STRESS DISTRIBUTION IN BONDED JOINTS

by

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Lehigh University
Bethlehem, Pennsylvania

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SUMMARY

The stress distribution in plates and tubes bonded through stepped joints is analyzed. The problem is solved under the assumption of generalized plane stress. Two numerical examples are worked out on specific plate and tube geometries and a material combination of aluminum vs. boron-epoxy composite. The effect of step ends is separately studied. As a limiting case, the solution for bonded plates with smoothly tapered joints is also given.

SYMBOLS

\( a_{1j}, a_{2j}, \alpha_j, \beta_j \) \hspace{1cm} constants defined in the text
\( A_j, B_j \) \hspace{1cm} integration constants
\( d_j \) \hspace{1cm} thickness of the adhesive at the \( j \)th step end
\( E_1, \nu_1 \) \hspace{1cm} elastic constants for the isotropic medium
\( E_{2x}, E_{2z} (E_{2\theta}) \) \hspace{1cm} elastic constants for the orthotropic medium
\( \nu_{2x}, \nu_{2z} (\nu_{2\theta}), G_2 \)
\( E_3, G_3, \nu_3 \) \hspace{1cm} elastic constants of the adhesive
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_1$, $h_{1j}$</td>
<td>thicknesses for the isotropic medium</td>
</tr>
<tr>
<td>$h_2$, $h_{2j}$</td>
<td>thicknesses for the orthotropic medium</td>
</tr>
<tr>
<td>$h_3$</td>
<td>thickness of the adhesive</td>
</tr>
<tr>
<td>$K_j$</td>
<td>total tensile force acting on the $j$th step end</td>
</tr>
<tr>
<td>$L = \lambda_{j+1} - \lambda_j$</td>
<td>step length</td>
</tr>
<tr>
<td>$\lambda_j$</td>
<td>distance of the $j$th step from the origin $x = 0$</td>
</tr>
<tr>
<td>$p(x)$, $q(x)$</td>
<td>contact stresses in smoothly tapered joint</td>
</tr>
<tr>
<td>$p_0$</td>
<td>total load (in tubes, and per unit width in plates)</td>
</tr>
<tr>
<td>$p_{1j}(x)$, $p_{2j}(x)$</td>
<td>total loads in media 1 and 2</td>
</tr>
<tr>
<td>$r$, $\theta$, $x$</td>
<td>cylindrical coordinates (in tubes)</td>
</tr>
<tr>
<td>$r_{1j}^o$, $r_{1j}^i$</td>
<td>outer and inner radii in stepped aluminum tube</td>
</tr>
<tr>
<td>$r_{2j}^o$, $r_{2j}^i$</td>
<td>outer and inner radii in boron-epoxy tube</td>
</tr>
<tr>
<td>$S_{1j}$, $S_{2j}$</td>
<td>cross-sectional areas of the tubes</td>
</tr>
<tr>
<td>$u_{1i}$, $u_{2i}$</td>
<td>displacements in media 1 and 2</td>
</tr>
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<td>$x$, $y$, $z$</td>
<td>rectangular coordinates</td>
</tr>
<tr>
<td>$\varepsilon_{1k}$, $\varepsilon_{2k}$</td>
<td>components of strain ($k = x,y,z$ or $r,\theta,x$)</td>
</tr>
<tr>
<td>$\sigma_{1k}$, $\sigma_{2k}$</td>
<td>components of stress ($k = x,y,z$ or $r,\theta,x$)</td>
</tr>
</tbody>
</table>
1. **INTRODUCTION**

The main purpose of this study is to develop a model for the calculation of stresses in bonded overlapped joints in plates and tubes. In a previous study on reinforcing cover plates bonded to an elastic base plate, it was shown that if the adhesive layer is not taken into account as a third medium, the load transfer between the two plates takes place along the boundary of the bond area only [1]. It was also shown that a more realistic solution of the problem can be obtained if one considers the adhesive layer as a third elastic medium acting basically as a shear spring between the two plates. In this study this concept will be used to analyze the stresses in the stepped joints between aluminum and boron-epoxy plates and tubes. The particular configurations studied are shown in Figures 1 and 2.

2. **PLATES BONDED THROUGH A STEPPED JOINT**

Let an isotropic plate, 1, be bonded to an orthotropic plate, 2, through a stepped joint as shown in Figure 1. The problem is that of determining the stresses $\sigma_{1x}(x)$ and $\sigma_{2x}(x)$ in the plates and the shear stress $\tau(x)$ on the interface.
under a uniform tensile force $p_0$ applied to the plates away from the joint. The problem will be solved under the following approximating assumptions:

a) The thicknesses $h_1, h_{1j}, h_2, h_{2j}$ and $h_3$ are small compared to the other dimensions of the composite structure so that the individual layers and the composite plate may be considered to be under generalized plane stress (i.e., $\sigma_{1y} = 0 = \sigma_{2y}$).

b) The thickness variation of the stresses in the plates will be neglected under the usual assumption that the surface shear transmitted through the adhesive layer acts as a body force.

c) In $z$ direction (see Figure 1), it will be assumed that either $\bar{\varepsilon}_z = \varepsilon_{1z} = \varepsilon_{2z} = 0$ or $\bar{\sigma}_z = 0$, $\bar{\sigma}_z$ being the average stress in the composite.*

For the $i$th portion of the stepped joint (i.e., $\lambda_i < x < \lambda_{i+1}$), let $h_{1i}, h_{2i}$ and $h_3$ be the thicknesses, $p_{1i}(x)$ and $p_{2i}(x)$ be the resultant forces per unit width acting in $x$ direction, $u_{1i}(x)$ and $u_{2i}(x)$ be the displacements in $x$ direction, and $\tau_i(x)$ be the adhesive shear stress. Neglecting the adhesive forces acting on the $x = \lambda_i$ parts of

*The first refers to wide plates under "fixed grip" type of loading. In the second the loads are applied to the plates at locations sufficiently far from the joint and the sides $z = \text{constant}$ are traction-free.
the steps (such as AB in Figure 1), from the equilibrium of
the plate 2 and the adhesive layer we obtain

\[ p_{2i}(x) = p_{2i-1}(x_{i-1}) + \int_{x_{i-1}}^{x} \tau_i(x) \, dx, \quad (i = 1, \ldots, n) \quad (1) \]

\[ \tau_i(x) = \frac{G_3}{h_3} (u_{2i} - u_{1i}), \quad (i = 1, \ldots, n) \quad (2) \]

where \( G_3 \) is the shear modulus of the adhesive. It is assumed
that the materials of plate 1 is isotropic with constants \( E_1, v_1 \) and plate 2 is orthotropic with constants \( E_{2x}, v_{2x}, E_{2z}, v_{2z}, G_2 \).

a) \( \varepsilon_z = \varepsilon_{1z} = \varepsilon_{2z} = 0 \)

Under this assumption \( \sigma_{1z} = v_1 \sigma_{1x}, \sigma_{2z} = v_2 \sigma_{2x} \) and
from the stress-strain relations (with \( \sigma_{1y} = 0 = \sigma_{2y} \)) and

\[ p_{1i} = p_0 - p_{2i}, \quad \sigma_{1x} = \frac{p_{1i}}{h_{1i}}, \quad \sigma_{2x} = \frac{p_{2i}}{h_{2i}} \quad (3) \]

we obtain

\[ \varepsilon_{1x}(x) = \frac{1 - v_1^2}{E_1 h_{1i}} [p_0 - p_{2i}(x)] \quad (4) \]

\[ \varepsilon_{2x}(x) = \frac{1 - v_{2x} v_{2z}}{E_{2x} h_{2i}} p_{2i}(x) \]

Noting that
from (1), (2) and (4) we obtain

\[ \frac{d^2}{dx^2} p_{2i} - \alpha_i^2 p_{2i} = \beta_i p_0, \quad (i = 1, \ldots, n) \]  

(6)

where \( n \) is the number of steps and

\[ \alpha_i^2 = \frac{\mu_3}{h_3} \left[ \frac{1 - \nu_i^2}{E_1 h_{1i}} + \frac{1 - \nu_{2i} \nu_{2j}}{E_2 h_{2j}} \right] \]  

(7)

\[ \beta_i = - \frac{\mu_3}{h_3} \frac{1 - \nu_{1i}^2}{E_1 h_{1i}} \]

b) \( \bar{\sigma}_z = 0 \)

In this case we have

\[ h_{1i} \sigma_{1z} + h_{2i} \sigma_{2z} = 0, \quad \varepsilon_{1z} = \varepsilon_{2z} \]  

(8)

Using (8) and the stress-strain relations with \( \sigma_{1y} = 0 = \sigma_{2y} \) we obtain

\[ \varepsilon_{1x} (x) = \frac{1}{E_1 h_{1i}} \left[ p_0 (1 - \nu_{1a_{2i}}) - (1 - \nu_{1a_{1i}}) p_{2i} (x) \right] \]  

(9)

\[ \varepsilon_{2x} (x) = \frac{1}{E_2 h_{2i}} \left[ p_0 \nu_{2x a_{2i}} + (1 - \nu_{2x a_{1i}}) p_{2i} (x) \right] \]
where

\[ a_{1i} = \left( v_1 + v_{2z} \frac{E_1 h_{1i}}{E_{2z} h_{2i}} \right) / \left( 1 + \frac{E_1 h_{1i}}{E_{2z} h_{2i}} \right) \]

\[ a_{2i} = v_1 / \left( 1 + \frac{E_1 h_{1i}}{E_{2z} h_{2i}} \right) \]

From equations (1), (2), (5) and (9) we again obtain the differential equation (6) for which the constants \( \alpha_i \) and \( \beta_i \) are given by

\[ \alpha^2_i = \frac{\mu_3}{h_3} \left( \frac{1 - v_{2x} a_{1i}}{E_{2x} h_{2i}} + \frac{1 - v_{1x} a_{1i}}{E_1 h_{1i}} \right) \]

\[ \beta_i = \frac{\mu_3}{h_3} \left( \frac{v_{2x} a_{2i}}{E_{2x} h_{2i}} - \frac{1 - v_{1x} a_{2i}}{E_1 h_{1i}} \right) \]

After solving (6) for \( p_{2i}(x) \), the stresses in \( x \) direction in the plates 1 and 2, and the interface shear may be obtained from

\[ \sigma_{1x_i}(x) = \frac{1}{h_{1i}} [p_o - p_{2i}(x)], \sigma_{2x_i}(x) = \frac{1}{h_{2i}} p_{2i}(x) \]

\[ \tau_i(x) = \frac{d}{dx} p_{2i}(x) \]

The solution of the differential equation (6) may be written as
The 2n integration constants $A_i$, $B_i$ are determined from the following 2n conditions\(^*\):

$$
\begin{align*}
\frac{d}{dx} p_{2i}(\lambda_i) &= \frac{d}{dx} p_{2i+1}(\lambda_i), \quad i = 1, \ldots, n-1 \\
\frac{d}{dx} p_{2n}(\lambda_n) &= p_0
\end{align*}
$$

3. **REINFORCED TUBE UNDER AXIAL LOAD**

The configuration for this axially symmetric problem is shown in Figure 2. The problem is quite similar to that discussed in Section 2b. It is solved here under the assumption that the thickness of the composite tube is sufficiently small compared to its radii to justify the assumptions that a) the radial stress $\sigma_r$ is negligible, and b) the $r$-dependence of the stresses may be ignored.

In this case, assuming that $p_0$, $p_1$, $p_2$ stand for the total axial loads in, respectively, the composite, the isotropic (inner) tube, and the orthotropic reinforcing shell, we find

\(^*\)The last set of equations in (14) follows from (2), third equation in (12) and the continuity of the displacements at $x = \lambda_i$, namely $u_{2i}(\lambda_i) - u_{1i}(\lambda_i) = u_{2i+1}(\lambda_i) - u_{1i+1}(\lambda_i)$. 

---
where

\[ \alpha_j^2 = \frac{\mu_3}{h_3} \left[ \frac{1}{E_{2x}} \left( \frac{1}{S_{2j}} - \frac{\nu_{2x} a_{1j}}{h_{2j}} \right) + \frac{1}{E_{1}} \left( \frac{1}{S_{1j}} - \frac{\nu_{1} a_{1j}}{h_{1j}} \right) \right] \]

\[ \beta_j = \frac{\mu_3}{h_3} \left[ \frac{\nu_{2x} a_{2j}}{E_{2x} h_{2j}} - \frac{1}{E_{1}} \left( \frac{1}{S_{1j}} - \frac{\nu_{1} a_{2j}}{h_{1j}} \right) \right] \]

\[ a_{1j} = \left( \frac{\nu_{1}}{E_{1} S_{1j}} + \frac{\nu_{2x}}{E_{2x} S_{2j}} \right) / \left( \frac{1}{E_{1} h_{1j}} + \frac{1}{E_{2x} h_{2j}} \right) \]

\[ a_{2j} = \left( \frac{\nu_{1}}{E_{1} S_{1j}} \right) / \left( \frac{1}{E_{1} h_{1j}} + \frac{1}{E_{2x} h_{2j}} \right) \]

In the tube problem \( x = \lambda_n \) is a plane of symmetry. Thus, in solving the differential equations (15), the only change in the boundary conditions (14) will be to change the condition at \( x = \lambda_n \) from \( p_{2n}(\lambda_n) = p_0 \) to

\[ \frac{d}{dx} p_{2n}(\lambda_n) = 0 \quad (16) \]

4. EFFECT OF STEP ENDS

In the foregoing analysis the effect of the step ends (such as AB in Figure 1) was neglected by assuming that these surfaces \( x = \lambda_i \), \( (i = 0,---,n) \) are traction-free. However, these effects can easily be introduced into the analysis by assuming that the adhesive layer at \( x = \lambda_i \) acts as a tension spring. Defining the total contact loads in \( x \) direction.
acting on the steps $x = \xi_i$ ($i = 0, 1, \ldots, n$) by $K_i$, the equilibrium equation (1) may be modified as

$$p_{2i}(x) = p_{2i-1}(\xi_{i-1}) + \int_{\xi_{i-1}}^{x} \tau_i(x) dx + K_{i-1} \quad (i = 1, \ldots, n)$$  \hspace{1cm} (17)

From the derivation of the differential equation (6) it is easily seen that (6) will remain valid with the same coefficients $\alpha_i$ and $\beta_i$. However, there are $n+1$ new unknown constants $K_i$. The additional conditions to account for these constants are obtained by considering the equilibrium of the adhesive layers at $x = \xi_i$, $(i = 0, 1, \ldots, n)$. For example, at $x = \xi_i$ we may write

$$\varepsilon_3 = \frac{1 - \nu_3^2}{E_3} \frac{K_i}{h_{i+1} - h_i} = \frac{u_2(\xi_i) - u_1(\xi_i)}{d_i} = \frac{h_3}{G_3 d_i} \tau_i(\xi_i)$$

$$= \frac{h_3}{G_3 d_i} \frac{d}{dx} p_{2i}(\xi_i)$$  \hspace{1cm} (18)

Thus the boundary conditions for this problem in bonded plates become

$$p_{2i}(0) = K_0, \quad p_{2n}(\xi_n) = p_0 - K_n$$

$$p_{2i+1}(\xi_i) = p_{2i}(\xi_i) + K_i, \quad (i = 1, \ldots, n-1)$$

$$\frac{d}{dx} p_{2i+1}(\xi_i) = \frac{d}{dx} p_{2i}(\xi_i), \quad (i = 1, \ldots, n-1)$$

$$\frac{d}{dx} p_{2i}(\xi_i) = \frac{(1 - \nu_3)d_1}{2h_3(h_{i+1} - h_i)} K_i, \quad (i = 0, 1, \ldots, n)$$
with \( h_{20} = 0 \). Equations (19) replace (14) in solving (6). Similar conditions can be derived for the reinforced tube.

5. SMOOTHLY TAPERED JOINT

Let the two plates be bonded through a smoothly tapered joint as shown in Figure 3. This is the limiting case of the configuration given in Figure 1 in which \( n \to \infty \) while

\[
\tan \alpha = \frac{(h_{2i+1} - h_{2i})/(l_{i+1} - l_i)}{1 + \tan \alpha}
\]

remains constant. In this section the solution of this problem will be given under the restrictive assumptions that, a) the thicknesses \( h_1, h_2 \) are very small compared to other dimensions of the plate, so that the generalized plane stress assumption is valid for the plates as well as their tapered parts, b) \( \sigma_{1y} = 0 = \sigma_{2y} \) throughout the plates, c) the contact stresses act on the tapered parts of the plates as body forces, d) through-the-thickness distribution of the stresses are negligible, and e) the adhesive acts as a combination of shear and tension springs.

Let \( h_3 \) be the thickness, \( G_3, E_3 \) be the elastic constants of the adhesive layer, \( p(x) \) and \( q(x) \) be the normal and shear components of the contact force, and \( \phi(x) \) be the total force (per unit width) acting on the plate 2 (see Figure 3). From the equilibrium of plate 2 we find (Figure 3b)

\[
\phi(x) = \int_{0}^{X} (p(t) \sin \alpha + q(t) \cos \alpha) \frac{dt}{\cos \alpha}
\]

\[
\int_{0}^{X} (p(t) \cos \alpha - q(t) \sin \alpha) \frac{dt}{\cos \alpha} = 0, \quad (0 < x < \lambda)
\]
Since the second equation of (20) is valid for all x in 0 < x < l, we have

\[ p(t)\cos \alpha = q(t)\sin \alpha \]  \hspace{1cm} (21)

and (20) becomes

\[ \phi(x) = (1 + \tan^2 \alpha) \int_0^x q(t) \, dt \]  \hspace{1cm} (22)

Now let the total relative displacement vector (at point t) be \( \Delta \hat{u} \), with normal and tangential components \( \Delta u_n \), \( \Delta u_t \) and x, y components \( \Delta u_x \), \( \Delta u_y \) (Figure 3c). From the equilibrium of the adhesive we may write

\[ \Delta u_n = \frac{h_3}{E_3} p(x), \quad \Delta u_t = \frac{h_3}{G_3} q(x) \]  \hspace{1cm} (23)

Using (21) and (23), from Figure 3c, it follows that

\[ u_{2x} - u_{1x} = \Delta u_x = h_3 \left( \frac{1}{G_3} + \frac{\tan^2 \alpha}{E_3} \right) q(x) \cos \alpha \]  \hspace{1cm} (24)

Thus, assuming \( \epsilon_z = \epsilon_{1z} = \epsilon_{2z} = 0 \), and observing that

\[ \sigma_{2x}(x) = \frac{\phi(x)}{x \tan \alpha}, \quad \sigma_{1x}(x) = \frac{p_0 - \phi(x)}{h_1 - x \tan \alpha} \]  \hspace{1cm} (25)

from (22), (24) and (25) we find
\[ \frac{d^2 \phi}{dx^2} - f(x)\phi(x) = g(x) \]

\[ f(x) = \frac{1}{c} \left[ \frac{1 - \nu_2^2}{E_2} \frac{2z}{xtan\alpha} + \frac{1 - \nu_1^2}{E_1} (h_1 - x tan\alpha) \right] \]

\[ g(x) = - \frac{(1 - \nu_1^2) p_o}{cE_1 (h_1 - x tan\alpha)} \]

\[ c = \frac{h_3 \cos\alpha}{1 + \tan^2\alpha} \left( \frac{1}{G_3} + \frac{\tan^2\alpha}{E_3} \right) \]

Equation (22) is solved subject to the following boundary conditions

\[ \phi(0) = 0, \phi(\ell) = p_0 \]

From (25) we observe that at \( x = 0 \) \( \sigma_2 \) and (if \( h_1 = h_2 = \ell tan\alpha \)) at \( x = \ell \) \( \sigma_1 \) are not defined which may be expressed as

\[ \sigma_2 (0) = \frac{\phi'(0)}{tan\alpha}, \sigma_1 (\ell) = \frac{\phi'(\ell)}{tan\alpha} \]

6. EXAMPLES

In the solution of the problems shown in Figures 1, 2 and 3 it will be assumed that the material 1 is aluminum, the material 2 is boron-epoxy composite and material 3 (the adhesive) is epoxy. The following elastic constants will be
Aluminum: $E_1 = 10^7$ psi, $v_1 = 0.3$

Boron-Epoxy: $E_{2x} = 32.4 \times 10^6$ psi, $v_{2x} = 0.23$, $E_{2z} = E_{2\theta} = 3.5 \times 10^6$ psi, $v_{2z} = v_{2\theta} = 0.03$, $(G_2 = 1.23 \times 10^6$ psi)

Epoxy: $G_3 = 1.65 \times 10^5$ psi, $E_3 = 4.45 \times 10^5$ psi

a) Bonded Plates

The following data were used in the solution of the plate problem:

$h_3 = 0.001$ in., (0.00075 in.)

$L = \ell_j - \ell_{j-1} = 0.3, 0.4, 0.5$ in., (j = 1,---,5)

$n = 5, \ell_0 = 0$

\[
\begin{array}{ccccccc}
  i & 1 & 2 & 3 & 4 & 5 & 6 \\
  h_{1i} \text{ (in.)} & 0.03 & 0.0245 & 0.0190 & 0.0135 & 0.008 & 0 \\
  h_{2i} \text{ (in.)} & 0.0055 & 0.0110 & 0.0165 & 0.0220 & 0.0275 & 0.033 \\
\end{array}
\]

*The materials and dimensions used in this study correspond to that used by W. Illg in his experimental work at NASA, Langley.

-14-
b) **Reinforced Tube**

The configuration of the reinforced tube is shown in Figure 2. Here the tapered portion of the aluminum tube is represented by two steps. The dimensions used in the solution of the problem are given below:

\[ h_3 = 0.001 \text{ in.}, \ n = 6, \text{ and} \]

<table>
<thead>
<tr>
<th>( j )</th>
<th>( \lambda_j ) (in.)</th>
<th>( r^o_{1j} ) (in.)</th>
<th>( r^i_{1j} ) (in.)</th>
<th>( r^o_{2j} ) (in.)</th>
<th>( r^i_{2j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.7445</td>
<td>0.567</td>
<td>0.750</td>
<td>0.7445</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>0.7390</td>
<td>0.567</td>
<td>0.750</td>
<td>0.7390</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>0.7335</td>
<td>0.567</td>
<td>0.750</td>
<td>0.7335</td>
</tr>
<tr>
<td>4a</td>
<td>2.1</td>
<td>0.7280</td>
<td>0.567</td>
<td>0.750</td>
<td>0.728</td>
</tr>
<tr>
<td>4b</td>
<td>(2.0)</td>
<td>(0.7280)</td>
<td>(0.567)</td>
<td>(0.750)</td>
<td>(0.728)</td>
</tr>
<tr>
<td>5a</td>
<td>2.35</td>
<td>0.728</td>
<td>0.677</td>
<td>0.750</td>
<td>0.728</td>
</tr>
<tr>
<td>5b</td>
<td>(2.25)</td>
<td>(0.7280)</td>
<td>(0.642)</td>
<td>(0.750)</td>
<td>(0.728)</td>
</tr>
<tr>
<td>6</td>
<td>3.0</td>
<td>0.728</td>
<td>0.706</td>
<td>0.750</td>
<td>0.728</td>
</tr>
</tbody>
</table>

The dimensions in parentheses in the table above refer to an alternate representation of the tapered section of the aluminum tube. The areas \( S_{ij} \) and the thicknesses \( h_{ij} \), used in the analysis are calculated from

\[ S_{1j} = \pi[(r^o_{1j})^2 - (r^i_{1j})^2], \]

\[ S_{2j} = \pi[(r^o_{2j})^2 - (r^i_{2j})^2], \]

\( h_{1j} = r^o_{1j} - r^i_{1j}, \) and \( h_{2j} = r^o_{2j} - r^i_{2j}. \)
7. THE RESULTS AND DISCUSSION

The results for the bonded plates, in which the loads $K_j$ at the step ends are ignored, are shown in Figures 4-11. The results are obtained for the external tensile load $p_0 = 1$, hence they have to be multiplied by the given value of $p_0$ (the load per unit width). Each figure contains four plots which are a) the shear stress $\tau_j(x)$ acting on the adhesive, b) total load $p_{2j}(x)$ acting on the boron-epoxy composite, c) the tensile stress $\sigma_{2x}$ in boron-epoxy composite, and d) the tensile stress in aluminum.

The plate results were obtained for three values of step length ($L = 0.5, 0.4, 0.3$ in.) and for the cases $\vec{\varepsilon}_z = 0$ and $\vec{\sigma}_z = 0$. The figures indicate that the difference between the two extreme assumptions regarding the boundary conditions in $z$ direction, namely $\vec{\varepsilon}_z = 0$, the fully-constrained case and $\vec{\sigma}_z = 0$, the free boundaries, is not significant. The figures also show that for the three values of $L$ considered in this paper, the variation in the results are again negligible. In Figures 4-9, the thickness of the adhesive was assumed to be $h_3 = 0.001$ in. In Figures 10 and 11, we have $h_3 = 0.00075$ in. Comparison of the results shown in Figures 4 and 5 with that of Figures 10 and 11 show that the difference in the stresses and the load $p_{2j}$ caused by this change in the adhesive thickness is not very significant.
In general, the figures show that in the plates bonded through a stepped joint

a) The adhesive shear stress is concentrated around the step ends \( x = \lambda_j \) \((j = 0, 1, \ldots, n)\), its maximum is at \( x = 0 \) and its next highest peak is at \( x = \lambda_n \) which is approximately \( 1/3 \) of the maximum,

b) The load \( p_{2j}(x) \) varies between 0 and \( p_o \) in a stepped fashion and the height of steps decreases as \( x \) increases,

c) The peak stress in boron-epoxy composite is at \( x = \lambda_1 \) (at the end of the first step) which is approximately equal to \( 2.8 \, p_o/h_2 \), \( h_2 \) being the full thickness of the boron-epoxy plate and,

d) The peak stress in the aluminum plate is at \( x = 0 \) which is equal to \( \left( \frac{h_1}{h_1} \right) / \left( \frac{p_o}{h_1} \right) = 1.18 \, p_o/h_1 \), \( h_1 \) being the full thickness of the aluminum plate.

Table I shows a summary of some of the significant results for the bonded plates. The results given in the table were obtained from \( h_3 = 0.001 \) in. except for the first two rows, in which \( h_3 = 0.00075 \) in. was used. Note that when the thickness \( h_3 \) of the adhesive is reduced, there is some increase in the peak values of the shear stress \( \tau(x) \), whereas the other quantities of interest remain basically unchanged. In the table, in all but the last two rows, it was assumed that \( K_j = 0, (j = 0, 1, \ldots, n) \). The last two rows will be dis-
<table>
<thead>
<tr>
<th>L (in.)</th>
<th>$\bar{\varepsilon}_z$, $\bar{\sigma}_z$</th>
<th>$\frac{p_{2\text{max}}}{p_0}$</th>
<th>$\frac{\tau_{\text{max}}}{(p_0/h_2)}$</th>
<th>$\frac{\sigma_{1\text{max}}}{(p_0/h_1)}$</th>
<th>$\frac{\sigma_{2\text{max}}}{(p_0/h_2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$\bar{\varepsilon}_z = 0$</td>
<td>1.0</td>
<td>0.506</td>
<td>1.18</td>
<td>2.72</td>
</tr>
<tr>
<td>(h_3 = 0.00075)</td>
<td>$\bar{\sigma}_z = 0$</td>
<td>1.0</td>
<td>0.543</td>
<td>1.18</td>
<td>2.85</td>
</tr>
<tr>
<td>(K_J = 0)</td>
<td>$\bar{\varepsilon}_z = 0$</td>
<td>1.0</td>
<td>0.440</td>
<td>1.18</td>
<td>2.72</td>
</tr>
<tr>
<td>(K_J = 0)</td>
<td>$\bar{\sigma}_z = 0$</td>
<td>1.0</td>
<td>0.470</td>
<td>1.18</td>
<td>2.85</td>
</tr>
<tr>
<td>0.4</td>
<td>$\bar{\varepsilon}_z = 0$</td>
<td>1.0</td>
<td>0.439</td>
<td>1.18</td>
<td>2.72</td>
</tr>
<tr>
<td>(K_J = 0)</td>
<td>$\bar{\sigma}_z = 0$</td>
<td>1.0</td>
<td>0.470</td>
<td>1.18</td>
<td>2.85</td>
</tr>
<tr>
<td>0.3</td>
<td>$\bar{\varepsilon}_z = 0$</td>
<td>1.0</td>
<td>0.438</td>
<td>1.18</td>
<td>2.72</td>
</tr>
<tr>
<td>(K_J = 0)</td>
<td>$\bar{\sigma}_z = 0$</td>
<td>1.0</td>
<td>0.470</td>
<td>1.18</td>
<td>2.85</td>
</tr>
<tr>
<td>0.3</td>
<td>$\bar{\sigma}_z = 0$</td>
<td>0.994</td>
<td>0.441</td>
<td>1.16</td>
<td>2.87</td>
</tr>
<tr>
<td>(d_3 = 0.01)</td>
<td>$\bar{\sigma}_z = 0$</td>
<td>0.964</td>
<td>0.285</td>
<td>1.01</td>
<td>3.12</td>
</tr>
<tr>
<td>(d_3 = 0.001)</td>
<td>$\bar{\sigma}_z = 0$</td>
<td>0.964</td>
<td>0.285</td>
<td>1.01</td>
<td>3.12</td>
</tr>
</tbody>
</table>
cussed later in the paper.

The results for the reinforced tube are shown in Figures 12 and 13. The difference between the two figures is in the representation of the tapered part of the aluminum tube by different steps (see Figure 2). The quantities which seem to be more sensitive to this difference are the adhesive shear stress \( \tau(x) \) and the tensile stress in the aluminum tube which have sharp peaks at the locations of assumed discontinuity. In the actual case, in which there is no discontinuity in the tube thickness, the shear stress and the stress in the aluminum tube are likely to be of the form shown by dashed lines in Figure 12a and Figure 12d.

In this problem the maximum shear acting on the adhesive is also at \( x = 0 \). The stress in boron-epoxy composite has its maximum at \( x = \ell_n \), (that is, the mid-plane) where approximately 77% of the total load \( P_0 \) is carried by this material. The peak stress in aluminum tube is around \( x = \ell_{n-1} \), which is roughly the end of the tapered section.

To study the effect of the bonding loads \( K_j \), \( j = 0, \ldots, n \) at the step ends, the problem described in Section 4 was solved for the plates with \( L = 0.3 \) in., \( h_3 = 0.001 \) in., \( \overline{\sigma}_z = 0 \), \( P_0 = 1 \) and for two values of \( d_3 \), \( d_3 = 0.01 \) in. and \( d_3 = 0.001 \) in. The results are shown in Figures 14 and 15. The values of the end loads \( K_j \) were found to be as follows:
It is seen that at the two extreme ends $K_0$ and $K_5$ are
tensile and the remaining $K_j$'s are compressive. Also, by
decreasing $d_3$ by one order of magnitude, it is seen that
$K_j$'s are increased also by approximately one order of magni­
tude. Comparison of Figures 9 and 14 shows that for $d_3 = 0.01$
the solutions obtained by assuming $K_j = 0$ and $K_j \neq 0$ are
roughly the same. On the other hand, as $d_3$ decreases the
differences in the calculated stresses may be significant.
This can also be seen from Table I by comparing the values
given in the last three rows where the last two rows show
the results of the analysis with $K_j \neq 0$. For the smaller
value of $d_3$, even though there is a slight increase in the
stress concentration factor in boron-epoxy plate, there is
also a significant decrease in $\tau_{max}$. In this example $d_3$ was
assumed to have the same value at all step ends. However,
the analysis is general and $d_3$ may have different values at
different step ends. It is clear that the ends with smaller
adhesive thickness $d_3$ would carry larger loads.

Finally, Figures 16 and 17 show the results for aluminum
and boron-epoxy plates bonded through a smoothly tapered joint

<table>
<thead>
<tr>
<th>$d_3$</th>
<th>$K_0$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.0226</td>
<td>-0.00684</td>
<td>-0.00425</td>
<td>-0.00319</td>
<td>-0.00279</td>
<td>0.00599</td>
</tr>
<tr>
<td>0.001</td>
<td>0.1462</td>
<td>-0.0949</td>
<td>-0.0577</td>
<td>-0.0443</td>
<td>-0.0414</td>
<td>0.0363</td>
</tr>
</tbody>
</table>
shown in Figure 3. Here (26) is solved with

\[ h_3 = 0.001 \text{ in.}, \tan \alpha = \frac{0.0055}{L}, L = 0.3, 0.5 \text{ in.} \]

\[ h_1 = h_2 = 0.033 \text{ in.}, p_0 = 1 \]

It is obvious that these results have all the basic trends of the solution of the stepped joint problem, except that, as it is expected, all the quantities are smooth functions of \( x \). The main results of this solution can be summarized as follows:

a) In the joint, the greater portion of the load is carried by the stiffer material (i.e., the boron-epoxy plate). For \( 0 < x < \ell \) the stress in aluminum is always less than, and the stress in boron epoxy plate is always greater than the average stress \( p_0/h_2 \) acting outside the joint. In fact for \( h_1 = h_2 \) we have \( \sigma_{2\text{min.}} = \frac{p_0}{h_2} = \sigma_{1\text{max.}} \), \( \sigma_1 \) and \( \sigma_2 \) being the stresses in aluminum and boron-epoxy, respectively.

*This result is consistent with that found in [2], where it is shown that in bonded isotropic wedges with \( h_3 = 0, \nu_1 = \nu_2 \) and total angle = \( \pi \), if the modulus \( E_1 \) of the wedge with the smaller angle is less than \( E_2 \), the modulus of the wedge with the greater angle, the stresses at the apex are finite, and if \( E_1 > E_2 \) the stresses at the apex have a singularity. In the present problem \( x = \ell \) correspond to the former and \( x = 0 \) correspond to the latter case. Thus, it is expected that the stresses in the neighborhood of \( x = 0 \) would be much higher than that around \( x = \ell \). As seen from Figures 16 and 17, this is clearly the case. In this problem too the consideration of the adhesive as a separate layer removes the singularity and renders the stresses finite throughout the body.
b) The contact shear stress \( q(x) \) has only one peak (at \( x = 0 \)) and is a rather smooth function of \( x \). As a result, the maximum value of \( q(x) \) is much smaller than that corresponding to the stepped joints, in which the load transfer takes place largely around the end points of the steps (see Figures 4-15). For \( h_1 = h_2 = 0.033 \) in., the following table shows the extreme values of the stresses:

<table>
<thead>
<tr>
<th>( x ) (in.)</th>
<th>( q_{\text{max}} ) ((p_0/h_2))</th>
<th>( q_{\text{min}} ) ((p_0/h_2))</th>
<th>( \sigma_{1\text{max}} ) ((p_0/h_2))</th>
<th>( \sigma_{1\text{min}} ) ((p_0/h_2))</th>
<th>( \sigma_{2\text{max}} ) ((p_0/h_2))</th>
<th>( \sigma_{2\text{min}} ) ((p_0/h_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8</td>
<td>0.053</td>
<td>0.00612</td>
<td>1.0</td>
<td>0.338</td>
<td>2.897</td>
<td>1.0</td>
</tr>
<tr>
<td>3.0</td>
<td>0.032</td>
<td>0.00367</td>
<td>1.0</td>
<td>0.338</td>
<td>2.910</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Note that the stress concentration in boron-epoxy in the tapered joint is roughly the same as that in the stepped joint. However, there is an order of magnitude difference in the maximum values of the adhesive shear stress.

REFERENCES


*The normal component of the contact stress, \( p(x) \), is not given in the Figures or the Table. It can simply be obtained from (21), i.e., \( p(x) = q(x) \tan \alpha \).
FIGURE 1. BONDED PLATES WITH A STEPPED JOINT

FIGURE 2. REINFORCED TUBE WITH A STEPPED JOINT
FIGURE 3. BONDED PLATES WITH A SMOOTHLY TAPERED JOINT
STRESS IN B-E VS. DISTANCE (L=0.5)

STRESS IN AL. VS. DISTANCE (L=0.5)

FIGURE 4  BONDED ALUMINUM-BORON EPOXY PLATES WITH $\varepsilon_z = 0$, $L = 0.5$ IN, $k_3 = 0$, $h_3 = 0.001$ IN, $P_0 = 1$.
FIGURE 5  BONDED ALUMINUM-BORON EPOXY PLATES WITH $\bar{u}_2 = 0$, $L = 0.5$ IN, $x_j = 0$, $y_2 = 0.005$ IN, $P_0 = 1$
Figure 6 Bonded Aluminum-Boron Epoxy Plates with $z_2 = 0$, $L = 0.4$ in, $h_2 = 0$, $h_3 = 0.001$ in.
FIGURE 7 BONDED ALUMINUM-BORON EPOXY PLATES WITH $E_2 = 0, L = 0.4$ IN., $R_2 = 0, R_3 = 0.001$ IN.

(a) SHEAR STRESS VS. DISTANCE ($L = 0.4$)

(b) FORCE IN B-E VS. DISTANCE ($L = 0.4$)

(c) STRESS IN B-E VS. DISTANCE ($L = 0.4$)

(d) STRESS IN AL. VS. DISTANCE ($L = 0.4$)
FIGURE 8  BONDED ALUMINUM-BORON EPOXY PLATES WITH $\varepsilon_x = 0$, $L = 0.3$ IN, $K_j = 0$, $h_3 = 0.001$ IN.

(a) SHEAR STRESS VS. DISTANCE (L=0.3)

(b) FORCE IN B-E VS. DISTANCE (L=0.3)

(c) STRESS IN B-E VS. DISTANCE (L=0.3)

(d) STRESS IN AL. VS. DISTANCE (L=0.3)
FIGURE 9  BONDED ALUMINUM-BORON EPOXY PLATES WITH $\overline{Q}_2 = 0$, $L = 0.3$ IN, $K_2 = 0$, $h_3 = 0.001$ IN.
Figure 10 Bonded aluminum-boron epoxy plates with $x_1 = 0$, $L = 0.5$ in, $K_1 = 0$, $h_2 = 0.00075$ in, $p_0 = 1$.
Figure 11. Bonded aluminum-boron epoxy plates with $\frac{d}{2} = 0$, $L = 0.5$ in, $R_1 = 0$, $k_2 = 0.00375$ in, $P_0 = 1$. 

Shear Stress vs. Distance (L=0.5) 

Force in B-E vs. Distance (L=0.5) 

Stress in B-F vs. Distance (L=0.5) 

Stress in Al. vs. Distance (L=0.5)
FIGURE 12  REINFORCED ALUMINUM-BORON EPOXY TUBE WITH $K_3 = 0$, $h_3 = 0.001$ IN, $p_0 = 1$ (STEPS A)
FIGURE 13  REINFORCED ALUMINUM-BORON EPOXY TUBE WITH $k_f = 0$, $h_3 = 0.001$ IN, $p_b = 1$
(ALTERTATE STEPS B)
FIGURE 14 BONDED PLATES WITH $T_1 = 0$, L = 0.3 IN, $h_2 = 0.001$ IN, $d_3 = 0.01$ IN, $x_j \neq 0$, $p_0 = 1$
FIGURE 15 BONDED PLATES WITH $F_x = 0$, $L = 0.3$ IN, $h_1 = 0.001$ IN, $d_3 = 0.001$ IN, $K_0 = 0.0, P_0 = 1$
Figure 16  Bonded plates with a smoothly tapered joint, $E_2 = 0$, $h_2 = 0.001$ in, $L = 1.0$ in, $\rho_o = 1$.
FIGURE 17  BONDED PLATES WITH A SMOOTLY TAPERED JOINT. $r_2 = 0$, $r_3 = 0.001$ IN., $L = 3.0$ IN., $r_0 = 1$. 

(a) Force in B-E vs. Distance ($L=0.5$)

(b) Stress in B-E vs. Distance ($L=0.5$)

(c) Stress in Al vs. Distance ($L=0.5$)

(d) Shear stress vs. Distance ($L=0.5$)
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