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METHODS FOR ESTIMATION OF BULLEN TURBULENCE SPECTRUM PARAMETERS

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SUMMARY

In NASA Contractor Report 3463 [Ref. 1], maximum likelihood and constrained least-squares methods were developed for estimating the parameters of the von Karman transverse and longitudinal spectra and autocorrelation functions. However, during the application of these methods to recorded turbulence measurements [NASA Contractor Report 3464, Ref. 2] it was discovered that a small percentage of such records are better modeled by the Bullen spectra or autocorrelation functions. The present report extends the above mentioned maximum likelihood and constrained least-squares methods to Bullen transverse and longitudinal spectra and autocorrelation functions.

INTRODUCTION

In Ref. 1, a general methodology was developed for characterizing atmospheric turbulence velocity records, estimating the parameters of the characterizations from such records, and computing aircraft response statistics from the characterizations. The characterization methods were implemented in Ref. 2 using several turbulence velocity records measured during the NASA MAT (Measurement of Atmospheric Turbulence) program. The turbulence model used in Refs. 1 and 2 assumes that a turbulence velocity record w(t) can be expressed as the sum of "slow" and "fast" components $w_s(t)$ and $w_f(t)$ – i.e.

$$\begin{aligned} w(t) &= w_s(t) + w_f(t) \\ &= w_s(t) + \sigma_f(t)z(t), \end{aligned} \tag{1.1}$$

where

$$w_f(t) = \sigma_f(t)z(t) \tag{1.2}$$

with

$$\sigma_f(t) \geqslant 0 \tag{1.3}$$

and

$$E\{z(t)\} = 0, \quad E\{z^2(t)\} = 1, \tag{1.4}$$

where $E\{\cdots\}$ denotes the mathematical expectation or ensemble average value of the quantity within the braces. The three random processes $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$ are assumed to be mutually statistically independent. Furthermore, $\{z(t)\}$ is assumed to be an ergodic Gaussian process. The process $\{\sigma_f(t)\}$ is not assumed to be Gaussian. Realizations of $\{\sigma_f(t)\}$ are assumed to vary slowly in comparison with realizations of $\{z(t)\}$, and realizations of $\{w_s(t)\}$ are assumed to vary slowly in comparison with realizations of $\{w_f(t)\}$. Hence, $w_s(t)$ has the appearance of a slowly varying additive component to the process $w_f(t)$, and $w_f(t)$ has the appearance of ordinary turbulence with a temporally varying envelope that is governed by the behavior of $\sigma_f(t)$. A more complete description of the above model can be found in Ref. 1.

It has been shown in Ref. 3 that whenever the scale of fluctuations of $\sigma_f(t)$ is of the order of ten or more times the integral scale of $z(t)$, the velocity spectrum of $w_f(t)$ will, for practical purposes, coincide with the velocity spectrum of $z(t)$ except for a constant multiplicative factor. Thus, if $z(t)$ is regarded as ordinary turbulence with a von Karman transverse or longitudinal spectrum, then to within an excellent approximation $w_f(t)$ will possess this same spectral form. Hence, in Refs. 1 and 2, it has been assumed that the velocity spectral densities or autocorrelation functions of $w(t)$ are the superposition of the spectral densities or autocorrelation functions of $w_s(t)$ and $\overline{\sigma_f^2}\, z(t)$, where $\overline{\sigma_f^2}$ denotes a temporal average or expectation. This assumption can be justified from the assumed mutual statistical independence of $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$ and the slowly varying behavior of $\{\sigma_f(t)\}$ relative to $\{z(t)\}$.

Since the spectrum of $w_s(t)$, by assumption, contains predominantly low frequency or long wavenumber components, the asymptotic slope of the spectrum of $w(t)$ should be governed by the process $z(t)$ which has been assumed to be described by the von Karman spectral form. The von Karman transverse and longitudinal spectral forms both possess the asymptotic -5/3 slope when plotted on log-log coordinates as required by classical turbulence theory [4]. However, when the methods developed in Ref. 1 were applied to 7 real turbulence records in Ref. 2, it was found that the asymptotic slopes of the spectra of 2 out of the 7 records were distinctly different from -5/3. These are the spectra of the lateral turbulence records from Flight 32 Run 4 and Flight 30 Run 8 of the MAT program shown in Figs. 5 and 11 respectively in Ref. 2. Thus, an extension of the methods developed in Ref. 1 to accommodate asymptotic slopes different from -5/3 would seem to be useful.

Two methods were developed in Ref. 1 for estimating "spectrum" properties of velocity records. The first method is applicable to cases where the slow component $w_s(t)$ in Eq. (1.1) is negligible in comparison with $w_f(t)$. This first method utilizes the maximum likelihood method to estimate the two parameters of the von Karman transverse or longitudinal spectral forms. The second method is applicable to cases where $w_s(t)$ in Eq. (1.1) is not negligible. This second method utilizes a constrained least-squares integral fit of an autocorrelation function model to the autocorrelation function of the measured turbulence record. Both of these methods are extended in this report to models where the asymptotic slope of the spectrum of the component z(t) is not constrained to be the classical value of -5/3.

MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS IN BULLEN TURBULENCE VELOCITY SPECTRA

In Sec. 3 of Ref. 1, the maximum likelihood method was used to develop equations for estimating the integral scale and intensity of the von Karman transverse and longitudinal spectral forms. In this section, this method is extended to the Bullen spectral forms given on pp. 200-202 of Ref. 5. The Bullen spectral forms are determined by three parameters rather than two as in the von Karman case. The third parameter controls the asymptotic slope of the spectra when plotted on log-log coordinates. When this asymptotic slope is set equal to -5/3, the Bullen spectral forms reduce to the von Karman forms.

The Bullen transverse and longitudinal *two-sided* wavenumber spectra are [p. 200 of Ref. 5]

$$\Phi_{BT}(k) = \sigma^2 L \frac{1+8\pi^2\ell^2k^2(n+1)}{(1+4\pi^2\ell^2k^2)^{n+3/2}} , \tag{2.1}$$

and

$$\Phi_{BL}(k) = \sigma^2 L \frac{2}{(1+4\pi^2\ell^2k^2)^{n+1/2}} \tag{2.2}$$

where the above forms are one-half of the values given in Ref. 5 because $\Phi_{BT}(k)$ and $\Phi_{BL}(k)$ are two-sided spectra rather than one-sided as in Ref. 5, and where the parameter a in Ref. 5 has been replaced by ℓ to avoid later notational confusion. L is the scale of the turbulence, σ^2 is the mean-square value, and ℓ is related to n and L by

$$\ell = \frac{\Gamma(n)}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})} L , \tag{2.3}$$

where $\Gamma(n)$ is the gamma function. When n = 1/3, Eqs. (2.1) and (2.2) reduce to the von Karman forms given by Eqs. (3.18) and (3.19) of Ref. 1.

In the maximum likelihood procedure to be developed below, we shall derive likelihood equations for the three parameters $\sigma^2 L$, ℓ, and n in Eqs. (2.1) and (2.2). When solutions are arrived at for these three parameters, one can readily compute from n and ℓ

$$L = \frac{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})}{\Gamma(n)}\ \ell\ , \tag{2.4}$$

according to Eq. (2.3), and then $\sigma^2 = \sigma^2 L/L$, which yield the scale L and mean-square value σ^2.

Let S_j, $j = 1,2,\cdots,N$ denote the periodogram (unsmoothed spectrum) samples described on pp. 74-80 of Ref. 1. The mathematical expectation of a typical spectrum sample S_j can be expressed as

$$E\{S_j\} \equiv \overline{S_j} = \sigma^2 L F_j(\ell,n)\ , \qquad j=1,2,\cdots N \tag{2.5}$$

where *for the Bullen transverse spectrum* one has from Eq. (2.1)

$$F_j(\ell,n) = \frac{1+8\pi^2\ell^2k_j^2(n+1)}{(1+4\pi^2\ell^2k_j^2)^{n+3/2}}\ , \tag{2.6}$$

and *for the Bullen longitudinal spectrum* one has from Eq. (2.2)

$$F_j(\ell,n) = \frac{2}{(1+4\pi^2\ell^2k_j^2)^{n+1/2}}\ , \tag{2.7}$$

where k_j is the wavenumber at the location of the jth spectrum sample S_j.

Turning now to derivation of the likelihood equations for estimating the spectrum parameters, when Eq. (2.5) is substituted into Eq. (3.15) of Ref. 1, one has for the loga- of the joint probability density function of the N spectrum samples S_j, $j=1,2,\cdots,N$

$$\ell n[p(S_1,S_2,\cdots,S_N)] =$$

$$= -\left\{N\ell n(\sigma^2 L)+\ell nF_1(\ell,n)+\ell nF_2(\ell,n)+\cdots+\ell nF_N(\ell,n) + \frac{1}{\sigma^2 L}\left[\frac{S_1}{F_1(\ell,n)} + \frac{S_2}{F_2(\ell,n)} + \cdots + \frac{S_N}{F_N(\ell,n)}\right]\right\} \quad (2.8)$$

where Eq. (3.12) of Ref. 1 also has been used. Equation (2.8) is to be maximized with respect to the three parameters $\sigma^2 L$, ℓ, and n. Thus, differentiating Eq. (2.8) with respect to $\sigma^2 L$, ℓ, and n and setting the resulting three expressions equal to zero yields after minor manipulations:

$$\sigma^2 L = \frac{1}{N}\sum_{j=1}^{N}\frac{S_j}{F_j(\ell,n)} \quad (2.9)$$

$$\sum_{j=1}^{N}\left[\frac{\partial}{\partial \ell}\ \ell nF_j(\ell,n)\right]\left[\frac{S_j}{F_j(\ell,n)} - \sigma^2 L\right] = 0 \quad (2.10)$$

and

$$\sum_{j=1}^{N}\left[\frac{\partial}{\partial n}\ \ell nF_j(\ell,n)\right]\left[\frac{S_j}{F_j(\ell,n)} - \sigma^2 L\right] = 0\ . \quad (2.11)$$

Equations (2.9) through (2.11) are three nonlinear simultaneous algebraic equations for the spectrum parameters $\sigma^2 L$, ℓ, and n in terms of the periodgram samples S_1, S_2, $\cdots$, S_N and the functions $F_j(\ell,n)$ given by Eqs. (2.6) and (2.7) for the Bullen transverse and longitudinal spectrum forms respectively. However, Eq. (2.9) can be substituted into Eqs. (2.10) and (2.11) so that only two nonlinear equations need to be solved simultaneously. Let us first define

$$G_{j\ell}(\ell,n) \triangleq \frac{\partial}{\partial \ell}\ \ell n\ F_j(\ell,n) \quad (2.12)$$

and

$$G_{jn}(\ell,n) \triangleq \frac{\partial}{\partial n} \ln F_j(\ell,n) \ , \tag{2.13}$$

which apply to either the transverse or longitudinal Bullen spectral forms as appropriate. Then substituting Eq. (2.9) into Eqs. (2.10) and (2.11) and using the definitions (2.12) and (2.13), we obtain instead of Eqs. (2.10) and (2.11),

$$\sum_{j=1}^{N} G_{j\ell}(\ell,n)\left[\frac{S_j}{F_j(\ell,n)} - \frac{1}{N}\sum_{i=1}^{N}\frac{S_i}{F_i(\ell,n)}\right] = 0 \tag{2.14}$$

and

$$\sum_{j=1}^{N} G_{jn}(\ell,n)\left[\frac{S_j}{F_j(\ell,n)} - \frac{1}{N}\sum_{i=1}^{N}\frac{S_i}{F_i(\ell,n)}\right] = 0 \ . \tag{2.15}$$

Equations (2.14) and (2.15) are to be solved simultaneously for the two spectrum parameters ℓ and n. These solutions ℓ and n are then to be substituted into Eq. (2.9) to yield the parameter $\sigma^2 L$. Equation (2.4) is then to be used to evaluate the scale L, and σ^2 then can be evaluated by $\sigma^2 = \sigma^2 L/L$. Thus, Eqs. (2.9), (2.14), and (2.15) are the solution to the problem of maximum likelihood estimation of the Bullen spectrum parameters $\sigma^2 L$, ℓ, and n which yield σ^2, L, and n.

The solution to Eqs. (2.14) and (2.15) can be readily carried out by numerical means along the lines illustrated in Appendix F of Ref. 1. To carry out these solutions, the functions $G_{j\ell}(\ell,n)$ and $G_{jn}(\ell,n)$ are required for the Bullen transverse and longitudinal spectra. By differentiating the natural logarithms of Eqs. (2.6) and (2.7), one can readily show *for the Bullen transverse spectrum* that

$$G_{j\ell}(\ell,n) = \frac{8\pi^2\ell^2k_j^2(n+\frac{1}{2})}{\ell}\left[\frac{1-8\pi^2\ell^2k_j^2(n+1)}{1+4\pi^2\ell^2k_j^2(2n+3)+32\pi^4\ell^4k_j^4(n+1)}\right] \tag{2.16}$$

and

$$G_{jn}(\ell,n) = \frac{8\pi^2\ell^2k_j^2}{1+8\pi^2\ell^2k_j^2(n+1)} - \ln(1+4\pi^2\ell^2k_j^2) \,, \tag{2.17}$$

and *for the Bullen longitudinal spectrum that*

$$G_{j\ell}(\ell,n) = -\frac{1}{\ell}\,\frac{8\pi^2\ell^2k_j^2(n+\frac{1}{2})}{1+4\pi^2\ell^2k_j^2} \tag{2.18}$$

and

$$G_{jn}(\ell,n) = -\ln(1+4\pi^2\ell^2k_j^2) \,. \tag{2.19}$$

CONSTRAINED LEAST-SQUARES ESTIMATION OF TURBULENCE AUTOCORRELATION FUNCTION PARAMETERS

First Method

In this section, the method developed in Sec. 4 of Ref. 1 is extended to the case where the spectrum and autocorrelation functions of component $w_f(t)$ in Eq. (1.1) are better modeled by the Bullen forms rather than the von Karman forms used in the cited reference. The parameter estimation method developed below is applicable to situations where the component $w_s(t)$ in Eq. (1.1) is not negligible.

The autocorrelation function model used here is

$$\underset{\sim}{\phi}(\xi) \triangleq \overline{\sigma_f^2}\, \phi_B(\xi;\ell,n) + \sum_{i=0}^{m} a_i \xi^i, \qquad 0 \leq \xi \leq \xi_H \tag{3.1}$$

which replaces Eq. (4.1) of Ref. 1. The quantity $\overline{\sigma_f^2}$ is the mean-square value of the "fast" component $w_f(t)$ in Eq. (1.1), $\phi_B(\xi;\ell,n)$ is either the Bullen transverse or longitudinal autocorrelation function, as appropriate, with $\phi_B(0;\ell,n) = 1$, and

$$\underset{\sim}{\phi}_{w_s}(\xi) = \sum_{i=0}^{m} a_i \xi^i, \qquad 0 \leq \xi \leq \xi_H \tag{3.2}$$

is a polynomial model of the autocorrelation function of the "slow" component $w_s(t)$ in Eq. (1.1) assumed valid within the range $0 \leq \xi \leq \xi_H$ as in Sec. 4 of Ref. 1.

The quantities $\overline{\sigma_f^2}$, ℓ, n, and a_i, $i=0,1,\cdots$, m are to be evaluated from the autocorrelation function $R(\xi)$ of a measured turbulence record $w(t)$ by minimizing the quantity

$$E \triangleq \int_0^{\xi_H} \left\{ R(\xi) - \overline{\sigma_f^2}\phi_B(\xi;\ell,n) - \sum_{i=0}^{m} a_i \xi^i \right\}^2 d\xi, \tag{3.3}$$

where the minimization procedure is constrained by the two additional relations:

$$\sum_{j=1}^{N} G_{jn}(\ell,n)\left[\frac{S_j}{F_j(\ell,n)} - \frac{1}{N}\sum_{i=1}^{N}\frac{S_i}{F_i(\ell,n)}\right] = 0 \qquad (3.4)$$

and

$$\overline{\sigma_f^2}L = \frac{1}{N}\sum_{j=1}^{N}\frac{S_j}{F_j(\ell,n)}, \qquad (3.5)$$

where periodogram samples S_j and S_i in the wavenumber range $k_j \geq k_{\ell'}$ and $k_i \geq k_{\ell'}$ only are to be included in the summations in Eqs. (3.4) and (3.5) as explained below.

Ordinarily, the wavenumber spectrum content of the "slow" component $w_s(t)$ in Eq. (1.1) is concentrated in the low wavenumber region. Thus, we assume there exists a wavenumber $k_{\ell'}$ such that for wavenumbers $k \geq k_{\ell'}$, the spectrum content of a turbulence record w(t) is dominated for all $k \geq k_{\ell'}$ by contributions from the "fast" component $w_f(t)$ only. It is advantageous to choose the smallest value of $k_{\ell'}$ that satisfies this condition. The two constraint equations (3.4) and (3.5) are to be evaluated using periodogram samples S_j and S_i taken only in the region $k \geq k_{\ell'}$ where the wavenumber content of the process $w_f(t)$ in Eq. (1.1) is everywhere dominant. By hypothesis, then, within this region $k \geq k_{\ell'}$, the wavenumber spectrum is well represented by the appropriate (transverse or longitudinal) Bullen spectral form. Equations (3.4) and (3.5) are two of the three likelihood equations available for evaluating the three Bullen spectrum parameters.

To examine why the third likelihood equation (2.14) is often not useful in the region $k \geq k_{\ell'}$, we examine the behavior of $G_{j\ell}(\ell,n)$ for large values of k_j for both the transverse and longitudinal cases. For large values of k_j, one can readily see from Eq. (2.16) that *for the case of the Bullen transverse spectrum* one has

$$G_{j\ell}(\ell,n) \sim \frac{8\pi^2\ell^2k_j^2(n+\frac{1}{2})}{\ell}\left[\frac{-8\pi^2\ell^2k_j^2(n+1)}{32\pi^4\ell^4k_j^4(n+1)}\right]$$

$$= -\frac{2(n+1)}{\ell}, \qquad k_j\to\infty, \tag{3.6}$$

and, similarly, from Eq. (2.18) *for the case of the Bullen longitudinal spectrum* one has approximately for large k_j

$$G_{j\ell}(\ell,n) \sim -\frac{1}{\ell}\,\frac{8\pi^2\ell^2k_j^2(n+\frac{1}{2})}{4\pi^2\ell^2k_j^2}$$

$$= -\frac{2(n+1)}{\ell}, \qquad k_j\to\infty. \tag{3.7}$$

Thus, for large values of k_j for both the transverse and longitudinal cases $G_{j\ell}(\ell,n)$ approaches the same *constant* value of $-2(n+1)/\ell$. Examination of Figs. 20 and 21 on pp. 90 and 91 of Ref. 1 shows that when $n = \frac{1}{3}$, this constant value is approximately reached for values of $k_j \geq 1/L$. Let us now suppose that k_ℓ, is approximately equal to $1/L$ or larger. In this case, then, in the region $k_j \geq k_\ell$, where the summation in Eq. (2.14) would be used, we see from Eqs. (3.6) and (3.7) that Eq. (2.14) would become, approximately,

$$-\frac{(2n+1)}{\ell}\sum_{j=1}^{N}\left[\frac{S_j}{F_j(\ell,n)} - \frac{1}{N}\sum_{i=1}^{N}\frac{S_i}{F_i(\ell,n)}\right] = 0,$$

or, dividing both sides of Eq. (3.8) by N,

$$- \frac{(2n+1)}{\ell} \left\{ \left[\frac{1}{N} \sum_{j=1}^{N} \frac{S_j}{F_j(\ell,n)} \right] - \frac{1}{N} \sum_{j=1}^{N} \left[\frac{1}{N} \sum_{i=1}^{N} \frac{S_i}{F_i(\ell,n)} \right] \right\} = 0 \;, \qquad (3.8)$$

which is satisfied for all values of ℓ and n. Hence, when the smallest useable value k_ℓ, of k_j is in the region where Eq. (3.6) or (3.7) is satisfied, the likelihood equation (2.14) is satisfied for all values of ℓ and n and therefore is useless for the purpose of determining these values. The functions $G_{jn}(\ell,n)$ given by Eqs. (2.17) and (2.19) do not approach constant values with respect to k_j as k_j gets arbitrarily large; hence, the likelihood equation given by Eq. (2.15) or (3.4) does not suffer from the above described limitation of Eq. (2.14).

To compare the present least-squares methodology with that in Sec. 4 of Ref. 1, we see that the Bullen autocorrelation functions $\phi_B(\xi;\ell,n)$ in Eq. (3.3) have one more parameter (n) than the von Karman autocorrelation functions $\phi_K(\xi;L)$ in Eq. (4.3) of Ref. 1. Furthermore, in the present least-squares procedure, there is one more equation of constraint, Eq. (3.4), than in the least-squares procedure described in Sec. 4 of Ref. 1. Hence, minimization of the quantity "E" given by Eq. (3.3) can be regarded as determining the single parameter σ_f^2 of the "fast" component $w_f(t)$ in Eq. (1.1). Similarly, in Sec. 4 of Ref. 1, minimization of "E" given by Eq. (4.5) of that reference also determined the single parameter σ_f^2 of the "fast" component $w_f(t)$. The reader is referred to Sec. 4 of Ref. 1 for a more thorough discussion of the least-squares procedure.

Derivation of Algebraic Equations for Autocorrelation Function Parameters

The equation of constraint (3.4) contains two unknown parameters ℓ and n. Thus, one can regard that equation as determining n as a function of ℓ — i.e., $n=n(\ell)$. Hence, assuming that this function $n=n(\ell)$ has been determined from Eq. (3.4), the second equation of constraint (3.5) can be expressed as

$$\sigma^2 = \frac{1}{L'[n(\ell)]\ell N} \sum_{j=1}^{N} \frac{S_j}{F_j[\ell, n(\ell)]}, \tag{3.9}$$

where the subscript f and the overbar have been left off $\overline{\sigma_f^2}$ in Eq. (3.9), and where the integral scale L has been expressed as

$$L = L'[n(\ell)]\ell , \tag{3.10}$$

where from Eq. (2.4), one has

$$L'[n(\ell)] = \frac{\sqrt{\pi}\ \Gamma[n(\ell)+\frac{1}{2}]}{\Gamma[n(\ell)]} \tag{3.11}$$

where in Eqs. (3.10) and (3.11) n has been expressed as a function of ℓ as determined by Eq. (3.4). Hence, Eqs. (3.4) and (3.9) together determine σ^2 as a function of ℓ — i.e., $\sigma^2 = \sigma^2(\ell)$. Assuming that σ^2 is a monotonic function of ℓ, Eqs. (3.4) and (3.9) also determine ℓ as a function of σ^2,

$$\ell = \ell(\sigma^2). \tag{3.12}$$

Thus, assuming that Eq. (3.4) has determined n as a function of ℓ, the function $\phi_B(\xi;\ell,n)$ has been reduced to dependence on only a single parameter ℓ which shall be denoted by

$$\phi_B^\dagger(\xi;\ell) \triangleq \phi_B[\xi;\ell,n(\ell)] . \tag{3.13}$$

Assuming further that ℓ has been determined as a function of $\sigma^2 = \overline{\sigma_f^2}$ through the additional relationship (3.5), Eq. (3.3) then can be expressed as

$$E = \int_0^{\xi_H} \left\{ R(\xi) - \sigma^2 \phi_B^\dagger[\xi;\ell(\sigma^2)] - \sum_{i=0}^{m} a_i \xi^i \right\}^2 d\xi, \tag{3.14}$$

where we have used the notation of Eq. (3.14) and the abbreviation $\sigma^2 = \overline{\sigma_f^2}$.

The problem, now, is to determine the values of σ^2 and a_i, $i=0,1,\cdots,m$ that minimize the value of E given by Eq. (3.14). This determination is carried out by finding value of σ^2 and a_i, $i=0,1,\cdots,m$ that satisfy

$$\frac{\partial E}{\partial a_j} = 0 \;, \quad j = 0,1,\cdots,m \qquad \frac{\partial E}{\partial \sigma^2} = 0 \;. \tag{3.15}$$

Differentiating Eq. (3.14) with respect to a_j, $j=0,1,\cdots,m$ and with respect to σ^2 and setting the resulting equations equal to zero yields the following set of m+2 nonlinear algebraic equations for a_0, a_1, $\cdots$, a_m and σ^2:

$$\sigma^2 \int_0^{\xi_H} \xi^j \phi_B^\dagger\left[\xi;\ell(\sigma^2)\right] d\xi + \sum_{i=0}^{m} a_i \int_0^{\xi_H} \xi^{i+j} d\xi = \int_0^{\xi_H} \xi^j R(\xi) d\xi,$$

$$j = 0,1,\cdots,m \tag{3.16}$$

and

$$\sigma^2 \int_0^{\xi_H} \left\{ \sigma^2 \frac{\partial \phi_B^\dagger[\xi;\ell(\sigma^2)]}{\partial \ell} \frac{d\ell}{d\sigma^2} + \phi_B^\dagger[\xi;\ell(\sigma)^2] \right\} \phi_B^\dagger[\xi;\ell(\sigma^2)] d\xi$$

$$+ \sum_{i=0}^{m} a_i \int_0^{\xi_H} \left\{ \sigma^2 \frac{\partial \phi_B^\dagger[\xi;\ell(\sigma^2)]}{\partial \ell} \frac{d\ell}{d\sigma^2} + \phi_B^\dagger[\xi;\ell(\sigma^2)] \right\} \xi^i d\xi$$

$$= \int_0^{\xi_H} \left\{ \sigma^2 \frac{\partial \phi_B^\dagger[\xi;\ell(\sigma^2)]}{\partial \ell} \frac{d\ell}{d\sigma^2} + \phi_B^\dagger[\xi;L(\sigma^2)] \right\} R(\xi) d\xi \;, \tag{3.17}$$

where the function $\ell(\sigma^2)$ is determined by Eqs. (3.4) and (3.9) as described below.

Unfortunately, the derivatives of $\phi_B^\dagger[\xi;\ell(\sigma^2)]$ with respect to ℓ turn out to be impractical to evaluate due, in part, to the dependence of the parameter n on ℓ that occurs in the expression for $L'[n(\ell)]$ given by Eq. (3.11). Hence, the *second method* described below is recommended instead.

Second Method

Let us examine the large k behavior of the Bullen transverse and longitudinal spectra, Eqs. (2.1) and (2.2). For large k, these spectra are asymptotically equal to

$$\Phi_{BT}(k) \sim \sigma^2 L \frac{8\pi^2\ell^2 k^2(n+1)}{(4\pi^2\ell^2 k^2)^{n+3/2}} \tag{3.18}$$

and

$$\Phi_{BL}(k) \sim \sigma^2 L \frac{2}{(4\pi^2\ell^2 k^2)^{n+1/2}} . \tag{3.19}$$

The logarithms of Eqs. (3.18) and (3.19) can be readily expressed as

$$\log\Phi_{BT}(k) \sim \log[2(n+1)\sigma^2 L] - (2n+1)\log(2\pi\ell) - (2n+1)\log k \tag{3.20}$$

and

$$\log\Phi_{BL}(k) \sim \log(2\sigma^2 L) - (2n+1)\log(2\pi\ell) - (2n+1)\log k. \tag{3.21}$$

Hence, when considered as a function of k, Eqs. (3.20) and (3.21) are both of the form

$$\log\Phi_B(k) \sim C - (2n+1)\log k, \tag{3.22}$$

where C is a constant. Thus, when plotted on log-log coordinates, both the Bullen transverse and longitudinal spectral forms are asymptotically linear for large k with a slope equal to $-(2n+1)$. In particular, for the von Karman transverse and longitudinal spectra, we have $n = 1/3$ which yields the well known asymptotic slope of $-5/3$. The form of Eq. (3.22) suggests the following well-conditioned procedure for determining the parameter n:

1. Tabulate and plot the logarithm of the empirically determined turbulence velocity spectrum as a function of log k.

2. Identify the region of approximate linearity of the data in the large k regime.

3. Form a least squares fit to the data using the functional form given by the right-hand side of Eq. (3.22) and solve for C and n.

4. Retain the value of n thus obtained.

Once n is evaluated as in steps 1 to 4 above, the Bullen spectral forms given by Eqs. (2.1) and (2.2) can be regarded as being dependent on only two additional parameters σ^2 and L [after substitution of Eq. (2.3) into Eqs. (2.1) and (2.2)]. Thus, after evaluation of n as above and substitution of Eq. (2.3) into Eqs. (2.1) and (2.2), the procedure for estimating σ_f^2 and L given on pp. 92 to 110 of Ref. 1 applies directly to the Bullen spectral forms $\Phi_{BT}(k)$ and $\Phi_{BL}(k)$ also. The only exception is that the von Karman transverse and longitudinal autocorrelation functions and their derivatives given by Eqs. (4.48) to (4.52) of Ref. 1 must be replaced by the corresponding Bullen forms which are [5]: *for the Bullen transverse spectra (vertical and lateral components):*

$$\bar{\Phi}_B(\bar{\xi}) = \frac{2}{\Gamma(n)}\left(\frac{\beta\bar{\xi}}{2}\right)^n \left[K_n(\beta\bar{\xi}) - \frac{\beta\bar{\xi}}{2} K_{n-1}(\beta\bar{\xi})\right] \tag{3.23}$$

$$\bar{\Phi}'_B(\bar{\xi}) = \frac{\beta}{\Gamma(n)}\left(\frac{\beta\bar{\xi}}{2}\right)^n \left[\beta\bar{\xi}K_n(\beta\bar{\xi}) - 2(n+1)K_{n-1}(\beta\bar{\xi})\right] , \tag{3.24}$$

for the Bullen longitudinal spectra:

$$\bar{\Phi}_B(\bar{\xi}) = \frac{2}{\Gamma(n)} \left(\frac{\beta\bar{\xi}}{2}\right)^n K_n(\beta\bar{\xi}) \qquad (3.25)$$

$$\bar{\Phi}'_B(\bar{\xi}) = - \frac{2\beta}{\Gamma(n)} \left(\frac{\beta\bar{\xi}}{2}\right)^n K_{n-1}(\beta\bar{\xi}), \qquad (3.26)$$

where, for both the transverse and longitudinal cases,

$$\beta = \frac{L}{\ell} = \frac{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})}{\Gamma(n)} \quad . \qquad (3.27)$$

Equations (3.23) through (3.27) reduce to Eqs. (4.48) through (4.52) of Ref. 1 when n is set equal to 1/3 which is the value of n for the von Karman case.

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