General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

Produced by the NASA Center for Aerospace Information (CASI)

NASA Technical Memorandum 83365
AIAA-83-0730

Sound Diffraction at Wall Impedance Discontinuities in a Circular Cylinder – Investigated Using Wiener-Hopf Technique

APR 1983
RECEIVED
NASA STI FACILITY
ACCESS DEPT.

Y.-C. Cho
Lewis Research Center
Cleveland, Ohio

(NASA-TM-83365) SOUND DIFFRACTION AT WALL IMPEDANCE DISCONTINUITIES IN A CIRCULAR CYLINDER, INVESTIGATED USING WIENER-HOPF TECHNIQUE (NASA) 27 p HC A03/MF A01 CSCL 20A G3/71
N83-23117
Unclas
09793

Prepared for the
Eighth Aeroacoustics Conference
sponsored by the American Institute of Aeronautics and Astronautics
Atlanta, Georgia, April 11-13, 1983

NASA

ORIGINAL PAGE IS
OF POOR QUALITY

SOUND DIFFRACTION AT WALL IMPEDANCE DISCONTINUITIES IN A CIRCULAR CYLINDER -- INVESTIGATED USING WIENER-HOPF TECHNIQUE

Y.-C. Cho*
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

E-1631

ABSTRACT

Rigorous solutions are presented for sound diffraction in a circular cylinder with axial discontinuities of the wall admittance (or impedance). Analytical expressions are derived for the reflection and the transmission coefficients for duct modes. The results are discussed quantitatively in the limits of small admittance shifts (Δ) and of low frequencies (ka). One of the more remarkable results is the low frequency behavior of the reflection coefficient $\left(R_{00}^{0}\right)$ of the fundamental mode. For the mode of a hardwall duct reflected from the junction with a softwall duct, $R_{00}^{0} \to -\left(1 - \sqrt{ka}\,\sqrt{2/i}\,\Delta\right)$; this result is in contrast to the frequency dependence of the reflection from the open end of a hardwall duct, for which $R_{00}^{0} \to -\left[1 - (ka)^2/2\right]$.

NOMENCLATURE

a	duct radius
c	sound speed
F_{mn}	see eq. (47)
$G(\vec{x},\vec{x}_0)$	Green's function; $\vec{x}$ and $\vec{x}_0$, respectively, being coordinates of observation and source points
J_m	Bessel function of order m
k	ω/c, free space wave constant
$k_{m\ell}$	propagation constant of (m,ℓ) mode
$L(\beta)$	see eq. (49)
$L_{\pm}(\beta)$	see eqs. (55) and (56)
$N_{m\ell}$	normalization factor, see eq. (7)

*Physicist.

This paper is declared a work of the U.S. Government and therefore is in the public domain.

ORIGINAL PAGE IS
OF POOR QUALITY

p	acoustic pressure
$Q_{m\ell}$	normalized eigenfunction, see eq. (6)
R^m_{nj}	reflection coefficient for (m,n) mode incident and (m,j) mode reflected
(r,φ,x)	cylindrical coordinates of observation point
(r_0,φ_0,x_0)	cylindrical coordinates of source point
S	surface, see fig. 2
$T^m_{n\nu}(1 \to 2)$	transmission coefficient for (m,ν) mode of duct 2 excited by incidence of (m,n) mode of duct 1
$T^m_{\nu n}(2 \to 1)$	transmission coefficient of (m,n) mode of duct 1 excited by incidence of (m,ν) mode of duct 2
t	time
$\hat{u}$	particle velocity
Y_m	Neumann function (Y - Bessel function) of order m
$\alpha_{m\ell}$	eigenvalue corresponding to (m,ℓ) mode
β	complex variable, real being conjugate to spatial coordinate variable x
Γ_m	Fourier transform of γ_m, see eq. (46)
γ_m	reduced Green's function, see eq. (28)
Δ	$\eta_2 - \eta_1$, admittance shift
$\delta(x)$	Dirac delta function
η	admittance ($\rho cu/p$)
$\kappa_{m\ell}$	see eq. (C6)
$\mu_{m\ell}$	see eq. (8)
ρ	density of medium
Φ_ℓ	Fourier transforms of Ψ_ℓ
Ψ	velocity potential

ORIGINAL PAGE IS
OF POOR QUALITY

$\Psi_{\bullet}$ see eqs. (33) and (34)

ω angular frequency

INTRODUCTION

Lined ducts have long been employed as silencers of noise generated in a fan duct system. The sound attenuation in a long, uniformly lined cylinder is adequately understood in terms of acoustic wave modes with complex propagation constants. However, in practice the liners are finite in length, and are sometimes made with segments thus providing discontinuities of the wall impedance. At the impedance discontinuities, a wave undergoes diffraction, being partly reflected and partly transmitted. The reflection accompanies the mode scattering, and the transmission involves the conversion of the mode to a new set of modes. In addition to the sound absorption by the liners it has been suspected that the reflection and the mode conversion can also contribute appreciably to the noise reduction.[1] Such a contribution still requires a comprehensive study. This paper presents a rigorous treatment using the Wiener-Hopf technique[2] of the diffraction of a mode due to the discontinuity of the wall impedance in a circular duct. The analysis involves an incident wave of an arbitrary mode, and can be expanded to a case of multiple diffractions which occur in a duct involving more than one axial discontinuity in the wall impedance.

Similar approaches were used previously in studies of the diffraction of the fundamental mode in rectangular ducts.[3-5] The formal solution involves infinite product terms, and can be easily manipulated to produce numerical results in some limiting cases with the fundamental mode incident. However, with a higher order mode incident, the numerical evaluation requires calculation of complex eigenvalues of many modes, and the infinite product terms converge weakly especially when the impedance discontinuity is large. In the present analysis the formal solution is obtained first, and is used only to determine the analytical properties of various functions involved in the intermediate stages of the computation.

FORMULATION AND SOLUTION OF WIENER-HOPF TYPE INTEGRAL EQUATION

Modal Solution

This section begins with a review of modal solutions in a circular duct. A duct mode corresponds to an eigensolution of the steady wave equation in a uniform duct which is subject to a homogeneous boundary condition on its wall. With simple harmonic time dependence $e^{-i\omega t}$ of the field, the wave equation is reduced to the scalar Helmholtz equation, which is written for cylindrical coordinates (r,φ,x) as follows:

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial b^2} + \frac{\partial^2}{\partial x^2} + k^2\right) \Psi = 0. \tag{1}$$

Here $k = \omega/c$, and Ψ is the acoustic velocity potential and accordingly, the particle velocity $\hat{u}$ and the acoustic pressure p are given by

ORIGINAL PAGE IS
OF POOR QUALITY

$$\hat{u} = \vec{\nabla}\Psi, \tag{2}$$

$$p = i\rho ck\Psi, \tag{3}$$

where ρ is the steady density of the medium. The wall boundary condition is specified in terms of the specific acoustic admittance η:

$$\left.\frac{\partial\Psi}{\partial r}\right|_{r=a} - ik\eta\Psi\Big|_{r=a} = 0. \tag{4}$$

An eigensolution to this problem is obtained as:

$$\Psi \sim e^{im\varphi}Q_{m\ell}(r)e^{ik_{m\ell}x}. \tag{5}$$

Here $Q_{m\ell}(r)$ is the normalized Bessel function:

$$Q_{m\ell}(r) = \frac{1}{N_{m\ell}} J_m\left(\alpha_{m\ell}\frac{r}{a}\right) \tag{6}$$

with the normalization factor given as

$$N_{m\ell} = \frac{aJ_m(\alpha_{m\ell})}{\sqrt{2}\,\mu_{m\ell}}, \tag{7}$$

where

$$\begin{aligned} \mu_{m\ell} &= 1 \qquad \text{for } \alpha_{m\ell} = 0, \\ &= \left[1 - \frac{m^2 + (ka\eta)^2}{(\alpha_{m\ell})^2}\right]^{-1/2} \qquad \text{for } \alpha_{m\ell} \neq 0. \end{aligned} \tag{8}$$

The solution in equation (5) corresponds to the (m,ℓ) mode, m and ℓ being integers called the circumferential and radial mode numbers, respectively. The constant $\alpha_{m\ell}$ is the corresponding eigenvalue that is obtained as the ℓth root of the equation

$$\alpha\frac{dJ_m(\alpha)}{d\alpha} - ika\eta J_m(\alpha) = 0, \tag{9}$$

and $k_{m\ell}$ is the propagation constant given by

$$k_{m\ell} = \sqrt{k^2 - \left(\frac{\alpha_{m\ell}}{a}\right)^2}. \tag{10}$$

ORIGINAL PAGE IS
OF POOR QUALITY

For an acoustically absorbing wall for which $\mathcal{R}e(\eta) > 0$, the phases of $\alpha_{m\ell}$ and $k_{m\ell}$ are chosen such that

$$\mathcal{R}e(\alpha_{m\ell}) > 0, \qquad \mathcal{I}m(\alpha_{m\ell}) < 0, \tag{11a,b}$$

$$\mathcal{R}e(k_{m\ell}) > 0, \qquad \mathcal{I}m(k_{m\ell}) > 0. \tag{12a,b}$$

It follows then from inequality (12a,b) that the solution in equation (5) corresponds to the (m,ℓ) mode propagating to the positive x-direction. For the propagation to the negative x-direction, $-k_{m\ell}$ is used in place of $k_{m\ell}$. Note that the lowest radial mode number is here chosen to be zero, corresponding to the zeroth (or smallest) root of equation (9).

The problem to be treated here is concerned with the diffraction of a duct mode due to the axial discontinuity of the wall boundary condition in a circular cylinder, as illustrated in figure 1. The cylinder is, for convenience, thought to be composed of two duct elements coupled at $x = 0$. The elements are designated by 1 and 2, respectively, for $x < 0$ and $x > 0$. The respective wall admittances are η_1 and η_2: that is, for $x < 0$

$$\left.\frac{\partial\Psi}{\partial r}\right|_{r=a} - ik\eta_1\Psi\Big|_{r=a} = 0, \tag{13}$$

and for $x > 0$

$$\left.\frac{\partial\Psi}{\partial r}\right|_{r=a} - ik\eta_2\Psi\Big|_{r=a} = 0. \tag{14}$$

The eigensolutions are

$$\Psi \sim e^{im\varphi}\, Q^{(1)}_{m\ell}(r)e^{ik^{(1)}_{m\ell}x} \qquad \text{for} \quad x < 0, \tag{15}$$

$$\Psi \sim e^{im\varphi}\, Q^{(2)}_{m\ell}(r)e^{ik^{(2)}_{m\ell}x} \qquad \text{for} \quad x > 0. \tag{16}$$

Here $Q_{m\ell}$ and $k_{m\ell}$ are superscripted with 1 and 2, respectively for the duct elements 1 and 2. The corresponding eigenvalues are denoted by $\alpha^{(1)}_{m\ell}$ and $\alpha^{(2)}_{m\ell}$. However, the circumferential mode number m is a conserved quantity because the cylindrical symmetry is maintained throughout the duct, and the angle dependence $e^{im\varphi}$ will be often omitted.

Consider an incident wave which is composed of a single mode of the duct element 1, propagating towards the center $(x = 0)$. Upon arrival at $x = 0$, the wave is partly reflected and partly transmitted. With the (m,n) mode incident, the resultant wave can be written as, for $x < 0$

ORIGINAL PAGE IS
OF POOR QUALITY

$$\Psi = Q_{mn}^{(1)}(r)e^{ik_{mn}^{(1)}x} + \sum_{j=0}^{\infty} R_{nj}^{m}(1)Q_{mj}^{(1)}(r)e^{-ik_{mj}^{(1)}x}, \tag{17}$$

and for $x > 0$

$$\Psi = \sum_{\nu=0}^{\infty} T_{n\nu}^{m}(1 \to 2)Q_{m\nu}^{(2)}(r)e^{ik_{m\nu}^{(2)}x}. \tag{18}$$

The first term in equation (17) is the incident wave, and the second term is the reflected wave, which is composed of many modes of the duct element 1. Equation (18) represents the transmitted wave which is composed of many modes of the duct element 2. The constants $R_{nj}^{m}(1)$ and $T_{n\nu}^{m}(1 \to 2)$ are the reflection and the transmission coefficients. The first subscript of the coefficients is the radial mode number of the incident wave, and the second subscript the radial mode number of a mode in the reflected or the transmitted wave. These coefficients will be here determined by using the Wirner-Hopf technique.

Wiener-Hopf Integral Equation

For the formulation of the Wiener-Hopf type integration, we consider the Green's function, $G(\vec{x},\vec{x}_0)$, which is a solution of the equation

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial x^2} + k^2\right) G(\vec{x},\vec{x}_0) = \frac{1}{r}\,\delta(r - r_0)\delta(\varphi - \varphi_0)\delta(x - x_0). \tag{19}$$

Let it be subjected to the boundary condition of the duct element 1; the Green's function is then constructed in terms of the eigensolutions for the duct element 1:[6]

$$G(\vec{x},\vec{x}_0) = \frac{-i}{4\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \sum_{\ell=0}^{\infty} \frac{Q_{m\ell}^{(1)}(r)Q_{m\ell}^{(1)}(r_0)}{k_{m\ell}^{(1)}}\, e^{ik_{m\ell}^{(1)}|x-x_0|}. \tag{20}$$

In the absence of sources, the acoustic field inside the duct can be expressed in the integral form

$$\Psi(\vec{x}) = \int_S \left[\Psi(\vec{x}_0)\vec{\nabla}_0 G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\vec{\nabla}_0\Psi(\vec{x}_0)\right] \cdot \hat{n}\, dS_0. \tag{21}$$

The boundary surface S is shown in figure 2 and $\hat{n}$ is the outward unit vector normal to the surface. It is convenient to divide the surface integral

ORIGINAL PAGE IS
OF POOR QUALITY

into three parts: the integrations on the duct wall, S_1, on the cross section at $x_0 = -\infty$, S_2, and on the cross section at $x_0 = \infty$, S_3, as follows:

$$\Psi = I_1 + I_2 + I_3, \tag{22}$$

$$I_1 = a \int_0^{2\pi} d\varphi_0 \int_{-\infty}^{\infty} dx_0 \left(\Psi \frac{\partial G}{\partial r_0} - G \frac{\partial \Psi}{\partial r_0} \right)_{r_0 = a}, \tag{23}$$

$$I_2 = \int_0^{2\pi} d\varphi_0 \int_0^{a} r_0 dr_0 \left(- \Psi \frac{\partial G}{\partial x_0} + G \frac{\partial \Psi}{\partial x_0} \right)_{x_0 = -\infty}, \tag{24}$$

$$I_3 = \int_0^{2\pi} d\varphi_0 \int_0^{a} r_0 dr_0 \left(\Psi \frac{\partial G}{\partial x_0} - G \frac{\partial \Psi}{\partial x_0} \right)_{x_0 = \infty}. \tag{25}$$

The integral I_3 is zero because Ψ and G decay exponentially as $x_0 \to \infty$, as one can see from equations (18) and (20). As for the integral I_2, Ψ in equation (17) is used. The contribution from the reflected wave is zero based on an argument similar to the argument for I_3. With the expression for the incident wave and equation (20) inserted into equation (24) one can readily obtain

$$I_2 = e^{im\varphi} Q_{mn}^{(1)}(r) e^{ik_{mn}^{(1)} x}. \tag{26}$$

This is none other than the incident wave. Now turn to the integral I_1. For $x_0 < 0$, Ψ and G satisfy the same boundary condition given in equation (13). Thus, the integrand is zero for $x_0 < 0$, and we have

$$I_1 = a e^{im\varphi} \int_0^{\infty} dx_0 \left(\Psi \frac{\partial \gamma_m}{\partial r_0} - \gamma_m \frac{\partial \Psi}{\partial r_0} \right)_{r_0 = 0}, \tag{27}$$

where γ_m is the reduced Green's function,

ORIGINAL PAGE IS
OF POOR QUALITY

$$\gamma_m(r,r_0;x-x_0) = -\frac{i}{2}\sum_{\ell=0}^{\infty}\frac{1}{k_{m\ell}^{(1)}}Q_{m\ell}^{(1)}(r)Q_{m\ell}^{(1)}(r_0)e^{ik_{m\ell}^{(1)}|x-x_0|}. \tag{28}$$

It is convenient to decompose Ψ as

$$\Psi = \Psi_i + \Psi_D. \tag{29}$$

Here Ψ_i is the incident wave. The function Ψ_D represents the reflected wave for $x < 0$, and also the transmitted wave minus the incident wave for $x > 0$. Note that for $x > 0$ neither Ψ_i nor Ψ_D satisfies the boundary condition in equation (14), although Ψ does. The condition for Ψ_D can be readily derived as, for $x > 0$

$$\left.\frac{\partial\Psi_D}{\partial r}\right|_{r=a} = ik\eta_2\Psi - ik\eta_1\Psi_i. \tag{30}$$

Using equations (29) and (30), and the fact that the incident wave and the Green's function satisfy the same boundary condition, one obtains

$$I_1 = -ika\,\Delta e^{im\varphi}\int_0^{\infty}\gamma_m(r,a;x-x_0)\Psi(a,x_0)dx_0 \tag{31}$$

where $\Delta = \eta_2 - \eta_1$. It follows from equations (22), (26), and (31) and $I_3 = 0$ that

$$\Psi_D(r,x) = -ika\,\Delta\int_0^{\infty}dx_0\gamma_m(r,a;x-x_0)\left[\Psi_i(a,x_0) + \Psi_D(a,x_0)\right]. \tag{32}$$

Introduce the functions $\Psi_\pm(x)$:

$$\Psi_+(x) = \begin{cases}\Psi_D(a,x) & \text{for } x \geq 0,\\ 0 & \text{for } x < 0.\end{cases} \tag{33}$$

$$\Psi_-(x) = \begin{cases}0 & \text{for } x \geq 0,\\ \Psi_D(a,x) & \text{for } x < 0.\end{cases} \tag{34}$$

Then Equation (32) can be written for $r = a$ as

$$\Psi_+(x) + \Psi_-(x) = -ika\,\Delta\int_0^{\infty}dx_0\gamma_m(a,a;x-x_0)\left[\Psi_i(a,x_0) + \Psi_+(x_0)\right]. \tag{35}$$

This is a Wiener-Hopf type integral equation.

ORIGINAL PAGE IS
OF POOR QUALITY

Fourier Transform Solution

First consider the Fourier transforms of $\Psi_{\pm}(x)$:

$$\Phi_+(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi_+(x) e^{i\beta x} dx, \tag{36}$$

$$\Phi_-(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi_-(x) e^{i\beta x} dx. \tag{37}$$

With the lowest radial mode number corresponding to the least attenuating mode, the field Ψ_D behaves asymptotically for $x \to \infty$, as

$$\Psi_D \xrightarrow{x \to \infty} e^{ik_{mo}^{(2)} x}. \tag{38}$$

Note that, if $\mathcal{Im}\left[k_{mn}^{(1)}\right]$ is less than $\mathcal{Im}\left[k_{mn}^{(2)}\right]$, $k_{mo}^{(2)}$ is replaced by $k_{mn}^{(1)}$ (the incident wave constant) here and in equations (40) and (42) below. And for $x \to -\infty$,

$$\Psi_D \xrightarrow{x \to -\infty} e^{-ik_{mo}^{(1)} x}. \tag{39}$$

It follows that $\Phi_+(\beta)$ and $\Phi_-(\beta)$ are analytic, respectively, in the upper half plane (UHP) and in the lower half plane (LHP) of the β space. Here the half planes are defined as follows:

$$\mathcal{Im}(\beta) > -\mathcal{Im}\left[k_{mo}^{(2)}\right] \qquad \text{for UHP}, \tag{40}$$

$$\mathcal{Im}(\beta) < \mathcal{Im}\left[k_{mo}^{(2)}\right] \qquad \text{for LHP}. \tag{41}$$

Note that the two half planes overlap in the region specified by

$$-\mathcal{Im}\left[k_{mo}^{(2)}\right] < \mathcal{Im}(\beta) < \mathcal{Im}\left[k_{mo}^{(1)}\right].$$

The functions $\Phi_+(\beta)$ and $\Phi_-(\beta)$ are both analytic in this region, and the inverse transforms are obtained as

$$\Psi_+(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_+(\beta) e^{-i\beta x} d\beta, \tag{43}$$

$$\Psi_-(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_-(\beta) e^{-i\beta x} d\beta. \tag{44}$$

ORIGINAL PAGE IS
OF POOR QUALITY

Note that the integral path is here the real axis, which is in the region specified by equation (42)

The Fourier transform of equation (35) yields

$$\phi_+(\beta) + \phi_-(\beta) = -ika\ \Delta\ \sqrt{2\pi}\ \Gamma_m(\beta)\left[F_{mn}(\beta) + \phi_+(\beta)\right]. \tag{45}$$

Here Γ_m, which is obtained in appendix A, is the Fourier transform of γ_m:

$$\Gamma_m(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\zeta\ e^{i\beta\zeta}\ \gamma_m(a,a;\zeta). \tag{46}$$

And

$$F_{mn}(\beta) \equiv \frac{iQ_{mn}^{(1)}(a)}{\sqrt{2\pi}\left[\beta + k_{mn}^{(1)}\right]}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dx\ \Psi_i(a,x)e^{i\beta x}. \tag{47}$$

The integral in equation (47) is defined only for $\mathcal{Im}(\beta) > -\mathcal{Im}\left[k_{mn}^{(1)}\right]$. However F_{mn} is defined in the entire β space except at the pole $\beta = -k_{mn}^{(1)}$, and constitutes the analytic continuation of the integral for $\mathcal{Im}(\beta) \leq -\mathcal{Im}\left[k_{mn}^{(1)}\right]$. Rearranging equation (45), one obtains

$$L(\beta)\left[\phi_+(\beta) + F_{mn}(\beta)\right] = -\phi_-(\beta) + F_{mn}(\beta), \tag{48}$$

where

$$L(\beta) = 1 + i\sqrt{2\pi}\ ka\ \Delta\ \Gamma_m(a,a;\beta). \tag{49}$$

As shown in appendix A, $\Gamma_m(a,a;\beta) \xrightarrow{|\beta| \to \infty} 0$, thus

$$L(\beta) \xrightarrow{|\beta| \to \infty} 1. \tag{50}$$

The next step of the Wiener-Hopf technique is to decompose equation (48) into two factors, one analytic in the UHP and the other analytic in the LHP. To this end, $L(\beta)$ shall be factorized first. With the substitution of equation (A11), equation (49) is written as

$$L(\beta) = \frac{\alpha J_m'(\alpha) - ika\eta_2 J_m(\alpha)}{\alpha J_m'(\alpha) - ika\eta_1 J_m(\alpha)}, \tag{51}$$

where

ORIGINAL PAGE IS
OF POOR QUALITY

$$\alpha = a\sqrt{k^2 - \beta^2}.$$

The numerator and the denominator in equation (51) are even functions of α for m even, and are odd functions of α for m odd. Thus, for m even the two factors are entire functions of β. For m odd, the two factors are concurrently multiplied by α to be made entire functions of β. In either case, the entire functins possess simple zeroes and equation (51) can then be written as[7]

$$L(\beta) = L_0 \prod_{\ell=0}^{\infty} \left(\left\{ \left[1 - \frac{\beta}{k_{m\ell}^{(2)}}\right] \left[1 + \frac{\beta}{k_{m\ell}^{(2)}}\right] \right\} \Big/ \left\{ \left[1 - \frac{\beta}{k_{m\ell}^{(1)}}\right] \left[1 + \frac{\beta}{k_{m\ell}^{(1)}}\right] \right\} \right) \tag{52}$$

where

$$L_0 = \frac{J_m'(ka) - in_2 J_m(ka)}{J_m'(ka) - in_1 J_m(ka)}. \tag{53}$$

The factorization of $L(\beta)$ immediately follows:

$$L(\beta) = L_+(\beta)/L_-(\beta) \tag{54}$$

where

$$L_+(\beta) = L_0 \prod_{\ell=0}^{\infty} \left\{ \left[1 + \frac{\beta}{k_{m\ell}^{(2)}}\right] \Big/ \left[1 + \frac{\beta}{k_{m\ell}^{(1)}}\right] \right\}, \tag{55}$$

$$L_-(\beta) = \prod_{\ell=0}^{\infty} \left\{ \left[1 - \frac{\beta}{k_{m\ell}^{(1)}}\right] \Big/ \left[1 - \frac{\beta}{k_{m\ell}^{(2)}}\right] \right\}. \tag{56}$$

Note that

$$L_-(-\beta) = L_0/L_+(\beta). \tag{57}$$

Equation (48) can now be written as

$$L_+(\beta)\left[\phi_+(\beta) + F_{mn}(\beta)\right] - L_-\left(-k_{mn}^{(1)}\right) F_{mn}(\beta)$$

$$= L_-(\beta)\left[-\phi_-(\beta) + F_{mn}(\beta)\right] - L_-\left(k_{mn}^{(1)}\right) F_{mn}(\beta). \tag{58}$$

ORIGINAL PAGE IS
OF POOR QUALITY

Here the term $L_-\left(-k_{mn}^{(1)}\right)\cdot F_{mn}(\beta)$ has been subtracted from both sides to remove the pole of $F_{mn}(\beta)$ from the right side.

The left and right sides of equation (58) are analytic, respectively, in the UHP and the LHP. The two sides are the same and analytic in the common analytic region specified in equation (42). Thus they are analytic continuations to each other,[4] and equation (58) defines a function which is analytic in the entire β space. As $\beta \to \infty$, $L_\pm(\beta) \to$ constant and $\phi_\pm(\beta) \to (1/\beta)$. The asymptotic behavior of $\phi_\pm(\beta)$ is inferred from the behavior of $\psi(x)$ near $x = 0$.[8] It follows then that both sides of equation (58) tend to be zero like $(1/\beta)$, as $\beta \to \infty$. According to the Liouville's theorem,[9] the function must equal zero everywhere, and one obtains

$$\phi_+(\beta) = \left[L_-\left(k_{mn}^{(1)}\right)\Big/L_+(\beta) - 1\right]F_{mn}(\beta), \tag{59}$$

$$\phi_-(\beta) = -\left[L_-\left(-k_{mn}^{(1)}\right)\Big/L_-(\beta) - 1\right]F_{mn}(\beta). \tag{60}$$

Reflected and Transmitted Waves

The reflected wave is determined from equation (44) with the substitution of equation (60). The integral in equation (44) can be replaced by a contour integral along a path C closed in the UHP as illustrated in figure 3:

$$\psi_-(x) = \frac{1}{\sqrt{2\pi}} \oint_C \phi_-(\beta)e^{-i\beta x}\, d\beta \qquad \text{for} \quad x < 0. \tag{61}$$

The function $\phi_-(\beta)$ is singular only at the simple poles $\beta = k_{mj}^{(1)}$ $j = 0,1,2,$ which corresponded to simple zeroes of $L_-(\beta)$. Thus, equation (61) involves a Cauchy type integral, which is readily evaluated as

$$\psi_-(x) = \sum_{j=0}^{\infty} R_{mj}^{m}(1)Q_{mj}^{(1)}(a)e^{-ik_{mj}^{(1)}x} \qquad \text{for} \quad x < 0, \tag{62}$$

where

$$R_{mj}^{m}(1) = \frac{Q_{mn}^{(1)}(a)}{Q_{mj}^{(1)}(a)} \cdot \frac{-k_{mj}^{(1)}L_-\left(-k_{mn}^{(1)}\right)}{k_{mj}^{(1)} + k_{mn}^{(1)}} \cdot \tag{63}$$

$$\prod_{\ell=0}^{\infty}\left[1 - \frac{k_{mj}^{(1)}}{k_{m\ell}^{(2)}}\right]\Bigg/\prod_{\ell\neq j}^{\infty}\left[1 - \frac{k_{mj}^{(1)}}{k_{m\ell}^{(1)}}\right].$$

ORIGINAL PAGE IS
OF POOR QUALITY

Equation (63) determines the reflection coefficients, which have been defined in equation (17).

The transmitted wave is determined from equation (43) with the substitution of equation (59). The integral in equation (43) is replaced by a contour integral along a path C' shown in figure 3. The function $\Phi_+(\beta)$ is singular only at the simple poles at $\beta = -k_{m\nu}^{(2)}$, $\nu = 0,1,2$ corresponding to the simple zeroes of $L_+(\beta)$. One can then readily obtain

$$\Psi_+(x) = \sum_{\nu=0}^{\infty} T_{n\nu}^{m}(1 \to 2) Q_{m\nu}^{(2)}(a) e^{ik_{m\nu}^{(2)}x} - Q_{mn}^{(1)}(a) e^{ik_{mn}^{(1)}x} \qquad \text{for} \quad x > 0, \tag{64}$$

where

$$T_{n\nu}^{m}(1 \to 2) = \frac{Q_{mn}^{(1)}(a)}{Q_{m\nu}^{(2)}(a)} \cdot \frac{L_-\left(-k_{mn}^{(1)}\right)}{L_0} \cdot \frac{k_{m\nu}^{(2)}}{k_{mn}^{(1)} - k_{m\nu}^{(2)}} \cdot \tag{65}$$

$$\prod_{\ell=0}^{\infty} \left[1 - \frac{k_{m\nu}^{(2)}}{k_{m\ell}^{(1)}}\right] \Bigg/ \prod_{\ell \neq \nu}^{\infty} \left[1 - \frac{k_{m\nu}^{(2)}}{k_{m\ell}^{(2)}}\right].$$

The second term in the right side of equation (64) is none other than the incident wave, thus for $x > 0$ the net field is, from equation (29),

$$\Psi(a,x) = \sum_{\nu=0}^{\infty} T_{n\nu}^{m}(1 \to a) Q_{m\nu}^{(2)}(a) e^{ik_{m\nu}^{(2)}x} \qquad \text{for} \quad x > 0. \tag{66}$$

This is the transmitted wave, and equation (65) determines the transmission coefficient, which has been defined in equation (18).

DISCUSSION: RECIPROCITY AND LIMITING CASES

Foregoing detailed numerical calculations, we will discuss here the limiting cases of a small admittance shift Δ and of low frequencies. The discussion will also include the reciprocity relations.[10]

Reciprocity Properties

As in appendix B, the infinite products that are explicitly comprised in equations (63) and (65) can be expressed in terms of the factor L_+ which can be evaluated numerically.[2] The reflection and the transmission coefficients can then be written as

ORIGINAL PAGE IS
OF POOR QUALITY

$$R^m_{nj}(1) = \frac{-ik\,\Delta}{ak^{(1)}_{mj}\left[k^{(1)}_{mn} + k^{(1)}_{mj}\right]} \cdot \frac{L_0\mu^{(1)}_{mj}\mu^{(1)}_{mn}}{L_+\left(k^{(1)}_{mn}\right) L_+\left(k^{(1)}_{mj}\right)} \tag{67}$$

$$T^m_{n\nu}(1 \to 2) = \frac{ik\,\Delta}{ak^{(2)}_{m\nu}\left[k^{(2)}_{m\nu} - k^{(1)}_{mn}\right]} \cdot \frac{L_+\left(k^{(2)}_{m\nu}\right)\mu^{(1)}_{mn}\mu^{(2)}_{m\nu}}{L_+\left(k^{(1)}_{mn}\right)} \tag{68}$$

where we have also used equation (57).

The reciprocity relations can be readily established from equations (67) and (68):

$$k^{(1)}_{mj}R^m_{nj}(1) = k^{(1)}_{mn}R^m_{jn}(1). \tag{69}$$

$$k^{(2)}_{m\nu}T^m_{n\nu}\ (1 \to 2) = k^{(1)}_{mn}T^m_{\nu n}\ (2 \to 1). \tag{70}$$

Here $T^m_{\nu n}(2 \to 1)$ is the transmission coefficient for the (m,n) mode generated in the duct 1 upon the incidence of the (m,ν) mode from duct 2. Recall that reciprocity is a characteristic of a system linear vibration. For a more detailed discussion, one may refer to reference 10.

Small Admittance Shift

When the admittance shift is small such that $|ka\,\Delta| \ll 1$, equations (67) and (68) become simple expressions: As detailed in appendix C, the reflection and the transmission coefficients are written as

$$R^m_{nj}(1) \cong \frac{-ik\,\Delta\,\mu^{(1)}_{mn}\mu^{(1)}_{mj}}{ak^{(1)}_{mj}\left[k^{(1)}_{mn} + k^{(1)}_{mj}\right]} + 0\left[(ka\,\Delta)^2\right], \tag{71}$$

$$T^m_{n\nu}(1 \to 2) \cong 1 - \frac{ik\,\Delta}{2a}\left[\frac{\mu^{(1)}_{mn}}{k^{(1)}_{mn}}\right]^2 + 0\left[(ka\,\Delta)^2\right] \qquad \text{for } \nu = n, \tag{72a}$$

$$\cong \frac{ik\,\Delta\,\mu^{(1)}_{mn}\mu^{(1)}_{m\nu}}{ak^{(1)}_{m\nu}\left[k^{(1)}_{m\nu} - k^{(1)}_{mn}\right]} + 0\left[(ka\,\Delta)^2\right] \qquad \text{for } \nu \neq n. \tag{72b}$$

As expected, a very small admittance shift causes very little reflection or very little mode scattering in the transmission. As Δ increases with a fixed

ORIGINAL PAGE IS
OF POOR QUALITY

η_1, the reflection coefficient increases linearly with Δ. The behavior of the transmission coefficient may be discussed in two separate cases. For $\nu = n$, the transmission coefficient deviates linearly with Δ from unity. For $\nu = n$, the transmission coefficient linearly increases with Δ, and is inversely proportional to $k_{m\nu}^{(1)} - k_{mn}^{(1)}$.

Low Frequency Limit: $ka \ll 1$ or $|\Delta|$

In this case, one may be interested in the incidence of the fundamental mode only. For simplicity, we also assume $\eta_1 = 0$, that is, the duct element 1 is hard-walled. The reflection and the transmission coefficients are then approximated as

$$R_{00}^{0}(1) \cong -\left[1 - \sqrt{\frac{2ka}{i\,\Delta}} + 0\left(\frac{ka}{\Delta}\right)\right], \tag{73}$$

$$T_{00}^{0}(1 \to 2) \cong \sqrt{\frac{2ka}{i\Delta}} + 0\left(\frac{ka}{\Delta}\right). \tag{74}$$

As $ka \to 0$, the wave will be completely reflected from the impedance discontinuity. As the frequency increases from zero, the amplitude of the reflection coefficient decreases from unity as $\sqrt{ka}$. It is interesting to compare this with the low frequency behavior of the reflection from the open end of the duct. For the latter case,[11] the reflection coefficient is

$$R_{00}^{0} \cong -\left[1 - \frac{(ka)^2}{2}\right]. \tag{75}$$

CONCLUDING REMARKS

The Wiener-Hopf technique has been employed to study modal scattering of sound in a circular cylinder subjected to an axial discontinuity of the wall impedance. Analytical expressions have been derived for the reflection and the transmission coefficients. The analytical expressions display some symmetry properties such as the reciprocity relations, and can be immediately evaluated by numerical integration of the factors in the reflection and the transmission coefficients without calculating the complex modal eigenvalues.

The quantitative discussion was confined to the limiting cases of small admittance shift, and of low frequency. With the small admittance shift, the reflection and the transmission coefficients depend linearly on the admittance shift.

The case of low frequency includes the fundamental mode that is incident from a hard-wall duct and reflected at the junction with a soft-wall duct. At the very low frequency, the reflection coefficient is almost unity in the amplitude and with phase change of 180 degrees. This result is similar to the low frequency limit for the reflection of the fundamental mode from an open end of a hard-wall duct. However, the deviation from unity with increasing frequency is quite different for the two cases. For the present problem, the deviation is proportional to the square root of the frequency, whereas, for

the reflection from the open end, the deviation is proportional to the square of the frequency.

Another low frequency characteristic is that the deviation of the reflection coefficient from unity is inversely proportional to the square root of the admittance of the soft wall.

ORIGINAL PAGE IS
OF POOR QUALITY

APPENDIX A

FOURIER TRANSFORM OF GREEN'S FUNCTION

The function Γ_m can be obtained directly from equation (28) by using equation (46). The procedure is, however, cumbersome. We employ a more elegant method in which Γ_m is obtained as a solution of the Fourier transformed equation. The equation for the reduced Green's function is

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial x^2} + k^2\right) \gamma_m(r,r_0;x - x_0) = \frac{\delta(r - r_0)\delta(x - x_0)}{r}. \tag{A1}$$

Consider the integration

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \text{eq. (A1)} e^{i\beta\zeta}\, d\zeta \tag{A2}$$

where $\zeta = x - x_0$. Effecting this integration, one obtains

$$\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{m^2}{r^2} + \frac{\alpha^2}{a^2}\right) \Gamma_m(r,r_0;\beta) = \frac{\delta(r - r_0)}{\sqrt{2\pi}\, r}, \tag{A3}$$

where

$$\alpha^2 = (k^2 - \beta^2)a^2. \tag{A4}$$

The solution can be expressed as

$$\Gamma_m(r,r_0;\beta) = \frac{1}{\sqrt{2\pi}\, r_0 W(y_1,y_2)\Big|_{r=r_0}} \begin{cases} y_1(r_0)y_2(r) & \text{for } r_0 < r \\ y_1(r)y_2(r_0) & \text{for } r < r_0 . \end{cases} \tag{A5}$$

Here y_1 and y_2 are homogeneous solutions of equation (A3), and W is the Wronskian. It follows from equation (A5) that $y_1(r)$ should be analytic at $r = 0$, and thus one has

$$y_1(r) = J_m\left(\alpha \frac{r}{a}\right). \tag{A6}$$

The solution y_2 is linearly independent of y_1, and is here chosen as

$$y_2 = J_m\left(\alpha \frac{r}{a}\right) + BY_m\left(\alpha \frac{r}{a}\right). \tag{A7}$$

The constant B is determined from the boundary condition at $r = a$ as given in equation (13), which was imposed on the Green's function:

ORIGINAL PAGE IS
OF POOR QUALITY

$$\frac{dy_2}{dr} - ikn_1 y_2 = 0 \qquad \text{at} \quad r = a. \tag{A8}$$

On inserting equation (A7) into equation (A8), one obtains

$$B = -\frac{\alpha J_m'(\alpha) - ikan_1 J_m(\alpha)}{\alpha Y_m'(\alpha) - ikan_1 Y_m(\alpha)}. \tag{A9}$$

The Wronskian is obtained as

$$W = \frac{2B}{\pi r_0}. \tag{A10}$$

Summarizing equations (A5) to (A10), one obtains for $r = r_0 = a$

$$\Gamma_m(a,a;\beta) = \frac{-1}{\sqrt{2\pi}} \frac{J_m(\alpha)}{\alpha J_m'(\alpha) - ikan_1 J_m(\alpha)}. \tag{A11}$$

Using the asymptotic form of J_m for a large argument,[12] one can show

$$\Gamma_m(a,a;\beta) \xrightarrow{|\beta| \to \infty} 0\left(\frac{1}{\beta}\right). \tag{A12}$$

ORIGINAL PAGE IS
OF POOR QUALITY

APPENDIX B

INFINITE PRODUCT TERMS

The infinite products contained explicitly in equations (63) and (65) will be expressed in terms of the factor L_+.

Let D_1 denote the infinite product in equation (63):

$$D_1 = \prod_{\ell=1}^{\infty} \left[1 - \frac{k_{mj}^{(1)}}{k_{m\ell}^{(2)}}\right] \Bigg/ \prod_{\ell \neq j}^{\infty} \left[1 - \frac{k_{mj}^{(1)}}{k_{m\ell}^{(2)}}\right]. \tag{B1}$$

Compared with equation (56), this product can be written as

$$D_1 = \lim_{\beta \to k_{mj}^{(1)}} \frac{1}{L_-(\beta)} \left[1 - \frac{\beta}{k_{mj}^{(1)}}\right]. \tag{B2}$$

Using equation (54) and $L_+(\beta)$ being analytic in the UHP, one can write equation (B2) as

$$D_1 = \frac{1}{L_+\left(k_{mj}^{(1)}\right)} \lim_{\beta \to k_{mj}^{(1)}} L(\beta) \left[1 - \frac{\beta}{k_{mj}^{(1)}}\right]. \tag{B3}$$

As noticed from equation (51) or (52), $L(\beta)$ possesses a simple pole at $\beta = k_{mj}^{(1)}$. Near the pole, $L(\beta)$ can be expanded as

$$L(\beta) \xrightarrow{\beta \to k_{mj}^{(1)}} \frac{\left[\mu_{mj}^{(1)}\right]^2 \left[\alpha_{mj}^{(1)} J_m'\left(\alpha_{mj}^{(1)}\right) - ika\eta_2 J_m\left(\alpha_{mj}^{(1)}\right)\right]}{a^2 k_{mj}^{(1)} J_m\left(\alpha_{mj}^{(1)}\right)\left[\beta - k_{mj}^{(1)}\right]}. \tag{B4}$$

On inserting equation (B4) into equation (B3), one obtains

$$D_1 = \frac{ika\,\Delta}{L_+\left(k_{mj}^{(1)}\right)} \cdot \left[\frac{\mu_{mj}^{(1)}}{aK_{mj}^{(1)}}\right]^2 \tag{B5}$$

where we have used

$$\alpha_{mj}^{(1)} J_m'\left(\alpha_{mj}^{(1)}\right) - ika\eta_1 J_m\left(\alpha_{mj}^{(1)}\right) = 0. \tag{B6}$$

Consider now the infinite product in equation (65):

ORIGINAL PAGE IS
OF POOR QUALITY

$$D_2 = \prod_{\ell=0}^{\infty} \left[1 - \frac{k_{m\nu}^{(2)}}{k_{m\ell}^{(1)}}\right] \Bigg/ \prod_{\ell\neq\nu}^{\infty} \left[1 - \frac{k_{m\nu}^{(2)}}{k_{m\ell}^{(2)}}\right]. \tag{B7}$$

Compared with equation (56), this product is written as

$$D_2 = \lim_{\beta \to k_{m\nu}^{(2)}} L_-(\beta) \left[1 - \frac{\beta}{k_{m\nu}^{(2)}}\right]. \tag{B8}$$

Using equation (54), one obtains

$$D_2 = L_+\left(k_{m\nu}^{(2)}\right) \lim_{\beta \to k_{m\nu}^{(2)}} \frac{1}{L(\beta)} \left[1 - \frac{\beta}{k_{m\nu}^{(2)}}\right]. \tag{B9}$$

Using the expansion of $L(\beta)$ near the simple zero $\beta = k_{m\nu}^{(2)}$, one obtains

$$D_2 = -ika\,\Delta\, L_+\left(k_{m\nu}^{(2)}\right) \left[\frac{\mu_{mj}^{(2)}}{ak_{m\nu}^{(2)}}\right]^2. \tag{B10}$$

ORIGINAL PAGE IS
OF POOR QUALITY

APPENDIX C

SMALL ADMITTANCE EXPRESSIONS

Here we will determine the limiting values of the eigenvalues, the propagation constants, and the factor L+. These limits have been used to derive equations (71) to (74) from equations (67 and (68).

Small Impedance Jump: $|ka\,\Delta| << 1$.

Consider the eigenvalue equations derivable from equations (13) and (14):

$$\alpha^{(1)} J_m'\left(\alpha^{(1)}\right) - ikan_1 J_m\left(\alpha^{(1)}\right) = 0, \tag{C1}$$

$$\alpha^{(2)} J_m'\left(\alpha^{(2)}\right) - ikan_2 J_m\left(\alpha^{(2)}\right) = 0. \tag{C2}$$

Set

$$\alpha^{(2)} - \alpha^{(1)} = \epsilon. \tag{C3}$$

In the limit of $|ka\,\Delta| << 1$, one expects that

$$|\epsilon| << \left|\alpha^{(1)}\right|. \tag{C4}$$

To determine ϵ, equation (C2) is expanded around $\alpha^{(1)}$ and equation (C1) is used to obtain an algebraic equation for ϵ. With the first order expansion, one obtains

$$\epsilon_{m\ell} \cong \frac{-ika\,\Delta\left[\mu_{m\ell}^{(1)}\right]^2}{\alpha_{m\ell}^{(1)}} \tag{C5}$$

where the mode numbers (m,ℓ) are used as the subscripts on ϵ and $\alpha^{(1)}$. Equation (C5) obviously satisfies the inequality (C4). The equation is not valid for the case of $\alpha_{m\ell}^{(1)} = 0$ which is separately treated later. Also it should be mentioned that the derivation of equation (72a) requires the second order expansion, which will not be discussed here.

For the propagation constants, one sets

$$a\left[k_{m\ell}^{(2)} - k_{m\ell}^{(1)}\right] = \kappa_{m\ell}. \tag{C6}$$

Using equations (10), (C3), and (C5), one obtains

ORIGINAL PAGE IS
OF POOR QUALITY

$$\kappa_{m\ell} \approx \frac{ika\,\Delta\left[\mu_{m\ell}^{(1)}\right]^2}{ak_{m\ell}^{(1)}}. \tag{C7}$$

Unless the frequency is close to the cutoff frequency of the (m,ℓ) mode, equation (C7) satisfies the inequality

$$\left|\kappa_{m\ell}\right| << \left|ak_{m\ell}^{(1)}\right|. \tag{C8}$$

Set

$$L_+(\beta) = L_0 g_2(\beta)/g_1(\beta) \tag{C9}$$

where

$$g_s(\beta) = \prod_{\ell=0}^{\infty}\left[1 + \frac{\beta}{k_{m\ell}^{(s)}}\right], \qquad s = 1,2. \tag{C10}$$

Using equation (C6), one may obtain

$$\left.\begin{aligned} g_2(\beta) &\approx \prod_{\ell=0}^{\infty}\left\{1 + \frac{\beta}{k_{m\ell}^{(1)}} - \frac{\beta K_{m\ell}}{a\left[k_{m\ell}^{(1)}\right]^2}\right\} \\ &\approx g_1(\beta)\left\{1 - \sum_{\ell=0}^{\infty}\frac{K_{m\ell}}{ak_{m\ell}^{(1)}\left[1 + k_{m\ell}^{(1)}/\beta\right]}\right\}. \end{aligned}\right\} \tag{C11}$$

With the substitution of equation (C7), equation (C11) is written as

$$g_2(\beta) \approx g_1(\beta)\left[1 - ika\,\Delta\,x(\beta)\right], \tag{C12}$$

where

$$x(\beta) = \sum_{\ell=0}^{\infty}\left[\frac{\mu_{m\ell}^{(1)}}{k_{m\ell}^{(1)}}\right]^2 \Bigg/ \left[1 + \frac{k_{m\ell}^{(1)}}{\beta}\right]. \tag{C13}$$

The function $x(\beta)$ is a finite and well behaved function of β in the UHP. For equation (53),

ORIGINAL PAGE IS
OF POOR QUALITY

$$L_0 = 1 - \frac{ika\,\Delta}{akJ_m'(ka) - ikan_1 J_m(ka)}. \tag{C14}$$

From equations (C9), (C12), and (C14),

$$L_+(\beta) = 1 - ika\,\Delta\left[x(\beta) - \frac{1}{kaJ_m'(ka) - ikan_1 J_m(ka)}\right]. \tag{C15}$$

On inserting equation (C15) into equations (67) and (68), one obtains equations (71) and (72b). As mentioned earlier, the second order expansion is required for equation (72a). For equation (72b), we have also used

$$\mu_{m\nu}^{(2)} \cong \mu_{m\nu}^{(1)}\left[1 + O(ka\,\Delta)\right]. \tag{C16}$$

Low Frequency Limit: $ka \ll 1$ and $|\Delta|$

As in the text, we consider the case of $n_1 = 0$ and of the incidence of the (o,o) mode (the plane wave mode). We have then

$$\Delta = n_2,\ \alpha_{00}^{(2)} = 0,\ k_{00}^{(1)} = k. \tag{C17}$$

Set $\alpha_{00}^{(2)} = \epsilon$, then the expansion of equation (C2) yields

$$\epsilon \cong \sqrt{-2ika\,\Delta} \tag{C18}$$

and

$$ak_{00}^{(2)} \cong \sqrt{2ika\,\Delta}, \tag{C19}$$

$$L_0 \cong \frac{2in}{ka}, \tag{C20}$$

$$L_+\left(k_{00}^{(1)}\right) \cong \frac{i\,\Delta}{ka}\left(1 + \sqrt{\frac{ka}{2in}}\right), \tag{C21}$$

$$L_+\left(k_{00}^{(2)}\right) \cong 2\,\frac{\sqrt{2in}}{ka}\left(1 - \sqrt{\frac{ka}{2in}}\right), \tag{C22}$$

$$\mu_{00}^{(2)} \cong \mu_{00}^{(1)} = 1. \tag{C23}$$

ORIGINAL PAGE IS
OF POOR QUALITY

REFERENCES

1. Heidelberg, L. J., Rice, E. J., and Homyak, L., "Acoustic Performance of Inlet Suppressors on an Engine Generating a Single Mode," AIAA Paper 81-1965, Oct. 1981.
2. Noble, B., Methods Based on the Wiener-Hopf Technique, Pergamon Press, New York, 1958.
3. Heins, A. E. and Feshbach, H., "The Coupling of Two Acoustical Ducts," Journal of Mathematics and Physics, Vol. 26, 1947, pp. 143-155.
4. Morse, P. M. and Feshbach, H., Methods of Theoretical Physics, McGraw-Hill Book Co., New York, 1953, Ch. 11.
5. Koch, W., "Attenuation of Sound in Multi-element Acoustically Lined Rectangular Ducts in the Absence of Mean Flow," Journal of Sound and Vibration, Vol. 52, No. 4, 1977, pp. 459-496.
6. Morse, P. M. and Feshbach, H., Methods of Theoretical Physics, McGraw-Hill Book Co., New York, 1953, Ch. 7.
7. Morse, P. M. and Feshbach, H., Methods of Theoretical Physics, McGraw-Hill Book Co., New York, 1953, p. 385.
8. Morse, P. M. and Feshbach, H., Methods of Theoretical Physics, McGraw-Hill Book Co., New York, 1953, p. 462.
9. Morse, P. M. and Feshbach, H., Methods of Theoretical Physics, McGraw-Hill Book Co., New York, 1953, p. 381.
10. Cho, Y.-C., "Reciprocity Principle in Duct Acoustics," Journal of Acoustical Society of America," Vol. 67, No. 5, May 1980, pp. 1421-1426.
11. Levine, H. and Schwinger, J., "On the Radiation of Sound from an Unflanged Circular Pipe," Physical Review, Vol. 73, No. 4, Feb. 1948, pp. 383-406. See eq. (VII.1).
12. McLachlan, N. W., Bessel Functions for Engineers, 2nd ed., Clarendon Press, New York, 1955.

ORIGINAL PAGE IS
OF POOR QUALITY

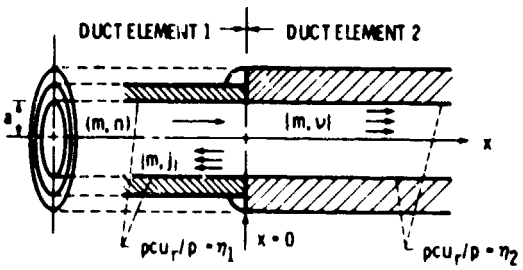
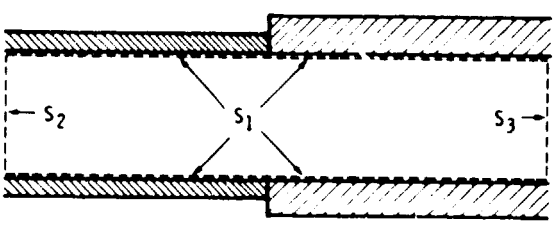


Figure 1. - Circular cylinder with admittance shift ($\Delta = \eta_2 - \eta_1$) at $x = 0$, (m, n) mode incident; $\{m, j\}$ modes reflected; $\{m, \nu\}$ modes transmitted.



$S = S_1 + S_2 + S_3$

Figure 2. - Integration surface.

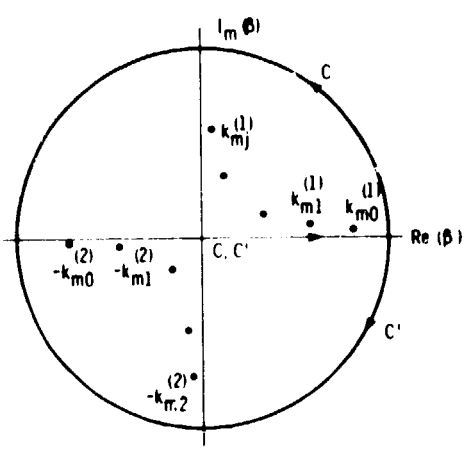


Figure 3. - Integration paths C and C' and simple poles.