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ABSTRACT

The method of matched asymptotic expansions is used to study the generation of Tollmien-Schlichting waves by free stream disturbances incident on a flat plate boundary layer. Near the leading edge the motion is governed by the unsteady boundary layer equation, while farther downstream it is governed (to lowest order) by the Orr-Sommerfeld equation with slowly varying coefficients. It is shown that there is an overlap domain where the Tollmien-Schlichting wave solutions to the Orr-Sommerfeld equation and an appropriate asymptotic solution of the unsteady boundary layer equation match, in the matched asymptotic expansion sense. The analysis leads to a set of scaling laws for the asymptotic structure of the unsteady boundary layer.

1. INTRODUCTION

It is well known that laminar to turbulent transition in boundary layers is strongly influenced by unsteady disturbances in the free stream. This is often the result of a sequence of events that begins with the excitation of spatially growing Tollmien-Schlichting waves by the free-stream disturbances. This so-called receptivity[1] phenomenon was

[1]The term 'receptivity' was first introduced by Morkovin (ref. 1).

discussed in a recent review article by Reshotko (ref. 2). When the free-stream disturbances are periodic in time and of sufficiently small amplitude, the Tollmein-Schlichting waves will also be periodic.

The receptivity problem differs from classical stability theory in that the former is a boundary value problem while the latter is an eigenvalue problem. Since the time harmonic Tollmien-Schlichting waves are eigenvalues of the Orr-Sommerfeld equation, which applies in the downstream region where the mean flow is nearly parallel, one can always add an arbitrary multiple of these waves to the solution of the boundary value problem and still satisfy the boundary conditions and the governing equations, unless an upstream boundary condition (i.e., an initial condition) is imposed at the start of the boundary layer. Thus, the Tollmien-Schlichting waves will be decoupled from the solution to the boundary value problem in a time-stationary flow, unless this upstream boundary condition is imposed.

An analogous situation arises in the temporal stability problem (ref. 3). Here a prescribed initial disturbance field can be coupled to temporally growing instability waves by imposing appropriate initial conditions. The solution is usually required to satisfy causality in the sense that it is assumed to be identically zero before the disturbance is 'turned on' at some prescribed initial time. There have been attempts to apply causality arguments to time-stationary (i.e., steady state) flows in order to explain the observed coupling between instability waves and the external disturbance field, but, as argued by Rienstra (ref. 4), it is hard to see how initial conditions imposed in the distant past can affect the steady-state solution. The upstream initial condition appears to be the appropriate one to impose in this case.

However, the resulting spatial initial value problem is considerably more difficult than the temporal problem. Near the leading edge of the boundary layer (actually within a region that occupies the first few wavelengths of the boundary layer), the divergence of the mean flow has a first order effect on the unsteady motion, rather than being a higher order effect that can be treated as a 'slowly varying' correction to classical parallel flow stability theory. In this region, inertia terms involving the cross-stream component of the mean flow velocity have to be included in the lowest order equation for the unsteady flow. However, one can neglect unsteady pressure fluctuations across the mean boundary layer, which is still relatively thin (on a wavelength scale). The flow is then governed by the unsteady boundary layer equation rather than by an Orr-Sommerfeld equation with slowly varying coefficients.

This latter equation, whose eigensolutions are the Tollmien-Schlichting waves, is only valid further downstream. The upstream initial condition for the solution to this equation should therefore be that it 'match', in the 'matched asymptotic expansion' sense (ref. 5), onto a solution of the unsteady boundary layer equation in some intermediate region that overlaps the unsteady boundary layer and Orr-Sommerfeld regions.

The main purpose of this paper is to carry out this matching in a formal but systematic way. This allows us to determine the dominant Reynolds number scaling of the amplitudes of the Tollmien-Schlichting waves; i.e., to determine their amplitudes to within an O(1) constant. The determination of this constant, which involves the numerical solution of the unsteady boundary layer equation, is beyond the scope of this paper. It will be carried out in a subsequent paper by analytically continuing the unsteady boundary layer solution into the complex plane.

In order to reduce this problem to its simplest terms we restrict our attention to a two-dimensional incompressible flow over an infinitely thin flat plate. The amplitude of the disturbance field is assumed to be small relative to the mean free-stream velocity, U_∞, and the equations are linearized about the mean flow. Since the unsteady flow is assumed to be time stationary only a single harmonic component of the disturbance field, say of frequency ω, need be considered. Finally, we assume that the Reynolds number based on the 'convective' wavelength U_∞/ω of the disturbance is large. Even though we assume that the plate is inifinitely thin, we ultimately show that our principal results apply to any flat plate whose 'nose radius' is $O(U_\infty/\omega)$. We could easily have considered the finite thickness flat plate at the outset, but this would have complicated the presentation.

Our approach is to take the reciprocal of this Reynolds number, which we denote by ϵ^6, as a small parameter and obtain a uniformly valid asymptotic expansion in this parameter. We suppose that the streamwise wave number of the imposed disturbance (i.e., its inverse spatial scale) is[2] $O(\omega/U_\infty)$. Then allowing $\epsilon \to 0$ while assuming that $x = \omega x^\dagger/U_\infty$ is order one, where $x^\dagger$ denotes the stream-wise distance from the leading edge (fig. 1), one obtains the unsteady boundary layer equation to lowest order of approximation (ref. 6).

Lighthill (ref. 7) obtained asymptotic solutions to this equation that are valid close to and far from the leading edge and connected them using the Karman-Pohlhausen method. In later work by Rott and Rosenzweig (ref 8), Lam and Rott (ref. 9), and Ackerberg and Phillips (ref. 10), the two asymptotic solutions were joined by analytical series and numerical methods.

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[2]Notice that this includes the zero wave number disturbance corresponding to a uniform oscillation of the stream (as a plane acoustic wave in the incompressible limit).

However, one only needs to know the asymptotic form of the unsteady boundary layer solution as $x \to \infty$ (i.e., far downstream) in order to show that it can be matched onto the Tollmien-Schlichting wave solution of the Orr-Sommerfeld equation. This asymptotic behavior was studied by Rott and Rosenzweig (ref. 8), Lam and Rott (ref. 9), and Ackerberg and Phillips (ref. 10). Their work shows that the solution develops a double layer structure in this downstream region (actually this structure begins to develop when $x \gtrsim 1$; see fig. 1). The inner layer is a Stokes shear-wave flow to lowest order, and the outer flow is a modified Blasius motion. In fact, this flow will be identical to a Stokes shear wave to lowest order (ref. 10) whenever the inviscid velocity perturbation becomes independent of x for large x.

Ackerberg and Phillips (ref. 10) show that this limiting asymptotic behavior is not approached monotonically, but rather through a progressively more damped oscillation about the asymptotic Stokes wave-type solution. They argue that these oscillations are represented mathematically by a discrete set of exponentially small 'asymptotic eigensolutions' of the unsteady boundary layer equation, which were originally discovered by Lam and Rott (ref. 9). Like the Stokes-type solution, these asymptotic eigensolutions exhibit a double layer structure. Ackerberg and Phillips (ref. 10) used the method of matched asymptotic expansions to obtain expressions for these eigensolutions that are uniformly valid in η for $0 \leq \eta < \infty$, where η is the Blasius variable; i.e., it is the cross-stream coordinate divided by the local thickness of the steady boundary layer. However, it turns out that these expressions are not quite correct as they stand. Thus, it is shown in section 3 that they still satisfy the unsteady boundary layer equations to the same degree of approximation when they are multiplied by x^{τ} for any constant τ. But, it is also shown that there is only one value of exponent τ for which the next order equations can be solved, and this solvability condition uniquely (i.e., to within a constant factor) determines the lowest order asymptotic eigensolutions.

Ackerberg and Phillips (ref. 10) and Lam and Rott (ref. 9) point out that the Stokes-type solution is essentially 'incomplete' because it is uniquely determined independently of the upstream conditions that must always be imposed when solving a parabolic partial differential equation. The downstream unsteady boundary layer solution, therefore, consists of the Stokes layer-type solution supplemented by a set of asymptotic eigensolutions. The arbitrary constants that multiply these solutions can be determined from the upstream conditions. This was done numerically by Ackerberg and Phillips (ref. 10), who used the asymptotic eigensolutions of Lam and Rott (ref. 9). These solutions, being exponentially small compared with the Stokes-type solution (which is an asymptotic power series in $x^{-n/2}$), are negligible in the strict Poincere' sense, which, while not explicitly excluding exponentially small terms, does not specify or confirm their existence. But table makers have long known that such terms must often be included in order to obtain numerically satisfactory results (see e.g., ref. 11 and ref. 12, pp. 266-268), and Ackerberg and Phillips argue that the Lam and Rott eigenfunctions are numerically significant in the downstream part of the boundary layer. Moreover, it is now generally recognized that strict Poincere' expansions are quite often incomplete, even when the general term in the expansion is known. Modern researchers often interpret asymptotic expansions in the complete sense of Watson (ref. 12, p. 543 and ref. 13) rather than in the Poincere' sense. Dingle (ref. 14, pp. 19 and 20) points out that retention of exponentially small terms is not only necessary for computational purposes, but also for ensuring that the expansions have the same analytical properties as the original function. Although no entirely satisfactory definition of completeness has yet been given (ref. 14, pp. 19 and 20; ref. 11 and ref. 12, p. 543), it is clear that asymptotic eigenfunctions must be included in any complete solution to the present problem. However, we cannot be sure that the resulting solution will then be complete in the sense that no other functionally independent asymptotic eigensolutions are possible. In fact, Brown and Stewartson (ref. 14) found an entirely different set of discrete asymptotic eigensolutions of the unsteady boundary layer equation, which, they argue, better represent the physics of the flow. There may also be asymptotic eigensolutions with a continuous spectrum. However, Brown and Stewartson's asymptotic eigensolutions may not be independent of the asymptotic eigensolutions of Lam and Rott, and either set of eigensolutions may, therefore, be used to obtain a complete solution; i.e., it may be possible to express any linear combination of one set as a linear combination of the other. After all, the wave equation has a complete set of plane wave solutions and a complete set of cylindrical wave solutions in any region that does not include the origin. An even more relevant example is provided by Murdock and Stewartson (ref. 16), who show that the asymptotic solution to a model equation, not unlike the unsteady boundary layer equation, can alternately be expressed as the sum of two different sets of eigensolutions and that each of these sums can be re-expanded to yield the other. However, this example tends to indicate that it is the solutions belonging to the continuous spectrum of the asymptotic unsteady boundary layer equation that can be re-expanded in terms of the asymptotic eigenfunctions of Brown and Stewartson. This issue will be discussed more fully in a forthcoming paper.

The 'asymptotic eigensolutions' of Lam and Rott, which are proportional to $\exp - \lambda_0 x^{3/2}$, where λ_0 is a complex constant, oscillate with a wavelength ($\sim x^{-1/2}$) that decreases with increasing x, while the mean boundary layer thickness increases at the same rate. Thus, the wavelength of the unsteady motion must ultimately become comparable to the boundary layer thickness, and the cross-stream pressure fluctuations, which are neglected in the unsteady boundary layer approximation, will then become important. The Lam and Rott eigensolutions, which are based on this approximation, will then be invalid (i.e., they will not be asymptotic solutions to the full Navier-Stokes equations).

We obtain new solutions, which apply further downstream than asymptotic eigensolutions of the unsteady boundary layer equation, by using a generalization of the method of multiple scales (ref. 17, pp. 276-282) and considering the limiting form of the governing equation as $\epsilon \to 0$ with $x_1 \equiv \epsilon^2 x$ (rather than x) held fixed. This leads to solutions that apply when $x = O(\epsilon^{-2})$ (fig. 1). They are essentially the classical large-Reynolds-number, small-wave-number approximation to the Tollmien-Schlichting wave solutions of the Orr-Sommerfeld equation, appropriately corrected for slow variation in boundary layer thickness. Thus, they decay exponentially fast in the downstream direction when x_1 is relatively small and at least one of them exhibits exponential growth when x_1 is sufficiently large. Since most of the asymptotic Tollmien-Schlichting theory predates the systematic use of matched asymptotic expansions[3], and since the classical theory does not account for the variations in boundary layer growth, it is necessary to rederive

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[3]Graebel (ref. 18) has made some progress toward introducing the method of matched asymptotic expansions into the theory, but his results cannot be used for the purposes required herein. Smith (ref. 19) obtained a solution that is related to the one obtained herein, but his scaling is different and his results cannot be used directly.

many results from the modern point of view - though we do attempt to use existing results as much as possible. The present approach leads to a set of asymptotic scaling laws for the unsteady motion, which we summarized in figure 1. The treatment of the nonparallel flow effects is somewhat different from that used in previous studies of this effect (see, e.g., Saric and Nayfeh, ref. 20; Gaster, ref. 21). In this regard, it is worth noting that both the amplitude and phase of the present solution are unambiguously determined to within an arbitrary constant.

The present solution corresponds to the case where the 'critical' layer is near the wall so that there is only a single viscous layer. But there are two inviscid regions outside this wall layer - a main inviscid region where the unsteady velocity is quasi-steady and, therefore, has the same cross-stream profile as the mean flow, and an outer region where unsteady effects are important, but where the mean flow is near uniform (fig. 1). This three-level structure is similar to the triple-deck structure found in steady boundary layers (Stewartson, ref. 22; Messiter, ref. 23; Smith, ref. 19).

Unlike the classical Tollmien-Schlichting solution, the present result is multiplied by a slowly varying function of x that is determined by a solvability condition for the next order solution.

Finally, we show (in section 5) that there exists an overlap domain (where x is large and x_1 is small) in which each of these outer Tollmien-Schlichting-like solutions match, in the matched asymptotic expansion sense (ref. 5), with a Lam and Rott asymptotic eigensolution of the unsteady boundary layer equation. However, only the lowest order asymptotic eigensolution matches onto a Tollmein-Schlichting wave that exhibits spatial growth.

The amplitude of each Tollmien-Schlichting wave is equal to ϵ^s times a function whose order of magnitude is unity. Matching with the asymptotic eigensolutions allows us to determine the exponent s. The O(1) function is only determined to within a constant factor whose evaluation involves the numerical solution of the unsteady boundary layer equation, but this constant is of much less importance than the ϵ^s factor. The implications are discussed at the end of section 6.

The progressive wavelength reduction of the Lam and Rott asymptotic eigensolutions that we referred to above is a sort of 'tuning' mechanism, which allows free-stream disturbances to couple with Tollmein-Schlichting waves, even when the streamwise wavelengths of the former are vastly different from those of the latter. In section 5, we show that this reduction occurs because the asymptotic eigensolutions can produce no pressure fluctuations and must, therefore, behave somewhat like convected disturbances propagating into a region of decreasing streamwise velocity. Since a convected disturbance is one with zero convective derivative, $(\partial/\partial t) + U(\partial/\partial x^+)$, where U is the mean velocity, its phase must be $t - \int(dx^+/U)$ and its wavelength must therefore decrease in the streamwise direction if U does. The importance of explaining this wavelength reduction mechanism was emphasized by Reshotko (ref. 2, p. 344).

The Stokes-type solution, of the boundary layer equation, remains uniformly valid in the downstream region and is, therefore, completely decoupled from the Tollmien-Schlichting waves. On the other hand, there may be other asymptotic eigensolutions of the unsteady boundary layer equation that do not have this property, and these may give rise to additional Tollmien-Schlichting waves. However, neither the eigenfunctions associated with the continuous spectrum nor the discrete eigenfunctions of Brown and Stewartson undergo enough wavelength reduction to become

Tollmien-Schlichting waves, and these are the only sets of asymptotic eigenfunctions that have been identified thus far. We do not pursue this question here, but merely note that the present results show that a spatially growing Tollmien-Schlichting wave will be generated if the lowest order Lam and Rott asymptotic eigensolution constitutes a functionally independent component of the asymptotic expansion of the solution to the unsteady boundary layer equation[4], no matter how small it may be and no matter what other components may be present. We believe that the present analysis then provides a complete description of the physics of the generation process, including the spatial evolution of the Tollmien-Schlichting waves before they reach the lower branch of the neutral stability curve.

There have been some previous studies of the boundary layer receptivity problem. Murdock (ref. 24) studied the response of an incompressible flat plate boundary layer to a plane uniformly pulsating stream by numerically solving the fully nonlinear parabolized Navier-Stokes equations in a limited region of the downstream boundary layer. He imposed an upstream boundary condition at the forward edge of this region that required his solution to equal the Stokes layer solution at this point.

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[4]In the forthcoming paper alluded to above, we show that the lowest order Lam and Rott eigensolution actually does constitute such a component. This is done by analytically continuing the solution of the unsteady boundary layer equation (obtained by a very accurate numerical procedure) into the region of the complex plane where the lowest order Lam and Rott eigensolution is dominant. This also allows us to fix the constant that multiplies this solution, and, consequently, the constant that multiplies the unstable Tollmien-Schlichting wave.

The present work implies that free-stream disturbances that enter the mean boundary layer near the leading edge (i.e., within about a wavelength), where the effect of mean flow divergence is very important, can induce unsteady motion in the boundary layer that decays exponentially as it propagates downstream until cross-stream inertia effects destabilize the flow and the motion turns into a spatially growing instability wave. However, the unsteady disturbance takes on the character of a decaying Tollmien-Schlichting wave (i.e., it has the same cross-stream velocity profile and represents a propagating wave) well within the unsteady boundary layer region, that is, at a point where $x = 1$. It is shown that the cross-stream inertia effects ultimately inhibit the progressive wavelength reduction, which is responsible for producing them, and a sort of quasi-equilibrium is reached.

Professor Eli Reshotko of Case Western Reserve University and Professor Franklin K. Moore of Cornell University made many helpful comments during the course of this work. Bruce Auer carried out the numerical computations.

2. FORMULATION

We consider a two-dimensional incompressible flow of density ρ and kinematic viscosity ν over a semi-infinite flat plate as shown schematically in figure 1. Far upstream the motion consists of a uniform flow with velocity U_∞ plus a small amplitude harmonic perturbation of frequency ω. We suppose that the velocity $\vec{v} = \{u,v\}$ has been nondimensionalized by U_∞. We also suppose that the time t has been nondimensionalized by ω^{-1} and that the Cartesian coordinates $\vec{x} = \{x,y\}$ have been nondimensionalized by U_∞/ω. The plate is assumed to be located at $y = 0$, $x > 0$.

In the absence of viscosity, the velocity at the surface of the plate would be of the form

$$u = 1 + u_\infty(x)e^{it}, \qquad v = 0, \qquad x > 0 \tag{2.1}$$

where u_∞ is assumed to be much less than one. In the general case the motion is governed by the two-dimensional momentum and continuity equations, which can be written in terms of the dimensionless vorticity $-\Omega$ and stream function Ψ as

$$\frac{\partial\Omega}{\partial t} + \frac{\partial(\Omega,\Psi)}{\partial(x,y)} = \epsilon^6\Delta\Omega \tag{2.2}$$

$$\Omega = \Delta\Psi \tag{2.3}$$

where $\partial(\Omega,\Psi)/\partial(x,y)$ denotes the Jacobian $(\partial\Omega/\partial x)(\partial\Psi/\partial y) - (\partial\Psi/\partial x)(\partial\Omega/\partial y)$, Δ denotes the Laplacian $(\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, and

$$\epsilon^6 \equiv \nu\omega/U_\infty^2 \tag{2.4}$$

is the reciprocal of the characteristic Reynolds number of the problem. The velocity components are given by

$$u = \frac{\partial\Psi}{\partial y}, \qquad v = -\frac{\partial\Psi}{\partial x} \tag{2.5}$$

so that on the surface of the plate

$$\Psi = \frac{\partial\Psi}{\partial y} = 0, \qquad y = 0, \qquad x > 0 \tag{2.6}$$

We are interested in the case where

$$\epsilon << 1 \tag{2.7}$$

$$\frac{du_\infty}{dx} = 0(u_\infty) \tag{2.8}$$

and

$$x^{-1} = 0(1)$$

Then x/ε^6, the Reynolds number based on the distance from the leading edge, will be large[5] and the viscous effects will be confined to a relatively thin region near the surface of the plate. Hence, the boundary condition at infinity is that Ψ match smoothly onto the inviscid solution at large values of y/ε^3. In fact, when the perturbation u_∞ is identically zero the resulting steady flow is given by the extended Blasius series (Goldstein, ref. 25, p. 142)

$$\Psi_{Bl.} = \varepsilon^3 \sqrt{2}\, \xi \left\{F(\eta) + 0\left(\varepsilon^6 \xi^{-2} \ln(\xi\varepsilon^{-3})\right)\right\} \tag{2.9}$$

where ξ, η are parabolic coordinates defined in the usual way by

$$z = x + iy = \chi^2 \tag{2.10}$$

$$\chi \equiv \xi + \frac{i\varepsilon^3}{\sqrt{2}} \eta \tag{2.11}$$

and F is the Blasius function, which is a solution of

$$F''' + FF'' = 0 \tag{2.12}$$

$$F(0) + F'(0) = 0 \tag{2.13}$$

$$F'(\eta) \to 1 + \text{exp. small terms} \qquad \text{as} \quad \eta \to \infty \tag{2.14}$$

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[5]We are excluding a relatively small region near the leading edge, but we shall see that this need not concern us here.

where the prime denotes differentiation with respect to η. Even though ω appears in the nondimensionalization of (2.2) and (2.3), it is clear that it does not appear in the Blasius solution (2.9).

Since we are interested in small amplitude motion, it is natural to linearize the solution about the Blasius solution (2.9). We only retain the linear terms in this amplitude expansion, but it will be necessary to consider higher order terms in the expansion in ϵ. However, these will all be of lower order than $\epsilon^6(\eta^2/\xi^2)$. Then it follows from (2.9) that we can approximate the steady solution by the Blasius solution $\sqrt{2}\epsilon^3\xi F(\eta)$ and seek a solution of the form

$$\Psi = \epsilon^3 \left[\sqrt{2}\ \xi F(\eta) + \psi(\xi^2,\eta)e^{-it} + \ . \ . \ .\right] \tag{2.15}$$

where

$$|\psi| << |\xi F(\eta)|$$

Moreover, equations (2.10) and (2.11) imply that

$$x = \xi^2\left[1 + O(\epsilon^6\eta^2/\xi^2)\right] \tag{2.16}$$

$$\left|\frac{dz}{dx}\right|^2 = 4\left|x\right|^2 = 4\xi^2\left[1 + O(\epsilon^6\eta^2/\xi^2)\right] \tag{2.17}$$

and that, to within this approximation,

$$\eta = y\epsilon^{-3}/\sqrt{2x} \tag{2.18}$$

Substituting (2.10), (2.11), and (2.15) into (2.2) and (2.3), subtracting (2.12), and neglecting quadratic terms in ψ, we obtain, upon using the approximation (2.17) to eliminate $|dz/dx|$ and the approximation (2.16) to reinsert the variable x in place of ξ,

$$-i\tilde{\Delta}\psi + \sqrt{x}\left[\frac{\partial\left(x^{-1}\tilde{\Delta}\psi,\ \sqrt{x}\,F\right)}{\partial(x,\eta)} + \frac{\partial\left(x^{-1/2}F'',\psi\right)}{\partial(x,\eta)}\right]$$

$$= \tilde{\Delta}\left(\frac{1}{2x}\,\tilde{\Delta}\psi\right) + O(\psi\epsilon^6\Lambda); \qquad \eta, x > 0 \tag{2.19}$$

where

$$\tilde{\Delta} \equiv \frac{\partial^2}{\partial\eta^2} + 2\epsilon^6 x\,\frac{\partial^2}{\partial x^2} + \epsilon^6\,\frac{\partial}{\partial x} \tag{2.20}$$

and

$$\Lambda \equiv \text{Max}\left\{\frac{\eta^2}{x},\ x^{-1}\right\}$$

This equation is sufficiently accurate to serve as a starting point for the present analysis. It must be solved subject to the boundary conditions

$$\psi = \frac{\partial\psi}{\partial\eta} = 0 \qquad \text{for} \quad \eta = 0,\ x > 0$$

since equation (2.18) shows that $\eta = 0$ on the surface of the plate. The solution to (2.19) must match onto the inviscid solution for large η.

3. UNSTEADY BOUNDARY LAYER REGION

We first consider the limit $\epsilon \to 0$ with $x = O(1)$. With the present nondimensionalization, this corresponds to letting the disturbance Reynolds number become infinite while keeping the streamwise distance at about a wavelength U_∞/ω from the leading edge. Then, in view of (2.8),

$$\psi = \psi_0(x,\eta) + O(\varepsilon^6) \tag{3.1}$$

and ψ_0 satisfies

$$-i\psi_{0\eta\eta} + \frac{\partial^2}{\partial\eta\partial x}(F'\psi_{0\eta} - F''\psi_0) - \frac{1}{2x}(F\psi_{0\eta})_{\eta\eta} = \frac{1}{2x}\psi_{0\eta\eta\eta\eta}$$

This equation can be integrated with respect to η to obtain the <u>linearized unsteady boundary equation</u>

$$\left(-i + F'\frac{\partial}{\partial x}\right)\psi_{0\eta} - F''\frac{\partial\psi_0}{\partial x} - \frac{1}{2x}\frac{\partial}{\partial\eta}(F\psi_{0\eta}) - \frac{1}{2x}\psi_{0\eta\eta\eta} = h(x) \tag{3.2}$$

where $h(x)$ is determined by the free stream pressure distribution. In fact, since this equation must be solved subject to the free stream boundary condition (see (2.1) and (2.15))

$$u_1 \equiv \varepsilon^3\psi_y = (2x)^{-1/2}\psi_{0\eta} \to u_\infty(x) + \text{exp. small terms} \qquad \text{as } \eta \to \infty, \tag{3.3}$$

it follows from (2.14) that

$$h(x) = \sqrt{2x}\left(\frac{\partial}{\partial x} - i\right)u_\infty(x) \tag{3.4}$$

On the surface of the plate, ψ_0 satisfies

$$\psi_0 = \psi_{0\eta} = 0, \qquad \eta = 0 \tag{3.5}$$

Equation (3.2), being essentially the boundary layer approximation to Navier-Stokes equation, neglects pressure variations across the layer, but accounts for the divergence or nonparallelism of the mean flow in that it retains inertia terms, like $V(\partial^2\psi/\partial y^2)$, where V is the mean cross-stream velocity.

As we indicated in the introduction, our interest here is in showing that the solution to this equation, which applies when $x = O(1)$, will match in the 'matched asymptotic expansion' sense (ref. 5) onto a Tollmien-Schlichting wave solution as x becomes large. For this purpose we need only know the large x asymptotic expansion of the former solution, which, as we already indicated, consists of a Stokes-type solution plus a number of 'asymptotic eigensolutions' which account for the effects of the upstream conditions. The Stokes-type solution remains uniformly valid in the downstream region, and is, therefore, effectively decoupled from the Tollmien-Schlichting wave that exists in that region. We therefore need only consider the asymptotic eigensolutions.

We consider only the asymptotic eigensolutions of Lam and Rott (ref. 9). Their work precedes the use of matched asymptotic expansion, but Ackerberg and Phillips (ref. 10) rederive their results using this approach. We refer mainly to this more modern work [6], which clearly shows that the asymptotic eigensolutions provide asymptotic solutions to the unsteady boundary layer equations, which are uniformly valid in η for $0 \le \epsilon \le \infty$ and exhibit a two-layer structure with adjustment to the wall boundary conditions taking place across a thin inner layer of the same thickness as the Stokes layer.

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[6]The Ackerberg and Phillips analysis was restricted to the case of constant u_∞, but the asymptotic eigensolutions do not depend on u_∞, and therefore remain unchanged in the more general case considered herein. Of course, the arbitrary constants that multiply these eigensolutions will be strongly dependent on the precise nature of $u_\infty(x)$.

In fact, in the present notation, their formula for the asymptotic eigenfunctions becomes[7]

$$\psi_{A.\&\,P.} = Cg_0(x,\eta)e^{-\lambda(2x)^{3/2}/3U_0'} \qquad \text{as} \quad x \to \infty \tag{3.6}$$

where C denotes an arbitrary constant,

$$g_0 = \begin{cases} \dfrac{iU_0'}{\lambda} + \sqrt{2x}\,F'(\eta) + O(x^{-3/2}) & \text{in the main boundary layer; } \eta = O(1) \\[2ex] \dfrac{U_0' \displaystyle\int_0^{\sigma} (\sigma - \tilde{\sigma})w(\tilde{\sigma})d\tilde{\sigma}}{\displaystyle\int_0^{\infty} w(\tilde{\sigma})d\tilde{\sigma}} + O(x^{-3/2}) & \text{for} \quad \eta = O(x^{1/2}) \end{cases} \tag{3.7}$$

F' is defined by (2.12) to[8] (2.14),

$$U_0' \equiv F''(0) = 0.4696\ldots \tag{3.8}$$

$$\lambda \equiv e^{-7\pi i/4} \Big/ \zeta_n^{3/2} \tag{3.9}$$

$$w(\sigma) \equiv Ai(\zeta_b) \tag{3.10}$$

$$\zeta_b \equiv (1 - i\sigma\lambda)\zeta_n \tag{3.11}$$

$$\sigma = \sqrt{2x}\,\eta = y/\epsilon^3 \tag{3.12}$$

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[7] It is important to note that Ackerberg and Phillips use an e^{it} time dependence, while we use e^{-it}, so our results are essentially the complex conjugates of theirs.

[8] Note that we set $\beta = +2/3\,\pi$ rather than $-2/3\,\pi$, as was done by Ackerberg and Phillips.

and ζ_n denotes the n^{th} root of

$$Ai'(\zeta_n) = 0 \qquad \text{for} \quad n = 1, 2, 3, \ldots \tag{3.13}$$

Here Ai and Ai$'$ denote the Airy function and its derivative in the usual notation (Abramowitz and Stegun, ref. 26, pp. 446 and 448). Since the roots of (3.13) all lie along the negative real axis, we can put

$$\zeta_n = \rho_n e^{-i\pi} \qquad \text{with} \quad \rho_n > 0. \tag{3.14}$$

It is easy to see that (3.6) oscillates with increasing rapidity (i.e., the wavelength of the oscillation decreases like $x^{-1/2}$) as $x \to \infty$.

Ackerberg and Phillips (ref. 10) show that this result is a homogeneous solution of the unsteady boundary layer equation (3.2)[9] to within an error that is smaller than (3.6) by a factor $O(x^{-3/2})$. However, as shown in appendix A, $\psi_c = x^\tau \psi_{A.\&P.}$ also provides a solution with this property for any constant τ.

The reason for this ambiguity is that, since (3.6) involves both exponential and algebraic functions, and since exponentially small functions are always negligible compared with algebraically small functions, (3.6) must really be treated as an expansion of the form $Cg_0 e^{-\theta_{B.L.}}$ where

$$\theta_{B.L.} = -\lambda(2x)^{3/2}/3U_0' + \tau \ln x + o(\ln x)$$

In this sense, the amplitude function is determined by higher order terms in the expansion. Ackerberg and Phillips only worked out the lowest order term in their expansion, while the exponent τ is determined by the higher order terms.

[9] Our dependent variable differs from that of Ackerberg and Phillips by a factor of $x^{1/2}$.

One way to determine this quantity is to begin with an expansion of the form

$$\psi_c = x^{\tau}\left(g_0 + x^{-3/2}g_1 + \ldots\right)e^{-\lambda(2x)^{3/2}/3U_0'} \qquad \text{as} \quad x \to \infty, \tag{3.15}$$

where g_0 is given by (3.7), and show that there is only one value of τ that leads to an equation for g_1 that possesses a solution. This amounts to using the method of multiple scales (ref. 17, pp. 276-282) and treating x^{τ} as the slowly varying amplitude function, which is determined solvability condition of the higher order problem. This procedure is carried out in appendix A, where it is shown that τ is given by

$$\tau = -\frac{\displaystyle\int_0^{\infty}\left[\sigma^2 g_0'' - 2\sigma g_0' + \lambda\,\frac{\sigma^3}{3}\left(\frac{\sigma g_0'}{4} - g_0\right)\right]w'\,d\sigma}{4\displaystyle\int_0^{\infty}(\sigma g_0' - g_0)w'\,d\sigma} \tag{3.16}$$

and g_0 and w are given by the second line of (3.7) and by (3.10), respectively.

4. THE ORR-SOMMERFELD REGION

Since arg. $\lambda = -\pi/4$, equation (3.15) represents a downstream propagating wave whose wavelength is decreasing like $x^{-1/2}$. Then the cross-stream velocity perturbation

$$v_1 \equiv -\epsilon^3\,\frac{\partial\psi}{\partial x} \tag{4.1}$$

will eventually become large relative to the streamwise velocity perturbation

$$u_1 = \varepsilon^3 \frac{\partial \psi}{\partial y} \tag{4.2}$$

This produces a significant cross-stream pressure fluctuation through the transverse momentum equation, and the unsteady boundary layer equation (3.2), from which (3.15) is derived, becomes invalid. In fact, substituting (3.15) in $\tilde{\Delta}\psi$, where $\tilde{\Delta}$ is defined by (2.20), shows that the second term in the resulting expression will eventually be proportional to

$$\left(2\lambda x \varepsilon^3 / U_0'\right)^2$$

and therefore when $x = O(\varepsilon^{-3})$, it will certainly not be negligible compared with the first term, as was assumed in deriving the unsteady boundary layer equation (3.2). However, it turns out that (3.15) breaks down at even smaller values of x. This occurs because $\partial u_1/\partial y = \varepsilon^3\, \partial^2\psi/\partial y^2$ is small in the outer portion of the boundary layer, and $\partial v_1/\partial x = -\varepsilon^3\, \partial^2\psi/\partial x^2$ therefore becomes significant at smaller values of x there.

We therefore seek a solution to (2.19) that is valid when

$$x_1 = \varepsilon^r x; \qquad 0 < r \leq 3 \tag{4.3}$$

remains of order one as $\varepsilon \to 0$ and which extends the asymptotic unsteady boundary layer solution (3.15) into this region; that is, which matches the latter solution asymptotically as $x_1 \to 0$.

The form of (3.15) suggests that this solution will be of the form

$$\psi = \varepsilon G^S(x_1, \eta, \varepsilon) e^{(i/\varepsilon^b) \int_0^x \kappa(x_1, \varepsilon) dx} \tag{4.4}$$

where κ and G are $O(1)$ and

$$b \equiv r/2 \tag{4.5}$$

and the constant s will be determined by the analysis.

Substituting (4.4) into (2.19) we obtain

$$\frac{1}{\alpha R}\mathscr{L}G = -\frac{\varepsilon^{3r/2}}{\kappa}\Bigg\{U\left(3\alpha\alpha_{x_1}G + 2\alpha^2 G_{x_1}\right) - c\left(\alpha\alpha_{x_1}G + 2\alpha^2 G_{x_1}\right) + U''G_{x_1}$$

$$- U(D^2 - \alpha^2)G_{x_1} + \frac{U}{x_1}(D^2 - \alpha^2)G + \frac{F}{2x_1}(D^2 - \alpha^2)DG + \frac{U'}{2x_1}DG\Bigg\} + O(\varepsilon^{6+r})$$

uniformly in η (4.6)

where $D \equiv \partial/\partial\eta$, the primes denote total derivatives with respect to η, the subscript x_1 denotes partial or total derivatives with respect to x_1,

$$R = R(x_1) \equiv \sqrt{2x}\big/\varepsilon^3 = \left(2x_1/\varepsilon^{r+6}\right)^{1/2}, \tag{4.7}$$

$$\alpha = \alpha(x_1) \equiv \varepsilon^{(3-r)}\sqrt{2x_1}\,\kappa(x_1), \tag{4.8}$$

$$c = c(x_1) = \varepsilon^{r/2}\big/\kappa(x_1), \tag{4.9}$$

$\mathscr{L}$ denotes the Orr-Sommerfeld operator

$$\mathscr{L} \equiv \left(D^2 - \alpha^2\right)^2 - i\alpha R\left[(U - c)(D^2 - \alpha^2) - U''\right] \tag{4.10}$$

and as can be seen from (2.5), (2.15), (2.16), and (2.18)

$$U = U(\eta) \equiv F'(\eta) \tag{4.11}$$

is the mean flow velocity in the direction along the plate.

The phrase 'uniformly in η, which appears at the end of equation (4.6), is intended to imply that we have retained sufficient terms to ensure that the approximation remains valid for all η, even in local regions where the terms involving higher derivatives with respect to η can become large and in regions where U and/or its derivatives become small (i.e., near the wall and at the outer edge of the Blasius boundary layer). Notice, for example, that even though the effective Reynolds number R will always be large and α and/or c will always be small, we have included terms involving these quantities on the left side of (4.6), which presumably contains only the lowest order terms.

Equation (4.6) is, then, to lowest order of approximation, just the Orr-Sommerfeld equation with coefficients α, c, and R that are slowly varying functions of x; that is, they are functions of the 'slow variable' x_1. However, α, c, and R are not all independent, but, as can be seen from (4.7) to (4.9), are related to each other and to x_1 by

$$\alpha c = \sqrt{2x_1}\, \varepsilon^{[3-(r/2)]} = \varepsilon^6 R \tag{4.12}$$

Of course, α and c correspond to the wave number and wave speed, respectively, in (4.10).

Since (4.6) has slowly varying coefficients, it is appropriate (ref. 20) to put

$$G = A(x_1)\gamma(\eta,x_1) \tag{4.13}$$

where A is a 'slowly varying' function of x_1 to be determined by the analysis. Then (4.6) becomes

$$\frac{1}{\alpha R}\mathscr{L}\gamma = -\frac{\varepsilon^{3r/2}}{\kappa}\left(H_1\frac{d\ \ln A}{dx_1} + H_2\right) + O(\varepsilon^{6+r}) \tag{4.14}$$

$$H_1 = \left[\alpha^2(3U - 2c) + U''\right]\gamma - UD^2\gamma \tag{4.15}$$

and

$$H_2 = \alpha\alpha_{x_1}(3U - c)\gamma + \left[U'' + \alpha^2(3U - 2c) - UD^2\right]\gamma_{x_1} + \frac{1}{2x_1}\left[D^2(FD\gamma) - \frac{\alpha^2}{F} D(F^2\gamma)\right] \tag{4.16}$$

It follows from (3.5), (4.4), and (4.13) that we must require

$$\gamma = D\gamma = 0 \quad \text{at} \quad \eta = 0 \tag{4.17}$$

and it is appropriate to require that $D\gamma$ vanish exponentially as

$$\eta \to \infty \tag{4.18}$$

Since the effective Reynolds number R is always large in the present approximation, it is only appropriate to consider the asymptotic (as $R \to \infty$) solution to (4.14). Fortunately, the asymptotic theory of the Orr-Sommerfeld equation has been highly developed by Tollmien (refs. 27 and 28), Lin (refs. 29 and 30), and many others.

As we indicated in the introduction, our interest is in showing that (4.14) has an eigensolution (i.e., a solution satisfying the homogeneous boundary conditions (4.17) and (4.18)) that matches the damped asymptotic eigensolution (3.15) when x_1 is small and develops into a growing (i.e., unstable) eigensolution when $x_1 = O(1)$).

Now it can be seen from (3.15), (4.4), and (4.8) that matching with (3.15) can only occur if $\alpha \sim x_1\epsilon^{3-r}$ as $x_1 \to 0$, which, in view of (4.3), implies that α must be small in this region. But it is also known (ref. 13, p. 306 or ref. 31) that α is small and in the vicinity of the neutral stability curve when R is large. Hence, it is appropriate to restrict our attention to the case where α is small.

However, it can be seen from (4.10) and (4.14) that R actually appears in the combination αR in the Orr-Sommerfeld equation, and the asymptotic theory for large R and small α must really be an asymptotic theory for large αR and small α. But it follows from (4.7) and (4.8) that

$$\alpha R = 2x_1\kappa(x_1,\epsilon)\Big/\epsilon^{(3r/2)} \tag{4.19}$$

will certainly be large in the present case.

It remains to choose the scaling exponent r. As we already indicated, we would like to do this so that x_1 will be of order one in the vicinity of the lower branch of the neutral stability curve. The asymptotic theory of the Orr-Sommerfeld equation for large αR and small α (ref. 13, pp. 279-281 or ref. 31) shows that $\alpha = O(c)$ in this region. Hence it follows from (4.8) and (4.9) that we must put

$$r = 2 \tag{4.20}$$

Then

$$\alpha = \epsilon\bar{\alpha}, \qquad c = \epsilon\bar{c} \tag{4.21}$$

where

$$\bar{\alpha} = \sqrt{2x_1}\,\kappa \tag{4.22}$$

and

$$\bar{c} = \kappa^{-1} \tag{4.23}$$

are $O(1)$.

It is important to notice that, even though κ, and consequently, $\bar{\alpha}$ and $\bar{c}$, are $O(1)$, κ <u>does</u> depend on ϵ, and therefore possesses an asymptotic expansion in this parameter. However, it is clear from (4.3) to (4.5) and (4.20) that this expansion need only be carried out up to, but not including, terms $O(\epsilon^4)$, since (see (4.3), (4.5), and (4.20)) the latter can always be incorporated into the 'slowly varying' function $A(x_1)$, which enters (4.4) via (4.13). Thus, in the present approach, the slowly varying amplitude function, which is usually introduced to account for nonparallel flow effects (Gaster, ref. 21; Saric and Nayfeh, ref. 20), is merely the natural continuation of the asymptotic expansion of κ.

At this point one might be tempted to simply use the results of the asymptotic theory of the Orr-Sommerfeld equation, but since our aim is to obtain a uniformly valid solution by the method of matched asymptotic expansions, and since most of the asymptotic Orr-Sommerfeld theory predates the introduction of this technique, it will be necessary to rederive some of the results from this point of view. However, we have tried to keep the analysis as close to the classical approach as possible and use existing results wherever we can.

Since α and c are both small, classical theory (ref. 31) suggests that the solution will exhibit a three-layer structure in the η direction. There will be (1) a viscous wall layer that contains the critical layer; (2) a main inviscid layer where the flow is quasi-steady and nearly parallel, since α and c appear only in the higher order terms here; and (3) an outer inviscid region where unsteady effects and streamwise variations are important, since α enters the lowest order solution in this region. We first consider the inviscid layers.

4.1 The Inviscid Region

The solution in the main inviscid layer corresponds to the limit $\epsilon \to 0$ with $\eta = O(1)$. We therefore seek an expansion of the form

$$\gamma = \gamma_0(\eta, x_1) + \epsilon\gamma_1(\eta, x_1) + O(\epsilon^2) \tag{4.24}$$

Since κ, and consequently $\bar{\alpha}$ and $\bar{c}$, depend on ϵ, the conventional approach would be to also expand these quantities in ϵ (ref. 17, pp. 68-71), but it turns out to be simpler to leave them unexpanded and determine their ϵ dependence during the course of the analysis. This introduces extraneous higher order terms into the analysis, but of course no additional error is incurred by retaining such terms. They can be eliminated at any stage of the analysis by re-expanding the solution, if this turns out to be desirable. Terms such as γ_0 and γ_1 will then depend on ϵ, as well as the indicated arguments but we simplify the notation by suppressing this dependence. Another reason for leaving $\bar{\alpha}$ and $\bar{c}$ unexpended is to keep the analysis as close as possible to classical stability theory.

Matching with the wall layer solution will require that the lowest order normal velocity component of this solution vanish at the wall, so we must take

$$\gamma_0(0, x_1) = 0 \tag{4.25}$$

The solution in the outer inviscid region corresponds to the limit $\epsilon \to 0$ with $\tilde{\eta} = O(1)$, where

$$\tilde{\eta} = \epsilon\eta \tag{4.26}$$

In this region the expansion must be of the form

$$\gamma = \tilde{\gamma}_0(\tilde{\eta}, x_1) + \epsilon\tilde{\gamma}_1(\tilde{\eta}, x_1) + O(\epsilon^2) \tag{4.27}$$

When condition (4.25) is imposed, the first two terms of these two expansions can be determined independently of the solution in the wall layer.

The procedure is straightforward and is relegated to appendix B. The result is that the solution in the main inviscid layer is given by

$$\gamma = U - (\bar{c}\epsilon) - \epsilon\bar{\alpha}U\left[\int_{\infty}^{\eta}\left(\frac{1}{U^2} - 1\right)d\eta + \eta\right] + O(\epsilon^2) \tag{4.28}$$

where we have set $K_2(x_1) = 0$, since it represents a higher order correction to the normalization 'constant', which we have already set equal to unity because it can be incorporated into the slowly varying function $A(x_1)$.

The more complete expansion is given to within an error of $O(\epsilon^5)$ by (B-16) and (B-20) through (B-24) with the functions Ω_1, Ω_2, Ω_3 appropriately re-expanded for small values of $c = \epsilon\bar{c}$. Since the latter expansion proceeds in powers of c, it is clear that the entire outer expansion is a power series to all orders in ϵ and that dropping terms $O(\alpha^n)$ introduces an error $O(\epsilon^n)$ uniformly in c. However, we have not as yet expanded the wave number κ, and consequently $\bar{c}$ and $\bar{\alpha}$, in powers of ϵ. This will introduce logarithmic terms into the expansion.

4.2. The Viscous Wall Layer and the Characteristic Equation

The solution in the viscous wall layer corresponds to the limit

$$\epsilon \to 0 \qquad \bar{\eta} = O(1).$$

where

$$\bar{\eta} \equiv \eta/\epsilon \tag{4.29}$$

We therefore seek an expansion of the form

$$\gamma = \epsilon b(x_1,\epsilon)\bar{\gamma}_0(\bar{\eta},x_1) + \epsilon^4\bar{\gamma}_4(\bar{\eta},x_1) + \ldots \tag{4.30}$$

where

$$b(x_1,\epsilon) = 1 + \epsilon b_1(x_1) + \epsilon^2 b_2(x_1) + \epsilon^3(\ln \epsilon) b_3(x_1)$$

Equations (4.19) and (4.20) show that

$$\bar{\beta} \equiv \epsilon(\alpha R)^{1/3} = (2x_1\kappa)^{1/3} \tag{4.31}$$

is of order one. Inserting (4.20), (4.21), and (4.29) to (4.31) into (4.14) to (4.16), using (4.11) and (A-5), and equating coefficients of like powers of ϵ yields

$$\mathscr{L}_w \bar{\gamma}_0 = 0 \tag{4.32}$$

$$\mathscr{L}_w \bar{\gamma}_4 = -U_0' \bar{c} \bar{\beta}^3 \left(\bar{H}_1 \frac{d \ln A}{dx_1} + \bar{H}_2 \right) \tag{4.33}$$

$$\vdots$$

where the operator $\mathscr{L}_w$ is defined by

$$\mathscr{L}_w \equiv \bar{D}^4 - i\bar{c}\bar{\beta}^3 \left(\frac{U_0' \bar{\eta}}{\bar{c}} - 1 \right) \bar{D}^2 \tag{4.34}$$

$$\bar{D} \equiv \frac{d}{d\bar{\eta}}$$

and the functions $\bar{H}_1$ and $\bar{H}_2$ are defined by

$$\bar{H}_1 \equiv \bar{D}(\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0), \qquad \bar{H}_2 \equiv \bar{D}\bar{H} \tag{4.35}$$

where

$$\bar{H} \equiv \frac{\partial}{\partial x_1} (\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0) + \frac{1}{4x_1} \bar{D}(\bar{\eta}^2 \bar{D}\bar{\gamma}_0) - \frac{iU_0'}{\bar{c}3!} \bar{\eta}^3 \left(\bar{\gamma}_0 - \frac{\bar{\eta}}{4} \bar{D}\bar{\gamma}_0 \right) \tag{4.36}$$

The boundary condition (4.17) implies that

$$\bar{\gamma}_0 = \bar{D}\bar{\gamma}_0 = 0 \qquad \text{at} \quad \bar{\eta} = 0 \tag{4.37}$$

and

$$\bar{\gamma}_4 = \bar{D}\bar{\gamma}_4 = 0 \qquad \text{at} \quad \bar{\eta} = 0 \tag{4.38}$$

Equation (4.32) was used by Lin (ref. 31) to describe the Tollmien-Schlichting waves in the neighborhood of the critical layer. Its solution is well known. Thus, introducing the new independent variable

$$\zeta \equiv \zeta_0\left(1 - \frac{U_0'\bar{\eta}}{\bar{c}}\right), \tag{4.39}$$

where

$$\zeta_0 \equiv e^{-5\pi i/6}\bar{c}\bar{\beta}/(U_0')^{2/3} = e^{-5\pi i/6}(\alpha R U_0')^{1/3} c/U_0', \tag{4.40}$$

into this result shows that $d^2\bar{\gamma}_0/d\zeta^2$ satisfies Airy's equation

$$\frac{d^2 Ai}{d\zeta^2} - \zeta Ai = 0. \tag{4.41}$$

It follows that

$$\bar{\gamma}_0 = a_1 + a_2\bar{\eta} + a_3 \int_{\infty_1}^{\zeta} d\zeta \int_{\infty_1}^{\zeta} Ai(\tilde{\zeta})d\tilde{\zeta} \tag{4.42}$$

where, in order to satisfy (4.37), the arbitrary functions of x_1, a_1, a_2, and a_3, must satisfy

$$a_1 = -a_3 \int_{\infty_1}^{\zeta_0} d\zeta \int_{\infty_1}^{\zeta} Ai(\tilde{\zeta})d\tilde{\zeta} \tag{4.43}$$

$$a_2 = \frac{U_0'}{\bar{c}} \zeta_0 a_3 \int_{\infty_1}^{\zeta_0} \mathrm{Ai}(\zeta)d\zeta \tag{4.44}$$

and the subscript 1 on ∞ is used to indicate that the path of integration tends to infinity in the sector $-\pi/3 < \arg \zeta < \pi/3$.

The inner expansion (4.30) must now be matched onto the solution in the main inviscid region. This will ultimately determine the expansion of κ in terms of ε that was alluded to above. For reasons that will become clear subsequently, it is convenient to simultaneously determine all terms through $O(\varepsilon^4 \ln \varepsilon)$ in this expansion. The $O(\varepsilon^4 \ln \varepsilon)$ term arises (see eq. (4.64) below and the equation preceding (4.65)) because the term

$$\varepsilon^4 \ln\left(-\frac{\bar{\eta}}{\bar{c}} U_0'\right) \frac{\bar{\alpha}\bar{c}^2}{2U_0'^3}\left(\bar{c} - U_0'\bar{\eta}\right)$$

appears in $\varepsilon^4\bar{\gamma}_4$ at large values of $\bar{\eta}$. It is therefore convenient to artificially introduce this term at an earlier stage of the expansion than it would normally occur. To this end we put

$$\bar{W} \equiv \varepsilon \frac{\left(\bar{\eta}U_0' - \bar{c}\right)\bar{D}\gamma - U_0'\gamma}{\bar{D}\gamma} = \varepsilon\bar{W}_1(\bar{\eta}) + \varepsilon^4\bar{W}_4(\bar{\eta}) + o(\varepsilon^4) \qquad \text{as} \quad \varepsilon \to 0,\ \bar{\eta} = O(1) \tag{4.45}$$

where

$$\bar{W}_1(\bar{\eta}) \equiv \frac{\left(\bar{\eta}U_0' - \bar{c}\right)\bar{D}\bar{\gamma}_0 - U_0'\bar{\gamma}_0}{\bar{D}\bar{\gamma}_0 - \varepsilon^3 \dfrac{\bar{\alpha}\bar{c}^2}{2U_0'^2} \ln\left(-\dfrac{\bar{\eta}}{\bar{c}} U_0'\right)} \tag{4.46}$$

and $\bar{W}_4(\bar{\eta})$ is $O(1)$ for $\bar{\eta} = O(1)$.

Since the integral in (4.42) vanishes exponentially fast as ζ, and hence $\bar{\eta}$, go to infinity, it follows that

$$\epsilon b\bar{\gamma}_0 \to \epsilon b(a_1 + a_2\bar{\eta}) + \text{exp. small terms} \qquad \text{as } \bar{\eta} \to \infty \qquad (4.45)$$

Introducing the outer variable η in place of $\bar{\eta}$ and re-expanding, we find that the 3 term outer expansion of the 3 term inner expansion of $\bar{W}$ (see ref. 32, p. 90) is given by

$$\epsilon \bar{W}_1 = - \frac{\epsilon\left(\bar{c} + \frac{a_1}{a_2} U_0'\right)}{1 + \frac{\epsilon^3}{ba_2} \frac{\bar{\alpha}\bar{c}^{-2}}{2U_0'^2} \ln(-\epsilon\bar{c}/U_0')} + O(\epsilon^4) \quad ; \quad \epsilon \to 0, \ \eta = O(1) \qquad (4.46)$$

which is independent of η and depends only on the slow variable x_1. Moreover, it is easy to show from (4.64) below that $\bar{W}_4$ is a polynomial in $\bar{\eta}$ at large values of $\bar{\eta}$ and, therefore, cannot introduce additional logarithmic terms into the outer expansion of $\bar{W}$.

We, therefore, carry out the matching in terms of the variable $\overline{W}$ rather than in terms of the basic variable γ.

Since (4.11) and (A-5) show that

$$U = U_0'\eta + O(\eta^4) \qquad \text{as} \quad \eta \to 0 \tag{4.47}$$

it is clear that, to the order of approximations being considered, $\overline{W}$ should match with the limiting value as $\eta \to 0$ of the <u>outer variable</u>[10]

$$W(\eta) \equiv \frac{(U - c)D\gamma - U'\gamma}{D\gamma}, \tag{4.48}$$

where γ is given by (B-16) with the $O(\varepsilon^4)$ terms omitted, i.e., by γ_H. The latter quantity is expressed in terms of Ω by (B-20) and Ω is, in turn, given by (B-21), in which the $O(\alpha^2)$ term can be omitted to the present level of approximation.

Inserting (B-20) and (B-21) into (4.48) we obtain

$$W = \frac{U - c}{1 - (U - c)U'\Omega} + O(\varepsilon^4) = -\frac{\alpha}{U'}\left\{(1 - c)^2 + \alpha\Omega_0^{\dagger} + \alpha^2\left[\Omega_0^{\dagger 2} - (1 - c)^4\Omega_1\right]\right\}$$

$$+ O(\varepsilon^4) \qquad \text{as } \varepsilon \to 0, \ \eta = O(1) \tag{4.49}$$

where

$$\Omega_1 = 2\int_\eta^\infty U^2\left[\int_\eta^\infty \left(U^2 - \frac{1}{U^2}\right) d\eta\right] d\eta + O(c) \tag{4.50}$$

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10 $w = u + iv = -c/W(0)$ is the 'inviscid function' that appears in the characteristic equation of classical stability theory (ref. 30, p. 37).

and we have put

$$\Omega_0^{+} \equiv (1 - c)^4\left[\frac{1}{U'(U - c)} - \Omega_0\right]$$

$$= \frac{1}{U'U} + \int_{\eta}^{\infty}\left(U^2 - \frac{1}{U^2}\right)d\eta + 2c\left[\frac{1}{2U'U^2} - \frac{2}{U'U} - \int_{\eta}^{\infty}\left(U + \frac{1}{U^3} - \frac{2}{U^2}\right)d\eta\right] + O(c^2)$$

(4.51)

In obtaining the expansions in (4.50) and (4.51) it is important to keep in mind that the outer limit corresponds to letting $\epsilon \to 0$ while $\eta \neq 0$ is held fixed so that $c = \epsilon\bar{c}$ must eventually become less than U in the integrands of (B-22) through (B-24).

It is easy to see from (4.47) through (4.51) that W is independent of η to within an error $O(\epsilon^4)$ when $\bar{\eta} = \eta/\epsilon = O(1)$. Moreover, comparing (4.45) with (4.28) and using (4.47) shows that

$$ba_2 = U_0' + O(\epsilon) \qquad (4.52)$$

Hence, it follows from (4.21), (4.43), (4.44), and (4.49) through (4.51) that the 3 term inner expansion of the 3 term outer expansion (of W) will equal the 3 term outer expansion of the 3 term inner expansion, which is given by (4.46), if we put

$$1 - \frac{\bar{\alpha}}{\bar{c}U_0'}\left[1 - \epsilon\left(2\bar{c} - \frac{\bar{\alpha}}{U_0'}J_1\right) + \epsilon^2\left(\bar{c}^2 + \frac{2\bar{\alpha}\bar{c}}{U_0'}J_2 + \frac{\bar{\alpha}^2}{U_0'^2}J_3\right) + \frac{1}{2}\epsilon^3\frac{\bar{\alpha}\bar{c}^2}{U_0'^3}\ln\left(-\frac{\epsilon\bar{c}}{U_0'}\right)\right] = F^{+}(\zeta_0) \qquad (4.53)$$

where

$$F^{\dagger}(\zeta_0) \equiv \frac{\int_{\infty_1}^{\zeta_0} d\zeta \int_{\infty_1}^{\zeta} Ai(\tilde{\zeta})d\tilde{\zeta}}{\zeta_0 \int_{\infty_1}^{\zeta_0} Ai(\zeta)d\zeta} \tag{4.54}$$

is the Tietjens function and J_1, J_2, J_3 are constants defined by

$$J_1 \equiv U_0' \int_0^{\infty} \left(U^2 - \frac{1}{U^2} + \frac{1}{U_0'^2 \eta^2}\right) d\eta \tag{4.55}$$

$$J_2 = -U_0' \int_0^{\infty} \left(\frac{1}{U^3} - \frac{2}{U^2} + U - \frac{1}{(U_0'\eta)^3} + \frac{2}{(U_0'\eta)^2}\right) d\eta \tag{4.56}$$

and

$$J_3 = J_1^2 - 2U_0'^2 \int_0^{\infty} U^2 \int_{\eta}^{\infty} \left(U^2 - \frac{1}{U^2}\right) d\eta \; d\eta$$

Not surprisingly, this is precisely the characteristic equation that is obtained from the classical large αR small α asymptotic solution to the Orr-Sommerfeld equation with the irrelevant higher order terms in c neglected (Lin,[11] ref. 31, see p. 294 of appendix and equation immediately following (12.5); also ref. 13, pp. 279-282). It applies when $c = O(\alpha)$. Its solutions are the eigenvalues of the Orr-Sommerfeld equation that correspond to the Tollmien-Schlichting instability waves.

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[11]There are some minor typographical errors in eq. (7) of Lin's appendix and a prime is missing in his eq. (12.5).

The neutral stability curve, which divides the region of growing instability waves from the region of decaying waves, corresponds to real values of α and c. The solution to (4.53) corresponding to the lower branch of this curve is given to lowest order in α and c (i.e., to lowest order in ϵ) by (ref. 31, eq. (12.7) and ref. 13, pp. 281-282)

$$c \simeq 2.296\,\alpha/U_0' \tag{4.57}$$

$$R \simeq 1.002(U_0')^5/\alpha^4 \tag{4.58}$$

The first of these shows that, as was anticipated, α is indeed of order c in the vicinity of this curve while the second shows that αR is large there.

The difference between the present result and that of conventional stability theory is that α, c and R are no longer independent, but are related to each other and to x_1 by equations (4.7) through (4.9) with $r = 2$. (see also (4.12)). Then since only κ and x_1 appear in the actual solution (4.4), it will be helpful to eliminate α, c and R and express the characteristic equation (4.53) entirely in terms of κ and x_1. To this end we integrate by parts to obtain

$$\int_{\infty_1}^{\zeta} d\zeta \int_{\infty_1}^{\zeta} Ai(\tilde{\zeta})d\tilde{\zeta} = \zeta \int_{\infty_1}^{\zeta} Ai(\zeta)d\zeta - \int_{\infty_1}^{\zeta} \zeta Ai(\zeta)d\zeta \tag{4.59}$$

Hence it follows from (4.41) that equation (4.54) can be written as

$$F^{\dagger}(\zeta_0) = 1 - \frac{Ai'(\zeta_0)}{\zeta_0 \displaystyle\int_{\infty_1}^{\zeta_0} Ai(\zeta)d\zeta} \tag{4.60}$$

On the other hand, it follows from (4.23) and (4.31) that (4.40) can be written as

$$\zeta_0 = e^{-5\pi i/6} \left(\sqrt{\tilde{x}_1}/\kappa\right)^{2/3} \tag{4.61}$$

where

$$\tilde{x}_1 \equiv 2x_1/U_0'^{\,2} \tag{4.62}$$

Inserting (4.22). (4.23), (4.60) and (4.61) into (4.53) yields

$$\begin{aligned}
\tilde{x}_1^{3/2} &+ \left(\epsilon e^{i\pi/4}\zeta_0^{3/2}\right) \tilde{x}_1 \left(2 - \frac{\tilde{x}_1^{3/2} J_1}{i\zeta_0^3}\right) \\
&+ \left(\epsilon e^{i\pi/4}\zeta_0^{3/2}\right)^2 \tilde{x}_1^{1/2} \left(1 + \frac{2\tilde{x}_1^{3/2} J_2}{i\zeta_0^3} - \frac{\tilde{x}_1^3 J_3}{\zeta_0^6}\right) \\
&- \frac{e^{i\pi/4}\left(\tilde{x}_1\zeta_0\right)^{3/2}\epsilon^3}{2\,U_0'^{\,2}} \ln\left(\frac{\epsilon \quad e^{i\pi/4}\zeta_0^{3/2}}{\tilde{x}_1^{1/2}\ U_0'}\right) \\
&= H(\zeta_0) \equiv \frac{e^{i5\pi/2}\zeta_0^2 Ai'(\zeta_0)}{\int_{\infty_1}^{\zeta_0} Ai(\zeta)d\zeta}
\end{aligned} \tag{4.63}$$

These equations determine the exponent κ in the solution (4.4) as a function of the 'slow variable' x_1 and the small parameter ϵ. They are accurate to within an error $O(\epsilon^4)$, but, as we already indicated, there is no need to determine κ with any greater precision because the higher order effects can now be accounted for by the slowly varying amplitude function $A(x_1)$.

4.3 Determination of Slowly Varying Amplitude Function Solvability

The amplitude function A, which enters the solution (4.4) through (4.13), is determined by the requirement that previously neglected $O(\epsilon^4)$ terms in the inner and outer expansions match in some overlap domain. We therefore have to find the $O(\epsilon^4)$ solution in the inviscid region and the solution $\overline{\gamma_4}$ in the viscous wall layer. It is shown in appendix C that (4.33), the boundary condition (4.38), and the requirement that it grow at most algebraically fast as $\overline{\eta} \to \infty$, determine $\overline{\gamma_4}$ to within an arbitrary multiple of $\overline{\gamma_0}$, which can only effect the complete solution by introducing an $O(\epsilon^4)$ change in its normalization. The solution in the main inviscid region, correct to $O(\epsilon^4)$, is determined by (B-16) and (B-20) through (B-24) to within an arbitrary normalization factor, which is equal to unity to lowest order in ϵ. Thus, it turns out that the solution in the main inviscid region will only match the wall layer solution $\overline{\gamma_4}$ when the slowly varying amplitude function $A(x_1)$ is appropriately adjusted.

The latter solution is given by (C-17). Although it is rather complicated, we only need to know its asymptotic behavior as $\overline{\eta} \to \infty$.

Introducing (4.39) into (C-17), expanding for large $\overline{\eta}$, using (4.45), (C-11), (4.52), and the equation preceding (4.31), introducing the new arbitrary function

$$\overline{e}_1 = e_1 + \frac{1}{U_0'}\left[e_2 + \frac{3}{4}\left(\frac{\overline{c}^2}{U_0'}\right)(a_1 + \overline{c})\right],$$

and neglecting $O(\epsilon)$ terms, we obtain

$$\overline{\gamma}_4 = \overline{e}_1(x_1)\overline{\gamma}_0 - \frac{U_0'^2}{4!}\left(\frac{\eta}{\varepsilon}\right)^4$$

$$- \frac{a_1 + \overline{c}}{2U_0'^2}\,\overline{c}^3 \left[\sum_{n=0}^{3} \frac{\text{sgn}(2n-3)}{n!}\left(\frac{U_0'\eta}{\varepsilon\overline{c}}\right)^n - \left(1 - \frac{U_0'\eta}{\overline{c}\varepsilon}\right)\ln\left(\frac{-\eta U_0'}{\varepsilon\overline{c}}\right)\right]$$

$$- i\overline{c}\left[\frac{a_1 + \overline{c}}{4x_1} - a_1\frac{d\ \ln A}{dx_1} - \frac{da_1}{dx_1}\right]$$

$$- \frac{a_1 + \overline{c}}{U_0'}\int_0^\infty \Gamma^+ \, d\overline{\eta} - i\overline{c}\pi Bi'(\zeta_0)\int_0^\infty \left(\overline{H}_1 \frac{d\ \ln A}{dx_1} + \overline{H}_2\right) Ai\ d\overline{\eta}$$

$$+ O(\overline{\eta}^{-1}) \qquad \text{as} \quad \overline{\eta} \to \infty \tag{4.64}$$

Now it is clear that any term involving η^n for $n \geq 0$ (including those involving the logarithms) can only be matched by the $O(\varepsilon^{4-n})$ term in the outer expansion (4.24) and it is not hard to prove that the matching actually occurs for all $n > 0$. The $\ln \varepsilon$ terms have already been accounted for and the term involving the arbitrary function $\overline{e}_1$ can be matched by adding (4.28) multiplied by $\overline{e}_1\varepsilon^3$ onto the outer expansion. This leaves the η-independent terms and the term involving $\ln \eta$ itself. They must clearly be matched by the $O(\varepsilon^4)$ term in the outer expansion, which is given by (B-16).

It is not hard to show that

$$\gamma_H = \frac{\overline{\alpha}\overline{c}^3}{2U_0'^3}\ln \eta - \frac{i\overline{\alpha}}{U_0'}\sum_{n=0}^{3} A_n \overline{c}^{n}\overline{\alpha}^{(3-n)} + O(\eta \ln \eta) \qquad \text{as} \quad \eta \to 0$$

where the A_n are constants (i.e., they are independent of x_1) whose exact form is unimportant for our purposes. Moreover, it follows from (4.47) that (4.28) is equal to $\epsilon[(\bar{\alpha}/U_0') - \bar{c}]$ at $\eta = 0$. Comparing this with (4.45) and using the equation preceding (4.31) we see that $a_1 = (\bar{\alpha}/U_0') - \bar{c} + O(\epsilon)$. It therefore follows from (4.47) that the η-independent terms in (4.64) and (B-16) can only match if

$$2\bar{\alpha}\,\frac{d\,\ln A}{dx_1} + \bar{\alpha}_{x_1} - \frac{\bar{\alpha}}{2x_1} + \frac{\bar{\alpha}}{\bar{c}} \sum_{n=0}^{3} \tilde{A}_n \bar{c}^{\,n} \bar{\alpha}^{-(3-n)}$$

$$= \pi U_0' Bi'(\zeta_0) \int_0^\infty \left(\bar{H}_1 \frac{d\,\ln A}{dx_1} + \bar{H}_2\right) Ai\, d\bar{\eta} - \frac{i\bar{\alpha}}{\bar{c}U_0'} \int_0^\infty \Gamma^+\, d\bar{\eta}, \qquad (4.65)$$

where the $\tilde{A}_n$ are $O(1)$ constants (related to the A_n) and the last integral is given by (C-16). This equation determines the slowly varying amplitude function $A(x_1)$.

5. MATCHING OF ASYMPTOTIC EIGENSOLUTION AND TOLLMIEN-SCHLICHTING WAVE

We must now show that as $x_1 \equiv \epsilon^2 x \to 0$, the downstream solution (4.4), which applies when $x_1 = O(1)$, matches the asymptotic eigensolution (3.15), which describes the limiting downstream behavior of the solution in the unsteady boundary layer region where $x = O(1)$. This will be done in three stages. First the exponential term in (4.4) will be shown to match the exponential term in (3.15). Then the function γ, which enters through (4.13), will be shown to match $g_0 x^{-1/2}$ and, finally, it will be shown that the slowly varying function A becomes equal to $x_1^{(\tau+1/2)}$. Then (4.4) and (3.5) will match if s is set equal to $-(2\tau+1)$.

5.1 Matching of Exponential Terms

As we already indicated, equations (4.61) and (4.63), which are only accurate to $O(\varepsilon^4)$, contain irrelevant higher order terms due to their implicit dependence on κ. We can eliminate them by expanding κ in an appropriate asymptotic series in ε and then re-expanding the result.

Since H is an analytic function of ζ_0, it is clear that κ must have an expansion of the form

$$\kappa = \kappa_0 + \varepsilon\kappa_1 + \varepsilon^2\kappa_2 + \varepsilon^3(\ln \varepsilon)\kappa_3 + O(\varepsilon^3) \tag{5.1}$$

Inserting this into (4.61) and (4.63), expanding H in a Taylor series about

$$\zeta_{00} \equiv e^{-5\pi i/6}\left(\sqrt{\tilde{x}_1}\Big/\kappa_0\right)^{2/3}, \tag{5.2}$$

and equating coefficients of like orders of ε we obtain

$$H(\zeta_{00}) = \tilde{x}_1^{3/2} \tag{5.3}$$

$$\frac{\kappa_1}{\kappa_0} = -\frac{3}{2}\, e^{i\pi/4}\zeta_{00}^{1/2}\tilde{x}_1\left(2 - \frac{\tilde{x}_1^{3/2}J_1}{i\zeta_{00}^3}\right)\Bigg/ H'(\zeta_{00}) \tag{5.4}$$

$$\begin{aligned}\frac{\kappa_2}{\kappa_0} = &-\frac{1}{3}\left[\frac{1}{2} - \frac{H''(\zeta_{00})\zeta_{00}}{H'(\zeta_{00})}\right]\left(\frac{\kappa_1}{\kappa_0}\right)^2 + 3e^{-i\pi/4}\left(\frac{\tilde{x}_1}{\zeta_{00}}\right)^{5/2} J_1\left(\frac{\kappa_1}{\kappa_0}\right)\Bigg/ H'(\zeta_{00}) \\ &- \frac{3i}{2}\,\zeta_{00}^2\tilde{x}_1^{-1/2}\left(1 + \frac{2\tilde{x}_1^{3/2}}{i\zeta_{00}^3}J_2 - \frac{\tilde{x}_1^3 J_3}{i\zeta_{00}^6}\right)\Bigg/ H'(\zeta_{00})\end{aligned} \tag{5.5}$$

$$\frac{\kappa_3}{\kappa_0} = \frac{3}{4U_0'^2}\, e^{i\pi/4}\zeta_{00}^{1/2}\tilde{x}_1^{3/2}\Big/ H'(\zeta_{00}) \tag{5.6}$$

where $H(\zeta_{00})$ is defined in (4.63) and the primes on H denote derivatives with respect to ζ_{00}.

Since

$$\int_{\infty_1}^{\zeta_{00}} Ai(\zeta)d\zeta$$

cannot become infinite as $\tilde{x}_1 \to 0$, it follows from (4.63) and (5.3) that either (1) $\zeta_{00} \to 0$ or (2) that $Ai'(\zeta_{00}) \to 0$ in this limit. But expanding (5.2) and (5.3) for small $\tilde{x}_1$ and ζ_{00} shows that condition (1) can only occur if arg. $\kappa_0 = 5\pi i/8$, which corresponds to an upstream propagating wave, or if arg. κ_0 has increased by more than a factor of 2π from its value at the neutral curve, where (4.57) and (4.58) hold. In the latter case, the eigensolution would have to exhibit growth upstream of the neutral curve - which certainly cannot occur. Hence we must conclude that

$$Ai'(\zeta_{00}) \to 0 \qquad \text{as} \quad \tilde{x}_1 \to 0 \tag{5.7}$$

It therefore follows that

$$\zeta_{00} \to \zeta_n \tag{5.8}$$

for some $n = 1,2,\ldots,$ where ζ_n is determined by (3.13), which can be thought of as the characteristic equation for the asymptotic eigensolution (3.15). Thus the limiting form of the characteristic equation for the Orr-Sommerfeld equation coincides with the characteristic equation for the asymptotic eigensolutions.

Equation (5.3) has been solved numerically. The results are shown in figure 2.

On the lower branch of the neutral stability curve the solution to (4.53) is given by (4.57) and (4.58) to lowest order of approximation in $\alpha = \epsilon\bar{\alpha}$ and $c = \epsilon\bar{c}$. These equations must therefore determine the neutrally stable solutions of (5.2) and (5.3) (i.e., the solutions corresponding to real κ_0) when α, c, and R are expressed in terms of κ_0 and $\tilde{x}_1$.

Substituting (4.57) and (4.58) into (4.12) and (4.40), using (5.1), and noting that $\zeta_0 = \zeta_{00} + O(\epsilon)$, we find that $\zeta_{00} = (2.296)\ (1.002)^{1/3}$ exp. $(-5\pi i/6)$ and $\tilde{x}_1 \equiv 2x_1/(U'_0)^2 = (2.296)^{4/3}\ (1.002)^{2/3} = 3.033$ on the lower branch of the neutral stability curve. The figure shows that the curves pass through this point. It also shows that ζ_{00} approaches -1.0188 as $\tilde{x}_1 \to 0$. As can be seen from the table on p. 478 of Abramowitz and Stegun (ref. 26), this is the smallest root of (3.13) (i.e., it is equal to ζ_1). This provides a numerical verification of our conclusion that (3.13) is the appropriate limiting form of the characteristic equation (5.3).

It now follows from (4.3), (4.20), (4.62), and (5.2) that

$$\kappa_0 \to i\ \sqrt{2x_1}\ \lambda/U_0' = i\epsilon\ \sqrt{2x}\ \lambda/U_0' \quad \text{as} \quad x_1 \to 0 \tag{5.9}$$

where λ is given by (3.9), or more precisely, $\kappa_0 = i\epsilon\sqrt{2x}\ \lambda/U_0' + 3C_0x_1^2 + o(x_1^2)$ as $x_1 \to 0$, where C_0 is an O(1) constant. On the other hand, it follows from (4.41), the right side of (4.63), and (5.7) that

$$H'(\zeta_{00}) \to e^{5\pi i/2}\zeta_{00}^3 Ai(\zeta_{00}) \Big/ \int_{\infty_1}^{\zeta_{00}} Ai(\zeta)d\zeta \qquad \text{as} \quad \tilde{x}_1 \to 0$$

and

$$H'(\zeta_{00})/H''(\zeta_{00}) \to \frac{1}{2}\zeta_{00}^3 \left[\frac{5}{2}\zeta_{00}^2 + iH'(\zeta_{00})\right]^{-1} \qquad \text{as} \quad \tilde{x}_1 \to 0.$$

Table 10.13 on p. 478 of Abramowitz and Stegun (ref. 26) and equation (5.7) therefore show that both $H'(\zeta_{00})$ and $H'\zeta_{00}/H''(\zeta_{00})$ are non-zero constants when $\tilde{x}_1 = 0$. Hence it follows from (5.4) through (5.6), (4.62), and (5.9) that

$$\left.\begin{array}{l} \kappa_1 \to \frac{5}{2} C_1 x_1^{3/2} \\ \kappa_2 \to 2\, C_2 x_1 \\ \kappa_3 \to 3\, C_3 x_1^2 \end{array}\right\} \tag{5.10}$$

as $x_1 \to 0$, where C_1, C_2, and C_3 are $O(1)$ constants. We therefore conclude from (4.3), (4.20), (5.1), and (5.9) that

$$e^{(i/\epsilon)\int_0^x \kappa(x_1,\epsilon)dx}$$

$$= \exp\left\{-\frac{\lambda(2x)^{3/2}}{3U_0'} + i\epsilon^3 x^2\left(C_0 x + C_1 x^{1/2} + C_2\right) + iC_3\epsilon^6(\ln\epsilon)x^3\right\} + O(\epsilon^4)$$

$$= \left[1 + i\epsilon^3 x^2\left(C_0 x + C_1\sqrt{x} + C_2\right) + O(\epsilon^4)\right] e^{-\lambda(2x)^{3/2}/3U_0'} \tag{5.11}$$

when $\epsilon \to 0$ and $x = O(1)$.

It therefore follows from (4.5) and (4.20) that <u>the exponential term in (4.4) does indeed match the exponential term in the asymptotic eigensolution (3.15)</u>.

Equations (4.61) and (4.63) were solved numerically to determine κ as a function of x_1. The results, which are plotted in figure 3, show that $\mathcal{Im}\,\kappa$ is zero when x_1 is on the neutral stability curve. The dashed curve is a plot of the real and imaginary parts of the right side of (5.9) with λ given by (3.9) for $n = 1$. (Note that $\mathcal{Re}\, i\lambda = \mathcal{Im}\, i\lambda = 0.6876$ in this case.) The $\epsilon = 0$ curves represent κ_0, the lowest order approximation to κ. Thus, the figure shows that (5.9) holds for the lowest order eigenvalue and that $\kappa \to \kappa_0$ as $x_1 \to 0$ with $\epsilon > 0$.

5.2 Matching of Amplitudes

We first consider the main inviscid region where $\eta = O(1)$. Inserting (4.23), (5.1), (5.9), and (5.10) into (4.28) shows that

$$\gamma \to U + \frac{iU_0'}{\lambda\sqrt{2x}} + O(\epsilon^3) \text{ as } x_1 \to 0$$

Comparing this with the first line of (3.7) and using (4.11) shows that

$$\gamma \to g_0/\sqrt{2x} \text{ as } x_1 \to 0 \text{ with } \eta = O(1) \tag{5.12}$$

Hence the amplitudes match to within a power of x, which will be accounted for when we match A and x^{τ}.

We now consider the solution in the wall layer where $\bar{\eta} = \eta/\epsilon = O(1)$ and $\gamma = \epsilon\bar{\gamma}_0 + O(\epsilon^2)$.

It follows from (4.23), (4.61), (5.1), (5.2), and (5.8) through (5.10) that

$$\bar{c} \to U_0' \Big/ \left(i\epsilon\lambda\sqrt{2x}\right) \quad \text{as } x_1 \to 0 \tag{5.13}$$

and

$$\zeta_0 \to \zeta_n \quad \text{as } x_1 \to 0 \tag{5.14}$$

Inserting these into (4.39) we find that

$$\zeta \to \zeta_b \quad \text{as } x_1 \to 0 \tag{5.15}$$

where ζ_b is defined in (3.11) with σ given by (3.12). Inserting this together with (5.13) into (D-1), introducing σ as a new variable of integration, and comparing the result with the second line of (3.7) shows that (5.12) also holds in the inner viscous layer where $\bar{\eta} = O(1)$.

5.3 Matching of Slowly Varying Amplitude

In view of (5.12) and the results of section 5.2, the matching of the slowly varying Tollmien-Schlichting wave, given by (4.4) and (4.13), and the asymptotic eigensolution (3.15) will be established once it is shown that

$$A(x_1) \to \sqrt{2}\, x_1^{(\tau + 1/2)} \qquad \text{as} \quad x_1 \to 0 \tag{5.14}$$

where τ is given by (3.16) and A is determined by (4.65).

For this purpose we can replace ζ_0, $\bar{c}$, and $\bar{\alpha}$ by their first approximations ζ_{00}, κ_0^{-1}, and $\sqrt{2x_1}\,\kappa_0$, respectively (see eqs. (4.22), (4.23), and (5.1)). Then since $Bi'(\zeta_0)Ai(\zeta)$, and $\zeta_0 Wi(\zeta)$, where Wi is defined by (C-6), are both $O(1)$ as $\zeta_0 \to \zeta_n$ and $\zeta \to \zeta_b$ (ref. 26, pp. 446 and 447), it follows from (C-16) and (5.9) that the second term on the right side of (4.65) will certainly be negligible relative to the first as $x_1 \to 0$.

Expanding (5.3) for small x_1 shows that $\partial\zeta_{00}/\partial x_1$ remains bounded as $x_1 \to 0$. On the other hand, it follows from (5.9) that

$$\frac{\partial \bar{c}}{\partial x_1} \to - \frac{1}{2x_1}\, \bar{c}$$

Hence it follows from (D-1) and the chain rule for partial differentiation that

$$\frac{\partial \bar{\gamma}_0}{\partial x_1} \to - \frac{1}{2x_1}\, \bar{\gamma}_0 + \frac{\partial \bar{\gamma}_0}{\partial \zeta}\, \frac{\partial \zeta}{\partial x_1} = - \frac{1}{2x_1}\, \bar{\gamma}_0 + (\bar{D}\bar{\gamma}_0)\, \frac{\partial\zeta/\partial x_1}{\partial\zeta/\partial\bar{\eta}}$$

Inserting $\bar{c} = \kappa_0^{-1}$ into (4.39) and using the result together with (5.9) to eliminate ζ now yields

$$\frac{\partial \bar{\gamma}_0}{\partial x_1} \to -\frac{1}{2x_1}(\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0)$$

Inserting this together with (3.12), (5.9) and $\bar{c} = \kappa_0^{-1}$ into (4.36) shows

$$\bar{H} \to -\frac{1}{2x_1}\left[(\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0) + \frac{1}{2}(\bar{\eta}^2\bar{D}^2\bar{\gamma}_0 - 2\bar{\eta}\bar{D}\bar{\gamma}_0) - \frac{1}{6}\sigma^3\lambda\left(\bar{\gamma}_0 - \frac{\bar{\eta}}{4}\bar{D}\bar{\gamma}_0\right)\right] \quad \text{as} \quad x_1 \to 0 \tag{5.16}$$

It follows from (4.35), (4.42), and the approximation (4.52) that

$$U_0'\pi Bi'(\zeta_0) \int_0^\infty \bar{H}_1 Ai \, d\bar{\eta}$$

is order one and will not go to zero as $x_1 \to 0$. Then since $\bar{\alpha}$ <u>does</u> go to zero, the first term on the left side (4.65) can be neglected. The remaining terms on the left side become equal to a constant and can, therefore, be neglected, since it follows from (4.35) and (5.16) that the first term on the right side of (4.65) increases at least as fast as x_1^{-1} as $x_1 \to 0$.

Hence (4.65) can be approximated by the first term on its right side. Inserting (4.35), integrating by parts, and using (4.37) and (4.39) this becomes

$$\int_0^\infty \left[\left(\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0\right)\frac{d \ln A}{dx_1} + \bar{H}\right] Ai \, d\bar{\eta} = 0$$

Inserting (5.16) into this result, using (4.24) and (5.12) to eliminate $\bar{\gamma}_0$ in terms of g_0 and introducing the function σ defined in (3.12) as a new variable of integration yields

$$\frac{d \ln A}{dx_1} \to \frac{2\tau + 1}{2x_1} \qquad \text{as} \quad x_1 \to 0$$

Integrating with respect to x_1 and making the appropriate choice for the integration constant shows that (5.14) is satisfied and that (4.4) does indeed match (3.15) as $x_1 \to 0$.

6. DISCUSSION OF RESULTS

The preceding analysis implies that Tollmien-Schlichting waves are generated by free stream disturbances in the relatively small region near the leading edge where the motion is governed by the unsteady boundary layer equation. These disturbances do not generate Tollmien-Schlichting waves in the downstream region where the motion is governed by the Orr-Sommerfeld equation. This is consistent with Murdock's (ref. 24) numerical results.

The remnant of the unsteady boundary layer solution oscillates about a Stokes shear-layer type of solution with progressively decreasing amplitude. Mathematically, these oscillations are represented by asymptotic eigensolutions. In this paper, we consider only the asymptotic eigensolutions of Lam and Rott (ref. 9), whose wave length decreases with increasing distance downstream (it decreases like $x^{-1/2}$). This reduction in spatial scale gives rise to cross stream inertia effects, which are absent in the unsteady boundary layer region and which can eventually (i.e., when $x = O(\epsilon^{-2})$) destabilize the flow - causing it to behave like a spatially growing Tollmien-Schlichting wave.

The reduction in wave length (and consequently in phase speed) allows free stream disturbances to couple with Tollmien-Schlichting waves even when the wave length of the former is very much larger than that of the latter.[12]

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[12] Reshotko (ref. 2, p. 344) states that "In all cases including that of tunnel sound, understanding of the mechanisms by which a forcing disturbance of a given frequency and prescribed phase velocity excites a free disturbance of the same or related frequency but different phase velocity is of great importance."

The reduction occurs because the asymptotic eigensolutions satisfy a homogeneous equation, which does not contain a free stream pressure term to balance the temporal acceleration term. Since the latter term cannot be entirely balanced by viscous effects, it must, in the main, be balanced by the convective acceleration term.

The temporal and convective acceleration terms would balance exactly if the phase Φ of the disturbance were

$$t - \int \frac{dx}{U}$$

Near the wall, $U \propto \eta \propto y/\sqrt{x}$, so that

$$\Phi - t \propto x^{3/2}$$

Thus, the wave length of this disturbance decreases like $x^{-1/2}$.

The wave length of the asymptotic eigensolution (3.6) decreases like $x^{-1/2}$ for a similar reason, i.e., because it behaves somewhat like a convected disturbance that penetrates into a region whose mean velocity is decreasing like $x^{-1/2}$.

The Lam and Rott asymptotic eigensolutions match onto Tollmien-Schlichting waves far downstream in the flow. The characteristic equation (4.63), which determines the eigenvalues of these waves, has one root for each of the asymptotic eigensolutions of the unsteady boundary layer equation. <u>Only the lowest order asymptotic eigensolution of the unsteady boundary layer equation turns into a spatially growing Tollmien-Schlichting wave</u>. The remaining eigensolutions match onto Tollmien-Schlichting waves that continue to decay.

Figure 3 shows that the imaginary part of the wave number κ of the former Tollmien-Schlichting wave decreases very rapidly with increasing downstream distance when $\tilde{x}_1 \equiv 2x_1/U_0'^2 > 0.3$. In this way, the lowest order asymptotic eigensolution, which is at first quite highly damped, eventually turns into a growing disturbance.

The figure also shows that the real part of the wave number (i.e., the reciprocal wave length) at first increases with x_1, reaches a maximum when $\tilde{x}_1 = 2x_1/(U_0')^2$ is near unity, and remains relatively constant thereafter.

Thus the initial wave length reduction, which occurs because of the absence of pressure fluctuations, ultimately produces the pressure fluctuations needed to keep the wave length relatively constant. One might say that a quasi-equilibrium condition is reached when x_1 is $O(1)$. Our numerical solution of (5.2) and (5.3) shows that $\kappa_0 / \sqrt{2x_1}$ is relatively independent of x_1 for the Tollmien-Schlichting waves corresponding to the remaining asymptotic eigensolutions.

The real part of the exponent in (4.4) is a measure of the amount of damping the Tollmien-Schlichting wave undergoes. This quantity multiplied by $-\varepsilon^3$ is, to the lowest order of approximation, equal to $\mathcal{Im}\int_0^x \kappa\,(x_1)dx_1$, which is plotted in figure 4 as a function of the normalized distance $\tilde{x}_1 = 2x_1/(U_0')^2$. It attains its maximum value at the neutral stability point, which occurs at $\tilde{x}_1 \approx 3.03$ when $\varepsilon = 0$. Beyond this point it begins to decrease until it becomes negative, which indicates that the instability wave has grown beyond its initial upstream value. The maximum damping is roughly equal to -3.62 when the characteristic Reynolds number ε^{-6} is 10^4.

It follows from (4.2), (4.4), (4.5), (4.13), (4.20) and the remarks at the end of the first paragraph of section 5 that the streamwise velocity fluctuation associated with the Tollmien-Schlichting wave is given by

$$u_1 = \epsilon^{-2\tau} C(2x_1)^{-1/2} A(x_1) \frac{\partial \Upsilon}{\partial \eta} e^{(i/\epsilon) \int_0^x \kappa(x_1, \epsilon) dx} \tag{6.1}$$

where τ is given by (3.16) and, in the main part of the boundary layer (see (4.28)),

$$\frac{\partial \Upsilon}{\partial \eta} = U' + O(\epsilon)$$

With the normalization of A implied by (5.14), C is the constant that multiplies the appropriate asymptotic eigensolution as the unsteady boundary layer equation.

Since the slowly varying function $A(x_1)/\sqrt{2x_1}$ is $O(1)$, the dominant Reynolds number dependence of the amplitude of the Tollmien-Schlichting wave is given by the factor $\epsilon^{-2\tau}$.

It is worth noting that the results of this paper are entirely independent of the nature of the free stream velocity perturbation u_∞. More importantly, they also apply to any finite thickness flat plate whose mean pressure gradient is sensibly zero in the downstream region where $x > 1$. The unsteady Blasius boundary layer equation (3.2) still holds in this region, but its asymptotic solution now corresponds to a different 'upstream boundary condition'. This will no doubt have an important effect on the constants that multiply the asymptotic eigensolutions, but as we have already indicated, we have not attempted to calculate these here.

APPENDIX A

AMPLITUDE CORRECTIONS FOR ASYMPTOTIC EIGENSOLUTIONS

It was shown by Lam and Rott (ref. 9) that (3.2) possesses the homogeneous solution (i.e., a solution with the free stream forcing term h put equal to zero).

$$\psi_c = (2x)^{1/2}\, p(x) - \frac{(2x)^{-1/2}}{i}\,[2xp_x + p(x)]F'(\eta) \tag{A-1}$$

where $p(x)$ can be any differentiable function of x and $F(\eta)$ is the Blasius function defined by (3.12) through (3.14).

Ackerberg and Phillips point out that (A-1) can be considered as an eigensolution for the outer flow since $\partial\psi_c/\partial\eta$ satisfies a homogeneous outer boundary condition. However it does not satisfy the wall boundary conditions (3.5) but, as shown by Ackerberg and Phillips (ref. 10), it can be asymptotically matched onto an 'inner solution' that does satisfy these conditions.

This latter solution is obtained by introducing the new independent variables σ (which is defined in (3.12)) and

$$\alpha = \xi^{-1} = x^{-1/2} \tag{A-2}$$

into (3.2) with $h = 0$ to obtain

$$4\tilde{\Psi}_{\sigma\sigma\sigma} + 2^{3/2}\alpha F\tilde{\Psi}_{\sigma\sigma} + 4i\tilde{\Psi}_{\sigma} - 2\alpha^2F'(\sigma\tilde{\Psi}_{\sigma\sigma} - \alpha\tilde{\Psi}_{\sigma\alpha})$$

$$+ \sqrt{2}\,\alpha^3F''(\sigma\tilde{\Psi}_{\sigma} - \alpha\tilde{\Psi}_{\alpha}) = 0 \tag{A-3}$$

where we have put

$$\tilde{\Psi}(\sigma,\alpha) \equiv \psi_c(\eta,x) \tag{A-4}$$

The variable σ is presumed to be of order one in the wall layer. Then it follows from (3.2) that η will be small and we can approximate the Blasius function by the first two terms in its Taylor series expansion

$$F = \frac{U_0'}{2}\eta^2 - \frac{(U_0')^2\eta^5}{5!} + \ldots \qquad \text{as} \quad \eta \to 0 \qquad \text{(A-5)}$$

Inserting this into (A-3) and dropping terms that are higher order than α^3 and $\alpha^7 \partial\tilde{\psi}/\partial\alpha$, we obtain

$$4\tilde{\psi}_{\sigma\sigma\sigma} + 4i\tilde{\psi}_\sigma + \sqrt{2}\,\alpha^4 U_0'(\sigma\tilde{\psi}_{\alpha\sigma} - \tilde{\psi}_\alpha) = \frac{U_0'}{2}\Bigg[\sqrt{2}\,\alpha^3(\sigma^2\tilde{\psi}_{\sigma\sigma} - 2\sigma\tilde{\psi}_\sigma)$$
$$+ \frac{U_0'}{8\cdot 3}\alpha^7\sigma^3(\sigma\tilde{\psi}_{\sigma\alpha} - 4\tilde{\psi}_\alpha)\Bigg] \qquad \text{(A-6)}$$

Inserting the expansion (3.15) into this result, we find upon equating coefficients of like powers of α that

$$L_p g_0 = 0$$

$$L_p g_1 = (U'_0/2^{5/2})\Big[4\tau(\sigma g_0' - g_0) + \sigma^2 g_0'' - 2\sigma g_0'$$
$$+ \frac{\lambda}{4\cdot 3}(\sigma g_0' - 4g_0)\sigma^3\Big] \qquad \text{(A-7)}$$

where we have put

$$L_p \equiv \frac{d^3}{d\sigma^3} + i\frac{d}{d\sigma} + \lambda\left(\sigma\frac{d}{d\sigma} - 1\right) \qquad \text{(A-8)}$$

and the primes now denote differentiation with respect to σ.

Equation (3.5) implies that g_0 and g_1 must satisfy the wall boundary conditions

$$g_0 = g_0' = 0 \quad \text{at} \quad \sigma = 0 \tag{A-9}$$

$$g_1 = g_1' = 0 \quad \text{at} \quad \sigma = 0 \tag{A-10}$$

Equation (A-6) is the same as Ackerberg and Phillips equation (4.4) for the σ dependent part of the inner solution. They showed that the solution to this equation that satisfies the wall boundary conditions (A-9) and does not grow exponentially fast as $\sigma \to 0$ (which is required in order that it be able to match a solution of the form (A-1)) is given by the second line of equation (3.7) above. This proves our contention that x^{τ} times the Ackerberg and Phillips solution satisfies the governing equations to within an error $O(\alpha^3)$ times that solution - at least in the inner region. We will complete the proof of this assertion when we show that matching with (A-1) requires p to be equal to x^{τ} times the p given by Ackerberg and Phillips plus terms that are smaller by a factor of α^3.

Ackerberg and Phillips show that

$$g_0 \sim U_0'\left(\frac{i}{\lambda} + \sigma\right) + \text{exp. small terms} \qquad \text{as} \quad \sigma \to \infty \tag{A-11}$$

The right side of (A-7) therefore behaves like

$$-(U_0')^2\left[\frac{4i\tau}{\lambda} + 2\sigma + \frac{\lambda\sigma^3}{3}\left(\frac{i}{\lambda} + \frac{3\sigma}{4}\right)\right] \varepsilon^{-5/2} + \text{exp. small terms} \qquad \text{as} \quad \sigma \to \infty$$

It follows that

$$g_1 \sim (U_0')^2\left(\frac{4i\tau}{\lambda^2} - \frac{\sigma^4}{3\cdot 4}\right)\varepsilon^{-5/2} + Kg_0 \tag{A-12}$$

where K is a constant.

On the other hand, inserting (A-2), (A-5) and (3.12) into (A-1), we find that for small η the outer solution behaves like

$$\psi_c \sim \frac{\sqrt{2}}{\alpha} p(\alpha) - \frac{\alpha}{\sqrt{2}i}(p - \alpha p_\alpha)\left[\frac{U_0'\alpha\sigma}{\sqrt{2}} - \frac{(U_0')^2(\alpha\sigma)^4}{4 \cdot 4!} + \ldots\right] \quad \text{as} \quad \eta \to 0$$

Hence inserting (A-11) and (A-12) into (3.15) we see that the inner and outer solutions will match if we put

$$p = p_0 + \alpha^3 p_1 \tag{A-13}$$

where

$$p_0 = \frac{\alpha^{(1-2\tau)}}{\lambda\sqrt{2}} U_0' i\, e^{-\lambda\, 2^{3/2}/(3\alpha^3 U_0')} \tag{A-14}$$

$$p_1 = \left(K + \frac{\tau U_0'}{\sqrt{2}\,\lambda}\right) p_0 \tag{A-15}$$

Inserting this into (A-1) we see that the lowest order solution is indeed equal to x^τ times Ackerberg and Phillips outer solution given by the first line of (3.7).

In order to determine τ we multiply both sides of (A-7) by $dw/d\sigma$ and integrate the result from zero to infinity. But since $w(\sigma)$ satisfies Airy's equation

$$\frac{d^2 w}{d\sigma^2} + (\lambda\sigma + i)w = 0 \tag{A-16}$$

we find upon integrating by parts and using (3.11), (3.13), and (A-10) that

$$\int_0^\infty \frac{dw}{d\sigma} L_p g_1 \, d\sigma = -\int_0^\infty \frac{d^2 w}{d\sigma^2}\frac{d^2 g_1}{d\sigma^2}\, d\sigma - \int_0^\infty (\lambda\sigma + i)w \frac{d^2 g_1}{d\sigma^2}\, d\sigma$$

$$- \left[i g_1'(0) - \lambda g_1(0)\right] Ai(\zeta_0) = 0$$

It follows that τ is given by (3.16)

APPENDIX B

CONSTRUCTION OF INVISCID ORR-SOMMERFELD SOLUTION

Inserting (4.19) through (4.23) into (4.14) through (4.16) we find

$$\mathscr{L}_R\gamma = \varepsilon^3 i\bar{c}\left[(UD^2 - U'')\left(\frac{d\ \ln A}{dx_1}\gamma + \gamma_{x_1}\right) - \frac{1}{2x_1}D^2(D^2 + FD)\gamma\right] + O(\varepsilon^5) \tag{B-1}$$

where

$$\mathscr{L}_R \equiv (U - \varepsilon\bar{c})(D^2 - \varepsilon^2\bar{\alpha}^2) - U'' \tag{B-2}$$

is the Rayleigh operator, which governs inviscid perturbations of a parallel shear flow.

Introducing the outer variable (4.26) into (4.14) and noting that $U \sim 1$ + exponentially small terms as $\eta \to \infty$, we obtain

$$\tilde{\mathscr{L}}_R\gamma = i\varepsilon^3\bar{c}\left[(D^2 - 3\bar{\alpha}^2)\left(\frac{d\ \ln A}{dx_1}\gamma + \gamma_{x_1}\right) - 3\bar{\alpha}\bar{\alpha}_{x_1}\gamma \right.$$
$$\left. - \frac{1}{2x_1}(1 + \tilde{D}\tilde{\eta})(\tilde{D}^2 - \bar{\alpha}^2)\gamma\right] + O(\varepsilon^4) \tag{B-3}$$

where

$$\tilde{\mathscr{L}}_R \equiv (1 - \varepsilon\bar{c})(\tilde{D}^2 - \bar{\alpha}^2) \tag{B-4}$$

This result applies when $\tilde{\eta} = O(1)$ as $\varepsilon \to 0$.

Inserting (4.24) into (B-1) and equating coefficients of like powers of ε, we obtain

$$UD^2\gamma_0 - U''\gamma_0 = 0 \tag{B-5}$$

$$(UD^2 - U'')\gamma_1 = \bar{c}D^2\gamma_0 \tag{B-6}$$

$$\vdots$$

The solution to (B-5) that satisfies (4.25) is to within an unimportant normalization constant

$$\gamma_0 = U \tag{B-7}$$

The solution to (B-6) is

$$\gamma_1 = -\bar{c} + UK_1\left[\int_\infty^\eta \left(\frac{1}{U^2} - 1\right)d\eta + \eta\right] + K_2U \tag{B-8}$$

where K_1 and K_2 can be any functions of the slow variable x_1. Notice that the integral will exist because $U \to 1$ + exponentially small terms as $\eta \to \infty$.

Inserting (4.27) into (B-3) and equating coefficients of like powers of ε we obtain

$$(\tilde{D}^2 - \bar{\alpha}^2)\tilde{\gamma}_0 = 0 \tag{B-9}$$

$$(\tilde{D}^2 - \bar{\alpha}^2)\tilde{\gamma}_1 = 0 \tag{B-10}$$

$$\vdots$$

where $\tilde{D} \equiv d/d\tilde{\eta}$.

For downstream propagating waves the real part of α will always be positive. Then the solutions to these equations that satisfy (4.18) are

$$\left.\begin{aligned} \tilde{\gamma}_0 &= \tilde{K}_1\, e^{-\bar{\alpha}\tilde{n}} \\ \tilde{\gamma}_1 &= \tilde{K}_2\, e^{-\bar{\alpha}\tilde{n}} \end{aligned}\right\} \qquad \text{(B-11)}$$

where $\tilde{K}_1$, $\tilde{K}_2$ are arbitrary functions of x_1.

Hence for small $\tilde{n}$

$$\tilde{\gamma} \sim \tilde{K}_1(1 - \epsilon\bar{\alpha}n) + \epsilon\tilde{K}_2 + O(\epsilon^2) \qquad \text{as} \quad \tilde{n} \to 0$$

Inserting (B-7) and (B-8) into (4.24) and expanding for large n yields

$$\gamma \sim 1 - \epsilon(\bar{c} - K_2) + \epsilon K_1 n + O(\epsilon^2) \qquad \text{as} \quad n \to \infty$$

These two solutions will match if we put

$$\left.\begin{aligned} \tilde{K}_1 &= 1 \\ \tilde{K}_2 &= -\bar{c} + K_2 \end{aligned}\right\} \qquad \text{(B-12)}$$

and

$$K_1 = -\bar{\alpha} \qquad \text{(B-13)}$$

Inserting (B-7), (B-8), and (B-13) into (4.24) yields (4.28).

Inserting (4.24) into the right side of (B-1) and using (2.12), (4.11), (B-5) through (B-8), and (B-13), we obtain

$$\mathscr{L}_R\gamma = i\epsilon^4\bar{c}\left\{\bar{c}U''\,\frac{d\,\ln(A\bar{c})}{dx_1} + \frac{\bar{\alpha}}{2x_1}\left[F\left(\frac{1}{U}\right)'\right]'\right\} + O(\epsilon^5) \qquad \text{(B-14)}$$

Similarly, inserting (4.27) into the right side of (B-3) and using (B-9) through (B-11) and (B-12) we obtain

$$\tilde{\mathscr{L}}_R \gamma = -i\epsilon^3 \bar{c}\bar{\alpha}\left[2\bar{\alpha}\frac{d \ln A}{dx_1} - (2\bar{\alpha}\tilde{\eta} - 1)\bar{\alpha}_{x_1}\right] e^{-\bar{\alpha}\tilde{\eta}} + O(\epsilon^4) \qquad \text{for} \quad \tilde{\eta} = O(1)$$

(B-15)

Hence, upon putting

$$\gamma = \gamma_H - i\epsilon^4 \bar{c}\left\{\bar{c}\frac{d \ln A\bar{c}}{dx_1} - \frac{F\bar{\alpha}}{6x_1 U^2} + \bar{\alpha}\left(\frac{1}{6x_1} - \frac{d \ln A}{dx_1}\right) U \left[\int_{\infty}^{\eta}\left(\frac{1}{U^2} - 1\right)d\eta + \eta\right]\right\}$$

(B-16)

when $\eta = O(1)$ and

$$\gamma = \tilde{\gamma}_H - i\epsilon^3 \tilde{\eta}\bar{\alpha}\bar{c}\left(\frac{\tilde{\eta}}{2}\ \bar{\alpha}_{x_1} - \frac{d \ln A}{dx_1}\right) e^{-\bar{\alpha}\tilde{\eta}} \qquad \text{(B-17)}$$

when $\tilde{\eta} = O(1)$, we see that

$$\mathscr{L}_R \gamma_H = O(\epsilon^5), \qquad \eta = O(1) \qquad \text{(B-18)}$$

and

$$\tilde{\mathscr{L}}_R \tilde{\gamma}_H = O(\epsilon^4), \qquad \tilde{\eta} = O(1) \qquad \text{(B-19)}$$

$\tilde{\gamma}_H$ vanishes at infinity and γ_H and $\tilde{\gamma}_H$ match asymptotically in some overlap domain (since the second members of (B-16) and (B-18) already have this property).

Thus, to within errors $O(\epsilon^5)$ and $O(\epsilon^4)$, respectively, γ_H and $\tilde{\gamma}_H$ are governed by the homogeneous Rayleigh equation with $\alpha = \epsilon\bar{\alpha}$ and $c = \epsilon\bar{c}$ both small and with $\tilde{\gamma}_H$ being the asymptotic extension of γ_H into the outer region where $\eta = O(\epsilon^{-1})$. A number of investigators obtained uniformly valid asymptotic solutions to this equation for the limit $\alpha \equiv \epsilon\bar{\alpha} \to 0$. The solution to the present problem is therefore most easily obtained by re-expanding such a solution for small values of $c = \epsilon\bar{c}$. Since we shall only need to know the logarithmic derivative of γ_H, the most convenient solution for the present purposes is probably the one given by Miles (ref. 33) which he obtained by transforming the inviscid Rayleigh equation into a Riccati equation. His result can be written as

$$\frac{D\gamma_H}{\gamma_H} = \frac{U'}{U - c} - \frac{1}{(U - c)^2 \Omega} + O(\alpha^5) \tag{B-20}$$

$$\Omega = \frac{1}{\alpha(1 - c)^2} + \Omega_0 + \alpha\Omega_1 + \alpha^2\Omega_2 \tag{B-21}$$

$$\Omega_0 = -\frac{1}{(1 - c)^2} \int_\eta^\infty \left[\frac{(U - c)^2}{(1 - c)^2} - \frac{(1 - c)^2}{(U - c)^2}\right] d\eta \tag{B-22}$$

$$\Omega_1 = -\frac{2}{(1 - c)^2} \int_\eta^\infty (U - c)^2 \Omega_0 \, d\eta \tag{B-23}$$

$$\Omega_2 = -\int_\eta^\infty (U - c)^2 \left[\frac{2\Omega_1}{(1 - c)^2} + \Omega_0^2\right] d\eta \tag{B-24}$$

A simple derivation is given in Reid (ref. 13, p. 279).

APPENDIX C

SOLUTION OF INHOMOGENEOUS WALL LAYER EQUATION

In this appendix we obtain the solution to (4.33) that satisfies (4.38) and does not grow exponentially fast $\bar{\eta} \to \infty$. Since (4.33) is an inhomogeneous Airy's equation for $\bar{D}^2\bar{\gamma}_4$, it follows from (4.32), (4.39), (4.40), and the asymptotic behavior of the Airy functions (Abramowitz and Stegun, ref. 26, pp. 446 and 449) that if $\bar{D}^2\bar{\gamma}_4$ is not allowed to grow exponentially fast as $\bar{\eta} \to \infty$, it must be given by

$$\bar{D}^2\bar{\gamma}_4 = e_1(x_1)\bar{D}^2\bar{\gamma}_0 + r(\bar{\eta}, x_1) \tag{C-1}$$

where we have put

$$r(\bar{\eta}, x_1) = \pi i U_0' \zeta_0 \left[Ai(\zeta) \int_{\zeta_0}^{\zeta} \left(\bar{H}_1 \frac{d \ln A}{dx_1} + \bar{H}_2 \right) Bi(\zeta) d\zeta \right.$$

$$\left. - Bi(\zeta) \int_{\infty_1}^{\zeta} \left(\bar{H}_1 \frac{d \ln A}{dx_1} + \bar{H}_2 \right) Ai(\zeta) d\zeta \right], \tag{C-2}$$

Bi is the second linearly independent Airy function, and e_1 is an arbitrary function of the slow variable x_1.

Using the lowest order of approximation for the coefficients in $\bar{\gamma}_0$ when evaluating $\bar{H}_1$ and $\bar{H}_2$ introduces an error $O(\varepsilon^5)$ into the analysis, which is negligible in the present approximation. Hence it follows from the equation preceding (4.31) and from (4.52) and (4.42) that

$$\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0 = a_1 + \text{exp. small terms} \qquad \text{as } \bar{\eta} \to \infty \tag{C-3}$$

and

$$\overline{H} = \frac{da_1}{dx_1} + \frac{U_0'\overline{\eta}}{2x_1} - \frac{iU_0'\overline{\eta}^3}{\overline{c}3!}\left(a_1 + \frac{3}{4}U_0'\overline{\eta}\right) + \text{exp. small terms} \qquad \text{as} \quad \overline{\eta} \to \infty \qquad \text{(C-4)}$$

where $\overline{H}$ is defined by (4.36).

Inserting these into (4.35), using the result together with (4.39) in (C-2), and noting the Ai and Bi both satisfy (4.41) we obtain upon integrating by parts and using (4.23), (4.31), and (4.40) to simplify the result

$$\Gamma = -\frac{(U_0'\overline{\eta})^2}{2} + \frac{\overline{c}(a_1 + \overline{c})}{2}\left[\frac{\zeta}{\zeta_0} - 2 - \zeta_0 Wi(\zeta)\right] + \text{exp. small terms} \qquad \text{as} \quad \overline{\eta} \to \infty$$

(C-5)

where we have put

$$Wi(\zeta) \equiv -\pi\left[Ai(\zeta)\int_{\zeta_0}^{\zeta} Bi(\zeta)d\zeta - Bi(\zeta)\int_{\infty_1}^{\zeta} Ai(\zeta)d\zeta\right] \qquad \text{(C-6)}$$

Using the asymptotic expansions of the Airy function (ref. 26, pp. 448 and 449; see also ref. 13, pp. 283 and 284 for a detailed discussion) we find

$$Wi(\zeta) = -\frac{1}{\zeta} + O(\zeta^{-4}) \qquad \text{as} \quad \overline{\eta} \to \infty \qquad \text{(C-7)}$$

Hence it follows from (C-1), (C-5), and (4.39) that the solution to (4.33) that grows algebraically as $\overline{\eta} \to \infty$ can be written as

$$\bar{\gamma}_4 = e_1(x_1)\bar{\gamma}_0 + e_2(x_1)\bar{\eta} + e_3(x_1) - \frac{U_0'^2\bar{\eta}^4}{4!} + \frac{\bar{c}^3(a_1 + \bar{c})}{2U_0'^2}\frac{\zeta}{\zeta_0}$$

$$\times \left[\ln\frac{\zeta}{\zeta_0} + \frac{\zeta}{\zeta_0}\left(\frac{1}{6}\frac{\zeta}{\zeta_0} - 1\right) - 1\right] + \int_\infty^{\bar{\eta}}\int_\infty^{\bar{\eta}} r^+(\bar{\bar{\eta}})d\bar{\bar{\eta}}\, d\bar{\eta} \qquad \text{(C-8)}$$

where

$$r^+(\bar{\eta}) \equiv r(\bar{\eta}) + \frac{\bar{\eta}^2U_0'^2}{2}\left[1 - \frac{(a_1 + \bar{c})}{\bar{c}}\frac{\zeta_0}{\zeta}\right] = O(\bar{\eta}^{-4}) \qquad \text{as } \bar{\eta} \to \infty \qquad \text{(C-9)}$$

and e_2 and e_3 are, as yet, arbitrary functions of the slow variable x_1. In order to satisfy (4.38) we must choose them to be

$$e_3 = \frac{11}{12}\frac{(a_1 + \bar{c})\bar{c}^3}{U_0'^2} - \int_\infty^0\int_\infty^{\bar{\eta}} r^+(\bar{\bar{\eta}})d\bar{\bar{\eta}}\, d\bar{\eta} \qquad \text{(C-10)}$$

$$e_2 = -\frac{3}{4}\frac{\bar{c}^2(a_1 + \bar{c})}{U_0'} + \int_0^\infty r^+\, d\bar{\eta} \qquad \text{(C-11)}$$

Integrating by parts and using (4.39) we obtain

$$\int_\infty^0\int_\infty^{\bar{\eta}} r^+(\bar{\bar{\eta}})d\bar{\bar{\eta}}\, d\bar{\eta} = \frac{\bar{c}}{U_0'}\int_\infty^0\left(\frac{\zeta}{\zeta_0} - 1\right) r^+ d\bar{\eta} \qquad \text{(C-12)}$$

Inserting (C-2) into (C-9) and using the fact that Ai and Bi both satisfy (4.41) to eliminate ζAi and ζBi, we obtain upon integrating by parts, using (4.39) and the fact that

$$\text{Ai Bi}' - \text{Bi Ai}' = \pi^{-1} \qquad \text{(C-13)}$$

(Abramowitz and Stegun, ref. 26, p. 446.)

$$\frac{\bar{c}}{U_0'\zeta_0}\int_{\infty}^{0}\zeta r^{+}\,d\bar{\eta}=\pi i\bar{c}\,Bi'(\zeta_0)\int_0^{\infty}\left(\bar{H}_1\frac{d\ln A}{dx_1}+\bar{H}_2\right)Ai\,d\bar{\eta}$$

$$-\lim_{\bar{\eta}\to\infty}\left[\left(\frac{\bar{c}}{U_0'\zeta_0}\right)^3\frac{dr}{d\bar{\eta}}+i\bar{c}\int_0^{\bar{\eta}}\left(\bar{H}_1\frac{d\ln A}{dx_1}+\bar{H}_2\right)d\bar{\eta}-\frac{U_0'^2}{8}\bar{\eta}^{-4}-\frac{U_0'a_1\bar{\eta}^{-3}}{6}\right] \tag{C-14}$$

Inserting (C-5), (C-7), and (4.35), carrying out the second integral over $\bar{\eta}$, and using (4.31), (4.37), (4.39), (4.40), (C-3), and (C-4) we obtain

$$\frac{\bar{c}}{U_0'\zeta_0}\int_{\infty}^{0}\zeta r^{+}\,d\bar{\eta}=i\bar{c}\left[\frac{a_1+\bar{c}}{4x_1}-a_1\frac{d\ln A}{dx_1}-\frac{da_1}{dx_1}\right.$$

$$\left.+\pi\,Bi'(\zeta_0)\int_0^{\infty}\left(\bar{H}_1\frac{d\ln A}{dx_1}+\bar{H}_2\right)Ai\,d\bar{\eta}\right] \tag{C-15}$$

Similarly it can be shown by integrating by parts that

$$\int_0^{\infty} r^{+}\,d\bar{\eta}=-i\zeta_0U_0'\int_0^{\infty}\left\{Wi\left[\bar{H}_1\frac{d\ln A}{dx_1}+\bar{H}_2-\frac{U_0'}{2x_1}\right.\right.$$

$$\left.\left.+\frac{iU_0'}{2\bar{c}}\bar{\eta}^{-2}(a_1+U_0'\bar{\eta})+\frac{\bar{c}(a_1+\bar{c})}{2iU_0'}\right]+\frac{\bar{c}(a_1+\bar{c})}{2iU_0'\zeta}\right\}d\bar{\eta} \tag{C-16}$$

where Wi is defined by (C-6).

Substituting (C-15) into (C-12), substituting the result into (C-10), and finally inserting that result along with (C-11) into (C-8) we obtain

$$\bar{\gamma}_4 = e_1(x_1)\bar{\gamma}_0 + e_2(x_1)\left(\bar{\eta} - \frac{\bar{c}}{U_0'}\right) - i\bar{c}\left[\frac{a_1 + \bar{c}}{4x_1} - a_1 \frac{d\ \ln A}{dx_1} - \frac{da_1}{dx_1}\right.$$

$$\left. + \pi\, Bi'(\zeta_0) \int_0^\infty \left(\bar{H}_1 \frac{d\ \ln A}{dx_1} + \bar{H}_2\right) Ai\, d\bar{\eta}\right] - \frac{U_0'^2 \bar{\eta}^4}{4!} + \frac{\bar{c}^3(a_1 + \bar{c})}{2U_0'^2}$$

$$\times \left\{\frac{\zeta}{\zeta_0}\left[\ln \frac{\zeta}{\zeta_0} + \frac{\zeta}{\zeta_0}\left(\frac{1}{6}\frac{\zeta}{\zeta_0} - 1\right) - 1\right] + \frac{1}{3}\right\}$$

$$+ \int_\infty^{\bar{\eta}} \int_\infty^{\bar{\eta}} \Gamma^+(\bar{\bar{\eta}})\, d\bar{\bar{\eta}}\, d\bar{\eta} \qquad \text{(C-17)}$$

APPENDIX D

TRANSFORMATION OF WALL LAYER SOLUTION

In this appendix we simplify the solution (4.42).

Inserting (4.43) and (4.44) into (4.42) and using (4.59) to eliminate the double integrals in the result yields

$$\bar{\gamma}_0 = a_3\left[\int_{\zeta}^{\zeta_0} \tilde{\zeta}Ai(\tilde{\zeta})d\tilde{\zeta} + \zeta\int_{\infty_1}^{\zeta} Ai(\zeta)d\zeta - \zeta_0\int_{\infty_1}^{\zeta_0} Ai(\zeta)d\zeta + \frac{U_0'\bar{n}\zeta_0}{\bar{c}}\int_{\infty_1}^{\zeta_0} Ai(\zeta)d\zeta\right]$$

Inserting (4.39) to eliminate $\bar{n}$ and using (4.44) and (4.52) to eliminate a_3 yields

$$\bar{\gamma}_0 = \frac{\bar{c}\displaystyle\int_{\zeta_0}^{\zeta} (\zeta - \tilde{\zeta})Ai(\tilde{\zeta})d\tilde{\zeta}}{\zeta_0\displaystyle\int_{\infty_1}^{\zeta_0} Ai(\tilde{\zeta})d\tilde{\zeta}} \tag{D-1}$$

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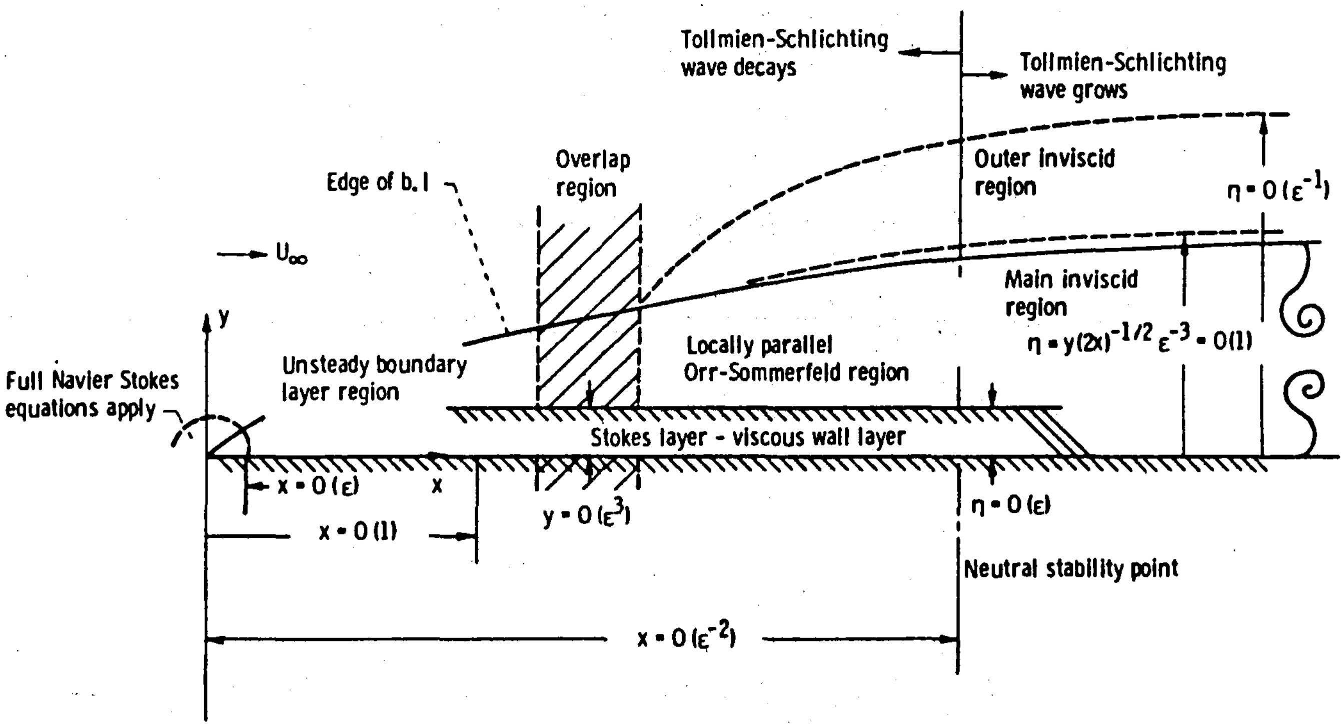


Figure 1. - Asymptotic structure of unsteady boundary layer; $\epsilon = (\nu\omega/U_\infty^2)^{1/6}$.

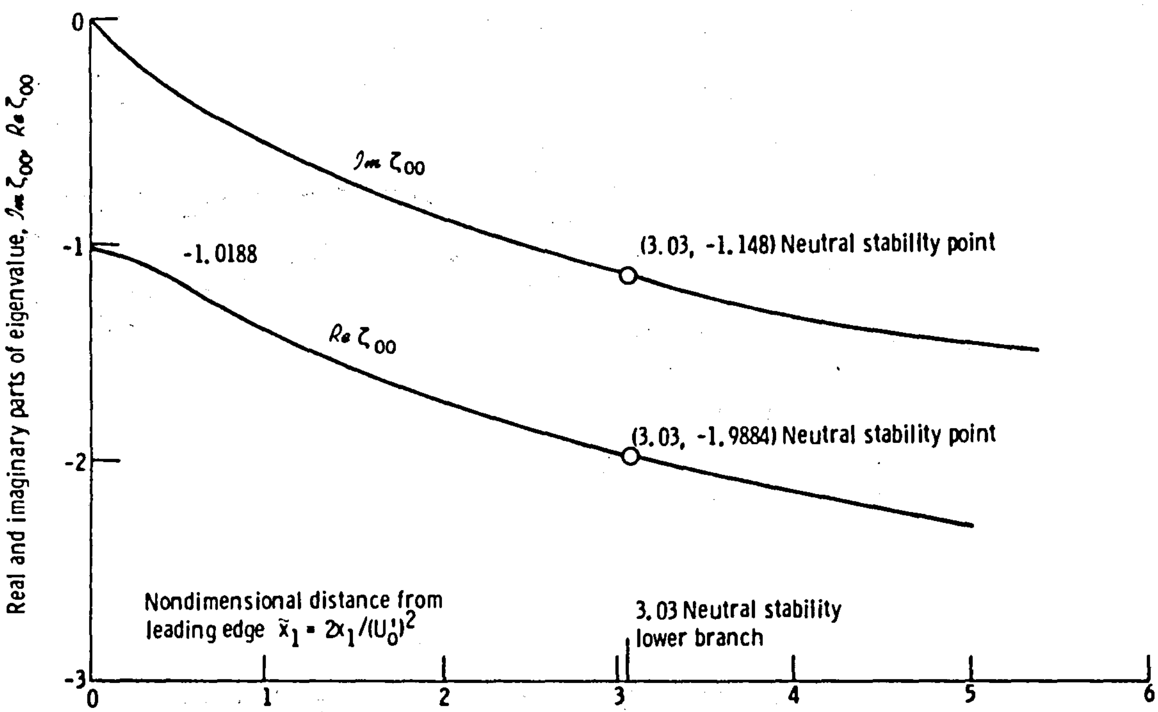
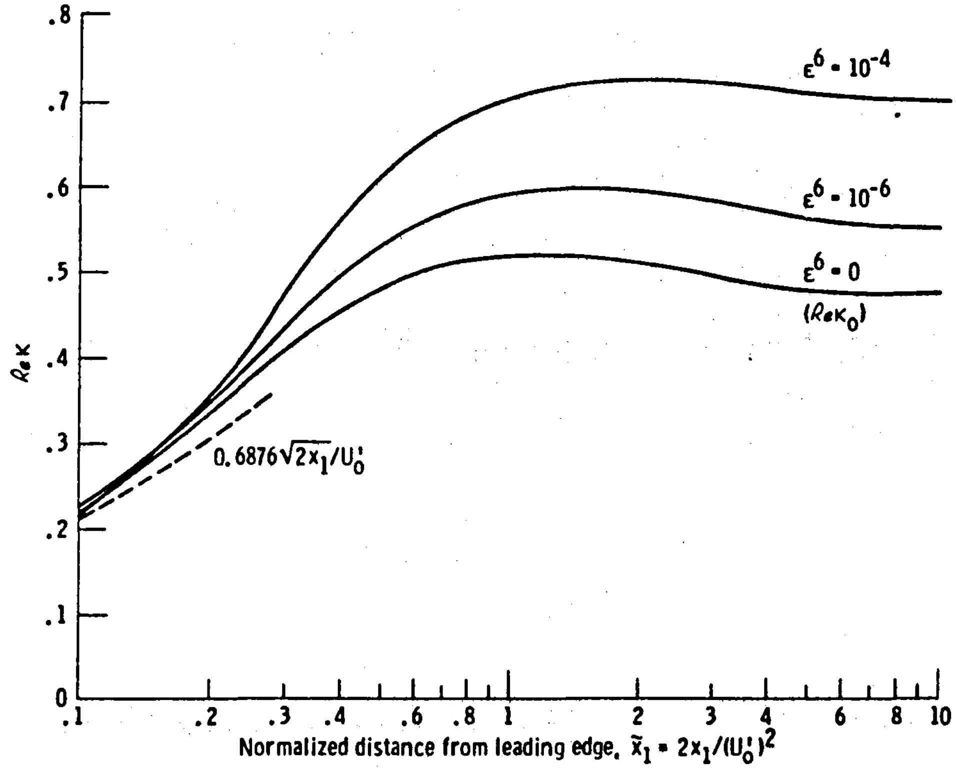
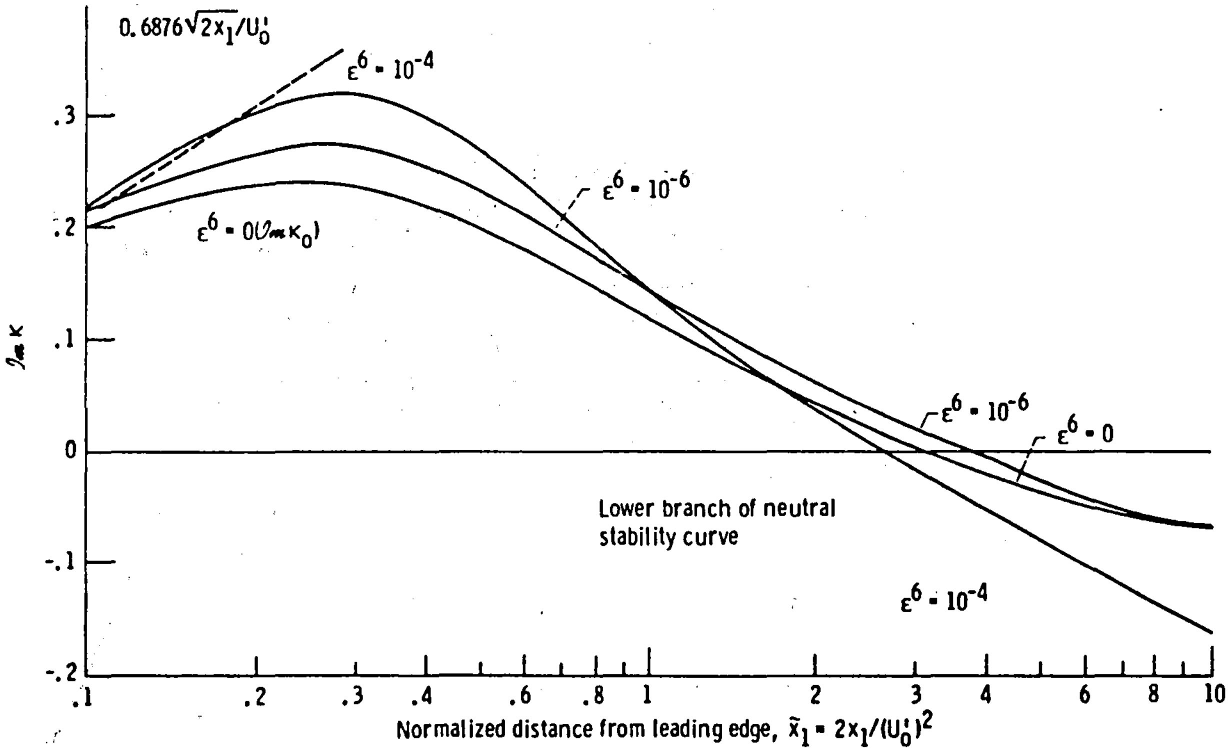


Figure 2. - Variation of complex eigenvalue ζ_{00} with nondimensional distance x_1.



(a) Real part.

Figure 3. - Variation of ε times the complex wave number with distance from leading edge.



(b) Imaginary part.

Figure 3. - Concluded.

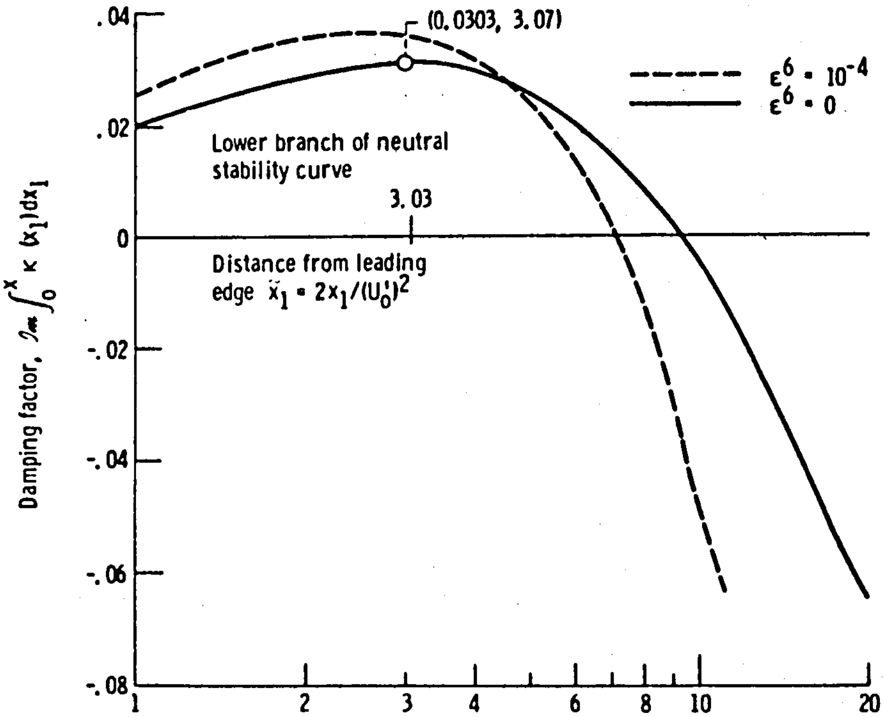


Figure 4. - Variation of damping factor with streamwise distance.

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16. Abstract

The method of matched asymptotic expansions is used to study the generation of Tollmien-Schlichting waves by free stream disturbances incident on a flat plate boundary layer. Near the leading edge the motion is governed by the unsteady boundary layer equation, while farther downstream it is governed (to lowest order) by the Orr-Sommerfeld equation with slowly varying coefficients. It is shown that there is an overlap domain where the Tollmien-Schlichting wave solutions to the Orr-Sommerfeld equation and an appropriate asymptotic solution of the unsteady boundary layer equation match, in the matched asymptotic expansion sense. The analysis leads to a set of scaling laws for the asymptotic structure of the unsteady boundary layer.

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