_{j1} + \alpha^2\ r_{12}^2} \qquad (10)$$

Substituting into (9) and performing the integration,

$$c\ \Delta t = r_{12} + \frac{2}{c^2} \sum_j \mu_j\ \ell n \left[\frac{r_2 + r_{12} + \mathbf{R}_{12} \cdot \mathbf{R}_{j1}/r_{12}}{r_1 + \mathbf{R}_{12} \cdot \mathbf{R}_{j1}/r_{12}} \right]$$

The argument of the logarithm can be shown to be equivalent to the argument of the logarithm in Eqn. (5), so that $c\ \Delta t$ can be written

$$c\ \Delta t = r_{12} + \frac{2}{c^2} \sum_j \mu_j\ \ell n \left[\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right] \qquad (11)$$

The total relativistic perturbation is thus equal to the sum of perturbations of the individual effects of the gravitational bodies. The largest effect for an earth orbiting satellite such as LAGEOS is the sun, but the earth is also significant at the centimeter level.

SECTION 4.0
STATION COORDINATE CORRECTIONS

The contraction of moving rods is well known from special relativity theory. In general relativity, a similar phenomenon occurs except that the term "rigid" must be defined. The subject of solar system barycenter coordinates for tracking stations fixed on a rigid earth* has been considered by Misner [1982], using the concept that the proper length $\int ds$ from the earth center world line to the tracking station world line remains constant, with the distance measured in a spacetime direction orthogonal to the earth's world line. This concept provides a scalar relationship between the solar system barycenter coordinates (which must be used in the data reduction program) and the station coordinates moving with the earth. On intuitive grounds, along with this scalar condition, Misner deduced the following relation between the two coordinate systems:

$$\mathbf{x}_T = \mathbf{X}_T - [U_E + (U_E \div 1/2\ V_E^{2})_{avg}]\ \mathbf{X}_T/c^2$$

$$- \frac{1}{2}\ (\mathbf{V}_E \cdot \mathbf{X}_T)\ \mathbf{V}_E/c^2 \qquad (12)$$

*Dynamic motion of stations is assumed to be otherwise accounted for.

where

X_T are the station coordinates in a coordinate system moving with the earth but non-rotating.

x_T are the station coordinates in the solar system frame.

V_E is the earth velocity in the solar system frame.

U_E is the gravitational potential due to the sun at the earth.

The net effect is a nearly constant scaling of the station coordinates by 2 1/2 parts in 10^8 and a daily variation which can be as large as $\pm$ 3 cm.

It should be emphasized that, while Eqn. (12) may be intuitively appealing and may be used with a reasonable degree of confidence, it has not yet been rigorously derived or verified.

SECTION 5.0
COORDINATE-ATOMIC TIME TRANSFORMATIONS

By definition, an interval of "proper" time measured by an atomic clock is proportional to the interval ds along its world line,

$$d\tau = \frac{ds}{c}, \tag{13}$$

The clock time interval is thus not proportional to the interval of coordinate time which is used for trajectory integration. However, the scale of the clock is, by convention, deliberately chosen so that atomic time and coordinate time are, on the average, colinear. The n-body line element given by Eqn. (4) can be used for ds in Eqn. (13), and the deviations between t and τ can then be calculated on the basis of the motion of the clock and the ephemerides of the gravitational bodies.

The coordinate time - Atomic time differences $(t-\tau)$ have been analyzed by various investigators, in particular by Moyer in two recent papers [Moyer: 1981a, 1981b]. Moyer's final result, in a form suitable for implementation in a computer program such as Geodyn in which the planetary ephemerides are conveniently available, is:

$$t - \tau = \Delta T_A + \frac{2}{c^2}(\dot{r}_B^S \cdot r_B^S) + \frac{1}{c^2}(\dot{r}_B^C \cdot r_E^B) + \frac{1}{c^2}(\dot{r}_E^C \cdot r_A^E) +$$

$$+ \frac{\mu_J}{c^2(\mu_S + \mu_J)}(\dot{r}_J^S \cdot r_J^S) + \frac{\mu_{SA}}{c^2(\mu_S + \mu_{SA})}(\dot{r}_{SA}^S \cdot r_{SA}^S) + \frac{1}{c^2}(\dot{r}_S^C \cdot r_B^S). \tag{14}$$

The subscripts and superscripts in this equation refer to:

A	=	location of atomic clock on Earth which reads International Atomic Time τ
E	=	Earth
B	=	Earth-Moon barycenter
M	=	Moon
S	=	Sun
C	=	Solar system barycenter
J	=	Jupiter
SA	=	Saturn

The quantity ΔT_A is a constant offset (32.184 seconds), the μ's are the masses of the subscript body, and r_B^S and $\dot{r}_B^S$ (e.g.) denote the position and velocity of the earth-moon barycenter (subscript) with respect to the sun (superscript).

The largest amplitude non-constant term on the right hand side of (14) is the term $2(\dot{r}_B^S \cdot r_B^S)/c^2$, with an amplitude of 1.658 msec and a 1 year period. Terms due to Jupiter and Saturn have maximum amplitudes on the order of 20 μsec and periods slightly greater than 1 year. The only short period term is $(\dot{r}_E^C \cdot r_A^E)/c^2$, which has a maximum amplitude of ~ 2 μsec and a period of 1 day.

It should be noted that the expression for t−τ above is approximate, and with the retained terms chosen on the basis of their influence on the NASA Deep Space Network observables (range and range rate). However, this criterion resulted in the retention of terms with amplitudes greater than 3.7 μsec, 0.11 μsec, and 1.3 μsec for periods of a day, a month, and a year, respectively [Moyer, 1981a]. Eqn. (12) should thus be more than adequate for the processing of earth-orbiting satellite data.

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SECTION 6.0
MEASUREMENT COMPUTATION IN BARYCENTER COORDINATES

As has been indicated, the utilization of the EIH solution for the satellite equations of motion has implied the adoption of a solar system barycenter coordinate system. In terms of the light time solution given in Section 3.0, this means that the difference between pulse transmission and return time at a tracking station is expressed as:

$$t_r - t_f = (r_{fb} + r_{br})/c$$

$$+ \frac{2GM_S}{c^3} \ln \left[\left(\frac{r_f^S + r_b^S + r_{fb}}{r_f^S + r_b^S - r_{fb}} \right) \left(\frac{r_b^S + r_r^S + r_{br}}{r_b^S + r_r^S - r_{br}} \right) \right] \tag{15}$$

where

t_f is the laser firing time

t_b is the laser receive ("bounce") time at the satellite

t_r is the receive time at the laser site

r_t^S is the range from the firing point to the sun at time t_f

r_b^S is the range from the satellite to the sun at time t_b

r_r^S is the range from the receiving site to the sun at time t_r

r_{fb} is the range from the transmitter point at time t_f to the satellite at time t_b

r_{br} is the range from the satellite at the time t_b to the receiving site at time t_r

For simplicity, relativistic effects of bodies other than the sun have been neglected in Eqn. (15). To relate r_{fb} + r_{br} to the range calculation normally performed in computer programs operating in earth-centered coordinates, we write out these ranges explicitly and expand all coordinates about their values at the satellite time t_b. First, we denote

X_L, Y_L, Z_L as the satellite coordinates

X_s, Y_s, Z_s as the tracking station coordinates

X_E, Y_E, Z_E as the coordinates of the center of the earth

All these coordinates are in the solar system barycenter system. The range sum r_{fb} + r_{br} is then defined by

$$r_{fb} + r_{br}$$

$$= \sqrt{[X_L(t_b)-X_s(t_f)]^2 + [Y_L(t_b)-Y_s(t_f)]^2 + [Z_L(t_b)-Z_s(t_f)]^2}$$

$$+ \sqrt{[X_L(t_b)-X_s(t_r)]^2 + [Y_L(t_b)-Y_s(t_r)]^2 + [Z_L(t_b)-Z_s(t_r)]^2} \tag{16}$$

Next consider the coordinate difference

$$X_L(t_b) - X_s(t_f) = X_L(t_b) - X_E(t_b) - [X_s(t_f)-X_E(t_b)]$$

$$= X_L(t_b)-X_E(t_b) - [X_s(t_b)-X_E(t_b)]$$

$$- \dot{X}_s(t_b)\,(t_f - t_b) - \ldots$$

$$\equiv X_{EL}(t_b) - X_{Es}(t_b) - \dot{X}_s(t_b)\,(t_f - t_b) - \ldots$$

$$\equiv X_{sL}(t_b) - \dot{X}_s(t_b)\,(t_f - t_b) - \ldots \tag{17}$$

Making use of (17) in (16), the latter becomes

$$r_{fb} + r_{br} = \sqrt{R - 2\,\mathbf{R} \cdot \dot{\mathbf{X}}_s (t_f - t_b) + (\dot{\mathbf{X}}_s)^2\,(t_f - t_b)^2}$$

$$+ \sqrt{R - 2\,\mathbf{R} \cdot \dot{\mathbf{X}}_s (t_r - t_b) + (\dot{\mathbf{X}}_s)^2\,(t_r - t_b)^2} \tag{18}$$

where

$$\mathbf{R} = [X_{sL}(t_b),\ Y_{sL}(t_b),\ Z_{sL}(t_b)]$$

$$\mathbf{X}_s = \mathbf{X}_s(t_b)$$

Expanding the square roots in (18), we obtain

$$r_{fb} + r_{br} \approx 2\,R - \frac{\mathbf{R} \cdot \dot{\mathbf{X}}_s}{R}\,[t_f - t_b + (t_r - t_b)]$$

$$+ \frac{1}{2}\,\frac{(\dot{\mathbf{X}}_s)^2}{R}\,[(t_f - t_b)^2 + (t_r - t_b)^2] \tag{19}$$

To first order we can set

$$t_f - t_b + t_r - t_b = - 2 \frac{\dot{X}_s \cdot R}{c^2}$$

and

$$(t_f - t_b)^2 + (t_r - t_b)^2 = 2 (R/c)^2$$

Eqn. (19) then reduces to

$$r_{fb} + r_{br} = 2 R + 2 R \left(\frac{R \cdot \dot{X}_s}{R c} \right)^2 + R \left(\frac{\dot{X}_s}{c} \right)^2 \tag{20}$$

Substituting this result into Eqn. (15) and making minor approximations in the logarithmic term, we obtain

$$\frac{c(t_r - t_f)}{2} = R + R \left[\left(\frac{R \cdot \dot{X}_s}{R c} \right)^2 + \left(\frac{\dot{X}_s}{c} \right)^2 \right]$$

$$+ \frac{2 \, GM_S}{c^2} \ln \left[\frac{r_s^S + r_L^S + R}{r_s^S + r_L^S - R} \right] \tag{21}$$

where r_s^S and r_L^S are the ranges from station and satellite, respectively, to the sun, all evaluated at the satellite time t_b.

Except that it is based on coordinate rather than atomic times, the left hand side of (21) is the normal measured range and the first term on the right hand side is the normal calculated range. The second term on the right hand side is a correction required because of the use of earth centered coordinates and the computation of a range based on an average set of station coordinates. The last term on the right hand side is a true relativistic correction.

It is important to recognize that the predominant part of $\dot{X}_s$ in the second term arises from the earth's barycentric velocity of ~ 30 km/sec, and is only slightly modified by the earth's rotational velocity of ~ 400 m/sec.

SECTION 7.0
SIMULATION RESULTS

To assess the magnitude of the relativistic perturbations in the reduction of LAGEOS laser tracking data, simulations were performed using a modified version of the ORAN error analysis program. With the exception that the range measurement corrections from Sections 3 and 6 were lumped together, all the perturbations were carried separately. The total effects on the estimated parameters are then the algebraic sum of the individual effects.

The simulations considered the following data set:

Laser Stations

- Goddard Space Flight Center (STALAS)
- Arequipa, Peru (ARELAS)
- Yaragadee, Australia (YARLAS)
- Owens Valley, California (OWENSV)
- Wetzell, West Germany (WETZEL)
- Haleakala, Maui, Hawaii (HOLLAS)

Station Visibility

- 50% of passes tracked

Elevation Cutoff - 20°

Arc Lengths: 2 weeks

Number of Arcs: 6, spaced two months apart in 1980

Two types of simulations were performed. First, noting that data reductions with the relativistic perturbations will require a different scaling from GM_E, all 6 arcs were processed with the common estimation of coordinates for all stations (with one station longitude held fixed), the common estimation of GM_E, plus the estimation of the 6 orbital elements for each arc. The results of this simulation are shown in Table 1. For the station positions, the dominant effects are those of the station coordinate modifications and measurement modeling on the station heights. These effects, however, almost cancel, leaving a net height effect on the order of a centimeter. For the estimated GM_E, the force model and measurement modeling produce the only significant effects. There is about 75% cancellation here, leaving a net change in GM_E of $-.00824$ km^3/sec^2. Since the sign convention in Table 1 is based on applying relativistic corrections versus not applying them, reduction of the simulated data set with the relativistic models will result in the estimation of a smaller value of GM_E than would be estimated using pure Newtonian models.

We next consider that the 6 arcs are used independently to estimate the same set of station coordinates, with a fixed value of GM_E. For comparison with a Newtonian solution, we assume that the relativistic solution uses a value of GM_E which is $.00824$ km^3/sec^2 smaller than that used in the Newtonian solution. Table 2 shows the results for one station for one of the 6 arcs. There are effects from all perturbations, except for timing,

Station		Force Model	Station Coordinates	A1-ET Timing	Measurement Modeling	Total
YARLAS	λ	1.14cm	-0.34cm	0.01cm	-0.46cm	0.35cm
	ϕ	-0.04 "	0.15 "	0.01 "	0.45 "	0.58 "
	H	0.77 "	16.84 "	0.03"	-18.55 "	-0.91 "
ARELAS	λ	0.31cm	-0.22cm	0.00cm	-0.13cm	-0.04cm
	ϕ	0.72 "	0.01 "	0.01 "	0.40 "	1.14 "
	H	0.29 "	16.82 "	-0.02 "	-18.66 "	-1.56 "
STALAS	λ	0.00cm	0.00cm	0.00cm	0.00cm	0.00cm
	ϕ	0.15 "	-0.62 "	-0.00 "	-0.52 "	0.32 "
	H	0.54 "	16.52"	0.01 "	-18.61 "	-1.54 "
OWENSV	λ	0.04cm	-0.14cm	0.01cm	-0.02cm	-0.11cm
	ϕ	1.07 "	-0.63 "	-0.00 "	-0.42 "	0.03 "
	H	1.07 "	16.50 "	0.01 "	-18.72 "	-1.14 "
WETZEL	λ	0.10cm	-0.12cm	-0.00cm	-0.11cm	-0.12cm
	ϕ	0.75 "	-0.56 "	0.01 "	-0.10 "	0.10 "
	H	1.27 "	16.32 "	-0.00 "	-18.88 "	-1.30 "
HOLLAS	λ	0.36cm	-0.10cm	-0.00cm	0.02cm	0.28cm
	ϕ	1.24 "	-0.56 "	0.01 "	-0.37 "	0.32 "
	H	0.91 "	16.68 "	-0.01 "	-18.64 "	-1.06 "
GM_E		0.02668	-0.00010	-0.00001	-0.03482	-0.00824 km^3/sec^2

TABLE 1. RELATIVISTIC EFFECTS ON STATION POSITIONS AND GM_e ESTIMATED USING 6 TWO WEEK ARCS OF LAGEOS, SPACED TWO MONTHS APART IN 1980

	FORCE MODEL	STATION COORDINATES	TIMING	MEASUREMENT EFFECTS	δGM	TOTAL
$\Delta\lambda$	3.66	- .65	.02	- 2.76	.58	.87 cm
$\Delta\phi$	- 1.54	- .62	-.03	2.93	-1.03	-.29 cm
ΔH	-35.29	16.58	-.35	28.69	-11.26	-1.63 cm

TABLE 2. RELATIVISTIC EFFECTS ON YARLAS COORDINATES
FOR ARC 5

on all coordinates in excess of half a centimeter, with a high degree of cancellation. The cancellation is most apparent for height, for which there are four different contributions with magnitudes in excess of 10 cm. The net effect on height is still only 1.6 cm.

Table 3 summarizes the individual arc simulations for station coordinate estimation, including the effect of using different values of GM_E for the relativistic and Newtonian data reductions. The last column in this table shows the arithmetic average of the effects on $\Delta\lambda$, $\Delta\theta$, and ΔH for each station. This column can be compared with the last column of Table 1, with which there should be good overall agreement. (The agreement should not be exact due to the slightly different weighting of each arc in the common estimation of station coordinates.) The rms agreement is .09 cm and the maximum difference is 0.2 cm, so the average station position change is quite consistent with the common parameter solution.

Examination of the month to month variations in Table 3, however, shows that they can be at the several centimeter level for all three coordinates. The largest month to month change is in the YARLAS longitude, which shows a 6.6 cm variation between January 1980 and March 1980. Although not enough arcs have been simulated to obtain a clear picture of the variation during the year, it is expected that the estimated station coordinates would show smooth month to month (or arc to arc) variations. Due to the motion of the LAGEOS node, the effects would not be expected to have the same pattern the following year.

Station		Arc 1 (Jan 1980)	Arc 2 (March 1980)	Arc 3 (May 1980)	Arc 4 (July 1980)	Arc 5 (Sept 1980)	Arc 6 (Nov 1980)	Avg.
YARLAS	$\Delta\lambda$	4.06cm	-2.58cm	0.09cm	0.67cm	0.87cm	-0.99cm	0.35
	$\Delta\phi$	0.36 "	-0.46 "	0.46 "	2.88 "	-0.28 "	-0.22 "	0.46
	ΔH	-1.52 "	-0.81 "	0.39 "	0.88 "	-1.63 "	-2.83 "	-0.92
ARELAS	$\Delta\lambda$	2.29cm	-2.41cm	0.03cm	0.16cm	2.13cm	-1.89cm	0.05
	$\Delta\phi$	1.30 "	0.30 "	2.33 "	2.38 "	-0.40 "	-0.25 "	0.94
	ΔH	-3.87 "	-1.81 "	-0.01 "	0.00 "	-0.89 "	-2.61 "	-1.53
STALAS	$\Delta\lambda$	0.00cm	0.00cm	0.00cm	0.00cm	0.00cm	0.00cm	
	$\Delta\phi$	1.59 "	1.02 "	-1.24 "	-1.97 "	1.40 "	1.19 "	0.33
	ΔH	-2.21 "	-0.46 "	-1.18 "	-0.54	-3.10 "	-2.15 "	-1.61
OWENSV	$\Delta\lambda$	1.49cm	-2.09cm	-0.87cm	0.62cm	2.22cm	-1.69cm	-0.05
	$\Delta\phi$	1.90 "	1.15 "	-2.07 "	-1.72 "	-0.50 "	0.23 "	-0.17
	ΔH	-0.08 "	-1.42 "	-1.08 "	-0.38 "	-2.52 "	-1.60 "	-1.18
WETZEL	$\Delta\lambda$	0.88cm	-1.76cm	0.46cm	1.00cm	0.22cm	-1.26cm	-0.08
	$\Delta\phi$	-0.02 "	1.26 "	-1.26 "	-1.10 "	0.15 "	0.64 "	-0.06
	ΔH	-0.15 "	-1.33 "	-1.04 "	-0.88 "	-2.54 "	-1.96 "	-1.32
HOLLAS	$\Delta\lambda$	1.67cm	-2.44cm	0.06cm	0.98cm	2.71cm	-1.37cm	0.27
	$\Delta\phi$	3.16 "	1.39 "	-0.69 "	-0.49 "	-1.90 "	0.00 "	0.25
	ΔH	-2.07 "	-1.46 "	0.51 "	0.26 "	-1.99 "	-2.10 "	-1.14

TABLE 3. NET RELATIVISTIC EFFECTS ON STATION POSITION ADJUSTMENTS BASED ON TWO WEEK LAGEOS ARCS (50% OF PASSES TRACKED), INCLUDING CHANGE IN GM_e BY $-.00824 km^3/sec^2$

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One other aspect of the simulations must also be considered, the extent to which the relativistic effects would be expected to produce systematic effects in measurement residuals. Figures 1 and 2 show the effects of the two most significant perturbations on measurement residuals for two particular passes selected from the 6 arcs with individual arc station adjustments. Figure 1 shows residual effects for an ARELAS pass in November 1980, with the force model and measurement effects tending to cancel throughout most of the pass. There is still a net effect of over 7 cm during part of the pass. In Figure 2, which shows residual effects for a WETZEL pass in September 1980, the situation is much worse. During most of the pass, the measurement effects and force model effects have the same sign, and their sum has values as large as -20 cm.

The interpretation of the total relativistic curves in Figures 1 and 2 is as follows. If the data set for a two week period is first processed using the Newtonian theory, one set of residuals will be obtained. Applying the set of relativistic perturbations will then produce changes in the residuals as shown by the curves with solid dots in Figures 1 and 2. If there were no other error sources, and the relativistic perturbations applied are correct (to within centimeter or so effects), then the residuals in the relativistic data reductions should be effectively zero. Thus, there must have been modeling errors (due to not including relativistic effects) in the Newtonian data reductions which produced residuals up to 20 cm or more. Such amplitudes far exceed the magnitudes of the station changes in Table 3. However, Table 3 is based on an average

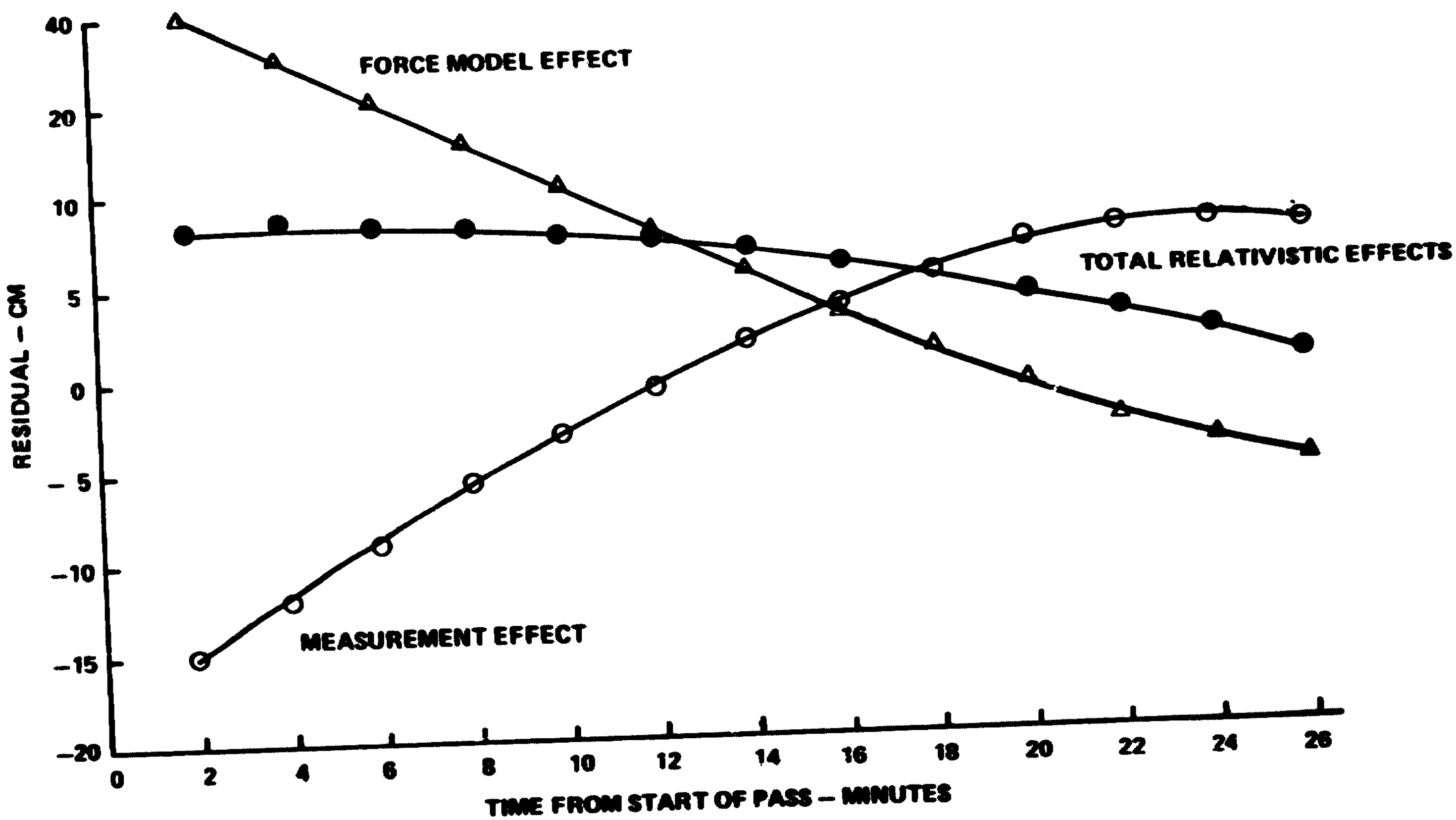

FIGURE 1. RELATIVISTIC EFFECTS ON ARELAS RESIDUALS ON NOVEMBER 12, 1980 (3^h 18^m) – 64° EL

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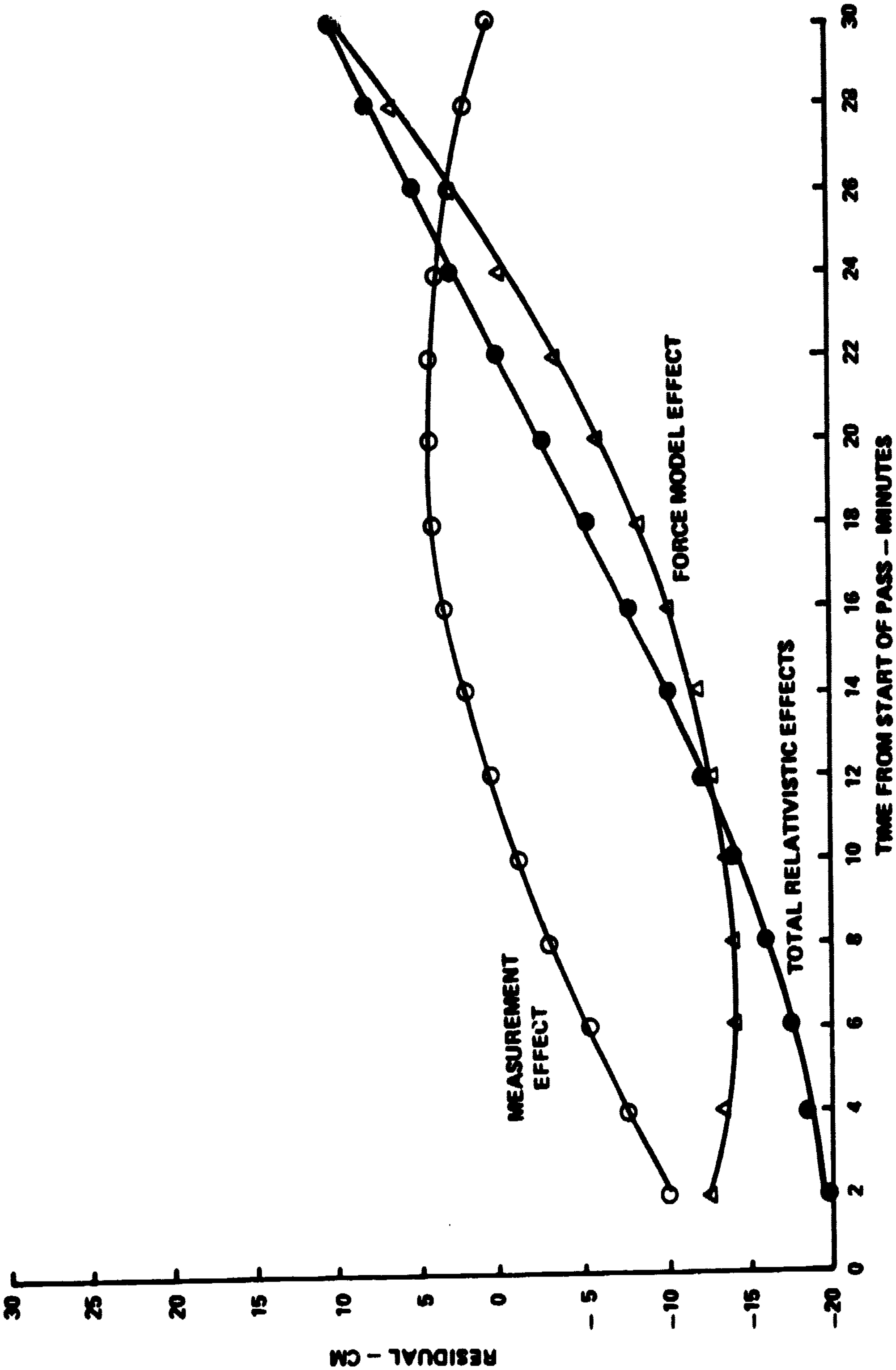

FIGURE 2. RELATIVISTIC EFFECTS ON WETZEL RESIDUALS ON SEPTEMBER 13, 1980 ($6^h 36^m$) – 51° EL

effect over a large number of passes over a 2 week period and 20 cm is a maximum residual effect on one pass, so the two results are not necessarily inconsistent. Nevertheless, a 20 cm residual is a sufficiently large effect that rms differences between relativistic data reductions and non-relativistic data reductions should show significant differences.

It should be emphasizd that the simulations discussed above have considered only two week arc lengths and have been based on simulated data and not data actually taken during 1980. Although it is believed that the results are qualitatively valid for other arc lengths and for the use of actual data, this has not been verified.

SECTION 8.0
SUMMARY AND CONCLUSIONS

Algorithms for the processing of orbital tracking data based on the general theory of relativity have been presented, assuming the existence of a computer program (such as Geodyn) implementing Newtonian theory. Laser ranging data, such as that currently being taken tracking the LAGEOS satellite, was the basic measurement type considered.

The differences between the relativistic theory and the Newtonian theory can be briefly summarized as follows:

1. The coordinate system used should, at least implicitly, be centered at (and moving with) the solar system barycenter, since this is the coordinate system used in deriving the relativistic equations of motion of the satellite and planetary bodies.

2. The differences between the atomic time, kept at the tracking sites, and coordinate time, used in the equations of motion, should be accounted for.

3. The relativistic equations of motion must be implemented.

4. The light time solution appropriate to the n-body metric must be used in relating the measured

round trip times to the coordinates of the satellite and tracking station.

5. Station coordinates used in the orbital solution must take account of the velocity of the stations relative to the integration coordinate system.

Of these areas, accounting for atomic time-coordinate time differences is the closest to being negligible for LAGEOS data reductions, but it should be included for completeness Item 5, the station coordinate transformation, is probably in greatest need of theoretical analysis, since the transformation proposed has not been rigorously derived.

Simulations show that use of the relativistic theory produces station position estimations which can vary at least several centimeters during the course of a year from estimates based on Newtonian theory. Averaging over a year, the differences are considerably reduced. The simulations also show differences in data reduction residuals (between the relativistic and Newtonian reductions) up to 20 cm in two week arcs in which all station positions are estimated.

It should be noted that orbital data reduction programs developed for processing tracking data for earth orbiting satellites have traditionally been developed in earth-centered coordinate systems, and no change in this procedure is recommended. However, the coordinate system used in the theoretical derivation of the relativistic equations of motion and light time solution is a solar system barycenter coordinate system. The integration process then produces the coordinates of the satellite in the barycenter coordinate system, minus the coordinates of the earth in the barycenter coordinate system. This procedure allows increased precision in satellite position and velocity coordinates relative to the earth. But satellite velocities of 6-7 km/sec relative to the earth should not be confused with the velocity of the satellite - and earth - in the barycenter coordinate system, which is of the order of 30 km/sec. For such a velocity, $(V/c)^2 \approx 10^{-8}$, and this scale factor applied to characteristic distances for LAGEOS ($\sim 6 \times 10^6$ m) corresponds to 6 cm. Relativistic effects of at least several centimeters are thus to be expected on the basis of the velocities involved, as well as from gravitational effects.

One modification (Section 6) has been developed for range measurement calculations to correct for an approximation originally made with the orbital program operating in an earth-centered coordinate system moving with the earth. It is, of course, possible that other approximations are also no longer valid for the barycenter coordinate system.

The ultimate test of the validity of a theory and its implementation is for it to produce better data fits and more consistent estimates of physical parameters. This is the obvious next step in the application of the Einstein gravitation theory to the reduction of laser ranging data to LAGEOS.

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