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DEPARTMENT OF MATHEMATICS
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NORFOLK, VIRGINIA

HIGHER MODES OF THE ORR-SOMMERFELD PROBLEM FOR BOUNDARY LAYER FLOWS

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Final Report
For the period ending August 31, 1982

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ABSTRACT

This work examines the discrete spectrum of the Orr-Sommerfeld problem of hydrodynamic stability for boundary layer flows in semi-infinite regions. Related questions concerning the continuous spectrum are also addressed. Emphasis is placed on the stability problem for the Blasius boundary layer profile.

A general theoretical result is given which proves that the discrete spectrum of the Orr-Sommerfeld problem for boundary layer profiles $(U(y), 0,0)$ has only a finite number of discrete modes when $U(y)$ has derivatives of all orders. This has been suspected for some time on the basis of numerical evidence, but the present result is the first theoretical proof of its type for unbounded flows.

Details are given of a highly accurate numerical technique based on collocation with splines for the calculation of stability characteristics. The technique includes replacement of "outer" boundary conditions by asymptotic forms based on the proper large parameter in the stability problem. Implementation of the asymptotic boundary conditions is such that there is no need to make apriori distinctions between subcases of the discrete spectrum or between the discrete and continuous spectrums. Typical calculations for the usual Blasius problem are presented. Results also show that there are not a large number of discrete temporal modes of the Blasius problem lying close to the temporal continuum.

The parallel flow assumption, which leads to the usual Orr-Sommerfeld problem, ignores the small V velocity component of the mean profile. If V is retained, a modified Orr-Sommerfeld equation involving first and third derivative terms is obtained. This modified equation has recently proved competitive with non-linear approaches in predicting instability for a model profile where V is stabilizing. The modified Orr-Sommerfeld problem is considered here for the Blasius problem in which V is destabilizing. Both the marginal stability curve and the higher modes of the modified problem are discussed. Critical parameters for instability are lowered by 3%, but this is not enough to reconcile the linear theory with experimental results. A conjecture is given as to when linear stability predictions with V included do not need further refinement from multiple scale methods.

INTRODUCTION

In hydrodynamic stability, a key assumption in the linear theory for the stability of small amplitude disturbances is that the disturbance stream function can be expanded in terms of a complete set of normal modes. These modes, in turn, are obtained from the eigenfunctions and eigenvalues of an appropriate boundary value problem. For example, for boundary layer type flows over a flat plate, the parallel flow assumption and consideration of two-dimensional disturbances leads to the usual Orr-Sommerfeld (O-S) problem for Tollmien-Schlitching waves. This problem involves a fourth-order linear differential equation and boundary conditions which come from the no-slip requirement on solid walls and appropriate conditions at infinity in the free stream.

Several basic questions arise naturally in the study of linear stability problems. The first question, of course, concerns the stability of the mean profile being considered. This is really a question about the first, or least stable, eigenvalue. As, in general, no closed-form solutions of the eigenvalue problem are possible, analytical work exploits the fact that the Reynolds number R is large for most flows of physical interest. Hence perturbation methods can be used to obtain asymptotic approximations to the eigenvalue relation, and a curve of marginal stability can be derived. On this curve, the disturbance wave number α, wave speed c, and frequency $\omega = \alpha c$ are all real, and hence the temporal and spacial instability problems coincide.

Inside the marginal stability curve, small amplitude disturbances will grow either spacially (α complex, ω real) or temporally (α real, c complex). Numerical methods can be used to obtain curves of constant amplification and constant phase speed in the (α,R) plane for the temporal problem or the (ω,R) plane for the spacial problem. Most work has focused on the temporal case where the eigenvalue c enters in a simple way. However, for comparison with experimental measurements, the relatively complicated spacial case is probably more appropriate.

Results for marginal stability and curves of constant amplification and phase speed are of great importance, both theoretically and for application to the practical engineering problem of transition prediction (Obremski, Morkovin, and Landahl, 1969). However, information of this sort is not adequate to attack the problem of boundary layer receptivity. Consider, for example, the Orr-Sommerfeld problem for the usual Blasius boundary layer profile on a flat plate ignoring the (R^{-1}) outflow velocity from the boundary layer. The receptivity problem requires the proper decomposition of an external disturbance into a superposition of linear stability modes. In particular, to use linear stability theory to study the evolution of a disturbance, we must know how the initial amplitudes of modal disturbances in the flow are caused by external disturbances imposed on the flow. Although few studies consider this basic question, the need to address it has been realized for some time. Mack (1977), in a discussion of the application of stability theory to transition prediction for boundary layers states that ". . . if there were no disturbances, there would be no transition and the boundary layer would remain laminar. Consequently, it is futile to talk about transition without in some way bringing in the disturbances which cause it . . . ." Further on, Mack states that ". . . the precise mechanism by which, say, free stream turbulence, sound and different types of roughness cause transition remains to be discovered."

Resolution of this type of question involves a form of wave packet analysis. The general problem is the solution of the initial-boundary value stability problem in which a disturbance which is initially localized in space and time interacts with the mean flow, propagates in space, and evolves in time. The theoretical question is to determine the distribution of energy from an arbitrary disturbance among the modes of the Orr-Sommerfeld equation and to calculate the subsequent evolution of the disturbance.

In theory the solution to this problem is quite straightforward. We simply determine the complete set of normal modes of the Orr-Sommerfeld equation and expand the initial disturbance in terms of them. The theory required to formally carry out this expansion has been recently derived by Salwen (1980). Once the disturbance has been expanded in terms of the normal modes, the initial amplitude of each mode is known and the evolution of the disturbance wave packet in the flow is completely determined by the evolution of each mode.

To carry out the expansion of an arbitrary disturbance in terms of normal modes of the Orr-Sommerfeld equation requires knowledge of the full spectrum for the boundary layer flows on unbounded domains. Unfortunately, key information on modes in the discrete spectrum has been lacking in both the temporal and spacial problems. It is, of course, well known that there is at least one discrete mode because a neutral stability curve exists. Grosch and Salwen (1978a,b) have shown that both the temporal and spacial stability problems in an unbounded domain always have a continuous spectrum. What is not known, and is vital in order to carry out the expansion, is how many discrete modes exist. This report contains a theoretical proof that there are only a finite number of discrete modes. The available evidence, numerical, further suggests that the number of discrete modes is relatively small, see, for example, Jordinson, 1970; Mack, 1976; Corner, Houston and Ross, 1976; Murdock and Stewartson, 1977. All of these investigators, and others, find a large number of possible discrete modes near the continuum. Most authors, however, interpret these modes as spurious discrete modes. The major exception is the work of Antar and Benek (1978) which claims to have found two families of discrete modes lying close to the temporal continuum.

To distinguish between a valid discrete eigenvalue and a continuum eigenvalue in either the spacial or temporal case requires accurate determination of the associated eigenfunction from the boundary out to a large distance into the free stream. If y denotes distance from the solid lower boundary, the amplitude $\phi(y)$ of the disturbance stream function for a discrete mode will be zero at $y = 0$, reach a maximum inside the boundary layer, and decay exponentially as $y \to \infty$. By contrast, a continuum eigenfunction is small in the boundary layer region, relatively large with oscillatory behavior above the boundary layer, and merely bounded as $y \to \infty$. For various reasons, both shooting and expansion techniques experience difficulties in computing eigenfunctions on unbounded domains. Consequently, it has not been clear that previously used numerical techniques could distinguish between discrete modes on or very near the continuum and continuum functions, so a large number of discrete modes near the continuum has remained a disputed possibility. The present work resolves this controversy by ruling out discrete modes near the continuum. The numerical technique employed here is local in nature, does not produce spurious modes, and allows accurate determination of eigenfunctions in both the spacial and temporal cases. Results are given in Section 4.

The Orr-Sommerfeld problem occupies a central place in the linear theory of hydrodynamic stability and has provided great insights into the nature of instabilities. However, it is important to note that the usual Orr-Sommerfeld situation is somewhat restrictive, even within the context of linear stability theory. This is due to the parallel flow assumption on the mean velocity profile in the derivation of the usual Orr-Sommerfeld equation. Under this assumption, the $O(R^{-1})$ mean outflow velocity is neglected entirely compared to the $O(1)$ mean streamwise flow. This amounts to ignoring all x-derivatives of the mean flow stream function $\bar{\Psi}(x,y,t)$. In an attempt to bridge the gap between the minimum critical Reynolds number R_c for instability predicted by the Orr-Sommerfeld problem and the lower value of Rc found experimentally, some authors e.g. Saric and Nayfeh (1975), have retained full x dependence in derivatives of $\bar{\Psi}$ and considered the stability problem using multiple scale techniques. There is, however, a middle course which remains within the framework of linear stability theory yet still partially relaxes the parallel flow assumption. Non-parallel effects due to outflow from the boundary layer region may be included in linear theory by retaining $\partial\bar{\Psi}/\partial x$ and neglecting only second and higher order x-derivatives of $\bar{\Psi}$. The accuracy of the mean flow representation in the stability problem now becomes the same as that of the Prandtl boundary layer equation. The resulting linear stability problem now involves a modified Orr-Sommerfeld equation with important first and third derivative terms attributable to the outflow velocities. For a test problem involving the asymptotic suction boundary layer profile, Lakin and Reid (1982) showed that including the $O(R^{-1})$ suction component in the mean profile significantly increased the minimum critical Reynolds number for instability. In the more realistic stability problem for the Blasius boundary layer, the effect of the outflow component of the mean profile should be destabilizing. Barry and Ross (1970) have examined the change in Rc induced by including the mean outflow component. However, very little work has been done on higher modes of the modified Orr-Sommerfeld equation for the Blasius problem including outflow. Corner, Houston, and Ross (1976) consider the higher spacial modes, but their results for the modified Orr-Sommerfeld problem are limited and inconclusive.

The highly accurate numerical procedure developed during the course of the present work has been used to re-examine the marginal stability curve for the Blasius problem including outflow. Results were also obtained for the higher modes of this problem in both the temporal and spacial cases. Details are given in Section 5 of this report.

THE NATURE OF THE DISCRETE TEMPORAL SPECTRUM

In order to use eigenfunction expansion techniques in stability calculations it is necessary to know the number and distribution of the eigenvalues of the Orr-Sommerfeld equation. In addition to the continuous spectrum (Grosch & Salwen, 1978; Salwen & Grosch, 1981), there is, of course, a discrete spectrum. The work of Jordinson (1971), Mack (1976), and Murdock & Stewartson (1977) suggests that the number of discrete eigenmodes is finite and, at finite Reynolds number, small.

Some light can be thrown on the question of the number of eigenvalues of the Orr-Sommerfeld equation by using a technique developed by Lidski and Sadovnick (1968). They considered the eigenvalue problem for the Orr-Sommerfeld problem in a finite domain and developed formulae for the sums of integer powers of the eigenvalues of the stability problem. They suggest that these formulae could be used to calculate the eigenvalues. It seems that this approach to the numerical problem is not very fruitful. However, their methodology can be used to address the question of the number of eigenvalues of the stability problem in the infinite domain.

Here we adapt the method of Lidski & Sadovnick, (hereafter L & S), with some slight change in notation. The Orr-Sommerfeld problem is, for the discrete modes,

$$L^2\phi = i\alpha R\{(U-C)L - U''\}\phi, \tag{1}$$

$$\phi(0) = \phi'(0) = 0, \ \phi \to \phi' \to 0 \quad \text{as} \quad y \to \infty \tag{2a-d}$$

with $$L = \frac{d^2}{dy^2} - \alpha^2, \tag{3}$$

and $U(y)$ the velocity profile.

Following L & S, define

$$r(y) \equiv - i\alpha R(U(y)-1), \tag{4}$$

$$q(y) \equiv i\alpha RU'', \tag{5}$$

$$z^2 = i\alpha R(c-1). \tag{6}$$

Now L & S show that if $r(y)$ and $q(y)$ have derivatives of all orders, then the four linearly independent solutions of equation (1) are given, formally, by the four series:

$$\phi_1 = e^{zy} \sum_{k=0}^{\infty} z^{-k} a_k(y), \tag{7 a}$$

$$\phi_2 = e^{-zy} \sum_{k=0}^{\infty} (-1)^k z^{-k} a_k(y), \tag{7 b}$$

$$\phi_3 = \sum_{k=0}^{\infty} z^{-k} b_{k,1}(y), \tag{7 c}$$

$$\phi_4 \sum_{k=0}^{\infty} z^{-k} b_{k,2}(y) \tag{7 d}$$

with z taken to be the root of (6) with positive real part.

The functions $\{a_k(y)\}$ are the solutions of the recurrence differential equations (primes denoting $\frac{d}{dy}$)

$$a_k' = -\left(\frac{5}{2}\right) a''_{k-1} + \frac{1}{2}\left(\alpha^2 - r\right) a_{k-1} - 2a'''_{k-2}$$

$$+ (2\alpha^2 - r)\, a'_{k-2} - \frac{1}{2}\, a'''_{k-3}$$

$$+ \left(\alpha^2 - \frac{1}{2} r\right) a''_{k-3} + \frac{1}{2}\left(\alpha^2 r - q - \alpha^4\right) a_{k-3}, \tag{8}$$

with $a_k(y) \equiv 0$ if $k < 0$, (9)

$$a_0(0) = 1; \quad a_k(0) = 0, \quad k \geq 1. \tag{10a,b}$$

The first few of the $\{a_k\}$ can be found quite easily. We have, for
$k = 0$: $a_0'(y) = 0, \quad a_0(0) = 1,$ (11a,b)

$$a_0(y) = 1; \tag{12}$$

$k = 1$: $a_1'(y) = \frac{1}{2}\left(\alpha^2 - r(y)\right), \quad a_1(0) = 0,$ (13a,b)

$$a_1(y) = \frac{1}{2}\left(\alpha^2 y - \int_0^y r(\xi)\,d\xi\right); \tag{14}$$

$k = 2$:

$$a_2'(y) = \left(\frac{5}{4}\right) r'(y) + \frac{1}{4}\left(\alpha^2 - r(y)\right)\left(\alpha^2 y - \int_0^y r(\xi)\,d\xi\right), \tag{15a}$$

$$a_2(0) = 0, \tag{15b}$$

$$a_2(y) = \left(\frac{5}{4}\right)\left(r(y) - r(0)\right) + \frac{1}{8}\alpha^4 y - \frac{1}{4}\alpha^2 y \int_0^y r(\xi)\,d\xi$$

$$+ \frac{1}{2}\alpha^2 \int_0^y \left[\int_0^\xi r(\xi)\,d\xi\right] d\xi + \frac{1}{4}\int_0^y r(\xi)\left[\int_0^\xi r(\xi)\,d\xi\right] d\xi. \tag{16}$$

The functions $\{b_{k,j}(y)\}$ are the solutions of the recurrence differential equations

$$Lb_{k,j}(y) = \left[L^2 + r(y)\,L + q(y)\right]b_{k-2,j}, \tag{17}$$

with $b_{k,j} \equiv 0$ if $k < 0$, (18)

and $b_{0,1}(0) = b'_{0,2}(0) = 1$, (19a,b)

$$b_{2\ell,j}(0) = b'_{2\ell,j}(0) = 0, \tag{20a,b}$$

$$b_{2\ell-1,j}(y) \equiv 0. \tag{21}$$

Again, the first few of the $\{b_{k,\ell}(y)\}$ can be found. We have, for $k = 0$,

$$Lb_{o,j} = 0 \tag{22}$$

and the boundary conditions (19a,b). Thus,

$$b_{0,1} = e^{-\alpha y}, \tag{23a}$$

$$b_{0,2} = \alpha^{-1} e^{\alpha y}. \tag{24a}$$

From (21)

$$b_{1,j} \equiv 0. \tag{25}$$

Finally, for $k = 2$

$$Lb_{2,j}(y) = \left[L^2 + rL + q\right]b_{0,j}, \tag{26}$$

with $b_{2,j}(0) = b'_{2,j}(0) = 0$. (27a,b)

Therefore

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$$b_{2,1}(y) = (2\alpha)^{-1} \int_0^y \left[e^{\alpha(y-\xi)} - e^{\alpha(\xi-y)}\right] e^{-\alpha\xi} q(\xi) d\xi, \tag{28}$$

$$b_{2,2}(y) = (2\alpha^2)^{-1} \int_0^y \left[e^{\alpha(y-\xi)} - e^{\alpha(\xi-y)}\right] e^{\alpha\xi} q(\xi) \delta\xi. \tag{29}$$

The structure of the $\{a_k(y)\}$ and the $\{b_{k,j}(y)\}$ is now clear. Of the four solutions $\{\phi_\ell\}$, $\ell = 1,\ldots,4$, we see that ϕ_1 and ϕ_4 grow exponentionally as $y \to \infty$ and ϕ_2, ϕ_3 decay exponentially as $y \to \infty$. Thus the solution to the Orr-Sommerfeld eigenvalue problem is a linear combination of ϕ_2 and ϕ_3.

The eigenvalue relation for c is therefore

$$\begin{vmatrix} \phi_2(0) & \phi_3(0) \\ \phi_2'(0) & \phi_3'(0) \end{vmatrix} = 0. \tag{30}$$

We have

$$\phi_2(o) = e^{-zy}\{ao(y) - a_1(y)/Z + a_2(y)/z^2 + 0(z^{-3})\}, \tag{31}$$

$$\phi_3(y) = b_{0,1}(y) + b_{2,1}(y)/z^2 + 0(z^{-4}). \tag{32}$$

Using the definitions of the $\{a_k(y)\}$ and $\{b_{k,1}(v)\}$ it is easy to see that

$$\phi_2(0) \equiv \phi_3(0) \equiv 1, \tag{31}$$

$$\phi_2'(0) = -z + \{[\alpha^2 - r(0)]/2z + 5\, r'(0)/4z^2 + 0\,(z^{-3})\}, \tag{32}$$

$$\phi_3'(0) \equiv -\alpha.$$

Therefore the eigenvalue relation (30) is

$$-\alpha + z - \{(\alpha^2 - r(0))/2z + 5\ r'(0)/4z^2\} + O(z^{-3})\} = 0$$

For $|z|$ large, equation (33) is an asymptotic approximation to the eigenvalue relation. From (6), $|z|$ may be large in two ways: either αR large or α and R fixed and $|c|$ large. Once z is obtained by solving (33) to the appropriate order, c is easily obtained from (6). For example, retaining terms in (33) to $O(z^{-1})$ gives

$$c = 1 - i\alpha/R \tag{34}$$

which is a single point on the temporal continuum. This is not a contradiction as the outer conditions (2c,d) that $\phi, \phi' \to 0$ as $y \to \infty$ automatically imply that Φ and Φ^1 remain bounded as $y \to \infty$, and these are outer conditions for the continuum. To the next order, (33) is

$$z^2 - \alpha z - \frac{1}{2}(z^2 - i\alpha R) = 0. \tag{35}$$

The condition $\mathrm{Re}(z) > 0$ now produces a single mode with $|c| = O(1)$ which does not lie on the continuum. The key point here is that equation (33) does not involve oscillatory or other type functions which might produce an infinite sequence of zeros, and hence, at each order, (33) will give only a finite number of modes. Further, if large $|z|$ is identified with large $|c|$, then the fact that (33) to $O(z^{-1})$ gives (34) shows that there are <u>no</u> discrete modes for fixed α and R outside of a circle with $O(1)$ radius centered at the origin in the complex c-plane. The eigenvalue relation is an entire function of α, c, and R and consequently, in the finite region of the c-plane inside the above circle, there can be only a finite number of discrete modes. Therefore, although this argument does not give the number of discrete modes, it does prove that the Orr-Sommerfeld problem for boundary layer profiles $U(y)$ (which have derivatives of all orders) has only a finite number of discrete temporal modes.

NUMERICAL METHODOLOGY AND ASYMPTOTIC BOUNDARY CONDITIONS

There have previously been two basic methods for numerically solving the Orr-Sommerfeld equation; the shooting method (Mack, 1965, 1976) and the expansion method (Grosch and Salwen, 1968; Orszag, 1971; Salwen and Grosch, 1972). The shooting method is quite efficient if the approximate location of the eigenvalue is known. There are, however, problems in calculating the eigenfunctions with this method because of the necessity of purifying or orthonormalizing the separate pieces of the eigenfunction. The higher discrete modes and the continuum eigenfunctions are especially difficult to calculate because they are rapidly oscillating functions of the distance from the boundary outside of the boundary layer and rapidly decaying inside the boundary layer. Shooting methods also explicitly require the imposition of the particular outer boundary conditions for either the discrete modes or the continuum functions. Finally the integrations must be iterated until the correct eigenvalue (the frequency in the temporal stability problem or the wave-number in the spatial stability problem) is obtained.

The expansion method presents problems of a different kind. The solution to the Orr-Sommerfeld equation is expanded in some set of functions and the expansion is truncated after a finite number of terms. This reduces the problem to finding the eigenvalues of a matrix. Once the eigenvalues are determined it is, at least in principle, straightforward to compute the matrix eigenvectors and the corresponding eigenfunctions of the Orr-Sommerfeld equations.

This method requires that the expansion functions be complete; and this requires, of course, the specification of some boundary conditions at the "outer" boundary. There are two ways to handle a boundary at infinity: either to apply the boundary conditions at some finite distance from the boundary where the base flow is approximately that at infinity or to map the infinite region into a finite region (Grosch and Orszag, 1977). In either case the boundary conditions for only the discrete modes or for only the continuum functions must be applied.

Then N eigenvalues of the N by N matrix are readily found, say by using the QR algorithm. Some of these N eigenvalues are approximations to the true discrete eigenvalues of the Orr-Sommerfeld equation and, if there are only a finite number of discrete modes, M, and $M < N$, the remaining N-M modes are spurious.

The numerical procedure used in the present work utilizes a solution process involving collocation with B-splines. The basis of this technique is the collocation code COLSYS developed by Ascher, Christiansen, and Russell (1978) for linear and non-linear boundary value problems. Because of the nature of the solution process, COLSYS does not suffer from the purification, normalization, and spurious mode problems which have plagued current shooting and expansion techniques. As implemented in the present study, there is also no need to treat discrete and continuum cases differently in terms of conditions at the outer boundary.

As developed by Ascher et al., COLSYS is a real variable code on a finite interval for problems not involving eigenvalue determination. By contrast, the O-S problem for boundary layer profiles is an eigenvalue problem involving complex variables on a semi-infinite interval. These differences help to explain why this collocation code has not previously been used in the stability context. To overcome these apparent limitations and produce a package which gives highly accurate numerical results for both eigenfunctions and eigenvalues in the boundary layer stability context, several strategies were used.

To enable the basic code to be used for eigenvalue problems, the eigenvalue was considered to be a varible and a scalar, first order equation ($\frac{dc}{dy} = 0$ or $\frac{d\alpha}{dy} = 0$) was appended to the usual Orr-Sommerfeld problem. This changes the formulation from a linear to a non-linear boundary value problem, but introduces no additional difficulties so far as COLSYS is concerned. Indeed, increasing the order of the problem allows specification of an additional condition which provides a natural vehicle for normalizing the numerical solutions.

The package of subroutines which comprise COLSYS involves only real arithmetic, and, because of the nature of these routines, COLSYS itself cannot be re-written to employ direct complex arithmetic. The complex-valued equations of the stability problem could be rewritten in real-valued form by explicitly deriving real and imaginary parts, but this procedure was rejected as being cumbersome, inflexible, and subject to manipulative errors. Instead, COLSYS itself was left untouched but the manner in which complex numbers are stored in computer memory was exploited so that complex arithmetic could be used directly in user-supplied subroutines specifying the eigenvalue problem. Real and complex derivatives were related using complex-valued Jacobians. Compatibility of COLSYS with the complex-arithmetic subroutines also required appropriate re-ordering of the dependent variables and their derivatives.

The principal difficulty in adapting COLSYS to boundary layer stability involves reduction of the computational domain from the semi-infinite physical interval $[0, +\infty]$ to the interval $[0,Y]$ where Y is not excessivly large, e.g. $Y < 20$ say. Simply putting a "top" on the problem is known to produce spurious modes in the discrete spectrum. The semi-infinite interval may also be mapped into a finite interval [Grosch and Orszag (1977)] but the resulting problem is quite complicated. The present work employs asymptotic outer boundary conditions which are equivalent to an asymptotic-numerical matching at Y of the computed eigenfunction for $y < Y$ and the proper asymptotic form of the eigenfunction for $y > Y$.

The heart of the asymptotic boundary conditions is the theoretical result of Lakin and Reid (1982) that unbounded domain effects induce higher order corrections to solutions of the eigenvalue relation for bounded values of y, right down to $y = 0$. Inclusion of these effects requires derivation and use of a proper large parameter which differs from the usual large parameter in the bounded domain case [Lakin and Reid (1970), Lakin, Ng, and Reid (1978)]. To obtain the proper parameter, consider the Orr-Sommerfeld equation in the explicit form

$$(D^2 - \alpha^2)^2\phi = i\alpha R\{(U-C)(D^2 - \alpha^2)\phi - U''\phi\} \tag{3.1}$$

where again $U = U(y)$ is the mean velocity profile, α is the wavenumber, c is the wave speed, $\phi(y)$ is the amplitude of the disturbance wave function, and $D = \frac{d}{dy}$. The Reynolds number R in (3.1) is based on the free stream velocity U_0 and the length scale L rather than the boundary layer thickness δ.* Length scale conversions can be made, if desired, using the relation

$$\delta * = 1.7207L. \tag{3.2}$$

In the present scaling, as $y \to \infty$, $U(y) \to 1$, $U''(y) \to 0$, and the O-S equation tends to a constant coefficient equation which can be written in the factored form

$$(D^2 - \alpha^2)(D^2 - \lambda^2)\phi = 0 \tag{3.3}$$

where

$$\lambda^2 = i\alpha R(1-c) + \alpha^2 \quad \text{and} \quad Re(\lambda) > 0. \tag{3.4}$$

Solutions of equation (3.3) which do not violate the outer boundary conditions are $e^{-\alpha y}$ and $e^{-\lambda y}$. For continuum modes, which have $Re(\lambda) \equiv 0$, the solution $e^{+\lambda y}$ is also admissible. The $e^{-\alpha y}$ solution can be identified with analytic continuations of solutions of the Rayleigh equation

$$(U-c)\ (D^2 - \alpha^2)\bar{\phi}^{(0)} - U''\bar{\phi}^{(0)} = 0 \tag{3.5}$$

to large y. As (3.5) is obtained from (3.1) by formally letting $R \to \infty$, these solutions exhibit inviscid-type behavior. By contrast, the $e^{-\lambda y}$ solutions have viscous-type behavior. This argument also indicates that the proper parameter in the usual O-S problem for boundary layer flows is λ^2 rather than $i\alpha R$. In particular, the scaling in λ is relative to the free stream velocity rather than the shear in the critical layer as in the case of bounded channel flows.

Asymptotic outer boundary conditions are based on solutions of equation (3.3) and avoid the spurious mode difficulties known to occur in the case of the discrete spectrum if a "top" is put on the flow. To examine the source of the spurious mode difficulties, suppose that a top is put on the flow at some (large) value $y = T$ (say) and the exact outer boundary conditions are replaced by the two conditions $\phi(T) = \phi'(T) = 0$. In effect, the unbounded domain has been replaced by a bounded domain (which could, for example, be rescaled to $[0,1]$ if desired). It is now well known that stability problems on bounded domains have an infinite number of discrete modes. Consequently, spurious discrete modes are to be expected in global methods which impose tops, as the problem they are actually solving does not provide a uniform approximation to the true problem for large y, no matter how large T is taken. The problem, of course, is that ϕ and ϕ' in the true O-S problem for boundary layer flows may be exponentially small at T, but unless $T \equiv +\infty$ (which is not practical computationally), $\phi(T)$ and $\phi'(T)$ are not identically zero.

The approach taken in the present work reduces the computational domain to $0 < y < Y$ yet avoids problems with spurious modes by matching the computed solution for $0 < y < Y$ to an appropriate functional form of the solution valid asymptotically for $y > Y$. Specifically, at $y = Y$, $\phi(y)$ is matched to an appropriate exponential solution of equation (3.5)

$$E(y,\gamma) = e^{-\alpha y} \tag{3.6}$$

where, consistent with equation (3.3), for discrete modes

$$\gamma(\alpha,c,R) = \begin{cases} \alpha & \text{if } Re(\alpha) < Re(\lambda) \\ \lambda & \text{if } Re(\lambda) < Re(\alpha) \end{cases}$$

Appropriate choices for γ in different situations are:

(1) Temporal discrete modes: In this case, α is real, $|\lambda|$ is large, and $\alpha < Re(\lambda)$. Consequently, the viscous contribution $e^{-\lambda y}$ to solutions of (3.3) dies out rapidly relative to the inviscid contribution $e^{-\alpha y}$. For later use in rejecting the possibility of discrete modes close to the temporal continuum, it may be noted that even if $1-cr$ is small but non-zero, $|\lambda^2|$ is still large for large αR, and hence the appropriate choice to (3.7) is always $\gamma = \alpha$ for discrete temporal modes.

(2) Spacial discrete modes: For these modes, α is complex while the frequency $\omega = \alpha c$ is real. In this case, it is convenient to re-write the large parameter as

$$\lambda^2 = iR(\alpha-\omega) + \alpha^2 \tag{3.8}$$

to decide if γ should be α or λ. Care is now required in assessing the relative speed with which viscous-type and inviscid-type solution of (3.3) die out as $y \to \infty$. For fixed values of ω and R, $Re(\alpha) < Re(\lambda)$ for most values of α. However, because of the complex nature of α, there <u>are</u> cases when $e^{-\lambda y}$ dies out slower than $e^{-\alpha y}$ and hence viscous effects penetrate relatively far outside the boundary layer. This may be seen by explicitly writing $\alpha = \alpha_r + i\alpha_i$ and separating λ^2 into real and imaginary parts to obtain

$$\lambda = i(\alpha_i R)^{1/2}\{1 - \frac{\alpha_r^2 - \alpha_i^2}{\alpha_i R} - i\,(\frac{2\alpha_r\alpha_i}{\alpha_i R} + \frac{\alpha_r - \omega}{\alpha_i})\}^{1/2}. \tag{3.9}$$

If R is large while α_r, α_i, and $\alpha_r - \omega$ are all order one, then

$$\lambda \simeq i(\alpha_i R)^{1/2}\,[1 - i\,\frac{(\alpha_r - \omega)}{\alpha_i}]^{1/2}. \tag{3.10}$$

Both real and imaginary parts of $[\lambda - i(\alpha_r - \omega)/\alpha_i]^{1/2}$ are order one and consequently $Re(\lambda) = O(\alpha_i R)^{1/2}$. Hence, $e^{-\lambda y}$ will die out rapidly compared to $e^{-\alpha y}$ and the latter exponential provides the appropriate asymptotic form of the eigenfunction for the asymptotic numerical matching. However, as noticed by Houston, Corner, and Ross (1976), if $\alpha_r = \omega$, to lowest order the imaginary part of λ is still large, but now

$$Re(\lambda) \cong \alpha_r(\alpha_i/R)^{1/2} \ll 1 \tag{3.11}$$

As $\alpha_r = O(1)$, the proper asymptotic form of solutions of (3.3) now involves $e^{-\lambda y}$ rather than $e^{-\alpha y}$.

A major strength of the present numerical technique is that the outer boundary conditions need not be specified apriori and are not intrinsically built into the main elements of the code, e.g. into the choice of expansion functions. Rather, boundary conditions enter the present technique through separate subroutines. This feature allows specification of the appropriate asymptotic functional form (3.6) of the eigenvalue at Y, but does not require that an apriori choice be made for $\gamma(\alpha,c,R)$ {or $\gamma(\alpha,\omega,R)$}. Rather, during the course of the stability calculation, the relative sizes of α_r and $Re(\lambda)$ can be compared, and γ can be suitably chosen. As indicated above, this flexibility is vital in the case of discrete spacial modes. However, it also allows treatment of continuum modes without the need to explicitly change from decaying to merely bounded outer boundary conditions. This can be done automatically when required based on the computed value of $Re(\lambda)$. More will be said about continuum modes in the next section. However, it is worthwhile emphasizing that the use of asymptotic outer boundary conditions not only consistently reduces the computational decision but also eliminates the need to treat discrete and continuum modes separately.

For discrete temporal modes, values of Y from 16 to 18 were found adequate to give seven decimal place accuracy in the imaginary part of the eigenvalue c. For the spacial stability problem, corresponding accuracy for the imaginary part of the eigenvalue α required slightly larger values of Y, e.g. 18 to 20. Some typical results of eigenfunctions in both the spacial and temporal problems are given in Tables 1 and 2. For the temporal problem, the standard test case $R = 580$ and $\alpha = .179$ is considered. The slightly unstable mode shown has $c = .36412 + i\ 0.0079597$. For the spacial problem, $R = 580$ and $\omega = 0.06520$. Table 2 gives the slightly unstable mode with $\alpha = .17933 - i\ 0.0033173$.

One further point regarding the present numerical results should be considered. Stability calculations are extremently sensitive to errors in specifying the mean velocity profile. This is not a problem in the case of bounded channel flows or model boundary layer-type flows, such as the asymptotic suction boundary layer profile, which can be specified analytically. For the Blasius problem, however, the profile must be obtained numerically as the solution of the boundary value problem

$$\left.\begin{aligned} & f'''(\eta) + f(\eta)\, f''(\eta) = 0 \\ & f(0) = f'(0) = 0, \quad f' \to 1 \quad \text{as} \quad \eta \to \infty \end{aligned}\right\} \qquad (3.12)$$

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Table 1. Unstable discrete temporal mode of the Blasius stability problem for alpha = 0.179 and R = 580. Asymptotic boundary conditions were applied at Y = 18. The computed value of c = 0.36412 + 0.0079597 i.

X	PHI		PHIP	
0.0000	0.	0.	0.	0.
1.8000	.36304E+00	-.38355E-02	.16114E+00	.59821E-02
3.6000	.47366E+00	.53418E-03	-.21310E-01	.15525E-03
5.4000	.37839E+00	.10759E-03	-.63697E-01	-.17671E-03
7.2000	.27558E+00	.20869E-05	-.49280E-01	-.56592E-05
9.0000	.19969E+00	.66046E-08	-.35744E-01	-.23961E-07
10.8000	.14468E+00	-.36693E-10	-.25899E-01	-.36911E-09
12.6000	.10483E+00	.61365E-10	-.18765E-01	-.35495E-10
14.4000	.75956E-01	-.29227E-11	-.13596E-01	.84633E-12
16.2000	.55034E-01	-.16641E-11	-.98511E-02	.23131E-12
18.0000	.39875E-01	-.15361E-48	-.71377E-02	.13512E-35

Table 2. Unstable discrete spacial mode for the Blasius stability problem with omega = 0.06520 and R = 580. Asymptotic boundary conditions were applied at Y = 20. The computed value of alpha = 0.17933 - .0033178 i corresponding to a wave speed c = 0.36346 + 0.0067246 i.

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Y	PHI		PHIP	
0.0000	-.75624E-24	.94971E-30	-.75224E-24	-.50304E-25
.1000	.41771E-02	-.32247E-02	.61965E-01	-.55077E-01
.2000	.15658E-01	-.95716E-02	.14626E+00	-.60966E-01
.3000	.33146E-01	-.15822E-01	.19431E+00	-.56156E-01
.4000	.54133E-01	-.20534E-01	.22284E+00	-.37480E-01
.5000	.77290E-01	-.23296E-01	.23857E+00	-.18005E-01
.6000	.10157E+00	-.24224E-01	.24587E+00	-.12562E-02
.7000	.12629E+00	-.23633E-01	.24797E+00	.11476E-01
.8000	.15106E+00	-.22072E-01	.24693E+00	.20075E-01
.9000	.17561E+00	-.19792E-01	.24366E+00	.24945E-01
1.0000	.19976E+00	-.17135E-01	.23925E+00	.26745E-01
1.1000	.22342E+00	-.14520E-01	.23324E+00	.26221E-01
1.2000	.24636E+00	-.11993E-01	.22584E+00	.24131E-01
1.3000	.26854E+00	-.97224E-02	.21711E+00	.21179E-01
1.4000	.28973E+00	-.77656E-02	.20715E+00	.17965E-01
1.5000	.30994E+00	-.61234E-02	.19621E+00	.14946E-01
1.6000	.32896E+00	-.47619E-02	.18454E+00	.12383E-01
1.7000	.34683E+00	-.36271E-02	.17245E+00	.10412E-01
1.8000	.36346E+00	-.26634E-02	.15019E+00	.90121E-02
1.9000	.37857E+00	-.18590E-02	.14796E+00	.80624E-02
2.0000	.39305E+00	-.10324E-02	.13588E+00	.74626E-02
2.1000	.40605E+00	-.36037E-03	.12403E+00	.70753E-02
2.2000	.41757E+00	.36494E-03	.11234E+00	.67442E-02
2.3000	.42853E+00	.10440E-02	.10087E+00	.64313E-02
2.4000	.43835E+00	.16702E-02	.89589E-01	.60868E-02
2.5000	.44849E+00	.22630E-02	.78496E-01	.57632E-02
2.6000	.45570E+00	.28046E-02	.67065E-01	.52696E-02

Table 2. continued.

Y	PHI		PHIP	
2.7000	.45499E+00	.33172E-02	.50943E-01	.46573E-02
2.8000	.46515E+00	.37312E-02	.40563E-01	.44227E-02
2.9000	.46930E+00	.42021E-02	.30443E-01	.39962E-02
3.0000	.47247E+00	.45316E-02	.20799E-01	.35858E-02
3.1000	.47465E+00	.49149E-02	.17512E-01	.31963E-02
3.2000	.47548E+00	.52210E-02	.80707E-02	.28304E-02
3.3000	.47643E+00	.54[illegible]E-02	.32761E-03	.24695E-02
3.4000	.47607E+00	.57198E-02	-.75675E-02	.21741E-02
3.5000	.47495E+00	.59225E-02	-.14603E-01	.18345E-02
3.6000	.47312E+00	.60976E-02	-.21553E-01	.16206E-02
3.7000	.47065E+00	.62475E-02	-.27752E-01	.13621E-02
3.8000	.46759E+00	.63748E-02	-.33388E-01	.11683E-02
3.9000	.46399E+00	.64819E-02	-.38472E-01	.97821E-03
4.0000	.45991E+00	.65712E-02	-.43012E-01	.81076E-03
5.0000	.40326E+00	.65321E-02	-.63778E-01	.22403E-04
6.0000	.34042E+00	.61131E-02	-.59954E-01	-.13506E-03
7.0000	.28489E+00	.56212E-02	-.51022E-01	-.24667E-03
8.0000	.23612E+00	.63221E-02	-.42719E-01	-.34416E-03
9.0000	.19901E+00	.54444E-02	-.35706E-01	-.40372E-03
10.0000	.16032E+00	.55204E-02	-.29845E-01	-.43812E-03
11.0000	.13900E+00	.50754E-02	-.24944E-01	-.44895E-03
12.0000	.11617E+00	.46276E-02	-.20848E-01	-.44442E-03
13.0000	.97684E-01	.41900E-02	-.17424E-01	-.42927E-03
14.0000	.81134E-01	.37714E-02	-.14562E-01	-.40712E-03
15.0000	.67604E-01	.33772E-02	-.12170E-01	-.38066E-03
16.0000	.56603E-01	.30130E-02	-.10171E-01	-.35192E-03
17.0000	.47352E-01	.26737E-02	-.85004E-02	-.32235E-03
18.0000	.39571E-01	.23060E-02	-.71040E-02	-.29360E-03
19.0000	.33066E-01	.20573E-02	-.59364E-02	-.26406E-03
20.0000	.27633E-01	.18354E-02	-.49615E-02	-.23762E-03

where η is a similarity variable. The form of (3.12) is awkward computationally, as it involves an outer condition at infinity. It has been known for some time (see, for example, Rosenhead (1963), page 222) that (3.12) can be converted to an equivalent initial value problem for a function $F(s)$ defined by $f(\eta) = aF(s)$ where a is a "homotopy" constant, $\delta = a\eta$, and the initial values are $F(0) = F;(0) = 0$, $F''(0) = 1$. The homotopy constant a is determined by the limiting behavior of $F'(s)$, i.e.

$$a = \{\lim_{s \to \infty} F'(5)\}^{1/2}. \tag{3.13}$$

The computation must be carried to sufficiently large s that a is determined to the required accuracy. Even slight errors in a may lead to a limiting behavior of $f'(\eta)$ which differs from unity, and this critically effects the stability characteristics.

While relatively simple itself, the system (3.12) provides a cannonical example of problems of boundary layer-type which are amenable to homotopy methods. It is therefore worth noting that the numerical determination of $f(\eta)$, $f'(\eta)$, and $f''(\eta)$ can be made more efficient through use of a "k-homotopy" constant. In particular, define

$$f(\eta) = bF(\xi) \quad \text{with} \quad \xi = b\eta \tag{3.14}$$

Then, $F(\xi)$ satisfies the same equation as $f(\eta)$, i.e.

$$F''' + FF'' = 0. \tag{3.15}$$

Suppose $F(\xi)$ has initial values

$$F(0) = F'(0) = 0 \quad \text{and} \quad F''(0) = K. \tag{3.16}$$

Then, (3.14) and the condition $F' \to 1$ as $\eta \to \infty$ give that

$$b = \{\lim_{\xi \to \infty} F'(\xi)\}^{-1/2} \quad \text{and}$$

$$f''(0) = K\,\{\lim_{\xi \to \infty} F'(\xi)\}^{-3/2} \tag{3.17}$$

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Another quantity of basic interest is the limiting value β_0 of the Blasius outflow velocity defined by

$$\beta_0 = \lim_{\eta \to \infty} \{\eta f' - f\} \tag{3.18}$$

In terms of the K-variables, this can be expressed as

$$\beta_0 = \{\lim_{\xi \to \infty} F'(\xi)\}^{-1/2} \{\lim_{\xi \to \infty} (\xi F' - F)\}. \tag{3.19}$$

For $K \equiv 1$, the K-variables reduce to the usual homotopy variables. For $K \neq 1$, the relations are

$$a = K^{1/3} \{\lim_{\xi \to \infty} F'(\xi)\}^{-1/2} \tag{3.20}$$

and

$$s = K^{1/3} \xi . \tag{3.21}$$

To see the advantage in using the K-homotopy technique, let $\xi_\infty(K)$ denote the value of ξ to which it is necessary to carry the initial value calculation in K-variables to achieve sufficient accuracy in the determination of the K-constant b. The advantage to choosing $K > 1$ is that ξ_∞ is a decreasing function of K. Consequently, by choosing a relatively large value of K, the computational domain can be significantly reduced for a given accuracy. The relationship between K and $\xi_\infty(K)$ for eight place accuracy is given in Table 3. Corresponding values of $f''(0)$ and β_0 were found to be

$$f''(0) = 0.46960020$$

and (3.22)

$$\beta_0 = 1.2167789$$

This value of $f''(0)$, together with $f(0) = f'(0) = 0$, can now be used to solve the Blasius equation as a straightforward initial value problem.

Table 3. The kappa-homotopy quantity $\xi_{\infty}(\kappa)$ for eight place accuracy in $f''(0)$.

κ	$\xi_{\infty}(\kappa)$
1.0	4.8
2.0	3.8
3.0	3.5
5.0	2.9
10.0	2.3

THE CONTINUOUS SPECTRUM AND THE POSSIBILITY OF NEARBY DISCRETE TEMPORAL MODES

For continuum modes of the Orr-Sommerfeld problem, the outer boundary conditions $\phi, \phi' \to 0$ as $y \to \infty$ are replaced by the weaker conditions

$$\phi(y), \phi'(y) \text{ bounded as } y \to \infty. \tag{4.1}$$

Grosch and Salwen (1978) have shown that on the continuum $\mathrm{Re}(\lambda) = 0$, and hence the proper functional form which describes the behavior of $\phi(y)$ for large y will involve both $e^{-\lambda y}$ and $e^{-\lambda y}$. Based on this observation, the technique of replacing the exact order conditions (4.1) by asymptotic conditions at Y in numerical calculations based on COLSYS is easily extended to the continuous spectrum. Indeed, this can be done in such a way that there is no need for apriori specification of whether a continuous or discrete mode is being computed, the decision being made automatically in the course of the computation. Physically, continuum modes may be identified with free stream vorticity. In particular, the eigenfunctions are relatively large outside the boundary layer region. An interesting observation, which has not been previously made, is that the appropriate parameter λ, which is large for the discrete spectrum, is not necessarily large on the continuum. For example, on the temporal continuum, Grosch and Salwen have shown that

$$c = 1 - i(1 + k^2)\,\alpha/R \tag{4.2}$$

where k is real and non-zero. If this expression is substituted into (3.4), the dependence of λ on R cancels identically and

$$|\lambda| = |k|\alpha \tag{4.3}$$

Thus, as k need be merely non-zero, for very lightly damped continuum modes, $|\lambda|$ may in fact be small.

While the temporal continuum is simply the straight line $c_r = 1$ and $c_i < 0$, the nature of the spacial continuum is more complicated due to the complex nature of the eigenvalue α. Corner, Houston, and Ross (1976) have found two higher discrete modes of the spacial problem for the Blasius profile which lie fairly close to the spacial continuum. However, the nearby continuous modes are easily distinguished from these discrete modes by the behavior of their eigenfunctions. An example of a spacial continuum eigenfunction for the Blasius profile is given in Table 4. For this mode, $R = 581.1$, $\omega = 0.04649$, and the computed eigenvalue is $\alpha = 0.047583 + 0.0085513$. The corresponding value of the complex wave speed for this mode is $c = .94645 - i\ 0.1701$.

In section 2 of this report, it was proved that there cannot be an infinite number of discrete modes in the temporal stability problem for boundary layer profiles. Although this is the first theoretical result to be proved to date, this fact has been accepted by most authors for some time based on numerical evidence. Further, most authors feel that for fixed α and R, the number of discrete modes is small (six or seven for Blasius). Previous numerical studies have produced modes near the continuum, but these were usually dismissed as spurious discrete modes attempting to mimic the continuum. However, there has been some room for doubt. For example, if there were, as perhaps might be inferred from the available numerical evidence, a number of discrete modes lying very close to the continuum it appears that neither shooting or expansion techniques could distinguish between these modes and the continuum functions. A case in point is provided by the work of Antar and Benek (1978). They numerically studied the spectrum of Blasius flow by imposing the boundary conditions at some finite distance from boundary and attempted to infer the nature of the spectrum for the semi-infinite region by systematically increasing the distance of the "upper" boundary from the lower boundary. Antar and Benek claim to have found two additional groups of discrete modes, which they label the P and S families, lying close to the temporal continuum. In particular, the P family has C_r close to 1 while the S family was initially found to lie along the line $C_r = 0.845$. Further, the S family was conjectured by them to contain an infinite number of modes.

The conjecture of Antar and Benek about the infinite extent of the S family is disproved by the theoretical results of section 2. The numerical procedure developed in the present work was also used to search for modes

Table 4. Spacial continuum mode of the Blasius stability problem for omega = 0.04649 and R = 581.1. Asymptotic boundary conditions were applied at Y = 20. The computed eigenvalue is alpha = 0.047583 + 0.0085517 i corresponding to a wave speed c = 0.94645 - 0.17010 i.

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Y	PHI		PHIP	
0.0000	0.	0.	0.	0.
.1000	-.74133E-04	-.26535E-04	-.13685E-02	-.57650E-03
.2000	-.25358E-03	-.12011E-03	-.21371E-02	-.12948E-02
.3000	-.48756E-03	-.26316E-03	-.24833E-02	-.19452E-02
.4000	-.74156E-03	-.50380E-03	-.25649E-02	-.24373E-02
.5000	-.99560E-03	-.76451E-03	-.24965E-02	-.27466E-02
.6000	-.12368E-02	-.10474E-02	-.23559E-02	-.28043E-02
.7000	-.14653E-02	-.13356E-02	-.21692E-02	-.28802E-02
.8000	-.16712E-02	-.16200E-02	-.19398E-02	-.27740E-02
.9000	-.18513E-02	-.18896E-02	-.16495E-02	-.26140E-02
1.0000	-.19982E-02	-.21430E-02	-.12714E-02	-.24594E-02
1.1000	-.21019E-02	-.23341E-02	-.78225E-03	-.23830E-02
1.2000	-.21507E-02	-.26251E-02	-.17608E-03	-.24726E-02
1.3000	-.21339E-02	-.28375E-02	.52302E-03	-.28276E-02
1.4000	-.20452E-02	-.32029E-02	.12482E-02	-.35516E-02
1.5000	-.18673E-02	-.36132E-02	.18807E-02	-.47361E-02
1.6000	-.16779E-02	-.41670E-02	.22466E-02	-.64420E-02
1.7000	-.14543E-02	-.49192E-02	.21214E-02	-.85748E-02
1.8000	-.12785E-02	-.59177E-02	.12447E-02	-.11360E-01
1.9000	-.12395E-02	-.72006E-02	-.65349E-03	-.14322E-01
2.0000	-.14516E-02	-.87819E-02	-.33171E-02	-.17272E-01
2.1000	-.20513E-02	-.10541E-01	-.84264E-02	-.19806E-01
2.2000	-.31653E-02	-.12713E-01	-.14522E-01	-.21422E-01
2.3000	-.50623E-02	-.14876E-01	-.22024E-01	-.21552E-01
2.4000	-.78278E-02	-.16954E-01	-.30636E-01	-.19514E-01
2.5000	-.11150E-01	-.18712E-01	-.39806E-01	-.15077E-01
2.6000	-.15594E-01	-.19869E-01	-.49024E-01	-.75272E-02

Table 4. Continued.

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Y	PHI		PHIP	
2.7000	-.20924E-01	-.20109E-01	-.57255E-01	.32592E-02
2.8000	-.26967E-01	-.19110E-01	-.63595E-01	.17257E-01
2.9000	-.33547E-01	-.16561E-01	-.67049E-01	.34151E-01
3.0000	-.40269E-01	-.12203E-01	-.66682E-01	.53319E-01
3.1000	-.46730E-01	-.56517E-02	-.61714E-01	.73846E-01
3.2000	-.52440E-01	.25732E-02	-.51611E-01	.94576E-01
3.3000	-.56673E-01	.13026E-01	-.36162E-01	.11417E+00
3.4000	-.59499E-01	.25321E-01	-.15530E-01	.13126E+00
3.5000	-.59625E-01	.39134E-01	.97261E-02	.14429E+00
3.6000	-.57431E-01	.54036E-01	.38666E-01	.15219E+00
3.7000	-.52011E-01	.69364E-01	.70007E-01	.15389E+00
3.8000	-.43401E-01	.84534E-01	.10221E+00	.14873E+00
3.9000	-.31596E-01	.98873E-01	.13356E+00	.13645E+00
4.0000	-.16777E-01	.11151E+00	.16229E+00	.11723E+00
5.0000	.16673E+00	.58941E-01	.84081E-01	-.18672E+00
6.0000	.93641E-01	-.42779E-01	-.13781E+00	.34920E-01
7.0000	.84695E-01	.53049E-01	.12899E+00	.71036E-01
8.0000	.19674E+00	.11640E-01	.14453E-01	-.12186E+00
9.0000	.13126E+00	-.32493E-01	-.66016E-01	.56585E-01
10.0000	.16469E+00	.32185E-01	.10934E+00	.10064E-01
11.0000	.22916E+00	-.22942E-01	-.16383E-01	-.70698E-01
12.0000	.18445E+00	-.30652E-01	-.15944E-01	.48194E-01
13.0000	.23513E+00	-.17501E-02	.33956E-01	-.18348E-01
14.0000	.26979E+00	-.43939E-01	-.23962E-01	-.31715E-01
15.0000	.24673E+00	-.37695E-01	.15068E-01	.27436E-01
16.0000	.30239E+00	-.33223E-01	.63130E-01	-.23584E-01
17.0000	.32640E+00	-.56926E-01	-.21991E-01	-.81697E-02
18.0000	.31504E+00	-.52018E-01	.34549E-01	.48356E-02
19.0000	.37232E+00	-.59437E-01	.48946E-01	-.13690E-01
20.0000	.38046E+00	-.65714E-01	-.13660E-01	-.12671E-03

close to the continuum. For $\alpha = .179$ and $R = 580$ (in the L scaling) values of c attributed to both the P and S families of temporal modes by Antar and Benek were used as initial guesses for calculations with ssymptotic boundary conditions applied at the relatively large value of Y = 20. The eigenfunctions obtained were all clearly continuum eigenfunctions rather than discrete eigenfunctions. A typical calculation is shown in Table 5. In light of the present results, it must be concluded that the P and S families are spurious discrete modes consequently, there are not large numbers of discrete modes lying close to the continuum.

BOUNDARY LAYER STABILITY INCLUDING OUTFLOW VELOCITIES

The parallel flow assumption of linear stability analysis assumes that streamwise amplifications of disturbance quantities takes place on a scale which is small relative to the scale for boundary layer growth. Consequently, streamwise gradients of mean flow quantities are ignored, e.g. all x derivatives of the streamfunction ψ for the mean velocity profile. This corresponds to neglecting the small outflow (or inflow) velocities of the boundary layer and in the case of boundary layer flows over a flat plate leads to the usual Orr-Sommerfeld problem. One of the most important flows of this type is, of course, the Blasius profile.

Early theoretical work on the Blasius problem was done by Tollmien (1929) and Lim (1945) using piecewise polynomial approximations to the exact velocity profile $f'(\eta)$. When the experimental data of Schubauer and Skramstad (1947) first became available, it was found to be in good agreement with the theoretical predictions. Unfortunately, later, more accurate numerical calculations by Jordinson (1971), who used numerical solutions rather than analytic approximations, for $f'(\eta)$, showed this apparent agreement to be illusory. Attempts were then made to restore agreement by taking into account the non-parallel character of the basic flow.

One approach which appeared promising was due to Barry and Ross (1970). They suggest that if $\psi(x,y,t)$ is the streamfunction of the mean flow, then effects of the boundary layer outflow velocity can be introduced by retaining $\partial\psi/\partial x$ and neglecting only second and higher streamwise derivatives of ψ. The accuracy of the mean flow representation in the stability problem

Table 5. Temporal continuum mode of the Blasius stability problem for alpha = 0.179 and R = 580. Asymptotic boundary conditions were applied at Y = 20. Values of c for a supposed discrete mode in the P family of Antar and Benek (1978) were used as starting values for the calculation. The computed value of c = 0.99760 - 0.022995 i.

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Y	PHI		PHIP	
0.0000	0.	0.	0.	0.
2.0000	-.15937E-04	.70793E-04	-.13254E-04	.47533E-04
4.0000	-.11453E-02	.74714E-03	-.17789E-02	.33014E-02
6.0000	.15925E-01	.90311E-02	.13513E-01	-.70393E-02
8.0000	.60747E-02	-.62005E-02	-.14025E-01	.43002E-02
10.0000	.11543E-01	.58669E-02	.12334E-01	-.30215E-02
12.0000	.77383E-02	-.40664E-02	-.10781E-01	.17897E-02
14.0000	.12722E-01	.33546E-02	.11926E-01	-.10575E-02
16.0000	.13985E-01	-.20046E-02	-.80313E-02	.42950E-03
18.0000	.20419E-01	.10913E-02	.13026E-01	-.15473E-03
20.0000	.27882E-01	0.	-.49906E-02	0.

is now the same as in the Prandtl boundary layer equation. The resulting stability problem involves a modified Orr-Sommerfeld equation of the form

$$(D^2-\alpha^2)\{(D^2-\alpha^2) - RVD\}\psi = i\alpha R\{(U-c)(D^2-\alpha^2) - U\}\psi - RV''\psi \tag{5.1}$$

where the mean profile is now $(U,V,0)$ and $V = O(R^{-1})$.

For the Blasius problem, inclusion of the mean boundary layer outflow velocity should be destabilizing. However, neutral stability calculations of Barry and Ross (1970) based on equation (5.1) led to only a slight reduction in the predicted value of R_c. Recently, interest in equation (5.1) has waned considerably in favor of the multiple-scale approach of Saric and Nayfeh (1975). Almost no work has been done on the higher discrete modes of the stability problem based on (5.1).

Recently, Lakin and Reid (1982) have provided a highly accurate treatment of the linear boundary layer stability problem based on uniform asymptotic expansions involving generalized Airy functions. As a test of the asymptotic theory, the stability characteristics of the asymptotic suction boundary layer profile, for which

$$U(y) = 1 - e^{-y} \quad \text{and} \quad V = -1/R \qquad 0 \leq y < +\infty \tag{5.2}$$

were considered on the basis of both equation (3.1) [V ignored] and equation (5.1) [V included]. Results showed that inclusion of the small suction component had a significant effect on stability increasing the value of the R_c from 47,152 to 54,405. The latter value of R_c was in agreement with Hocking's (1975) non-linear treatment of the stability problem including suction.

The approach to stability including V based on equation (5.1) is attractive because of its relative simplicity. Based on its competitiveness with other approaches in the case of the asymptotic suction boundary layer profile, it seemed appropriate to re-examine the stability of the Blasius problem based on equation (5.1). This was done using the present numerical technique and asymptotic outer boundary conditions involving the appropriate large parameter. If outflow is included, the analogue of equation (3.3) as $y \to \infty$ is

$$(D^2-\alpha^2)([D-\beta_0/4]^2-\lambda^2)\psi = 0 \tag{5.3}$$

where now

$$\lambda^2 = i\alpha R(1-c) + \alpha^2 + \beta_0^2/16 \tag{5.4}$$

and β_0 is as in (3.18) and (3.22). In particular, the asymptotic behavior of "viscous-type" solutions of (5.1) as $y \to \infty$ is now

$$e^{-\lambda\eta + \frac{\beta_0}{4}\eta} \tag{5.5}$$

rather than simply $e^{-\lambda\eta}$. Consequently, the appropriate form of the asymptotic boundary conditions for the discrete modes depends on the relative sizes of $Re(\alpha)$ and $Re(\lambda-\beta_0/4)$ when $Re(\lambda)$ is larger than $\beta_0/4$.

The marginal stability curve computed from equation (5.1) and the discrete boundary conditions are shown in figure 1. The computed values of the critical parameters are $Rc = 290.21312$ at $\alpha_c = 0.17776$ and $c_c = 0.40308$. The corresponding variation of c with α along the marginal stability curve is given in figure 2. The value of R_c in (5.6) is fractionally, but not significantly, lower than the value obtained by Barry and Ross (1970). For comparison, when the outflow velocity V is neglected, Davey (unpublished) has obtained the critical values $\tilde{R}_c = 301.64$, $\tilde{\alpha}_c = 0.17653$, and $\tilde{c}_c = 0.39664$ in the L-scaling. These values are marked with an x in figures 1 and 2. Experimentally observed values of R_c are 261 [Schubauer and Skramstad (1947)] and 232 [Ross et al. (1970)]. Figure 1 only shows the "nose" of the marginal stability curve. However, for values of R greater than 500, differences between neutral parameters including and neglecting outflow become slight. Further, good agreement is observed between the usual linear theory predictions and the experimental data.

Previous work on the higher modes of the Blasius problem based on equation (5.1) [Corner, Houston, and Ross (1976)] considered only the spacial case and results were inconclusive. Computations of the higher modes in the present work at the standard test values inside the nose of the neutral

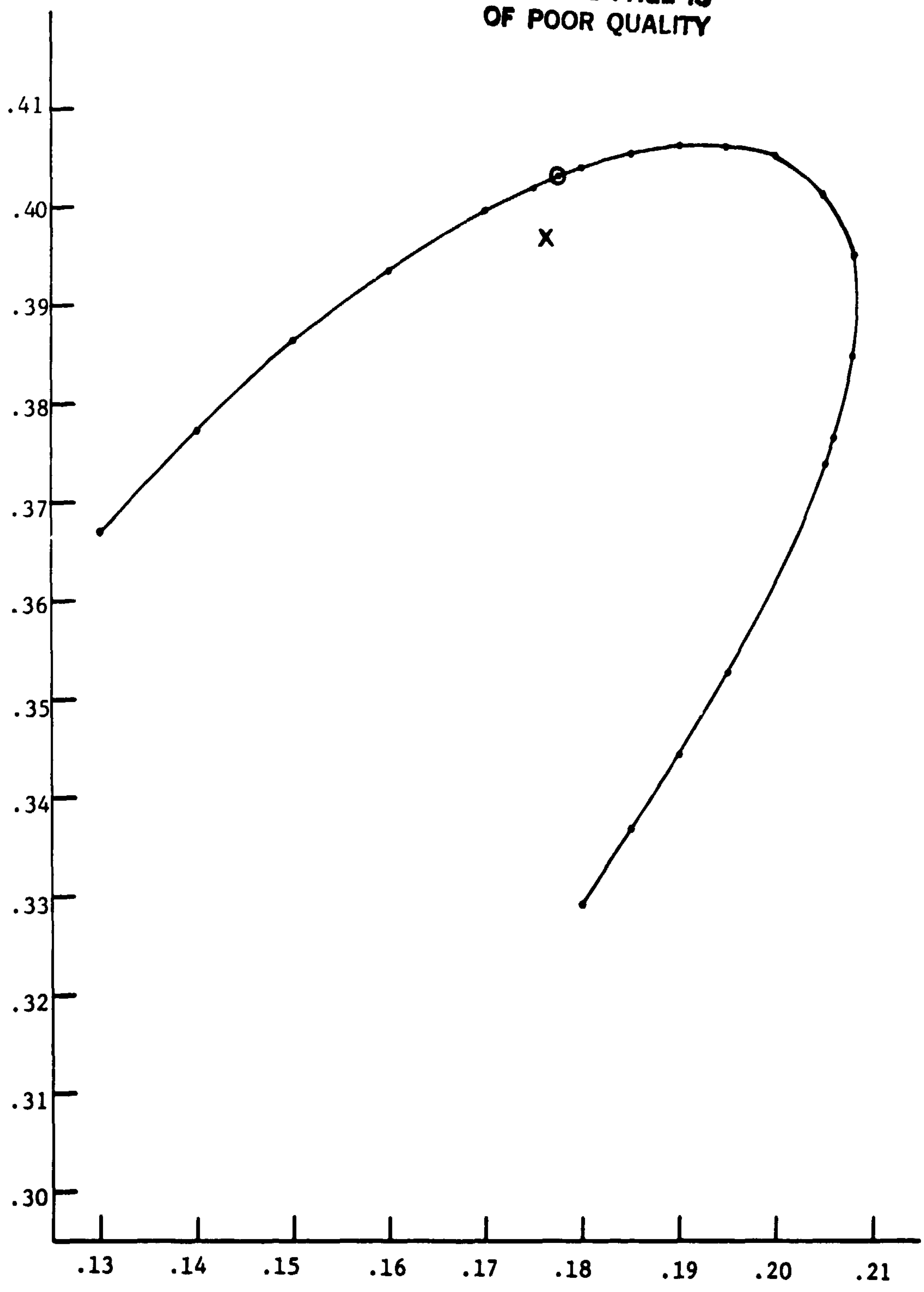

Figure 1. Marginal stability curve for the Blasius problem with proper outflow included from equation (5.1). The circled point marks the minimum critical parameters. The x marks the corresponding minimums with outflow ignored.

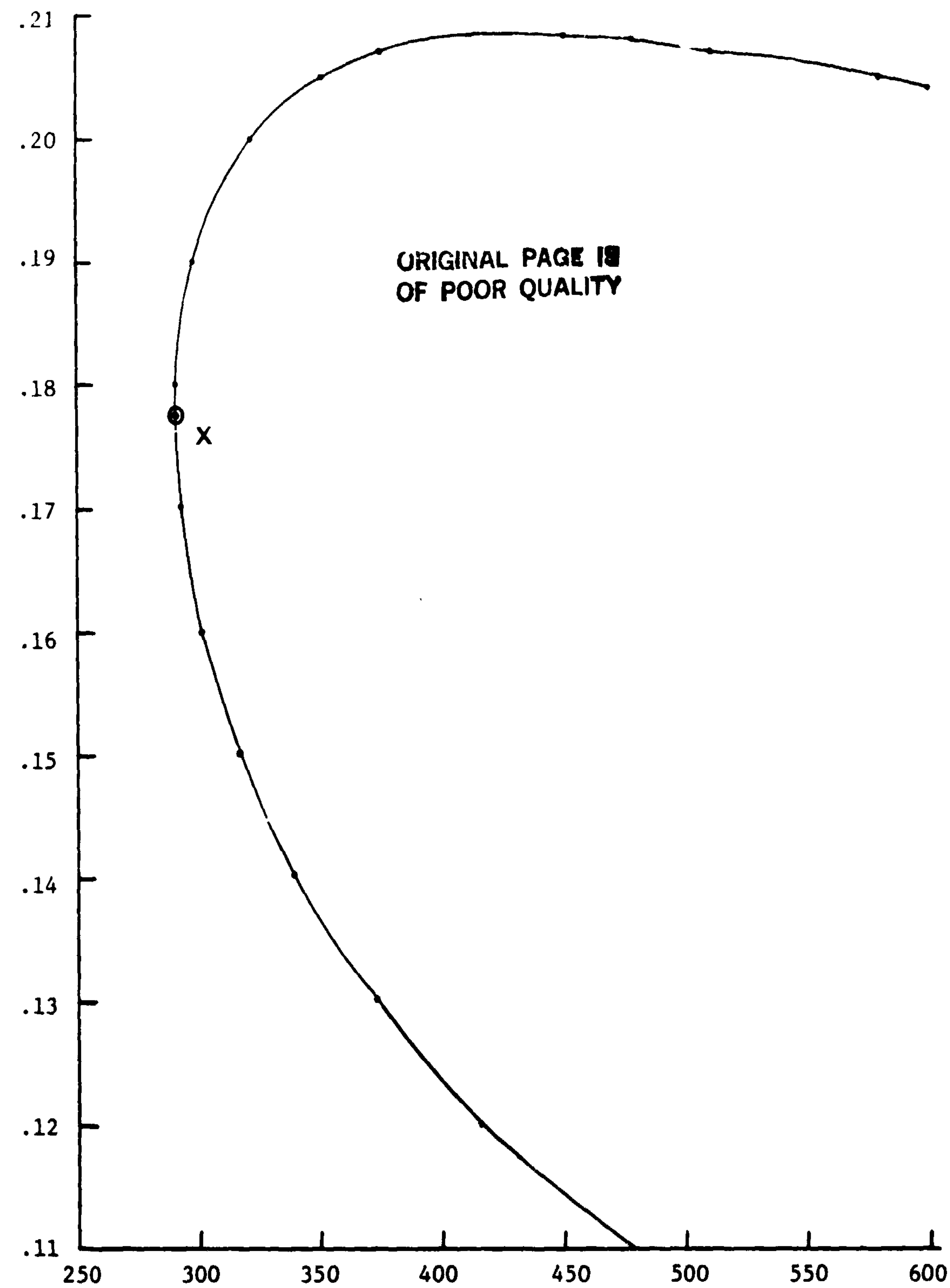

Figure 2. Variation of c with α along the marginal stability curve for the Blasius problem with proper outflow included. The circle denotes the values at R_c. The x marks the corresponding point when outflow is neglected.

curve did not suffer from convergence problems and showed very little difference in decay rates when V was retained or neglected. The eigenfunctions of mode 3 of the temporal stability problem for $\alpha = 0.179$ and $R = 580$ are given as an example in Table 6. With V ignored the computed eigenvalue is $c = 0.48393 - i.19207$ while with V included the eigenvalue is $c = 0.48698 - i.19208$.

The present results show that while linear stability theory is excellent for predicting general stability characteristics, the inclusion of the velocity V along the lines of equation (5.1) is not sufficient to reconcile theory and experimental results for boundary layer profiles involving outflow. When contrasted with the success of equation (5.1) in the case of the asymptotic suction boundary layer profile, a tentative conclusion may be drawn: For boundary layer flows where V is destabilizing, boundary layer growth effects may require the use of more complicated multiple-scale approaches to refine linear stability predictions. However, if V is stabilizing, good results can be expected from the linear stability problem based on equation (5.1). In particular, if suction is used on a flat plate (back from the leading edge) to retard instabilities and retain a laminar flow, the resulting local V velocity will be stabilizing and equation (5.1) should give accurate stability predictions.

Table 6. The effect of partially relaxing the parallel flow assumption on mode 3 in the discrete temporal spectrum of the Blasius stability problem with alpha = 0.179 and R = 580. Asymptotic boundary condition was applied at Y = 18.

A. The eigenfunction based on equation (3.1) where V is ignored. The computed eigenvalue is c = 0.48393 - 0.19207 i.

X	PHI		PHIP	
0.0000	0.	0.	0.	0.
1.8000	.32274E+00	.21776E+00	.44973E+00	-.22353E+00
3.6000	.47058E+00	.21880E-01	-.16989E-01	-.30361E-01
5.4000	.37835E+00	.92538E-03	-.63542E-01	-.19965E-02
7.2000	.27558E+00	.89177E-05	-.49282E-01	-.26830E-04
9.0000	.19969E+00	.18039E-07	-.35744E-01	-.70105E-07
10.8000	.14468E+00	.67144E-11	-.25899E-01	.49292E-10
12.6000	.10483E+00	-.38842E-11	-.18765E-01	.39744E-10
14.4000	.75956E-01	-.16147E-11	-.13596E-01	-.36406E-12
16.2000	.55034E-01	-.72366E-12	-.98511E-02	-.53543E-12
18.0000	.39875E-01	-.70906E-49	-.71377E-02	.62370E-36

Table 6. Continued.

B. The eigenfunction based on equation (5.1) where V is included. The computed eigenvalue is c = 0.48698 - 0.19208 i.

X	PHI		PHIP	
0.0000	0.	0.	0.	0.
1.8000	.31858E+00	.23063E+00	.49196E+00	-.21535E+00
3.6000	.47119E+00	.22723E-01	-.18270E-01	-.31558E-01
5.4000	.37839E+00	.97232E-03	-.63624E-01	-.20980E-02
7.2000	.27559E+00	.93304E-05	-.49283E-01	-.29122E-04
9.0000	.19969E+00	.18594E-07	-.35744E-01	-.72178E-07
10.8000	.14468E+00	.51932E-11	-.25899E-01	.59081E-10
12.6000	.10483E+00	-.44340E-11	-.18765E-01	.48448E-10
14.4000	.75956E-01	-.17983E-11	-.13596E-01	.96688E-12
16.2000	.55034E-01	-.80810E-12	-.98511E-02	-.93127E-13
18.0000	.39875E-01	-.11731E-48	-.71377E-02	.10319E-35

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