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Gravitational Models for Mission Planning

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SHUTTLE PROGRAM

Gravitational Models for Mission Planning

By: Alan C. Mueller, Computational Mechanics Service NAS 9/16546

JSC Task Monitor: Victor R. Bond, Software Development Branch

Approved: Eric N. McHenry

Eric N. McHenry, Chief
Software Development Branch

Approved: Ronald L. Berry

Ronald L. Berry, Chief
Mission Planning and Analysis Division

Mission Planning and Analysis Division

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1.0 INTRODUCTION

The gravitational geopotential is usually represented by the spherical harmonic expansion solution to Laplace's equation. The harmonic coefficients may be determined by reduction of satellite trajectory data, mean ocean surface altimeter measurements, or local accelerometer readings (1, 2). The zonal harmonics give rise to secular variations in the angular orbital elements but only to periodic changes in the momenta elements. The secular and sectorial harmonics cause no true secular variations, but the non-zero mean of the periodic variations in the mean motion induces a linear perturbation in the mean anomaly (3).

The normalized harmonic coefficients do not decrease in magnitude as the degree of the expansion increases. Also, each harmonic is premultiplied by a power of the ratio of the Earth radius and the position radius. Since for low Earth satellites this ratio is near unity, successive terms in the expansion do not tend to diminish. This necessarily requires the inclusion of a very large number of terms to accurately model the gravitational perturbations. Recursion algorithms give an efficient means for calculating the higher degree perturbations, but the computation times grow geometrically with the degree (4, 5). The computational environment may preclude the use of such expensive models, and the common trade-off is to simply truncate to only a few important terms.

The coefficients of the large degree models are determined from observations as a complete set, and it is not clear that simply truncating the model is the most appropriate action. Certainly, fitting the observations with only the truncated model would yield different numerical values for the harmonics. The intent of this study is to develop a fitted truncated model and analyze any differences between this "fitted" model and one derived by simply truncating. Based on the study, recommendations are made for an appropriate model for use in a mission planning environment.

2.0 ERROR NORMS

Fitting models implies a norm by which to measure the quality of the fit. The harmonics are generated by minimizing the residuals between the observations and the model predictions. Observations may be very indirect measurements of the model and may include errors and biases so that minimizing the residuals becomes no easy chore. To simplify matters a bit, assume that a reference geopotential model represents the real world exactly. This truth model then can be used to evaluate the accuracy of any other model and eliminate many difficulties resulting from real world observations. The most appropriate norm should measure the differences in the state vectors defined by the equations of motion derived from the different gravity models. Suppose also that an exact solution of the equations of motion is available. If one defines $\hat{\underset{\sim}{x}}(\underset{\sim}{x}_o,\underset{\sim}{v}_o,t)$ as the position vector at all times t for the truth model such that

$$\hat{\underset{\sim}{x}}(\underset{\sim}{x}_o,\underset{\sim}{v}_o,t=0) = \underset{\sim}{x}_o \tag{2.1}$$

and a similar solution to the truncated model, then the norm

$$J = \int_{\Omega}\int_{\dot{\Omega}}\int_{o}^{T} ||\hat{x}(\underset{\sim}{x}_o,\underset{\sim}{v}_o,t) - x(\underset{\sim}{x}_o,\underset{\sim}{v}_o,t)||^2 \, dt d\underset{\sim}{x}_o d\underset{\sim}{x}_o \tag{2.2}$$

gives a measure of the differences between potential models. Here, the $||\cdot||$ is the dot product norm and the domains Ω and $\dot{\Omega}$ represent all possible values of the initial state vectors $\underset{\sim}{x}_o$ and $\underset{\sim}{v}_o$. Such a norm is a good indicator of the propagation errors resulting from different models, but it is not an easily computable norm. More direct norms on the geopotential model itself may be a worthy substitute. For instance, the L^2 norm of the difference between the potential models V and $\hat{V}$ is:

$$J = \int_{\Omega} (V - \hat{V})^2 \, d\underset{\sim}{x} \tag{2.3}$$

For this norm to be a minimum, the first variation must vanish so that the error $e = V-\hat{V}$ must satisfy

$$\int_{\Omega} e\omega d\underset{\sim}{x} = 0 \quad \forall \; \omega \in L^2 \tag{2.4}$$

The test function ω may be thought of as the first variation of the potential V, and L^2 represents the space of all functions which are square integrable. Only a finite number of the basis functions are available in the truncated model so that the error can be tested to only a finite number of functions ω.

Since it is the gradients of the potential and not simply the potential which define the accelerations, a more appropriate norm may be the semi-norm

$$J = \int_{\Omega} \underset{\sim}{\nabla} e \cdot \underset{\sim}{\nabla} e \, d\underset{\sim}{x} \tag{2.5}$$

In this norm, the error must be orthogonal to every test function in the space of functions whose derivatives are square integrable.

$$\int_{\Omega} \underset{\sim}{\nabla} e \cdot \underset{\sim}{\nabla} \omega \quad d\underset{\sim}{x} = 0 \qquad \forall\ \omega \in H^1 \tag{2.6}$$

In spherical coordinates r, μ, λ the above equation reads

$$\int_{\Omega} \left[\frac{\partial e}{\partial r} \frac{\partial \omega}{\partial r} + \frac{(\mu^2 - 1)}{r^2} \frac{\partial e}{\partial \mu} \frac{\partial \omega}{\partial \mu} + \frac{1}{r^2(1 - \mu^2)} \frac{\partial e}{\partial \lambda} \frac{\partial \omega}{\partial \lambda} \right] dr d\mu d\lambda = 0 \tag{2.7}$$

This formulation is singular at the poles, but this may be easily averted by multiplying through by $r^2(1-\mu^2)$ to give

$$\int_{\Omega} \left[r^2(1 - \mu^2) \frac{\partial e}{\partial r} \frac{\partial \omega}{\partial r} + (1 - \mu^2)^2 \frac{\partial e}{\partial \mu} \frac{\partial \omega}{\partial \mu} + \frac{\partial e}{\partial \lambda} \frac{\partial \omega}{\partial \lambda} \right] dr d\mu d\lambda = 0 \tag{2.8}$$

which corresponds to the minimizing of the weighted norm

$$J = \tfrac{1}{2} \int_{\Omega} r^2(1 - \mu^2) \underset{\sim}{\nabla} e \cdot \underset{\sim}{\nabla} e \, d\underset{\sim}{x} \tag{2.9}$$

3.0 THE GEOPOTENTIAL EXPANSION AND TEST FUNCTIONS

The function V and $\hat{V}$ are represented by the usual spherical harmonic expansion

$$\hat{V} = \frac{1}{r} \sum_{n=0}^{\hat{N}} (R/r)^n \sum_{m=0}^{n} P_{n,m}(\mu)\{\hat{C}_{nm}\cos m\lambda + \hat{S}_{nm}\sin m\lambda\} \tag{3.1}$$

and

$$V = \frac{1}{r} \sum_{n=0}^{N} (R/r)^n \sum_{m=0}^{n} P_{n,m}(\mu)\{C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\} \tag{3.2}$$

where

$$C_{nm} = \hat{C}_{nm} + \overline{\Delta C}_{nm}/\overline{P}_{nm} \tag{3.3}$$

and

$$S_{nm} = \hat{S}_{nm} + \overline{\Delta S}_{nm}/\overline{P}_{nm} \tag{3.4}$$

Here $\overline{P}_{nm}$ is the average of the Legendre functi s over the entire latitude domain. $\hat{C}_{nm}$ and $\hat{S}_{nm}$ are given by the truth model, and the unknowns in the truncated model are the normalized harmonic deviations. Solving for the normalized deviations, instead of simply the harmonic coefficients themselves, conditions the matrices of the linear algebraic problem resulting from either equation (2.4) or (2.6).

The test functions are then

$$\omega = \left\{ \frac{1}{r}\left(\frac{R}{r}\right)^k \frac{P_{k\ell}}{\overline{P}_{k\ell}} e^{i\ell\lambda} \right\} \begin{array}{l} k = 0,N \\ \ell = 0,k \end{array} \tag{3.5}$$

and its associated derivatives are

$$\frac{\partial\omega}{\partial r} = \frac{-(k+1)}{r^2}\left(\frac{R}{r}\right)^k \frac{P_{k\ell}}{\overline{P}_{k\ell}} e^{i\ell\lambda} \tag{3.6}$$

$$\frac{\partial\omega}{\partial\mu} = \frac{1}{r}\left(\frac{R}{r}\right)^k \frac{dP_{k\ell}}{d\mu} \Big/ \bar{P}_{k\ell} \quad e^{i\ell\lambda} \tag{3.7}$$

and

$$\frac{\partial\omega}{\partial\lambda} = \frac{i\ell}{r}\left(\frac{R}{r}\right)^k \frac{P_{k\ell}}{\bar{P}_{k\ell}} \quad e^{i\ell\lambda} \tag{3.8}$$

In spherical coordinates, the minimum of the L^2 norm must satisfy

$$\int_{-\varepsilon}^{\varepsilon} \int_{\pi}^{\pi} \int_{R}^{R+h} e\ \omega\ dr\ d\lambda\ d\mu = 0 \tag{3.9}$$

where h defines the radius domain and ε the latitude domain.

4.0 THE LINEAR DISCRETE EQUATIONS

Replacing the error and test functions into equation (2.4), one arrives at

$$\int_{-\epsilon}^{\epsilon}\int_{-\pi}^{\pi}\int_{R}^{R+h}\left\{\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{1}{r^2}\left(\frac{R}{r}\right)^{n+k}\frac{P_{nm}P_{k\ell}}{\bar{P}_{nm}\bar{P}_{k\ell}}\left[\overline{\Delta C}_{nm}\cos m\lambda + \overline{\Delta S}_{nm}\sin m\lambda\right]e^{i\ell\lambda}\right\}dr\,d\mu\,d\lambda$$

$$= \int_{-\epsilon}^{\epsilon}\int_{-\pi}^{\pi}\int_{R}^{R+h}\left\{\sum_{n=N+1}^{\hat{N}}\sum_{m=0}^{n}\frac{1}{r^2}\left(\frac{R}{r}\right)^{n+k}\frac{P_{nm}P_{k\ell}}{\bar{P}_{nm}\bar{P}_{k\ell}}\left[\hat{C}_{nm}\cos m\lambda + \hat{S}_{nm}\sin m\lambda\right]e^{i\ell\lambda}\right\}dr\,d\mu\,d\lambda \tag{4.1}$$

but, due to the orthogonality of the transcendental functions, these equations may be considerably reduced. Defining

$$D_k^n = \int_R^{R+h}\frac{1}{r^2}\left(\frac{R}{r}\right)^{n+k}dr \tag{4.2}$$

$$E_k^{n,m} = \int_{-\epsilon}^{\epsilon}P_{nm}\,P_{km}\,d\lambda \tag{4.3}$$

and

$$A_k^{n,m} = D_k^n\,E_k^{n,m}\Big/\left(\bar{P}_{km}\,\bar{P}_{nm}\right) \tag{4.4}$$

the linear equations read

$$\sum_{n=m}^{N} A_k^{n,m}\overline{\Delta C}_{nm} = \sum_{n=N+1}^{\hat{N}} A_k^{n,m}\,\overline{P}_{nm}\,\hat{C}_{nm} \tag{4.5}$$

$$\sum_{n=m}^{N} A_k^{n,m}\overline{\Delta S}_{nm} = \sum_{n=N+1}^{\hat{N}} A_k^{n,m}\,\overline{P}_{nm}\,\hat{S}_{nm} \tag{4.6}$$

for $m = 0, \ldots N$
$k = m, N$

This is a system of linear algebraic equations in the unknowns $\overline{\Delta C}_{nm}$ and $\overline{\Delta S}_{nm}$. Only the harmonics of the same order are coupled. For the case where the fit is applied over the whole domain $\epsilon = 1$, then $A_k^{n,m} = 0$ except when $k = n$. In such a case, the problem is completely uncoupled and the harmonic deviations must vanish. It indicates that, in this particular norm, the simply truncated model is indeed the optimal fit. This is not the case for restricted domains or for other norms, as will be shown.

The weighted H^1 semi-norm also results in a linear system of equations. With the definitions

$$F_k^{n,m} = \int_{-\epsilon}^{\epsilon} (1 - \mu^2)^2 \frac{dP_{km}}{d\mu} \frac{dP_{nm}}{d\mu}\, d\mu \tag{4.7}$$

$$G_k^{n,m} = \int_{-\epsilon}^{\epsilon} (1 - \mu^2)P_{km}\, P_{nm}\, d\mu \tag{4.8}$$

and

$$A_k^{n,m} = D_k^n \left\{(k + 1)(n + 1)G_k^{n,m} + F_k^{n,m} + m^2E_k^{n,m}\right\} \Big/ (\overline{P}_{km}\overline{P}_{nm}) \tag{4.9}$$

the equations still retain the same form as those in equations (4.5) and (4.6). Since the derivatives of the Legendre functions are not orthogonal, these equations remain coupled even when fitting over the entire domain.

The linear system of equations may be solved using a Cholesky decomposition since they are positive-definite and symmetric.

5.0 THE QUADRATURES

The quadratures involving the radius may be computed analytically. Defining the ratio $\zeta = 1/(1 + h/R)$, then the parameter is given by

$$D_K^n = \frac{1}{R(1 + n + k)}(1 - \zeta^{n+K+1}) \tag{5.1}$$

The quadratures involving the Legendre functions may not be found analytically for $\varepsilon \neq 1$. A numerical Gauss point rule is a convenient and arbitrarily accurate method to evaluate the quadrature. An s^{th} order rule is given by

$$\int_a^b f(\mu)d\mu = \frac{b-a}{2} \sum_{i=1}^{s} \omega_i^s f(\mu_i^s) + R_s \tag{5.2}$$

where

$$\mu_i^s = \frac{b-a}{2} \xi_i^s + \frac{b+a}{2} \tag{5.3}$$

$\xi_i^s \equiv$ Gauss points

$\omega_i^s \equiv$ Gauss weights

The residual is given by

$$R_s = \frac{(b-a)^{(2s+1)}(s!)^4}{(2s+1)(2s!)^3} f^{(2s)}(\bar{\mu}) \;;\; u\epsilon(a,b) \tag{5.4}$$

Even though only s function evaluations are needed, the rule is accurate to order $2s + 1$. The domain defined by ε may be partitioned into smaller subdomains to take advantage of the factor

$$\left(\frac{b - a}{2}\right)^{2s+1}$$

Recursion formulas may be used to evaluate the Legendre polynomials and their derivatives at the Gauss points. These include

$$(n - m + 1)P_{n+1,m} = (2n + 1)\mu\, P_{nm} - (n + m)P_{n-1,m} \tag{5.5}$$

and

$$(\mu^2 - 1)\frac{dP_{nm}}{d\mu} = n\mu\, P_{nm} - (n + m)P_{n-1,m} \tag{5.6}$$

with starting values

$$P_{mm} = (1 - \mu^2)^{m/2}(1 \cdot 3 \cdot 5 \ldots (2m - 1)) \tag{5.7}$$

and

$$P_{m+1,m} = (2m + 1)\mu\, P_{mm} \tag{5.8}$$

Lastly, the mean values of the Legendre polynomials are

$$P_{no} = \frac{1}{\sqrt{2n + 1}} \tag{5.9}$$

and

$$\overline{P}_{nm} = \left[\frac{(n + m)!}{2(n - m)!(2n + 1)}\right]^{\frac{1}{2}} \qquad m \neq 0 \tag{5.10}$$

6.0 NUMERICAL EVALUATION OF THE FITTED TRUNCATED MODELS

A program has been developed to fit the truncated model to a reference model, in this case the GEM10 (1). Fit is according to the L^2 or the weighted H^1 semi-norm or a combination of the two. A ninth order Gauss integration rule on two subdomains is employed. To evaluate the quality of the fit, another program has been developed to propagate satellite orbits in a gravitational field. The recursion formulation described in Mueller (4) is employed to compute the accelerations; and an embedded Runge-Kutta method (6) is used to numerically integrate the cartesian equations of motion.

The state vector resulting from the fitted truncated model is compared to the state resulting from the truth model. As a control, the simply truncated model is also evaluated in this manner. The $L^2(T)$ error norm has been chosen to indicate the performance

$$J = \left[\int_0^T \| \hat{\underset{\sim}{x}}(\underset{\sim}{x}_0, \underset{\sim}{v}_0, t) - x(\underset{\sim}{x}_0, \underset{\sim}{v}_0, t) \|^2 dt \right]^{\frac{1}{2}} \tag{6.1}$$

A variety of initial conditions can be used to test the models thoroughly. The time interval has been selected as one day for all test cases.

6.1 TEST CASE 1

In the first case, a 4x4 model is fit to a reference 8x8 model. The fit is over the entire latitude domain and is up to 500 km in height. The norm used is the linear combination of the L^2 and weighted H^1 semi-norm. The numerical values of the fitted and simple normalized harmonics are shown in table 6-I.

The initial conditions are such that the satellite remains in the 500 km height band. The height of the semi-major axis is initially 300 km. The eccentricity and argument of perigee have the nominal values of $e = 0.01$ and $\omega = 0$. The inclination and ascending nodes have been varied to scan a number of initial conditions. The inclinations range from 0 to 30 degrees in increments of 6 degrees, while the node ranges from 0 to 360 degrees in 45 degree increments. The error norms for both the fitted and simply truncated models are shown in table 6-II. The results indicate that the fitted model shows little, if any, improvement over simple truncation.

TABLE 6-I.- FITTED AND SIMPLY TRUNCATED NORMALIZED HARMONICS ($\times 10^{-6}$)

C_{nm}	2	3	4	S_{nm}	2	3	4
0	-484.0791 -484.1654	0.9136 0.9584	0.6604 0.5411				
1	-0.0312 0.0010	2.0581 2.0286	-0.5011 -0.5352	1	-0.0542 -0.0024	0.3061 0.2520	-0.4957 -0.4693
2	2.4456 2.4340	0.5998 0.8927	0.3313 0.3521	2	-1.3560 -1.3991	-0.5279 -0.6235	0.8327 0.6640
3		0.7950 0.7003	1.0452 0.9885	3		1.4546 1.4125	-0.1976 -0.2018
4			-0.1791 -0.1953	4			0.3734 0.2988

GM = $3.98600657 \times 10^{14}$ ($m^3\ s^{-2}$)
$3.98600640 \times 10^{14}$

TABLE 6-II.- POSITION ERROR NORMS (FITTED/SIMPLE)

	$ms^{\frac{1}{2}}$	inclination (deg)					
		0	6	12	18	24	30
node deg	0	3.6/4.9	3.3/4.6	2.8/4.0	2.3/3.3	2.0/2.8	1.9/2.6
	45	3.5/2.6	3.0/2.3	2.4/1.6	1.8/1.0	1.5/0.8	1.5/1.0
	90	4.7/4.8	5.3/5.5	6.2/6.6	6.9/7.6	7.2/8.1	6.9/8.1
	135	6.4/8.0	5.9/7.4	5.0/6.3	4.0/5.0	3.5/4.2	3.6/4.0
	180	1.9/1.6	2.0/1.7	2.3/2.1	2.9/2.8	3.3/3.4	3.6/3.9
	225	5.1/5.8	5.1/5.6	4.8/5.3	4.3/4.8	3.7/4.1	3.2/3.4
	270	6.2/7.3	6.0/7.1	5.3/6.3	4.2/5.1	3.5/4.2	3.3/3.8
	315	6.2/5.7	6.4/6.0	7.3/7.1	8.6/8.6	9.6/9.9	9.9/10.0

6.2 TEST CASE 2

The second test case again uses a 4x4 model fit to an 8x8 reference. This time the latitude domain is restricted to 30 degrees for evaluating the tesseral and sectorial harmonic fit. The zonal harmonics, however, use the full 90-degree domain. The results of the fit are shown in table 6-III and the position error norms resulting from the fit are given in table 6-IV. Again, the fitted model shows no improvement over simple truncation.

TABLE 6-III.- FITTED AND SIMPLY TRUNCATED NORMALIZED HARMONICS ($\times 10^{-6}$)

C_{nm}	2	3	4	S_{nm}	2	3	4
0	-484.0791	0.9136	0.6604				
	-484.1654	0.9584	0.5411				
1	-0.0276	2.5849	-0.2356	1	-0.1072	0.6420	-0.4325
	0.0010	2.0286	-0.5352		-0.0024	0.2520	-0.4693
2	2.5043	0.3371	0.3866	2	-1.0655	-0.2916	1.2699
	2.4340	0.8927	0.3521		-1.3991	-0.6235	0.6640
3		0.9593	1.0819	3		1.5002	-0.2611
		0.7003	0.9885			1.4125	-0.2018
4			-0.1588	4			0.4852
			-0.1953				0.2988

GM = 3.98600657x10 ($m^3\ s^{-2}$)
3.98600640x10

TABLE 6-IV.- POSITION ERROR NORMS (FITTED/SIMPLE)

		inclination (deg)					
	$ms^{1/2}$	0	6	12	18	24	30
node deg	0	3.7/4.9	3.5/4.6	3.0/4.0	2.4/3.3	2.0/2.8	1.8/2.6
	45	4.9/2.6	4.7/2.3	4.0/1.6	3.3/1.0	2.7/0.8	2.4/1.0
	90	4.7/4.8	5.1/5.5	5.8/6.6	6.5/7.6	6.7/8.1	6.6/8.1
	135	5.0/8.0	4.7/7.4	4.0/6.3	3.2/5.0	2.8/4.2	3.0/4.0
	180	2.0/1.6	2.1/1.7	2.6/2.1	3.3/2.8	3.9/3.4	4.3/3.9
	225	5.7/5.8	5.5/5.6	4.9/5.3	4.2/4.8	3.4/4.1	2.8/3.4
	270	5.5/7.3	5.2/7.1	4.4/6.3	3.5/5.1	2.9/4.2	3.2/3.8
	315	6.5/5.7	6.7/6.0	7.5/7.1	8.5/8.6	9.2/9.9	9.2/10.0

7.0 RECOMMENDATIONS AND CONCLUSIONS

From the numerical results, it appears that there is no advantage to fitting truncated models (at least in the norms investigated). If truncation is a necessity in the computational environment, the question left open is at what degree truncation should occur. The magnitudes of each term in the expansion can be approximated by the normalized harmonics multiplied by the distance ration power or

$$\delta_{nm} = (R/r)^n \{\bar{C}_{nm}^2 + \bar{S}_{nm}^2\}^{\frac{1}{2}} \tag{7.1}$$

At a height of 300 km, only 4 terms have magnitudes greater than 10^{-6}. They are $n = 2, m = 0$; $n = 2, m = 2$; $n = 3, m = 1$; and $n = 3, m = 3$. If the tolerance is increased by one digit to 10^{-7}, then the number of additional terms increases dramatically to 33, or most of the terms of a 10x10 model.

The analyst also should consider other factors in selecting a model. Since in low Earth orbits the downtrack quadratic drag errors overwhelm the linear downtrack errors due to truncation, it is not important to include the tesseral and sectorial harmonics. These terms contribute secular variations only in the downtrack. The zonal harmonics, on the other hand, contribute radial and crosstrack secular perturbations which are much larger than the corresponding linear errors due to drag. A low degree model appears to be a very suitable model in the presence of strong drag (h < 300 km). In these strong drag cases, resonance is not an important factor since drag inhibits the period from resonating with any particular harmonic over any significant length of time. For the weak drag cases, the analyst should include only that particular resonating term. But recursion algorithms for computing the accelerations must build up a table starting from a low degree, so selecting a particular term is not always possible.

Another consideration in selecting a model is consistency and uniformity. The analyst should select models which are consistent with the realtime environment. For instance, if a high degree of accuracy is required, one should certainly select the same model which was used to generate the initial state from the observations. The analyst also may wish to select a simple model which is consistent with those used by other analysts in related work. In this way, a source of discrepancies can be eliminated.

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