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A STUDY OF DYNAMIC ENERGY EQUATIONS FOR STIRLING CYCLE ANALYSIS

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Abstract

This report covers an analytical and computer study of the dynamic energy equations that describe the physical phenomena that occurs in a Stirling cycle engine. The basic problem is set up in terms of a set of hyperbolic partial differential equations. The characteristic lines are determined. The equations are then transformed to ordinary differential equations that are valid along characteristic lines. Computer programs to solve the differential equations and to plot pertinent factors are described.

T A B L E O F C O N T E N T S

INTRODUCTION	1
PREVIOUS WORK IN THE FIELD	1
THE PHYSICAL PHENOMENA	2
ENGINEERING APPROACH TO THE PROBLEM.	3
INNOVATIONS IN THIS STUDY.	4
THE BASIC EQUATIONS.	6
AN ANALYTICAL APPROXIMATION FOR THE GAS VELOCITY AND ITS DERIVATIVE WITH RESPECT TO X.	8
CLASSIFICATION OF THE PARTIAL DIFFERENTIAL EQUATIONS	14
DERIVATION OF DIFFERENTIAL EQUATIONS ALONG THE CHARACTERISTICS.	16
THE COMPUTER PROGRAM	18
The MAIN Program of CDEE	19
The CHIC Subroutine.	21
The REGEN Subroutine	22
The RKF45 Runge-Kutta Integration Program.	22
DEEPP, a Plotting Program.	23
Computational Parameters	24
RESULTS.	26
Effects of Changed Operating Conditions.	36
Suggested Extensions and Modifications of this Research.	36
NOMENCLATURE	39
REFERENCES	40

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INTRODUCTION

The dynamic energy equations treated in this report describe the energy conversion phenomena that occurs in a Stirling cycle engine. Analytical and numerical techniques that are somewhat different from those used by previous investigators have been applied. The problem is rather complex in that it involves the non-steady oscillatory flow with compression and expansion of a gas in a heater, regenerator and cooler configuration. The basic problem for this study has been an analytical and numerical description of the energy conversion in a Stirling cycle engine by means of Dynamic Energy Equations, and a definition of the temperature, density and pressure as these variables change with time and with position during the cyclic operation of the machine.

PREVIOUS WORK IN THE FIELD

There is an extensive body of literature in the field of Stirling cycle machine analysis and design. Only a few of the basic references are listed in this paper. Walker¹ is a standard basic reference. Martini² provides a concise Stirling cycle engine design manual. Finkelstein³, Urieli⁴, Martini⁵, Tew, Jeffries and Miao⁶, Tew⁷, and Martini⁸ present various analytical and computer approaches. Rios⁹, Urieli¹⁰, and Kirkley¹¹ are Ph.D. thesis on various aspects of the Stirling cycle. Larson¹² and Larson¹³ cover some of the analytical concepts and computation techniques discussed in this report. Lorenzo and Daniele¹⁴ presents aspects of Stirling engine performance and control. The above references include ideal Schmidt cycle analysis, lumped control volume analysis, regenerator effectiveness studies, nodal analysis, and general thermodynamic studies. The authors cited have done a great deal

of Stirling cycle work and have written many other papers. Some of the cited references contain extensive bibliographies. The other references listed present mathematical, physical and computational information that is pertinent to this study. These latter references are cited at appropriate points in following pages of this report.

THE PHYSICAL PHENOMENA

The working gas in a Stirling cycle engine is sealed between a compression piston and an expansion piston. The gas oscillates through a heater, a regenerator and a cooler as the pistons reciprocate. In Stirling machines the expansion piston reciprocates with a phase difference from the compression piston so the gas is periodically compressed and expanded. Heating and cooling is provided so that power is produced from heat in an engine configuration (or heat energy produced from work for the heat pump configuration). The gas flow region for a typical configuration is illustrated schematically in Figure 1. The regenerator is a passive unit cyclically absorbing heat from, and releasing heat to, the working gas. The gas flow is driven by the expansion and compression pistons which move with an approximate sinusoidal motion with a phase difference (usually about 90°). In this study sinusoidal motion with a phase difference of 90 degrees has been assumed, with temperatures and phase relationship typical for an engine. A change in input, temperatures, and piston phase will provide heat pump analysis.

The cyclic phenomena occurs at the frequency of engine rotation. Under the quasi steady state conditions the gas conditions will repeat

every 360° of piston motion. The gas boundary is assumed to be at the respective constant wall temperatures in the heater and cooler. In the regenerator the mesh temperature varies with time and position. The time variation is cyclic. The position variation covers the range from near the cooler temperature at one regenerator boundary to near the heater temperature at the other end.

ENGINEERING APPROACH TO THE PROBLEM

The physical phenomena that occurs in a Stirling cycle engine can be described by a set of dynamic energy equations. In a following section of this report it is shown that those Dynamic Energy Equations, a set of partial differential equations, are of the hyperbolic type. The specification and numerical solution of the descriptive equations is the object of this study.

The engineering approach to the problem includes three major aspects:

1. A mathematical description of the phenomena that occurs with the oscillation of the gas. This description includes appropriate formulation of the laws for conservation of mass, conservation of energy, conservation of momentum and the equation of state for gas. Convective energy transfer is a major factor in the problem.
2. Formulation of the differential equations and analytical expressions in a form appropriate for digital computer solution. An important aspect of the engineering approach has been to formulate expressions that describe applicable conditions with sufficient accuracy for the problem and that will result in a tractable overall problem statement.

3. Development and selection of suitable computation techniques.

A fourth-fifth order Runge-Kutta numerical solution technique has been adapted to solve the ordinary differential equations that result when the characteristic directions of the hyperbolic system have been determined. The computation technique includes a subroutine to compute appropriate segment lengths for the cooler, regenerator and heater. The analytical, numerical and logical techniques are discussed in following sections of this report.

INNOVATIONS IN THIS STUDY

This study includes a number of techniques which, to the author's knowledge, have not been applied previously to the analysis of Stirling cycle machines. These include the partial differential equation classification with a characteristic equation approach to the problem; treatment of cooler, regenerator and heater as connected units with selectable transition regions (instead of a more conventional lumped parameter approach); and the use of heater and cooler increments that grow at a constant ratio as distance from the regenerator increases.

The Runge-Kutta integration program (RKF45) was selected from the literature¹⁵ and used with minor modifications for this study. The gas velocity is defined in terms of a linear Lagrange interpolation to define its velocity between the compression and expansion pistons. This provides an analytical expression for the gas velocity as well as analytical expressions for the velocity derivatives with respect to position and with respect to time. Details of these techniques are presented in following sections of this report.

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The simultaneous partial differential equations are shown to be of the hyperbolic type. Since the gas velocity is small relative to the sonic velocity in the gas, the wave phenomena can be neglected. It should be noted that the equations are of the hyperbolic type when wave phenomena is neglected. Thus the characteristics of this analysis are not the characteristic Mach lines of compressible flow. The characteristic lines, of variable slope, are functions of position and of time. The slope of the characteristics is dependent on the direction and velocity of the gas flow in the system components. The analytical expression for the gas velocity provides a good approximation to the actual gas velocity and its derivatives in the heater, regenerator and cooler and provides the necessary information to define the varying characteristic directions.

The computation takes advantage of the fact that along the characteristic lines the position variable is eliminated, thus transforming the partial differential equations into a set of simultaneous ordinary differential equations. The resulting ordinary differential equations are solved numerically using a fourth-fifth order Runge-Kutta integration technique. The computer program includes the determination of the characteristic directions that apply for the current step and an automatic computer adjustment of the step size. The Runge-Kutta integration along the characteristic lines provides a solution technique that is more stable than finite difference techniques frequently utilized in the solution of partial differential equations. An initial cosine curve assumption for the regenerator mesh temperature quickly produces a cyclic repetition of the computed gas temperature, density and pressure conditions. The temperature, the variable of primary interest, is automatically plotted using (off-line) computer graphics equipment.

THE BASIC EQUATIONS

The energy transfer in a Stirling cycle machine can be described by three basic principles: 1) conservation of mass, 2) conservation of energy, and 3) conservation of momentum. The general equations on a unit volume basis are

$$\partial \rho_g / \partial t + \nabla \cdot (\rho_g \bar{U}_g) = 0 \quad (1)$$

$$\partial / \partial t (\rho_g C_v T_g) / \partial t = - \nabla \cdot [\rho_g \bar{U}_g (h_c + \bar{U}_g \bar{U}_g / 2GJ) + K_g (\nabla T_g)] + \partial W / \partial t - n_c A_c (T_g - T_m) \quad (2)$$

$$\rho_g [(\bar{U}_g \cdot \nabla \bar{U}_g) + \partial \bar{U}_g / \partial t] + \nabla P_g = F_v \quad (3)$$

The momentum equation (3) will not be explicitly required in the set of simultaneous equations if an analytical expression for the gas velocity is available. In a following section of the paper an analytical approximation for the gas velocity is derived. It should be noted that while equation (3) is not utilized, the momentum factor is implicitly carried along in the computation through the gas density, velocity and pressure terms.

The enthalpy of the gas is typically much greater than the kinetic energy of the gas. Also the wave velocity in the gas is normally much greater than the gas velocity; therefore, the times of interest for the oscillatory flow are much greater than pressure wave propagation times. Assuming these fluid flow characteristics, the kinetic energy terms and the second derivative terms in equation (2) can be neglected. It should be noted that the compressibility of the gas is taken into account when

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computing volume, pressure, density and temperature levels and changes in these parameters.

In all following equations it is assumed that the flow is one-dimensional. Therefore the vector notation will be dropped. (For one-dimensional flow the differential operator ∇ is a scalar operator, $\nabla = \partial/\partial x$).

With the above assumptions, the partial differential equations defining the temperature and the density of the gas can be written in the matrix form

$$\begin{bmatrix} U_g & 1 & 0 & 0 \\ \gamma U_g T_g & T_g & \gamma p U_g & p_g \end{bmatrix} \begin{bmatrix} \rho_x \\ \rho_t \\ T_x \\ T_t \end{bmatrix} = \begin{bmatrix} -\rho_g U_{gx} \\ -\gamma p_g T_g U_{gx} - h_c A_c (T_g - T_m)/\rho_g C_v \end{bmatrix} \quad (4)$$

The subscripts x and t on the ρ , T and U terms indicate partial differentiation with respect to x and t respectively. Thus, for example, $\rho_t = \partial\rho/\partial t$ and $U_{gx} = \partial U_g/\partial x$ etc.

In addition to the equations describing gas conditions, an equation describing the regenerator material temperature is required.

$$\rho_m C_m (\partial T_m / \partial t) = h_c A_c (T_g - T_m) + K_m \nabla^2 T_m \quad (5)$$

In this equation the subscript m refers to the heat transfer material. The temperature of the material receiving and providing heat in the cooler and in the heater will be assumed to be constant.

AN ANALYTICAL APPROXIMATION FOR THE GAS
VELOCITY AND ITS DERIVATIVE WITH RESPECT TO X

A Stirling cycle engine will be taken as the specific type of machine in this paper. Simple changes in nomenclature and detail would permit generalization to other Stirling machines. The portion of a Stirling cycle engine of interest is illustrated schematically in Figure 1.

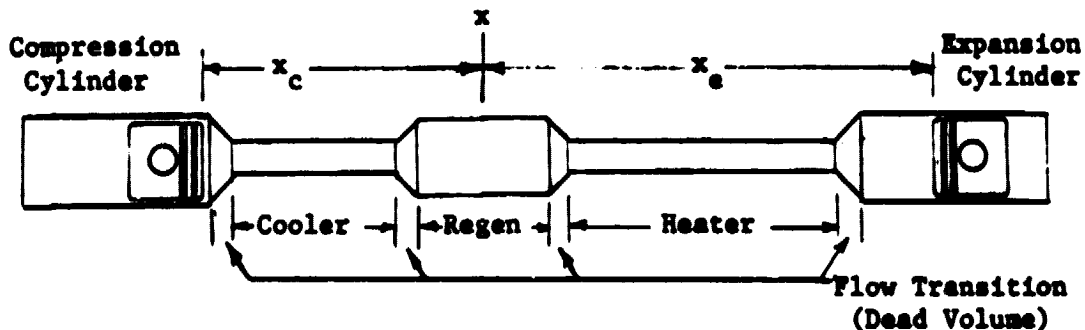


Figure 1. Schematic Diagram of Stirling Cycle Engine

The velocity of the gas is a function of the velocity of the compression and expansion pistons, the flow geometry, and the flow resistance. The pistons will be assumed to move with simple sinusoidal motion with the expansion piston lagging the compression piston by 90°

$$U_c = R_c \omega \sin \omega t \quad (6)$$

$$U_e = R_c \omega \sin(\omega t - 90^\circ) \quad (7)$$

An "ideal" gas velocity would occur with zero flow resistance. This is equivalent to assuming uniform gas pressure in the region between the pistons (flow pressure drop and pressure waves are neglected), and also assuming one-dimensional flow (even though the flow area changes). The gas

velocity between the pistons can be determined by a linear Lagrange interpolation. Note that if the flow area were uniform and equal to the piston area the ideal gas velocity would be

$$U_{gi} = (U_c x_e + U_e x_c) / (x_c + x_e) \quad (8)$$

where x_c and x_e are the distances from the point of interest to the expansion and compression pistons respectively. The geometry of a typical flow path between the pistons is rather complex. In that case the distances x_c and x_e from the pistons are not as significant as the gas volumes in determining the ideal gas velocity.

With the varying flow area generalizing equation (8) the ideal gas velocity (a function of x and t) will be approximately given by a "volume" Lagrange interpolation.

$$U_{gi} = (A_p/A_f)(U_c V_e + U_e V_c) / (V_c + V_e) \quad (9)$$

where V_c and V_e are the actual gas volumes from the point of interest, x , to the compression and to the expansion pistons respectively.

The ideal velocity flow corresponds to nonviscous or zero pressure drop flow. The actual flow results in a pressure drop. That pressure drop depends on the length of the flow path, the fluid velocity, the flow path configuration, and the Reynolds number. The periodic flow will result in a flow pressure drop that is a function of time and position.

For the present application it will be convenient to define the effect of flow resistance on the gas velocity. So that

$$U_g = U_{gi} - \Delta U_g \quad (10)$$

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Consider a point x along the flow path. To be specific assume x is in the regenerator. Because of flow resistance the pressure and the density will be greater at points in the direction of the piston "driving" the flow than it will be at points in the direction of the receding piston. The flow resistance-pressure-density characteristic will result in a gas velocity that is less in absolute value than the ideal at all points along the flow path of the oscillating fluid except at the ends in contact with a piston.

The pressure drop can be expressed

$$\Delta P = f \rho_g L U_g^2 / 2GD \quad (11)$$

The value of the friction factor (f) for laminar flow is

$$f = 64/Re = 64\mu / (\rho_g |U_g| D) \quad (12)$$

For turbulent flow the friction factor is dependent on the Reynolds number (Re) and on a tube roughness factor. Typically Re will be less than 15000. For Re values between 2100 and 15000 the friction factor can be approximated by the equation

$$f = C_{1f} + C_{2f}/Re \quad (13)$$

C_{1f} and C_{2f} are taken to approximate a selected curve. Note that in the laminar regime ($Re < 2100$) $C_{1f} = 0.0$ and $C_{2f} = 64.0$

From values tabulated in reference¹⁸ it appears that the viscosity (μ) for hydrogen and for helium can be approximated by linear functions of

temperature (for temperatures in the range $300 < T_g < 1100^\circ\text{K}$). With this approximation

$$\mu = a + bT_g \quad (14)$$

Using equation (13) in equation (11) and taking acceleration effects into account by adding the $\partial U_g / \partial t$ and $\partial U_g / \partial x$ terms, gives

$$\Delta P = -K_p(C_{1f} + C_{2f}/Re)\rho x_f |U_g| U_g / (2GD_f) - \rho(U_{gt} + U_g U_{gx})x_f / G \quad (15)$$

Considering the fluid as a spring, the force $\Delta P_g A_f$ to produce the flow is obtained by a "deflection" or decrease in x , say Δx . Let K_s be a "spring constant" then, for small Δx

$$\Delta P_g A_f = K_s \Delta x \quad (16)$$

The change in pressure from the uniform pressure (zero viscosity) case is given by a pair of integrals, one with limits from the compression piston to x (the point of interest), and the other with limits from x to the expansion piston. That pressure effect can then be interpreted in terms of its effect on the position in equation (16).

It is reasonable to neglect any change in temperature due to the relatively small flow pressure drop because the cooler, regenerator and heater are designed for high heat transfer. Then noting the total pressure drop along the flow path has a contribution from the point x to the compression piston, and one from x to the expansion piston

$$|\Delta P_c / P| + |\Delta P_e / P| = |\Delta V_c / V_c| + |\Delta V_e / V_e| \quad (17)$$

Then letting $A_f \Delta x = \Delta V_c = \Delta V_e$, and $\Delta P = \Delta P_c + \Delta P_e$

$$|\Delta P| = P A_f |\Delta x| (v_c + v_e) / v_c v_e \quad (18)$$

(The absolute values are used because the effect of the sign of ΔP and Δx will be taken into account in equations (20) and (21).)

The change in distance (Δx) in equations (16) and (18) results in a change in gas velocity (ΔU_g). Assume that

$$\Delta U_g / U_g = K \Delta x / x_f \quad (19)$$

Then equating the pressure drop as given by equations (15) and (18), inserting Δx from (19) and solving for ΔU_g (and designating the constant term as K_v) gives

$$\Delta U_g = K_v (C_{1f} + C_{2f} / Re) \rho_g x_f v_c v_e U_{g1}^3 / (2GD_f P_g (v_c + v_e)^2) \quad (20)$$

The partial derivative of U_g with respect to x is approximately

$$\begin{aligned} \frac{\partial U_g}{\partial x} = & \frac{A_p (U_e - U_c)}{(v_c + v_e)} \left[1 - \frac{K_v v_c v_e x_f}{2GD_f P_g (v_c + v_e)^2} (3C_{1f} \rho U_g^2 + 2C_{2f} \mu |U_{g1}|) \right] + \\ & - \frac{K_v (v_e - v_c) A_f \rho x_f}{2GD_f P_g (v_c + v_e)^2} [C_{1f} + C_{2f} / Re] U_{g1}^3 \end{aligned} \quad (21)$$

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Typical velocity curves are illustrated by Figure 2. The effect of the flow resistance on the gas velocity is exaggerated by the curves on the figure. The effect is most apparent in the slightly curved lines for 150° and for 330°.

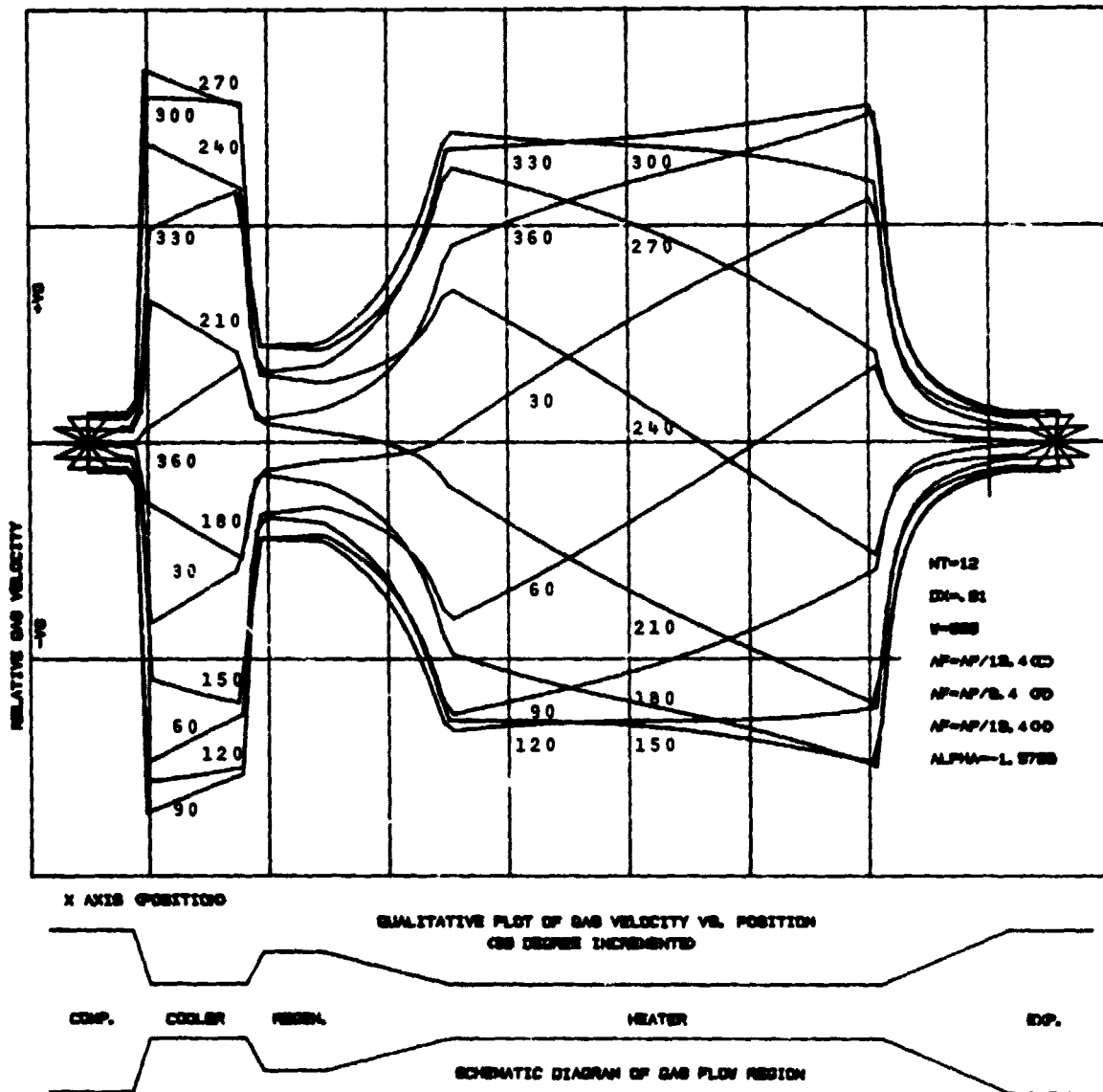


FIGURE 2 TYPICAL GAS VELOCITY PATTERN

CLASSIFICATION OF THE
PARTIAL DIFFERENTIAL EQUATIONS

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The analytical approximations to U_g and $\partial U_g / \partial x$ will be used in the sequel to simplify the treatment of the dynamic energy equations. The gas density and temperature are functions of position and of time. The total differentials of these terms are

$$d\rho_g = (\partial\rho_g/\partial x)dx + (\partial\rho_g/\partial t)dt \quad (22)$$

$$dT_g = (\partial T_g/\partial x)dx + (\partial T_g/\partial t)dt \quad (23)$$

Appending these equations to equation (4)

$$\begin{bmatrix} U_g & 1 & 0 & 0 \\ \gamma U_g T_g & T_g & \gamma \rho_g U_g & \rho_g \\ dx & dt & 0 & 0 \\ 0 & 0 & dx & dt \end{bmatrix} \begin{bmatrix} \rho_x \\ \rho_t \\ T_x \\ T_t \end{bmatrix} = \begin{bmatrix} -\rho_g U_g x \\ -\gamma \rho_g T_g U_g x - h A_c (T_g - T_m)/c_v \rho \\ d\rho_g \\ dT_g \end{bmatrix} \quad (24)$$

This system of partial differential equations is of the hyperbolic, parabolic, or elliptic type when the discriminant of the characteristic equation is positive, zero, or negative respectively. (Ames reference ¹⁶)

The characteristic equation of this set of simultaneous partial differential equations is obtained by setting the determinant of the coefficient matrix equal to zero. The characteristic equation is

$$-\gamma \rho_g U_g^2 dt^2 + \rho_g (\gamma + 1) U_g dt dx - \rho_g dx^2 = 0 \quad (25)$$

Dividing through by $\rho_g dx^2$ (ρ_g is nonzero) gives a quadratic equation in dx/dt . The discriminant of that equation is

$$B^2 - 4AC = U_g^2(\gamma-1)^2 \quad (26)$$

Note that this discriminant is greater than zero (except locally in time and space when U_g is zero. Therefore the set of equations is hyperbolic (except parabolic when $U_g = 0$).

CHARACTERISTIC DIRECTIONS

Solving the characteristic equation for dt/dx provides the characteristic directions. Then assigning α and β to the distinct directions

$$dt/dx)_\alpha = 1/U_g \quad (27)$$

$$dt/dx)_\beta = 1/\gamma U_g \quad (28)$$

Figure 3 is a computer drawn plot of a characteristic line network in the regenerator portion of the flow path.

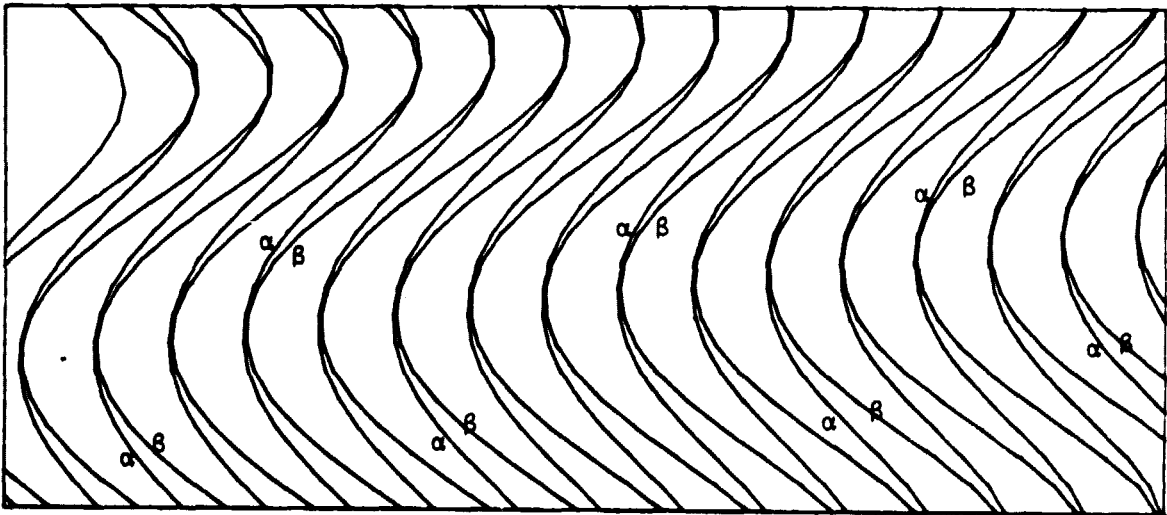


FIGURE 3 PLOT OF ALPHA AND BETA CHARACTERISTIC LINES

DERIVATION OF DIFFERENTIAL EQUATIONS
ALONG THE CHARACTERISTICS

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Along the characteristic lines the determinant is zero. The equations will have a solution only if the determinant is also zero when the right hand column vector is substituted for any column of the matrix. Then setting up the equation in terms of dt/dx and successively inserting the α and β characteristic values for dt/dx will reduce the partial differential equation to a set of ordinary differential equations. It is necessary to select appropriate columns for replacement by the right hand vector so as to get an ordinary differential equation for each of the dependent variables. (In some cases a trivial result may occur.)

Substituting the right hand column vector for the first column gives a determinant that must equal zero for a solution.

$$\begin{bmatrix} -\rho_g U_{gx} & 1 & 0 & 0 \\ -\gamma \rho_g T_g U_{gx} + & T_g & \gamma \rho_g U_g & \rho_g \\ -h_c A_c (T_g - T_m)/C_v \rho_g & & & \\ d\rho_g & dt & 0 & 0 \\ dT_g & 0 & dx & dt \end{bmatrix} = 0 \quad (29)$$

After substitution of $dt/dx)_\alpha = 1/U_g$ this determinant reduces to the equation

$$(\rho_g \partial U_g / \partial x + d\rho_g / dt)(\gamma \rho_g U_g - \rho_g U_g) = 0 \quad (30)$$

Since $\gamma \neq 1$ the first term must equal zero. Therefore the density is

defined along the α characteristic by the ordinary differential equation

$$d\rho_g/dt = -\rho_{g\alpha}(\partial U_g/\partial x) \quad (31)$$

Along the β characteristic the substitution $dt/dx)_\beta = 1/\gamma U_g$ gives

$$(\rho_g \partial U_g / \partial x + d\rho_g / dt)(\gamma \rho_g U_g - \gamma \rho_g U_g) = 0 \quad (32)$$

The second term is identically zero so another substitution is required. Substitution of the right column for the second column gives the same result as above, i.e., defines $d\rho_g/dt$ along the α characteristic, but does not define dT_g/dt .

Replacing either the third or fourth column of the matrix by the right hand vector gives an equation including dT_g/dt and $d\rho_g/dt$. Taking the value for $d\rho_g/dt$ along the α characteristic

$$d\rho_g/dt = -\rho_{g\alpha}(\partial U_g/\partial x) \quad (33)$$

and using the β value, $dt/dx)_\beta = 1/\gamma U_g$ provides an ordinary differential equation defining the gas temperature

$$dT_g/dt = -T_{g\beta} U_{gx\beta} (\gamma - \rho_{g\alpha} U_{gx\alpha} / \rho_{g\beta} U_{gx\beta}) - h_c A_c (T_{g\beta} - T_m) / C_v \rho_{g\beta} \quad (34)$$

In this equation the α and β subscripts indicate that the various functions are to be evaluated along the α and β characteristics respectively, thus

$$U_{gx\alpha} = \partial U_g / \partial x)_\alpha \quad (35)$$

THE COMPUTER PROGRAM

The computer program CDEE (characteristic dynamic energy equations) is designed to solve equations (3), (5), (33) and (34) using equations (27) and (28) to provide appropriate characteristic points, using equations (9), (10), (20), and (21) to provide gas velocity information. The gas velocity functions uncouple the momentum equation (3) from the system of equations. Note that equations (33) and (34) do not explicitly involve position, and include position derivatives only in the $\partial U_g / \partial x$ terms. This latter term is provided by the analytical expression (21). Many other mathematical equations and logical criteria are also utilized in the program. The Runge-Kutta RKF45 integration program utilizes computer-determined basic time increments (H) with weighted integration points at H/4, 3H/8, H/2, 12H/13 and H. It is accurate to fifth order (sixth order locally) according to reference¹⁷. The characteristic directions are utilized to determine appropriate "characteristic points" for evaluation of the variables.

The utilization of the characteristic directions and the analytical expression for the gas velocity provide a significant advantage for numerical solution of the dynamic energy equations. The advantage is that the usual finite difference solution of the partial differential equations can be replaced by a rather sophisticated Runge-Kutta integration routine, described in detail in reference¹⁷. Adams type predictor correction integration techniques could be used but have not been applied.

The major parts of the program are:

A. CDEE, Characteristic Dynamic Energy Equations

1. The MAIN program
2. The CHIC subroutine
3. The REGEN subroutine
4. The Runge-Kutta integration program RKF45 with subroutines, RKFS, and FEHL

B. DEEPP, Dynamic Energy Equation Plotting Program

The MAIN Program of CDEE

The main program provides general program control. This portion of the program provides for specification of the computation parameters, and logical and physical data. MAIN provides for specification and computation of derived initial and boundary conditions. The variables used in the program are defined by "comment" statements and are provided with default values. MAIN provides for all input and output. Input variations are entered through three NAMELISTs so that no special input FORMAT statements are required. In practice the NAMELISTs are used to change the default values to those desired for a specific analysis.

The program output is from MAIN. It contains all WRITE statements and all FORMAT statements. Error messages are handled by setting flags so that such output is also from MAIN.

The variables and parameters used in the program can be divided into the following four categories:

1. Computational data that define time and position increments, accuracy requirements, the extent of the computation, and other program control and bookkeeping functions
2. Logical data that can be set up to permit alternate conditions, alternate fluids and differences in operational or computational logic

3. Numerical data for the independent variables and parameters that define the physical configuration, the material properties and the operating conditions for a particular analysis
4. Dependent variables or computed values that comprise the computer output and provide the desired information on system characteristics

The first three groups of variables and parameters are included in three NAMELISTs: COMDAT, LOGDAT, and NUMDAT, respectively. Default values for all inputs are provided in MAIN so that only those requiring different values need be entered through the appropriate NAMELIST.

The computational data is listed alphabetically in a NAMELIST called COMDAT. This data includes such items as number of increments to be used in each part of the computation, initial and final time, or degrees of rotation. (The actual time increment used varies during the computation and is automatically adjusted in RKFS, the Runge-Kutta integration subroutine.)

The program provides for selection of different working gases, alternative computational logic, and various assumptions through the specification of selected logical data which are listed alphabetically in a NAMELIST called LOGDAT.

The numerical data and independent variables for the program are listed in a NAMELIST called NUMDAT. This includes data that describes the physical configuration, various material properties, initial conditions, boundary conditions, and the physical constants that are required for the computation.

All NAMELISTs are arranged alphabetically to facilitate checking. Virtually all input data is included in the NAMELISTs. However, because of the NAMELIST input characteristic (and because default values are provided), only specific items in the list to be changed need be typed into an input

NAMelist. All the NAMelists are read by MAIN before execution and all are printed out in the standard free format at the beginning of the program execution, and also at one point along with the formatted output. This provides a convenient check of the actual parameter values that were used for a particular computation.

The MAIN program includes segments to insure that total mass is conserved, to compute work output, and to enter data on tape for off-line plotting by DEEPP.

The CHIC Subroutine

The regenerator is the region of greatest interest. It has by far the greatest temperature variation, ranging from the temperature of the cooler to the temperature of the heater. The uniform increment length in the regenerator (LR/NINCR) is computed from the selected number of increments. Variable increment lengths are used in the heater and cooler to permit a reasonable total number of increments and still include all of the long, relatively uniform temperature heater and cooler. The subroutine CHIC (Cooler, Heater Increment Computation) is provided to adjust the size of the increments in a progressive manner. The increment on either side of the regenerator is the same length as those in the regenerator. A ratio (R) is computed such that a selected number of increments will cover the full length at the heater (or cooler). The length of the Ith increment (in the heater, for example) is

$$XINC(I) = XINCR * R^{*(I-1)} \quad (36)$$

The ratio R is computed from the equation

$$XINCR (1 + R + R^2 + \dots + R^{N-1}) = LH \quad (37)$$

The number of increments in the cooler, regenerator, and heater are specified independently in the computation data NAMELIST (COMDAT).

The printout of pertinent data in a controlled format at specified time (or angular rotation) increments and at specified evaluation positions is provided by MAIN.

The REGEN Subroutine

REGEN is the portion of CODEE in which the gas velocity is computed and the equations to be integrated are evaluated. This subroutine provides for computation of required characteristic points which do not in general coincide with the specific position increments into which the system is divided. REGEN utilizes a number of logical variables that permit a variety of computational assumptions. They include use of hydrogen or helium, various fluid flow area change configurations, and other flow characteristics. The area variations include linear area variation (LAV), linear velocity variation (LVV), and uniform velocity ratios (UVR) at successive evaluation points. The primary function of REGEN is evaluation of the gas velocity and evaluation of the time derivatives of the density, the gas temperature, and regenerator mesh temperature. REGEN is used as an EXTERNAL function which is called by arguments in the RKF45, RKFS and FEHL subroutines.

The RKF45 Runge-Kutta Integration Program

The Runge-Kutta integration is carried out by three subroutines described below. These subroutines, described in detail in reference¹⁷ were modified to permit computation using the varying network of characteristic points in place of the usual regular mesh network.

RKF45 is an interface subroutine that implements the fourth-fifth order Runge-Kutta integration of a system of first order ordinary differential equations. This subroutine is the bookkeeping portion of the overall integration program which includes RKFS and FEHL in addition to RKF45.

RKFS is a control subroutine that checks accuracy requirements and provides for automatic adjustment of the time step. The actual numerical integration is carried out by the subroutine FEHL which is called from RKFS.

FEHL integrates the system of first order equations (in REGEN). Initial values and the initial derivatives are specified at the starting time. (Initial in this context means simply the values at the start of a particular step.) FEHL advances the solution over one time step that is set by RKFS. The variable time step characteristic of the program is provided by RKFS so that FEHL receives a time step that is fixed (but may be different at another call).

DEEPP, a Plotting Program

The gas temperatures computed by CDEE are written into a library file. A particular library file called CEDATnnn is retrieved for off-line plotting by computer graphics equipment. The plotting program DEEPP (Dynamic Energy Equation Plotting Program) is an interactive program written in BASIC. The program is flexible so that it can be run with temperatures in degrees Rankine or in degrees Kelvin; the scale can be adjusted for the size plot desired. The plot can be either temperature versus time at a specified position or temperature versus position with crank rotation as a parameter.

Typically the cooler and heater are much longer than the regenerator, but the temperature variation in the regenerator is of greater interest than the relatively uniform cooler and heater temperature. To provide a good "picture" of the gas temperature profiles in the regenerator the major portion of the plot (actually the center half) is devoted to regenerator gas temperature. The program is set up to use different scales in the three regions, or for a true scale but with only a portion of the heater and cooler included. In the latter case the scale is set by the regenerator, which is half of the plotted abscissa, the part of the heater and cooler included on the plot is equal to half of the regenerator length. This BASIC program is interactive. The various alternatives are obtained by operator response to questions printed on the screen.

Computational Parameters

The numerical values associated with a particular physical configuration are very significant in an analysis. The determination of appropriate values is usually a routine computation, however occasionally the values and the effects are subject to interpretation. In some cases significant research efforts may be required to determine the effects.

The ratio of heat transfer area to flow volume (or perimeter to flow cross section) is a very significant parameter for the cooler, the heater and the regenerator. For units with simple round tubes the ratio is simply $4/\text{diam.}$ with an appropriate dimension. For a regenerator comprised of many layers of fine stainless steel screen the effective value is not obvious. For this study the following values were used:

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$$A_{cc} = 3700 \text{ m}^{-1}$$

$$A_{cr} = 32000 \text{ m}^{-1}$$

$$A_{cn} = 1300 \text{ m}^{-1}$$

The heat transfer was defined in terms of the Nusselt number ($Nu = hd/k$).

Laminar flow was assumed when the Reynolds number ($Re = \rho U d / \mu$) is less than 2100. It is assumed that the flow is turbulent when Re is greater than 2100. The following equations were utilized for convective heat transfer:

$$Nu = 3.65 \text{ for laminar flow}$$

$$Nu = 0.023 (Re)^{0.8} (Pr)^{0.4} \text{ for turbulent flow}$$

In the latter equation the Prandtl number ($Pr = C_p \mu / k$) was taken as constant ($Pr = 0.70$ for hydrogen and $Pr = 0.73$ for helium). The gas conductivity k and the gas viscosity (μ) were approximated by linear functions of gas temperature. These approximations provided values reasonably close to values as given in the handbook¹⁸ for the limited temperature range that is applicable for this study.

The pressure drop effect on gas velocity is assumed to vary with distance between the compression and the expansion pistons according to the relationship as given by equation (9), repeated here for convenience

$$U_{gI} = (A_p / A_f) (U_c V_e + U_e V_c) / (V_c + V_e)$$

Note that at the compression and at the expansion pistons the gas velocity will be equal to the respective piston velocity as it must be. Between the pistons the gas velocity is a gas volume Lagrange average.

S
Y

Using the equation (9) and the assumptions pointed out previously, the change in gas velocity from the ideal is given by equation (20). Note that this expression (that is the assumptions leading to this expression) results in a gas volume effect that takes the gas volume at each end into account. The maximum effect of this factor is at the center as would be expected. That is, the maximum decrease in gas velocity occurs at the mid point between the pistons (in terms of volume). The maximum value of the $V_c V_e / (V_c + V_e)^2$ term is 1/4. To bring the net gas velocity effect of the flow resistance up to the assumed value a pressure increment coefficient (PICO) is introduced as a multiplicative factor in the pressure drop and in the velocity drop equations. One additional factor is introduced because it appears that the pressure drop, particularly that in the regenerator, is understated by the classical pressure drop expression (equation 11). This seemed to be particularly significant for the helium cases. To compensate for this perceived effect a factor that will provide an appropriate nonlinear gas velocity effect (with greatest significance at the higher gas velocities) is needed. This has been provided by an increase in an exponent (PIEX) on the absolute gas velocity from 1.0 to PIEX with a value greater than one, typically a value of 1.25 has been used. PICO and PIEX are included in the NUMDAT NAMELIST so values other than the default values can be used when that is deemed advisable.

RESULTS

The variation of gas temperature with position and with time is one of the more distinctive characteristics of Stirling cycle engines. Typically temperature varies within a range of $\pm 30^\circ \text{C}$ of the cooler temperature in

the regions close to the compression cylinder. In the heater typical variations are within $\pm 130^{\circ}$ C of the heater temperature. (Specific typical values are given below.) This variation is due to the oscillation of the gas between the various components along the flow path as the gas is subject to periodic compression and expansion. It should be noted that typical volumes of the cooler, regenerator, heater and dead spaces are such that the gas does not traverse the complete path. For example, in the GPU-3 engine the total volume between the cylinder is approximately 1.6 times the total displacement volume of one cylinder. The relative volumes (with values comparable to those for a GPU-3) based on the total maximum volume, which for the assumed configuration occurs when the compression piston crank is at 135° , are as follows:

	At 135° (Max Vol)	At 315° (Min Vol)	At TDC	At BDC
Compression cylinder	25.5%	4.4%*	0%*	30%*
Dead volume cooler region	11.4	- Same as at left	-	-
Cooler	2.6	- Same as at left	-	-
Regenerator	5.3	- Same as at left	-	-
Heater	11.3	- Same as at left	-	-
Dead volume heater region	18.4	- Same as at left	-	-
Expansion cylinder	<u>25.5</u>	<u>4.4 *</u>	<u>15 *</u>	<u>15 *</u>
Total	100.0%	57.8%*	64%*	94%*

* All values are based on the total at the maximum volume position.

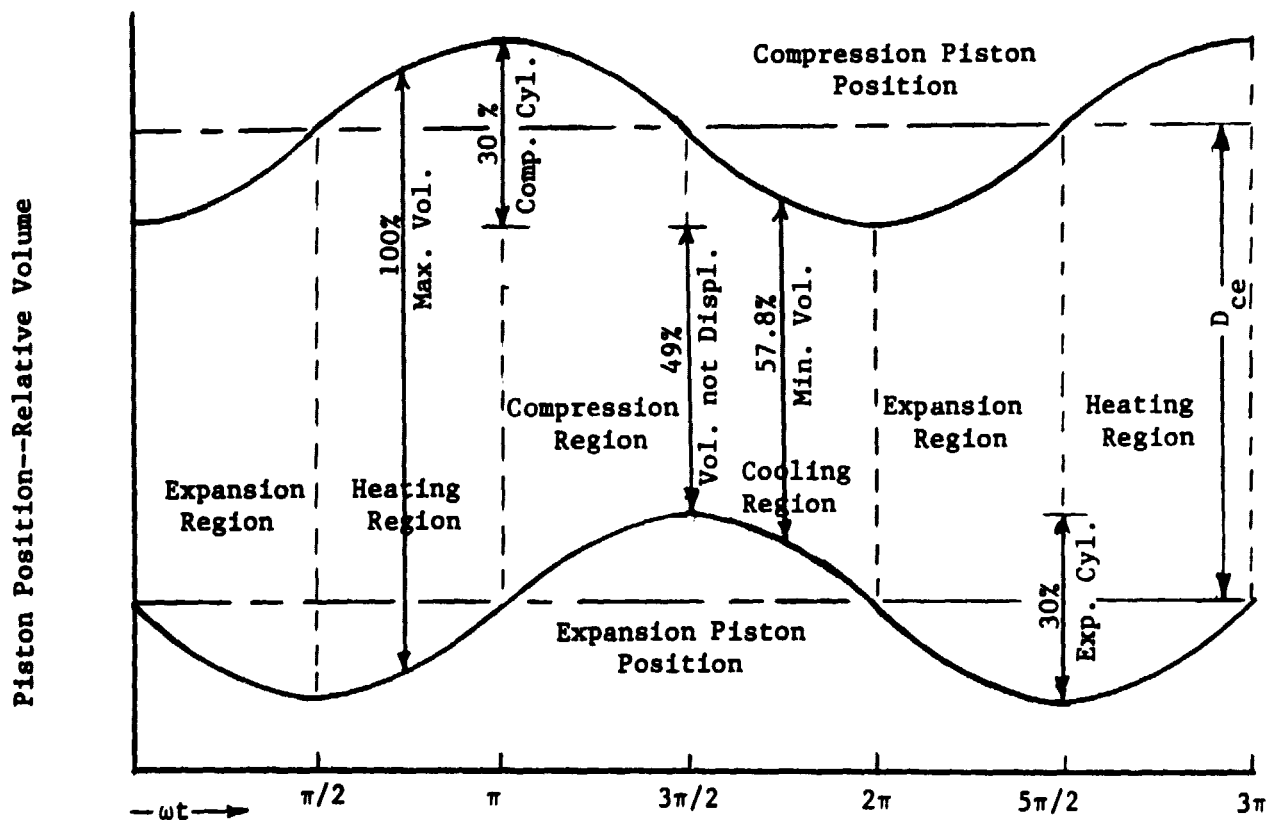
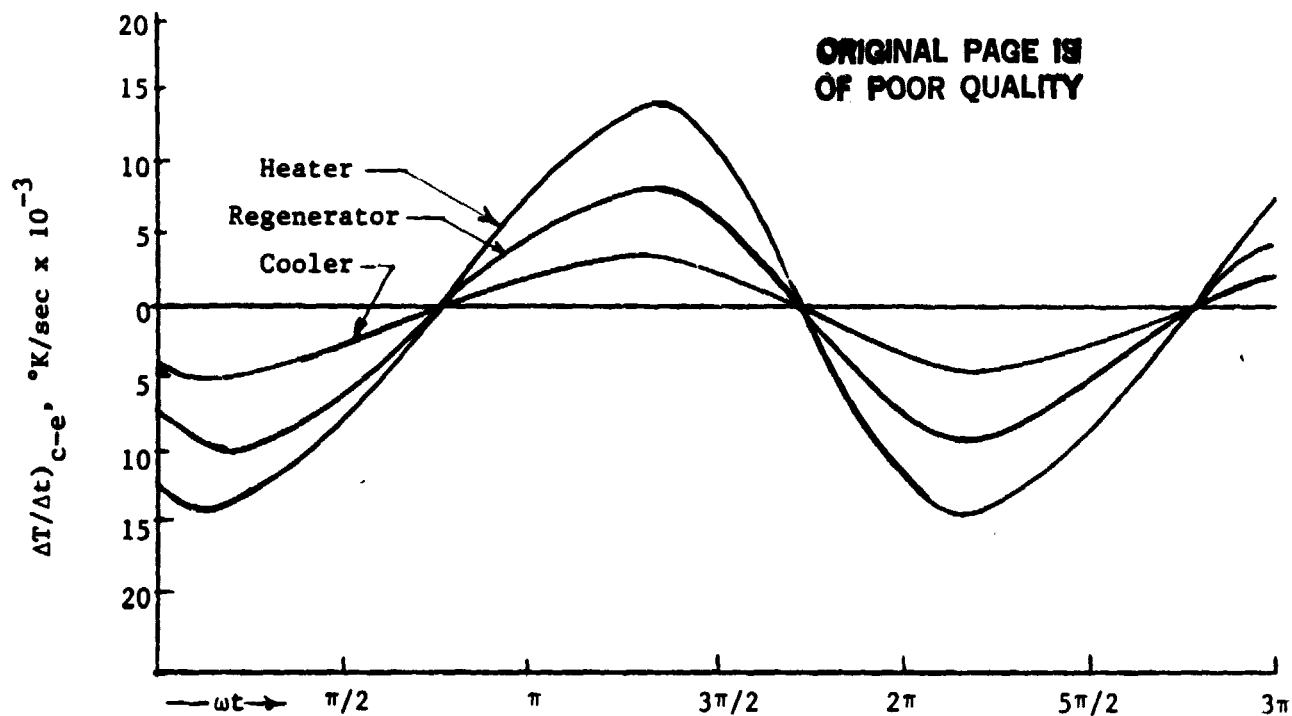
The approximate relative volumes are for a 90° phase difference. For the GPU-3 the ratio of maximum volume to minimum volume is approximately 1.73. The relative volume and the associated compression and expansion piston positions are illustrated schematically in Figure 5. The sine curves represent the position of the pistons with the compression piston leading the expansion piston by 90°. The vertical distance between the sine curves is approximately proportional to the gas volume. Note the change in volume with crank rotation as illustrated by the cyclical change as the time varies along the horizontal axis. The relative magnitudes of significant volumes are indicated on the figure.

The curves of Figure 4 indicate typical gas temperature variation $(\Delta T / \Delta t)_{c-e}$ due to the compression and expansion effects. The temperature of the gas is affected by factors other than compression and expansion during the cycle. Such factors are not included here. (See Figures 7, 10 and 11 and the related discussion.) The curves of Figure 4 represent the solution of the isentropic compression expansion equation

$$(\Delta T_g / \Delta t)_{ce} = (1-\gamma)(T_1/V_1)A_p R_c \omega (\sin(\omega t + \phi) - \sin \omega t) \quad (38)$$

The skewed shape of the curves is primarily due to the dependence on the gas temperature which has a skewed irregular shape as shown in Figure 7. The shape of the gas temperature curves is the result of the complex phenomena including convective heat transfer, periodic motion through the heater, regenerator and cooler regions, and compression and expansion.

The curves of Figure 6 illustrate the typical cyclic gas pressure characteristic in the regenerator. Note that the pressure curves have a



ORIGINAL PAGE 18
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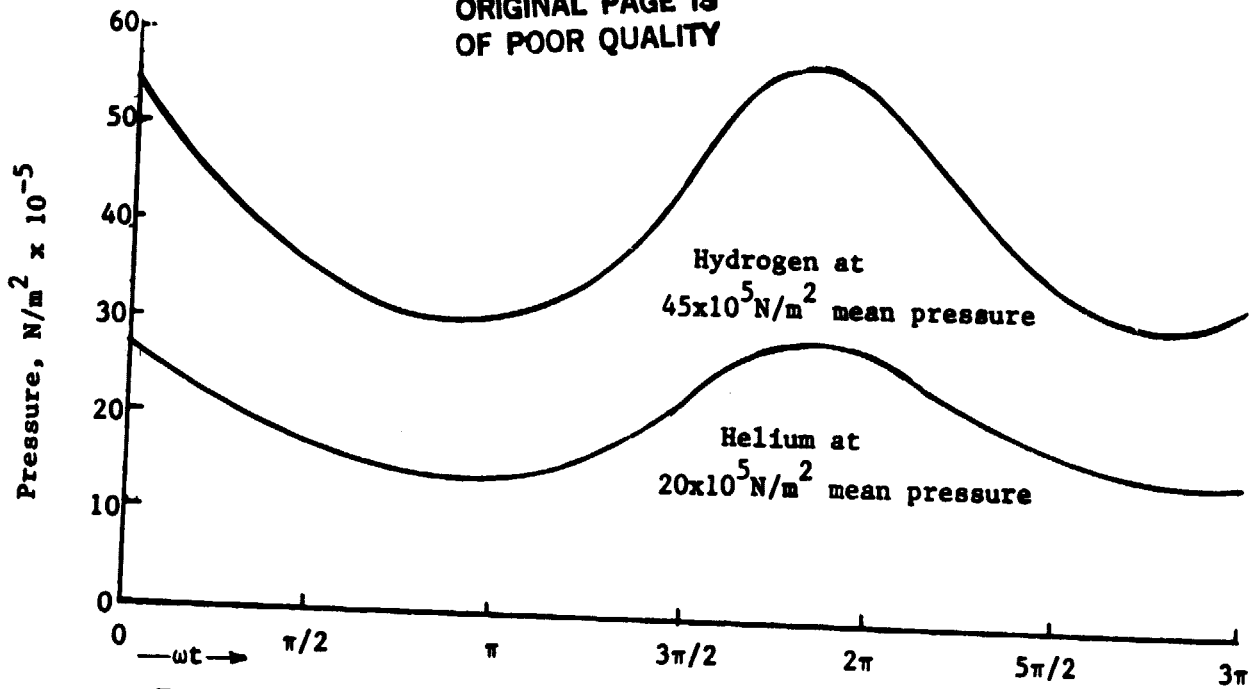


Figure 6. Typical Gas Pressure vs Crank Rotation (time) Curves

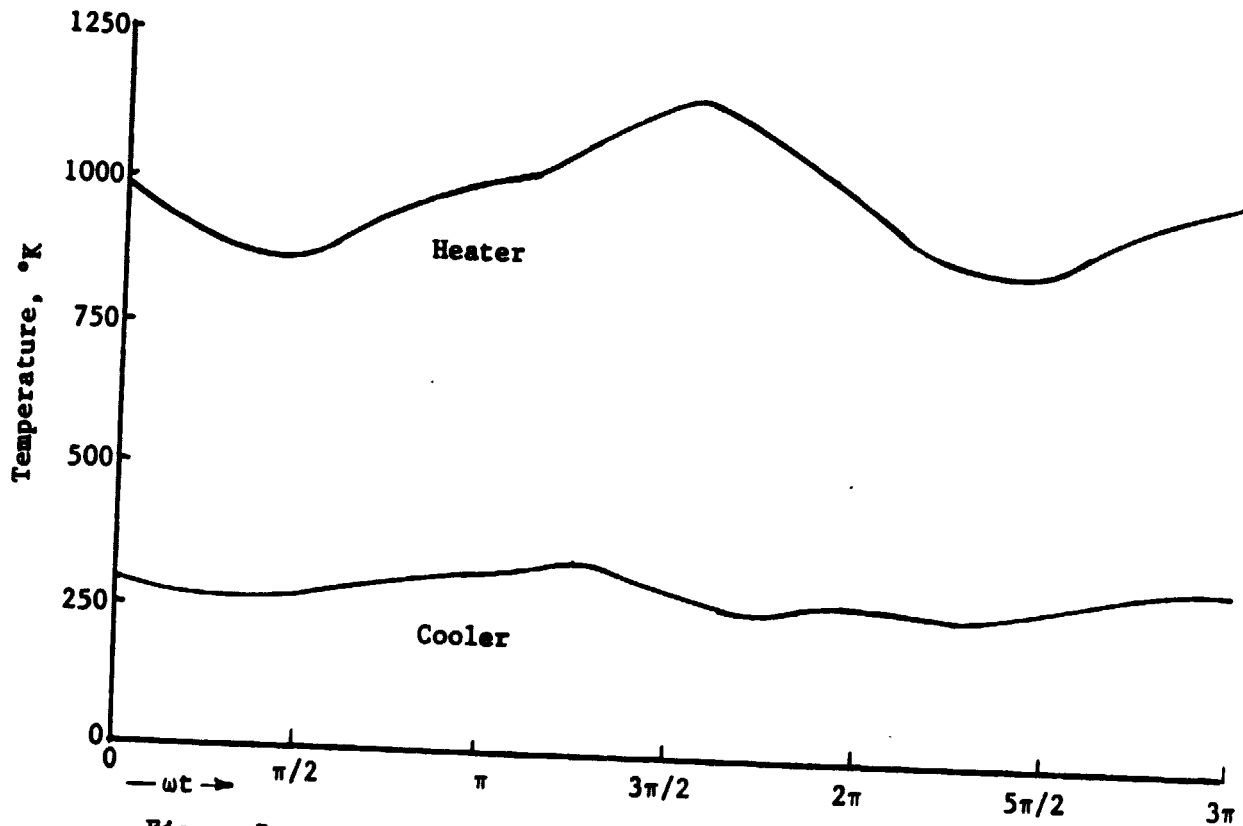


Figure 7. Typical Gas Temperature vs Crank Position (time) Curves

regular shape but are distorted from sine waves with broad "valleys" and smooth but narrower peak regions.

The cited curves are referred to as typical because they illustrate general characteristics. With minor modifications and scale changes they could be used for computations with different input data.

Figures 8a and 9a are pressure-volume plots for hydrogen and for helium as the working gas. The figures include separate closed curves for the compression and the expansion cylinders plotted in a conventional manner as, for example, in Ash¹⁷. The pairs of closed curves are plotted with the compression cylinder volume for the narrower curve, and with the expansion cylinder volume for the "fatter" curve. This author questions the significance of such curves because in a Stirling cycle engine the cylinders are connected without valving by the cooler, regenerator and heater. The curves do not represent separate portions of a cycle as do the p-v diagrams for an internal combustion engine.

The p-v curves of Figures 8b and 9b are believed to be more significant. They are plotted as pressure versus volume curves with the volume varying from the minimum volume (the volume not displaced plus the minimum volume of gas in one or the other or both cylinders) to the maximum volume. Figure 5, previously discussed, illustrates these points. As discussed in a previous section of this report, the maximum volume for the assumed configuration occurs at 135° for the compression piston, and the minimum volume occurs at the 315° compression piston position. For the plots of Figures 8b and 9b the abscissa is labeled in terms of the decimal fraction of maximum volume. For this type of p-v diagram, during the expansion portion from minimum to maximum clockwise (the upper curve) the lower of the compression cylinder

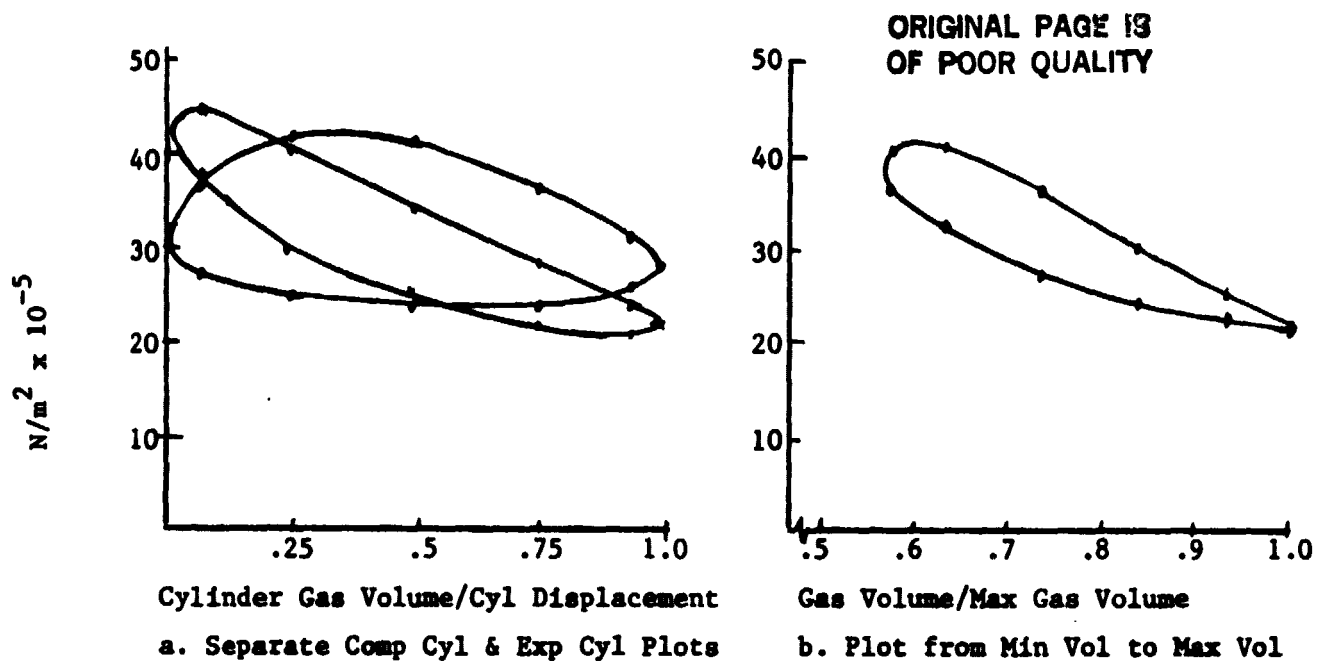


Figure 8. Typical Pressure-Volume Diagrams for Stirling Cycle
with Helium at $30 \times 10^5 \text{ N/m}^2$ Mean Pressure

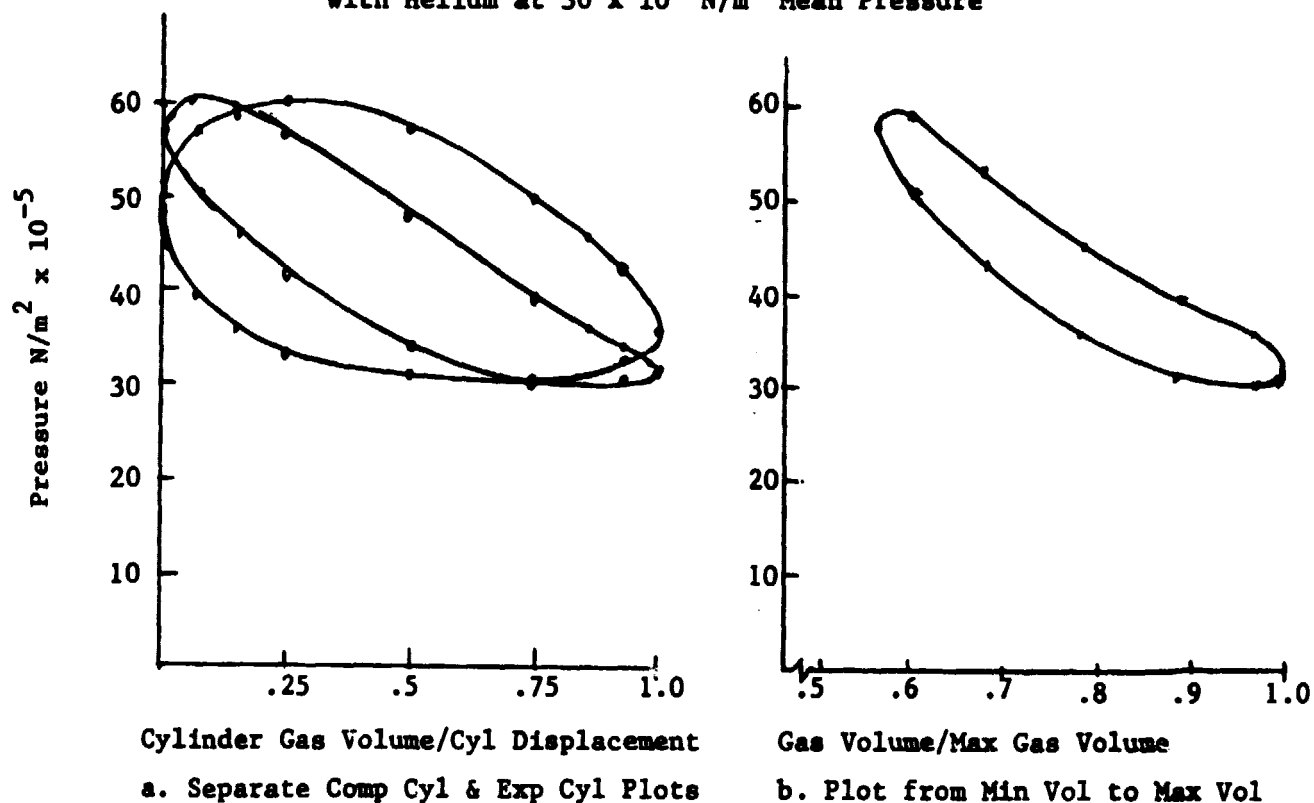


Figure 9. Typical Pressure-Volume Diagrams for Stirling Cycle
with Hydrogen at $45 \times 10^5 \text{ N/m}^2$ Mean Pressure

pressure and the expansion cylinder pressure should be used. During the compression portion (the lower part of the closed curve) the higher of the compression cylinder pressure and the expansion cylinder pressure should be plotted. Using the above logic the specific pressure point plotted will depend on the direction of the gas flow. The pressure-volume area (indicated work) will then exclude pressure drop due to fluid flow resistance.

A typical gas temperature versus position plot is presented in Figure 10. This data is for conditions selected to be similar to those used by NASA for experimental runs with the results listed in reference⁵. The plot drawn by DEEPP is from computer output data as stored in CEDAT 198.

Figure 11 is a "softened" gas temperature versus position plot drawn to provide a clearer picture of the temperature pattern as it varies in the flow region with position and with time. The system has been softened by reducing the effective heat transfer in the regenerator to about one-eighth of the computed typical value. This computer plot was made from data generated by the computer and stored in file CEDAT 63.

Power versus RPM curves are presented in Figure 12a. These curves for the engine operating with hydrogen have been plotted from data in CEDAT 194 to CEDAT 199. A plot of NASA experimental data as reported in reference⁵ is included in Figure 12a.

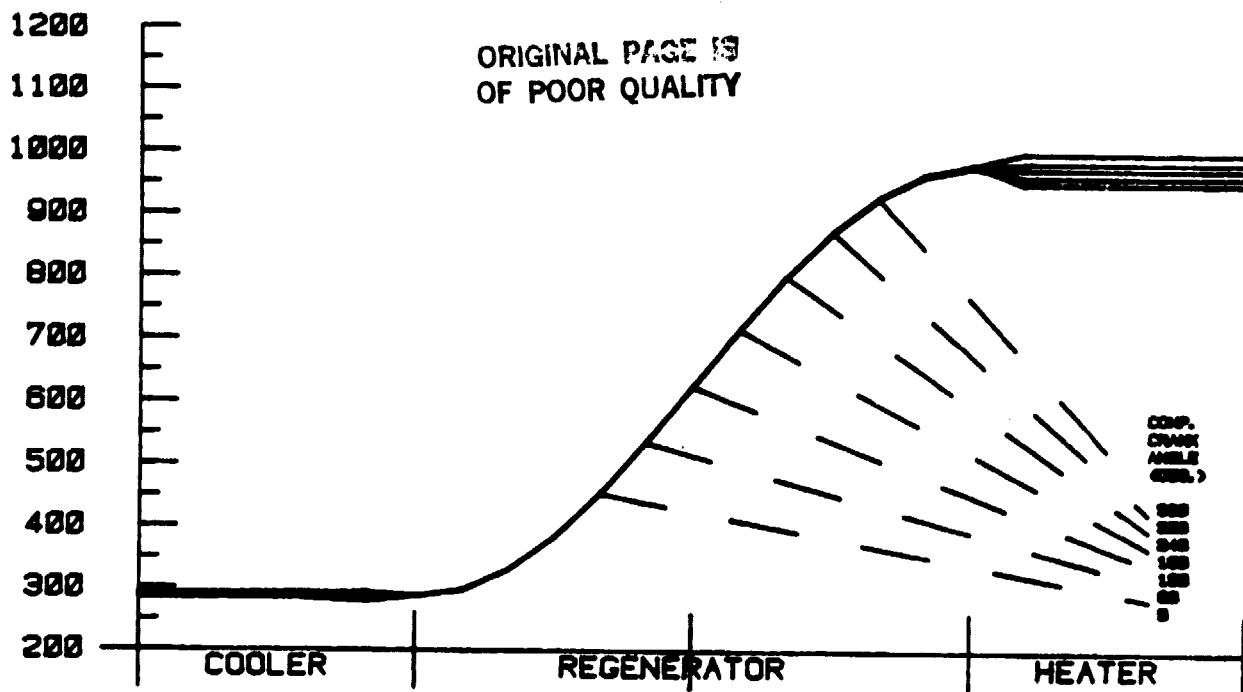


FIGURE 10 GAS TEMPERATURE VS. POSITION

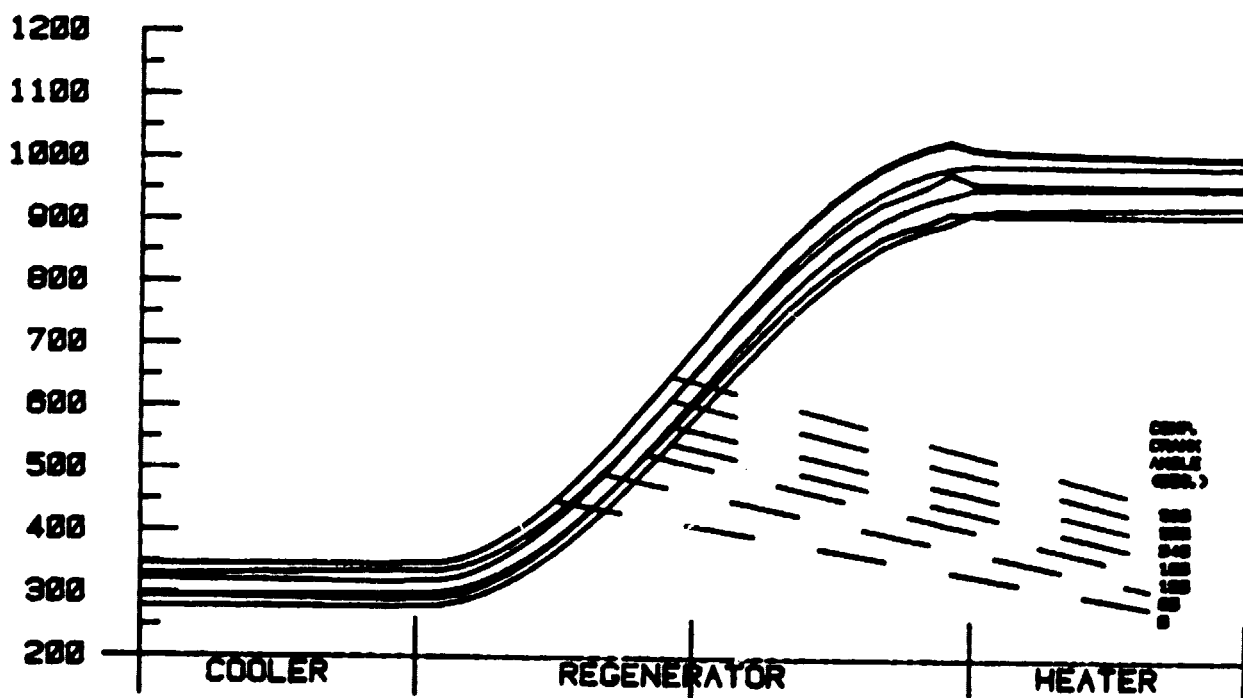
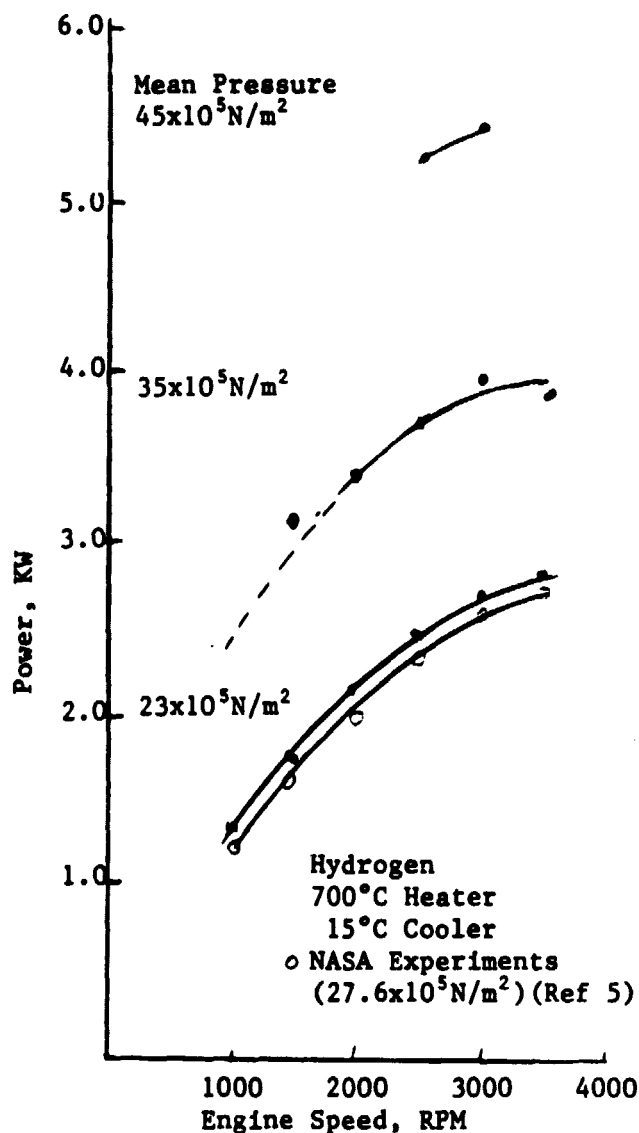


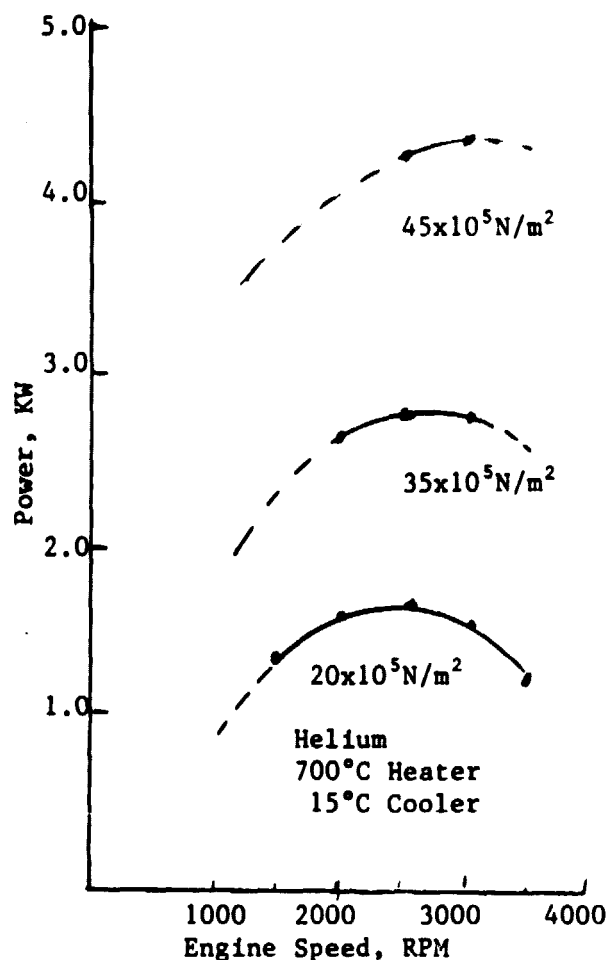
FIGURE 11 GAS TEMPERATURE VS. POSITION
(Softened System)

ORIGINAL PAGE 33
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The curves of Figure 12b are typical plots of power versus RPM for the assumed Stirling cycle engine configuration using helium as the working gas. In Figures 12a and 12b the dashed portion of the curves are based on runs with some variation in conditions. The dashed extensions while not simple extrapolations are not plotted directly from data and operating conditions fully consistent with the solid line portions of the curves.



a. Hydrogen



b. Helium

Figure 12. Power vs Engine Speed for Various Mean Pressures

Effects of Changed Operating Conditions

The most significant factor affecting computed power output is the gas used. Helium and hydrogen were considered. The computations indicated that for similar conditions the maximum power output is at considerably higher rotational speeds when operating with hydrogen. This is primarily due to the lower viscosity and the resultant lower flow pressure drop. Other properties of hydrogen also have favorable effects. The power output at any given speed is higher for hydrogen than for helium with similar assumed conditions.

Increasing the gas pressure has a direct effect on the computed power output. At relatively low pressures the effect appears to be greater than a simple linear effect; however the computations did not take into account leakage or other problems more pronounced with higher pressures. The effect of cooler and heater temperatures is as expected; however only a few variations of these parameters were made. Detailed parameter variation studies have not been made for other factors. One run was made with much lower temperatures and changing from expansion piston phase lag to expansion piston phase lead. This simulated a Stirling cycle heat pump application. The results were not analyzed in detail but they were consistent with expectations.

Suggested Extensions and Modifications of this Research

It is interesting to note that the temperature variations as computed by CDEE for the cooler or compression region are somewhat less than they would be for isentropic compression and expansion while the variations are somewhat more than the isentropic variations in the heater or expansion region. Typical values of the variations are as follows:

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1. With hydrogen as the working gas:

- a. Cooler ΔT_g (CDEE computation) $\pm 35^\circ \text{C}$
 ΔT_g (Isentropic) $\pm 45^\circ \text{C}$
- b. Heater ΔT_g (CDEE computation) $\pm 180^\circ \text{C}$
 ΔT_g (Isentropic) $\pm 150^\circ \text{C}$

2. With helium as the working gas

- a. Cooler ΔT_g (CDEE computation) $\pm 50^\circ \text{C}$
 ΔT_g (Isentropic) $\pm 65^\circ \text{C}$
- b. Heater ΔT_g (CDEE computation) $\pm 250^\circ \text{C}$
 ΔT_g (Isentropic) $\pm 210^\circ \text{C}$

The typical isentropic values cited are primarily dependent on the compression ratio and the value of $\gamma = C_p/C_v$, as the cooler and heater temperature levels will not vary drastically. Typically for this study $T_c = 15^\circ \text{C}$ and $T_h = 704^\circ \text{C}$. The values computed by CDEE are dependent on a number of parameters; the most significant ones are compression ratio, temperature levels, heat transfer area, the value of γ , and other gas properties values.

The variation from the isentropic case can be accounted for by the heat transfer effects. A check of the computed values indicates that effective polytropic "n" values with hydrogen as the working gas are approximately 1.3 for the cooler and 1.52 for the heater. These values can be compared with a " γ " value of 1.4. When helium is used as the working gas the polytropic "n" values are approximately 1.48 for the cooler and 1.85 for the heater, compared with a " γ " value of 1.67. The regenerator between the cooler and heater is relatively close to an isothermal region with rather small temperature changes with time. There is, of course, a large temperature gradient (variation with position) in the regenerator.

It might be possible to develop a simple, effective and reasonably accurate Stirling cycle analysis technique based on a generalization and extension of the observations in the preceding paragraph. This would probably entail the determination of effective polytropic exponents based on design factors and correlation with experimental data, followed by thermodynamic, fluid mechanics and general mechanical analysis techniques.

A modification in CDEE that might be worthwhile was tried but not carried out to a conclusive evaluation. The intent was to provide for a coarse initial computation by bypassing the computer adjusted time increment routine. This routine may impose higher accuracy criteria than is justified for some parts of the computation. The intent was to provide an optional QUICK logic routine. Perhaps a simple relaxation of intermediate accuracy requirements would be a more fruitful approach. This could be accomplished by a simple logic addition to permit variable accuracy requirements with a lower accuracy for an initial computation.

The program would be suitable for a detailed parametric study, and with some refinements for system design and optimization studies.

A study of transient effects and system control studies could be carried out with program simplification so that particularly pertinent data would be obtained with minimum computation. The computer program could be modified to provide for an iteration on the computed gas density, temperature and pressure values. Such a modification could permit larger time and position increments and provide greater accuracy.

NOMENCLATURE

A_f	area of flow
A^c	convective heat transfer area
C^c	specific heat at constant pressure
C^p	specific heat at constant volume
D^v	effective diameter
f	gas flow friction factor
F^v	viscous force
G^v	acceleration of gravity
h^c	convective heat transfer coefficient
I^c	increment number
K^p	pressure coefficient
K^v	flow velocity factor
L^v	effective length
LC	cooler length
LH	heater length
LR	regenerator length
$NINCR$	number of regenerator increments
P	pressure
R	increment length ratio
T	temperature ($^{\circ}K$)
t	time
U	velocity
U^c	velocity of compression piston
U^e	velocity of expansion piston
V^e	volume
W	work
x	position
x_c	distance to compression piston
x_e	distance to expansion piston
α	alpha characteristic line (or point)
β	beta characteristic line (or point)
γ	ratio of specif heats (C_p/C_v)
Δ	small increment
ρ	density
μ	viscosity
∂	partial derivative
∇	differential operator ($\bar{i}\partial/\partial x + \bar{j}\partial/\partial y + \bar{k}\partial/\partial z$)
ω	angular velocity (radian/sec)

Subscripts

c	compression end
e	expansion end
g	gas
h	heater
i	ideal
m	mesh
r	regenerator
t	partial differentiation with respect to t
x	partial differentiation with respect to x

REFERENCES

ORIGINAL PAGE 18 OF POOR QUALITY

1. Walker, G., "Stirling Cycle Machines," Oxford University Press, 1973.
2. Martini, W.R., "Stirling Engine Design Manual," NASA CR-135382, April 1978.
3. Finkelstein, T., "Generalized Thermodynamic Analysis of Stirling Engines," SAE paper 118A, 1960.
4. Urieli, I., "A Review of Stirling Cycle Machine Analysis," 14th IECEC, August 1979.
5. Martini, W.R., "An Isothermal Second Order Stirling Engine Calculation Method," 14th IECEC, August 1979.
6. Tew, R., Jeffries, K., Miao, D., "A Stirling Engine Computer Model for Performance Calculations," NASA TM-78884, July 1978.
7. Tew, R.C., Thieme, L.G., and Miao, D., "Initial Comparison of Single Cylinder Stirling Engine Computer Model Predictions with Test Results," NASA TM-79044, March 1979.
8. Martini, W.R., "A Simple Method of Calculating Stirling Engines for Engine Design Optimization, 13th IECEC, August 1978.
9. Rios, P.A., "An Analytical and Experimental Investigation of the Stirling Cycle," Sc.D. thesis, M.I.T., June 1969.
10. Urieli, I., "A Computer Simulation of Stirling Cycle Machines," Ph.D. thesis, Univ. of Witwatersrand, February 1977.
11. Kirkley, D.S., "An Investigation of the Losses Occurring in Reciprocating Hot Air Engines," Ph.D. thesis, University of Durham, Oct. 1963.
12. Larson, V.H., "Characteristic Dynamic Energy Equations for Stirling Cycle Analysis," 16th IECEC, August 1981.
13. Larson, V.H., "Computation Techniques and Computer Programs to Analyze Stirling Cycle Engines using Characteristic Dynamic Energy Equations," 17th IECEC, August 1982.
14. Lorenzo, C.F., and Daniele, C.J., "A Four-Cylinder Stirling Engine Controls Model," Automotive Technology Development Contractor Coordinator Meeting, November 1980.
15. Forsythe, G.E., Malcom, M.A., and Moler, C.B., "Computer Methods for Mathematical Computations," Prentice Hall, Englewood Cliffs, NJ, 1977.
16. Ames, W.F., "Nonlinear Partial Differential Equations in Engineering," Vol II, Academic Press, New York, 1972.
17. Ash, J.E., and Heames, T.J., "Comparative Analysis of Computer Codes for Stirling Engine Cycles," 16th IECEC, August 1981.
18. "Handbook of Chemistry and Physics," Chemical Rubber Publishing Co.