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ABSTRACT

We develop numerical procedures for constructing asymptotic solutions of certain nonlinear singularly perturbed vector two-point boundary value problems having boundary layers at one or both endpoints. The asymptotic approximations are generated numerically and can either be used as is or to furnish a general purpose two-point boundary value code with an initial approximation and the nonuniform computational mesh needed for such problems. The procedures are applied to a model problem that has multiple solutions and to problems describing the deformation of a thin nonlinear elastic beam that is resting on an elastic foundation.

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1. Introduction

Initial value problems for stiff systems of ordinary differential equations are now considered to be relatively tractible numerically (cf. Enright et al. [7]). However, codes for stiff (or singularly perturbed) boundary value problems are not readily available, even though these problems arise in a great many applications.

In this paper we consider asymptotic and numerical methods for singularly perturbed two-point boundary value problems of the form

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{y},t,\varepsilon) \quad , \quad \varepsilon\dot{\underset{\sim}{y}} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{y},t,\varepsilon) \tag{1.1}$$

$$\underset{\sim}{a}(\underset{\sim}{x}(0),\underset{\sim}{y}(0),\varepsilon) = \underset{\sim}{0} \quad , \quad \underset{\sim}{b}(\underset{\sim}{x}(1),\underset{\sim}{y}(1),\varepsilon) = \underset{\sim}{0} \quad , \tag{1.2a,b}$$

where $\underset{\sim}{x}$, $\underset{\sim}{y}$, $\underset{\sim}{a}$, and $\underset{\sim}{b}$ are vectors of dimension m, n, q, and $r = m + n - q$, respectively, and ε is a small positive parameter.

Although many special problems of this form can be solved by known asymptotic or numerical techniques, the general problem is very difficult and beyond our current understanding. The form of equations (1.1, 2) imply that, whenever $\underset{\sim}{g}$ is not small, $\underset{\sim}{y}$ varies rapidly relative to $\underset{\sim}{x}$. The behavior of the solution in these zones of rapid transition can be very complicated. For example, $\underset{\sim}{y}$ can "jump" abruptly in a narrow boundary layer near $t = 0$ and/or 1. These jumps can also occur at interior locations where solutions or their derivatives will become unbounded as $\varepsilon \to 0$. The locations of the interior layers are generally unknown and must be determined as part of the solution process. Examples of these and other phenomena are discussed in, e.g., O'Malley [20], Kevorkian and Cole [18], Pearson [24, 25], Hemker [15], and Flaherty and O'Malley [11].

The traditional numerical techniques for two-point boundary value problems all have difficulties with singularly perturbed problems unless the grid that is used for the discretization is appropriately fine, at least within boundary or interior layers. If the grid is not fine enough to resolve the layers, the computed solution typically exhibits spurious mesh oscillations. There are, however, special purpose schemes that can solve some singularly perturbed boundary value problems without using a fine discretization in transition regions. Most notable among these are the "upwind" or one-sided finite difference schemes (cf., e.g., Kreiss and Kreiss [19] or Osher [23]) and the exponentially weighted finite difference and finite element schemes (cf., e.g., Flaherty and Mathon [9] or Hemker[15]). These schemes must usually be either restricted to relatively simple problems or employ complicated algebraic transformations.

In view of these theoretical and computational difficulties, we simplify problem (1.1, 2) considerably by assuming, in addition to natural smoothness hypotheses, that (i) $\underset{\sim}{g}$, $\underset{\sim}{a}$, and $\underset{\sim}{b}$ are linear functions of the "fast" variable $\underset{\sim}{y}$, (ii) the n x n Jacobian

$$\underset{\sim}{G}(\underset{\sim}{x},t) := \underset{\sim}{g}_{\underset{\sim}{y}}(\underset{\sim}{x},\underset{\sim}{y},t,0) \tag{1.3}$$

has a strict hyperbolic splitting with $k > 0$ stable and $n - k > 0$ unstable eigenvalues for all $\underset{\sim}{x}$ and $0 \leq t \leq 1$, and (iii) $q > k$ and $r > n - k$. A corresponding theory for problems with quadratic dependence on $\underset{\sim}{y}$ is very limited (cf., e.g., Howes [17] which discusses second-order scalar equations). This, of course, limits extension of a numerical theory, but encourages further numerical experimentation.

The assumed hyperbolic splitting restricts any rapid variations in $\underset{\sim}{y}$ to occur in boundary layer regions near $t = 0$ and/or 1. Thus, we unfortunately have eliminated many important and challenging problems having interior or "shock" layers. Some numerical work on these problems was done

by Kreiss and Kreiss [19], Osher [23], and O'Malley [22].

In a series of three papers, Ascher and Weiss [2, 3, 4] show that symmetric, or centered, collocation schemes could be used on problems that satisfied assumptions similar to ours provided that appropriately fine meshes were used in the endpoint boundary layers. Our approach is somewhat different in that we use the assumed hyperbolic splitting to find an asymptotic solution of problem (1.1, 2) which is composed of a limiting outer solution $(\underset{\sim 0}{X}(t), \underset{\sim 0}{Y}(t))$ and boundary layer corrections near t = 0 and 1. The limiting solution satisfies a reduced system, which is obtained from (1.1) by formally setting ε to zero, i.e.,

$$\dot{\underset{\sim 0}{X}} = \underset{\sim}{f}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0) , \qquad \underset{\sim}{0} = \underset{\sim}{g}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0) \quad . \tag{1.4a,b}$$

Because $\underset{\sim}{G}$ is everywhere nonsingular, we can solve Eq. (1.4b) for $\underset{\sim 0}{Y} = \underset{\sim 0}{Y}(\underset{\sim 0}{X}, t)$ in a locally unique way, and there remains the m th order differential system (1.4a) for determining $\underset{\sim 0}{X}(t)$.

In order to completely specify the limiting solution, we must prescribe m boundary conditions for Eq. (1.4a). We do this in Section 2 by providing a "cancellation law" that selects a combination of q - k initial conditions (1.2a) and of r - n + k terminal conditions (1.2b) to be satisfied by $\underset{\sim 0}{X}(t)$. For more nonlinear problems, we note that such a cancellation law is much more difficult to specify (cf. O'Malley [21]). Boundary layer corrections are generally needed to compensate for the cancelled initial and terminal conditions, and these are easily determined once $\underset{\sim 0}{X}(t)$ and $\underset{\sim 0}{Y}(t)$ have been found (cf. Section 2).

In Section 3 we discuss a numerical procedure for calculating the asymptotic solution of Section 2. We implement the cancellation law by

using orthogonal transformations to reduce $\underset{\sim}{G}(\underset{\sim}{x}(t),t)$ to a block triangular form with its stable and unstable eigenspaces separated. We also use the general purpose two-point boundary value code COLSYS to solve the reduced problem and then add numerical approximations of the boundary layer corrections. This approximation is considerably less expensive to obtain than solving the full stiff problem numerically and it has the advantage of improving in accuracy, without any additional computational cost, as the small parameter ε tends to zero. However, when ε is only moderately small, our asymptotic approximation may not be sufficiently accurate for some applications, so we have developed a procedure for generating an improved solution by using COLSYS to solve the complete problem (1.1, 2) with our asymptotic approximation as an initial guess. In order for this approach to succeed, we must also provide COLSYS with an initial nonuniform mesh that is appropriately graded in the boundary layers, and we give an algorithm for constructing such a mesh in Section 3.

In Section 4 we apply our procedures to a third order model problem that has multiple solutions and to problems involving the deformation of a thin nonlinear elastic beam. These examples show that our methods can calculate accurate solutions of stiff problems for a very modest computational effort. While our algorithm for furnishing COLSYS with an initial guess and a nonuniform mesh does not seem to be optimal, it does offer some advantages over the more standard approach of continuation in ε, where one starts with a large value of ε (e.g., $\varepsilon = 1$) and a crude initial guess of the solution and reduces ε in steps so that the mesh is gradually concentrated into the boundary layer regions.

We also present two examples in Section 4 that are beyond the capabilities of our current methods because their solutions become unbounded as $\varepsilon \to 0$. We include numerical results for these problems in this paper in order to show some of the several challenging effects that can occur with singularly perturbed problems. Finally, in Section 5, we discuss our results and present some suggestions for future investigations.

2. Asymptotic Approximation

With the assumed hyperbolic splitting, we expect solutions of (1.1,2) to feature boundary layers in the fast $\underset{\sim}{y}$ variable near both endpoints as $\varepsilon \to 0$. Thus, it is natural (cf. O'Malley [21]) to seek bounded uniform asymptotic expansions of the form

$$\left.\begin{aligned} \underset{\sim}{x}(t,\varepsilon) &= \underset{\sim}{X}(t,\varepsilon) + \varepsilon\underset{\sim}{\xi}(\tau,\varepsilon) + \varepsilon\underset{\sim}{\eta}(\sigma,\varepsilon) \\ \underset{\sim}{y}(t,\varepsilon) &= \underset{\sim}{Y}(t,\varepsilon) + \underset{\sim}{\mu}(\tau,\varepsilon) + \underset{\sim}{\nu}(\sigma,\varepsilon) \end{aligned}\right\} , \quad 0 < t < 1 , \tag{2.1}$$

where the outer solution $(\underset{\sim}{X}(t,\varepsilon), \underset{\sim}{Y}(t,\varepsilon))$ represents the solution asymptotically within (0,1), the initial layer correction $(\varepsilon\underset{\sim}{\xi}(\tau,\varepsilon), \underset{\sim}{\mu}(\tau,\varepsilon))$ decays to zero as the stretched variable

$$\tau = t/\varepsilon \tag{2.2a}$$

tends to infinity, and the terminal layer correction ($\varepsilon\underset{\sim}{\eta}(\sigma,\varepsilon)$, $\underset{\sim}{\nu}(\sigma.\varepsilon)$) goes to zero as the stretched variable

$$\sigma = (1 - t)/\varepsilon \tag{2.2b}$$

approaches infinity. The outer solution and the boundary layer corrections are represented by expansions of the form

$$\begin{bmatrix} \underset{\sim}{X}(t,\varepsilon) \\ \underset{\sim}{Y}(t,\varepsilon) \\ \underset{\sim}{\xi}(\tau,\varepsilon) \\ \underset{\sim}{\mu}(\tau,\varepsilon) \\ \underset{\sim}{\eta}(\sigma,\varepsilon) \\ \underset{\sim}{\nu}(\sigma,\varepsilon) \end{bmatrix} \sim \sum_{j=0}^{\infty} \begin{bmatrix} \underset{\sim}{X}_j(t) \\ \underset{\sim}{Y}_j(t) \\ \underset{\sim}{\xi}_j(\tau) \\ \underset{\sim}{\mu}_j(\tau) \\ \underset{\sim}{\eta}_j(\sigma) \\ \underset{\sim}{\nu}_j(\sigma) \end{bmatrix} \varepsilon^j . \tag{2.3a-f}$$

The limiting uniform approximation is obtained from (2.1) by letting ε tend to zero, i.e.,

$$\underset{\sim}{x}(t,\varepsilon) = \underset{\sim}{X}_0(t) + \underset{\sim}{0}(\varepsilon), \quad \underset{\sim}{y}(t,\varepsilon) = \underset{\sim}{Y}_0(t) + \underset{\sim}{\mu}_0(\tau) + \underset{\sim}{\nu}_0(\sigma) + \underset{\sim}{0}(\varepsilon) . \tag{2.4}$$

At $t = 0$ the fast vector $\underset{\sim}{y}$ usually has a discontinuous limit, jumping from $\underset{\sim}{y}(0,0) = \underset{\sim}{Y}_0(0) + \underset{\sim}{\mu}_0(0)$ to $\underset{\sim}{Y}_0(0)$ at $t = 0^+$. An analogous Heaviside discontinuity generally occurs near $t = 1$.

The outer expansion (2.3a,b) must satisfy the full problem (1.1) within (0,1) as a power series in ε; thus, the limiting solution $(\underset{\sim 0}{X}, \underset{\sim 0}{Y})$ will satisfy the nonlinear and non-stiff reduced system (1.4). As previously noted, since $G(\underset{\sim}{X}_0, t)$ (cf. Eq. (1.3)) is nonsingular we can solve Eq. (1.4b) for $\underset{\sim 0}{Y} = \underset{\sim 0}{Y}(\underset{\sim 0}{X}, t)$ in a locally unique way, so there remains the m th order nonlinear system (1.4a) for $\underset{\sim 0}{X}(t)$. Later terms of the expansion (2.3a,b) satisfy linearized versions of the reduced system. For example, the coefficients of order ε give

$$\begin{aligned} \dot{\underset{\sim 1}{X}} &= f_{\underset{\sim}{X}}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0)\underset{\sim 1}{X} + f_{\underset{\sim}{Y}}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0)\underset{\sim 1}{Y} + \underset{\sim \varepsilon}{f}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0) \quad , \\ \dot{\underset{\sim 0}{Y}} &= g_{\underset{\sim}{X}}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0)\underset{\sim 1}{X} + G(\underset{\sim}{X}_0, t)\underset{\sim 1}{Y} + \underset{\sim \varepsilon}{g}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0) \quad . \end{aligned} \tag{2.4}$$

We can determine $\underset{\sim 1}{Y}(t)$ in terms of $\underset{\sim 0}{X}$, $\underset{\sim 0}{Y}$, and $\underset{\sim 1}{X}$ from (2.4b) and, once again, there remains the m th order linear system (2.4a) for $\underset{\sim 1}{X}$. Similarly, for each $j > 1$, we obtain a system of the form

$$\begin{aligned} \dot{\underset{\sim j}{X}} &= f_{\underset{\sim}{X}}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0)\underset{\sim j}{X} + f_{\underset{\sim}{Y}}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0)\underset{\sim j}{Y} + \underset{\sim j-1}{\alpha}(\underset{\sim 0}{X}, \cdots, \underset{\sim j-1}{X}, t) \quad , \\ \dot{\underset{\sim j-1}{Y}} &= g_{\underset{\sim}{X}}(\underset{\sim 0}{X}, \underset{\sim 0}{Y}, t, 0)\underset{\sim j}{X} + G(\underset{\sim}{X}_0, t)\underset{\sim j}{Y} + \underset{\sim j-1}{\beta}(\underset{\sim 0}{X}, \cdots, \underset{\sim j-1}{X}, t) \end{aligned} \tag{2.5}$$

with successively determined inhomogeneous terms.

In order to completely specify the outer expansion (2.3a,b), we must prescribe boundary conditions for the m-vectors $\underset{\sim}{X}_j(t)$. Most critically, we need to specify m boundary conditions for the limiting slow vector $\underset{\sim}{X}_0(t)$. It is natural to attempt to determine them by somehow selecting a subset of m combinations of the m + n boundary conditions (1.2) evaluated at $\varepsilon = 0$. For scalar higher order linear differential equations, the first such "cancellation law" was obtained by Wasow [29]. Harris [14] obtained a more complicated cancellation law for linear systems with coupled boundary conditions and Ferguson [8] developed a numerical procedure for corresponding linear problems. These early works suggest that we should seek a cancellation law that ignores an appropriate combination of k initial conditions and of n-k terminal conditions. To this end, we must examine the boundary layer corrections and we begin by considering the initial layer correction $(\varepsilon\underset{\sim}{\xi}, \underset{\sim}{\mu})$. Near t = 0, the terminal layer correction $(\varepsilon\underset{\sim}{\eta}, \underset{\sim}{\nu})$ may be neglected, so the representation of our asymptotic solution (2.1) requires the initial layer correction $(\varepsilon\underset{\sim}{\xi}, \underset{\sim}{\mu})$ to satisfy the nonlinear system

$$\begin{aligned} d\underset{\sim}{\xi}/d\tau &= d\underset{\sim}{x}/dt - d\underset{\sim}{X}/dt = \underset{\sim}{f}(\underset{\sim}{X}+\varepsilon\underset{\sim}{\xi}, \underset{\sim}{Y}+\underset{\sim}{\mu}, \varepsilon\tau, \varepsilon) - \underset{\sim}{f}(\underset{\sim}{X}, \underset{\sim}{Y}, \varepsilon\tau, \varepsilon) \quad , \\ d\underset{\sim}{\mu}/d\tau &= \varepsilon(d\underset{\sim}{y}/dt - d\underset{\sim}{Y}/dt) = \underset{\sim}{g}(\underset{\sim}{X}+\varepsilon\underset{\sim}{\xi}, \underset{\sim}{Y}+\underset{\sim}{\mu}, \varepsilon\tau, \varepsilon) - \underset{\sim}{g}(\underset{\sim}{X}, \underset{\sim}{Y}, \varepsilon\tau, \varepsilon) \quad , \end{aligned} \tag{2.6}$$

on $\tau > 0$ and to decay to zero as $\tau \to \infty$. Substitution of the asymptotic expansion (2.3c,d) into (2.6) provides successive differential equations for the coefficients $(\underset{\sim}{\xi}_j, \underset{\sim}{\mu}_j)$. In particular, when $\varepsilon = 0$, we have the limiting initial layer system

$$d\underset{\sim 0}{\xi}/d\tau = \underset{\sim}{f}(\underset{\sim 0}{X}(0),\underset{\sim 0}{Y}(0) + \underset{\sim 0}{\mu},0,0) - \underset{\sim}{f}(\underset{\sim 0}{X}(0),\underset{\sim 0}{Y}(0),0,0) \quad , \tag{2.7}$$

$$d\underset{\sim 0}{\mu}/d\tau = \underset{\sim}{g}(\underset{\sim 0}{X}(0),\underset{\sim 0}{Y}(0) + \underset{\sim 0}{\mu},0,0) - \underset{\sim}{g}(\underset{\sim 0}{X}(0),\underset{\sim 0}{Y}(0),0,0) \quad .$$

The decay requirement determines

$$\underset{\sim 0}{\xi}(\tau) = -\int_{\tau}^{\infty} (d\underset{\sim 0}{\xi}(s)/d\tau)ds \tag{2.8a}$$

as a functional of $\underset{\sim 0}{\mu}$, while $\underset{\sim 0}{\mu}$ satisfies the conditionally stable system

$$d\underset{\sim 0}{\mu}/d\tau = \underset{\sim}{G}(\underset{\sim 0}{X}(0),0)\underset{\sim 0}{\mu} \quad . \tag{2.8b}$$

We used (1.3) and the assumed linearity of $\underset{\sim}{g}$ in $\underset{\sim}{y}$ when obtaining (2.8b). If $\underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{y},t,\varepsilon)$ were not linear in $\underset{\sim}{y}$, the initial layer correction would satisfy a nonlinear differential equation which would generally be difficult to solve (cf. O'Malley [21]). Indeed, it would then be extremely difficult to specify what set of initial vectors $\underset{\sim 0}{\mu}(0)$ would lead to decaying solutions of the boundary layer system (2.7b). Here, Eq. (2.4) is readily integrated to give

$$\underset{\sim 0}{\mu}(\tau) = e^{\underset{\sim}{G}(\underset{\sim 0}{X}(0),0)\tau}\,\underset{\sim 0}{\mu}(0) \quad . \tag{2.9}$$

Thus, $\underset{\sim}{\mu}_0$ will decay to zero as $\tau \to \infty$ provided that

$$\underset{\sim}{\mu}_0(0) = \underset{\sim}{P}(\underset{\sim}{X}_0(0),0)\underset{\sim}{\mu}_0(0) \quad , \tag{2.10}$$

where $\underset{\sim}{P}$ is a projection onto the k dimensional stable eigenspace of $\underset{\sim}{G}(\underset{\sim}{X}_0(0),0)$.

Substituting (2.10) into (1.2a) and letting $\varepsilon \to 0$, we see that the q limiting initial conditions take the form

$$\underset{\sim}{a}(\underset{\sim}{X}_0(0),\underset{\sim}{Y}_0(0) + \underset{\sim}{P}(\underset{\sim}{X}_0(0),0)\underset{\sim}{\mu}_0(0),0) = \underset{\sim}{0} \; . \tag{2.11}$$

Now, using the linearity of $\underset{\sim}{a}$ in $\underset{\sim}{y}$, we let

$$\underset{\sim}{A}(x,t) = \underset{\sim}{a}_{\underset{\sim}{y}}(x,\underset{\sim}{y},t,0) \tag{2.12}$$

and further assume that $\underset{\sim}{A}(\underset{\sim}{X}_0(0),0)\underset{\sim}{P}(\underset{\sim}{X}_0(0),0)$ has its full and maximal rank k. Then we can uniquely determine $\underset{\sim}{\mu}_0(0)$ as a function of $\underset{\sim}{X}_0(0)$ from k of the equations (2.11). Having done this, initial conditions for the reduced problem can be determined from the remaining q - k conditions in (2.11). For the moment, we write these in the form

$$\underset{\sim}{\Phi}(\underset{\sim}{X}_0(0)) = \underset{\sim}{0} \; . \tag{2.13}$$

In Section 3, we discuss a numerical procedure for determining $\underset{\sim}{P},\underset{\sim}{\mu}_0(0)$ and $\underset{\sim}{\Phi}(\underset{\sim}{X}_0(0))$.

The terminal layer correction can be analyzed in an analogous manner. In particular, the leading term $\underset{\sim 0}{\nu}(\sigma)$ satisfies

$$\underset{\sim 0}{\nu}(\sigma) = e^{\underset{\sim}{G}(\underset{\sim 0}{X}(1),1)\sigma}\,\underset{\sim 0}{\nu}(0) \quad . \tag{2.14}$$

Now, $\underset{\sim 0}{\nu}$ will decay to zero as $\sigma \to \infty$ provided that

$$\underset{\sim 0}{\nu}(0) = \underset{\sim}{Q}(\underset{\sim 0}{X}(1),1)\,\underset{\sim 0}{\nu}(0) \quad , \tag{2.15}$$

where $\underset{\sim}{Q}$ is a projection onto the $n - k$ dimensional unstable eigenspace of $\underset{\sim}{G}(\underset{\sim 0}{X}(1),1)$. Substituting (2.15) into (1.2b) and letting $\varepsilon \to 0$ gives the r limiting terminal conditions as

$$\underset{\sim}{b}(\underset{\sim 0}{X}(1),\underset{\sim 0}{Y}(1) + \underset{\sim}{Q}(\underset{\sim 0}{X}(1),1)\,\underset{\sim 0}{\nu}(0),0) = \underset{\sim}{0} \quad . \tag{2.16}$$

We let

$$\underset{\sim}{B}(x,t) = \underset{\sim}{b}_{\underset{\sim}{y}}(x,y,t,0) \tag{2.17}$$

and assume that $\underset{\sim}{B}(\underset{\sim 0}{X}(1),1)\,\underset{\sim}{Q}(\underset{\sim 0}{X}(1),1)$ has full rank $n - k$. Then we can solve (2.17) for $\underset{\sim 0}{\nu}(0)$ and the remaining $r - n + k$ conditions specify terminal conditions for the limiting problem, which we denote by

$$\underset{\sim}{\Psi}(\underset{\sim 0}{X}(1)) = \underset{\sim}{0} \quad . \tag{2.18}$$

The reduced problem consists of the nonlinear reduced differential equation and the m separated nonlinear boundary conditions (2.13, 18). If it is solvable, it may have many solutions; however, corresponding to any of its isolated solutions $(\underset{\sim 0}{X}(t),\underset{\sim 0}{Y}(t))$, one can expect to find a solution of the original problem (1.1, 2) that converges to $(\underset{\sim 0}{X}(t),\underset{\sim 0}{Y}(t))$ on $0 < t < 1$

as $\varepsilon \to 0$. Sufficient hypotheses to obtain an asymptotic solution having the form of (2.1) are provided by Hoppensteadt [16] and others. For this reason, we shall merely indicate the considerations that are involved in obtaining further terms in the initial and terminal layer expansions and boundary conditions for the outer expansion.

Additional terms of the initial layer expansion (2.3c,d) are determined by equating the coefficients of ε^j in the nonlinear system (2.7), i.e.

$$\begin{aligned} d\underset{\sim}{\xi}_j/d\tau &= f_{\underset{\sim}{Y}}(\underset{\sim}{X}_0(0),\underset{\sim}{Y}_0(0) + \underset{\sim}{\mu}_0(\tau),0,0)\underset{\sim}{\mu}_j + \underset{\sim}{\gamma}_{j-1}(\tau) \quad , \\ d\underset{\sim}{\mu}_j/d\tau &= G(\underset{\sim}{X}_0(0),0)\underset{\sim}{\mu}_j + \underset{\sim}{\delta}_{j-1}(\tau) \quad , \end{aligned} \tag{2.19}$$

for $j > 1$, where the inhomogeneous terms are exponentially decaying as $\tau \to \infty$ because $\underset{\sim}{\xi}_{i-1}$ and $\underset{\sim}{\mu}_{i-1}$, $i = 1, \ldots, j-1$, and their derivatives behave in this manner. The linear system (2.19) may be integrated to yield

$$\begin{aligned} \underset{\sim}{\xi}_j(\tau) &= -\int_\tau^\infty (d\underset{\sim}{\xi}_j(s)/d\tau)ds \quad , \\ \underset{\sim}{\mu}_j(\tau) &= e^{G(\underset{\sim}{X}_0(0),0)\tau}\underset{\sim}{\mu}_j(0) + \int_0^\tau e^{G(\underset{\sim}{X}_0(0),0)(\tau-s)}\underset{\sim}{\delta}_{j-1}(s)ds \quad . \end{aligned} \tag{2.20}$$

We see that $\underset{\sim}{\xi}_j(\tau)$ decays as τ increases and $\underset{\sim}{\mu}_j(\tau)$ will decay provided that $\underset{\sim}{\mu}_j(0)$ lies in the unstable eigenspace of $G(\underset{\sim}{X}_0(0),0)$, i.e.,

$$\underset{\sim}{\mu}_j(0) = P(\underset{\sim}{X}_0(0),0)\underset{\sim}{\mu}_j(0) \quad . \tag{2.21}$$

Using (2.1) and (2.3a,b) we find that the coefficient of ε^j in the initial condition (1.2a) has the form

$$\underset{\sim}{a}_x(\underset{\sim}{X}_0(0),\underset{\sim}{Y}_0(0) + \underset{\sim}{\mu}_0(0),0)\underset{\sim}{X}_j(0) + \underset{\sim}{\underset{\sim}{A}}(\underset{\sim}{X}_0(0),0)[\underset{\sim}{Y}_j(0) + \underset{\sim}{\underset{\sim}{P}}(\underset{\sim}{X}_0(0),0)\underset{\sim}{\mu}_j(0)] = \underset{\sim}{\zeta}_{j-1} \quad . \tag{2.22}$$

Since $\underset{\sim}{\underset{\sim}{A}}(\underset{\sim}{X}_0(0),0)\underset{\sim}{\underset{\sim}{P}}(\underset{\sim}{X}_0(0),0)$ has its maximal rank k, we can determine $\underset{\sim}{\mu}_j(0)$ from k of these equations, and the remaining q - k equations determine linear equations for $\underset{\sim}{X}_j(0)$. The situation for the terminal layer correction is completely analogous; thus, $\underset{\sim}{\nu}_j(0)$ and the terminal conditions for $\underset{\sim}{X}_j(1)$ are determined from linear equations of the form

$$\underset{\sim}{b}_x(\underset{\sim}{X}_0(1),\underset{\sim}{Y}_0(1) + \underset{\sim}{\nu}_0(0),0)\underset{\sim}{X}_j(1) + \underset{\sim}{\underset{\sim}{B}}(\underset{\sim}{X}_0(1),1)[\underset{\sim}{Y}_j(1) + \underset{\sim}{\underset{\sim}{Q}}(\underset{\sim}{X}_0(1),1)\underset{\sim}{\nu}_j(0)] = \underset{\sim}{\theta}_{j-1} \quad . \tag{2.23}$$

To summarize, we have shown that the j th (j > 1) term in the outer expansion satisfies an m th order linear boundary value problem consisting of Eq. (2.5) and a set of m linear boundary conditions determined from (2.22) and (2.23). It is a linearization of the problem for $\underset{\sim}{X}_0(t)$.

3. Numerical Procedure.

In this section we discuss a numerical procedure for finding the limiting uniform asymptotic solution (2.4). It consists of solving the limiting outer problem (1.4, 2.13, 2.18) and determining boundary layer corrections from (2.9) and (2.14).

Our first task is to find the projections $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ and we do this by finding the Schur decomposition of the matrix $\underset{\sim}{G}$ at $t = 0$ and $t = 1$. In particular, at $t = 0$ we find an orthogonal matrix $\underset{\sim}{E}(x(0),0)$ such that

$$\underset{\sim}{G}(x(0),0)\underset{\sim}{E}(x(0),0) = \underset{\sim}{E}(x(0),0)\begin{bmatrix} \underset{\sim}{T}(x(0),0) & \underset{\sim}{U}(x(0),0) \\ \underset{\sim}{0} & \underset{\sim}{T}_{+}(x(0),0) \end{bmatrix} \tag{3.1}$$

where $\underset{\sim}{T}$ is k x k and upper triangular with the stable eigenvalues of $\underset{\sim}{G}(x(0),0)$, and $\underset{\sim}{T}_{+}$ is upper triangular with the remaining n - k unstable eigenvalues. The decomposition (3.1) can often be obtained analytically; however, when this is not possible or practical it can be determined numerically by using the QR algorithm (cf. Golub and Wilkinson [13], Ruhe [26], and Bjork [5] for specific procedures).

We partition $\underset{\sim}{E}$ after its k th column as

$$\underset{\sim}{E} = [\underset{\sim}{E} \quad \underset{\sim}{\bar{E}}] \tag{3.2}$$

and note that $\underset{\sim}{E}$ spans the stable eigenspace of $\underset{\sim}{G}$ at $t = 0$ and

$$\underset{\sim}{P} = \underset{\sim}{E}\,\underset{\sim}{E}^{T} \tag{3.3}$$

is the desired projection onto this eigenspace.

Substituting (3.3) into (2.11) gives

$$\underset{\sim}{a}(\underset{\sim}{X}_0(0), \underset{\sim}{Y}_0(0) + \underset{\sim\sim}{E}(\underset{\sim}{X}_0(0),0)\underset{\sim\sim}{E}^T(\underset{\sim}{X}_0(0),0)\underset{\sim}{\mu}_0(0),0) = \underset{\sim}{0} \quad . \tag{3.4}$$

as the equation for determining $\underset{\sim}{\mu}_0(0)$ and $\underset{\sim}{\Phi}(\underset{\sim}{X}_0(0))$. Since

$\underset{\sim\sim}{A}(\underset{\sim}{X}_0(0),0)\underset{\sim\sim}{E}(\underset{\sim}{X}_0(0),0)$ is of rank k, we construct a q x q matrix

$$\underset{\sim}{L}^T = [\underset{\sim\sim}{L}^T \; \underset{\sim\sim}{\bar{L}}^T] \tag{3.5a}$$

that reduces it to echelon form, i.e.,

$$\begin{bmatrix} \underset{\sim\sim}{L} \\ \\ \underset{\sim\sim}{\bar{L}} \end{bmatrix} \underset{\sim\sim}{A}(\underset{\sim}{X}_0(0),0)\underset{\sim\sim}{E}(\underset{\sim}{X}_0(0),0) = \begin{bmatrix} \underset{\sim\sim}{V} \\ \\ \underset{\sim}{0} \end{bmatrix} , \tag{3.5b}$$

where $\underset{\sim\sim}{V}$ is k x k and nonsingular. Multiplying Eq. (3.4) by $\underset{\sim}{L}$, using the linearity of $\underset{\sim}{a}$ in $\underset{\sim}{y}$, and Eq. (3.5) gives the initial layer jump $\underset{\sim}{\mu}_0(0)$ and the q - k initial conditions (2.13) for the reduced problem, respectively, as

$$\begin{aligned} \underset{\sim}{\mu}_0(0) &= -\underset{\sim\sim}{E}(\underset{\sim}{X}_0(0),0)\underset{\sim\sim}{V}^{-1}\underset{\sim\sim\sim}{L}\,\underset{\sim}{a}(\underset{\sim}{X}_0(0),\underset{\sim}{Y}_0(0),0) \quad , \\ \underset{\sim}{\Phi}(\underset{\sim}{X}_0(0)) &:= \underset{\sim\sim\sim}{\bar{L}}\,\underset{\sim}{a}(\underset{\sim}{X}_0(0),\underset{\sim}{Y}_0(0),0) = \underset{\sim}{0} \quad . \end{aligned} \tag{3.6}$$

We find the terminal layer jump and the r - (n-k) terminal conditions for the reduced problem in an analogous fashion with the exception that we define $\underset{\sim\sim}{E}(\underset{\sim}{x}(1),1)$ such that

$$\underset{\sim}{G}(x(1),1)\underset{\sim}{E}(x(1),1) = \underset{\sim}{E}(x(1),1)\begin{bmatrix} \underset{\sim}{\hat{T}}(x(1),1) & \underset{\sim}{\hat{U}}(x(1),1) \\ \underset{\sim}{0} & \underset{\sim}{\hat{T}}(x(1),1) \end{bmatrix} \tag{3.7}$$

which we partition after its (n - k) th column as

$$\underset{\sim}{E} = [\underset{\sim+}{E} \; \underset{\sim+}{\bar{E}}] \quad . \tag{3.8}$$

In parallel with Eqs. (3.1) and (3.2), the matrices $\underset{\sim}{\hat{T}}$, $\underset{\sim+}{\hat{T}}$, and $\underset{\sim+}{E}$ contain the k stable eigenvalues, the n - k unstable eigenvalues, and span the unstable eigenspace, respectively, of $\underset{\sim}{G}$ at t = 1. Our reasons for switching the positions of the matrices containing the stable and unstable eigenvalues of $\underset{\sim}{G}$ is that we are unaware of a simple and stable computational procedure for finding a set of vectors that span a given subspace and are not in the leading columns of an orthogonal matrix like $\underset{\sim}{E}$ (cf. Golub and Wilkinson [13]).

Now, following the procedure that we used for the initial layer, we take

$$\underset{\sim}{Q}(\underset{\sim0}{X}(1),1) = \underset{\sim+}{E}(\underset{\sim0}{X}(1),1)\underset{\sim+}{E}^{T}(\underset{\sim0}{X}(1),1) \tag{3.9}$$

as our projection onto the (n - k) dimensional unstable eigenspace of $\underset{\sim}{G}(\underset{\sim0}{X}(1),1)$ and construct an r x r matrix

$$\underset{\sim}{R}^{T} = [\underset{\sim+}{R}^{T} \; \underset{\sim+}{\bar{R}}^{T}] \tag{3.10a}$$

that reduces the rank n - k matrix $\underset{\sim}{B}(\underset{\sim0}{X}(1),1)\underset{\sim+}{E}(\underset{\sim0}{X}(1),1)$ to echelon form, i.e.,

$$\begin{bmatrix} \underset{\sim+}{R} \\ \underset{\sim+}{\bar{R}} \end{bmatrix} \underset{\sim}{B}(\underset{\sim0}{X}(1),1)\underset{\sim+}{E}(\underset{\sim0}{X}(1),1) = \begin{bmatrix} \underset{\sim+}{V} \\ \underset{\sim}{0} \end{bmatrix} \quad , \tag{3.10b}$$

where $\underset{\sim}{V}_+$ is (n-k) x (n-k) and nonsingular. Multiplying Eq. (2.16) by $\underset{\sim}{R}$ and using Eqs. (3.9) and (3.10), we find the terminal layer jump and terminal conditions for the reduced problem, respectively, as

$$\underset{\sim}{v}_0(0) = -\underset{\sim}{E}_+(\underset{\sim}{X}_0(1),1)\underset{\sim}{V}_+^{-1}\underset{\sim}{R}_+\underset{\sim}{b}(\underset{\sim}{X}_0(1),\underset{\sim}{Y}_0(1),0) \quad ,$$

$$\underset{\sim}{\Psi}(\underset{\sim}{X}_0(1)) := \overline{\underset{\sim}{R}}_+\underset{\sim}{b}(\underset{\sim}{X}_0(1),\underset{\sim}{Y}_0(1),0) = \underset{\sim}{0} \quad . \tag{3.11}$$

Since the reduced problem (1.4), (3.6b), and (3.11b) is not stiff, we can use any good code for two-point boundary value problems (cf. Childs et al.[6]) to solve it, and we have chosen to use the collocation code COLSYS of Ascher, Christiansen, and Russell [1]. The reduced problem is generally nonlinear and since COLSYS solves nonlinear problems using a damped Newton method, we have to supply formulas for evaluating the Jacobians of $\underset{\sim}{f}$, $\underset{\sim}{Y}$, $\underset{\sim}{\Phi}$, and $\underset{\sim}{\Psi}$ with respect to $\underset{\sim}{X}$. We do this, but introduce an error, by providing analytical formulas for these Jacobians that neglect the influence of the derivatives of $\underset{\sim}{E}$, $\underset{\sim}{L}$, $\underset{\sim}{R}$, and $\underset{\sim}{G}$. (These derivatives will be small when the related subspaces are nearly constant). This procedure failed to converge once on Example 1 of Section 4 and a minor modification to the Jacobian of $\underset{\sim}{\Phi}$ restored convergence; however, an alternative possibility would be to approximate the Jacobians by finite differences.

We start the Newton iteration with a uniform mesh and an initial guess $\underset{\sim}{X}_0^{(0)}(t)$ for $\underset{\sim}{X}_0(t)$. In section 4, we used the default initial guess that is provided by COLSYS for Example 2 and a constant initial guess for Example 1.

This latter choice was necessary as Example 1 has three solutions. At each iteration step, we calculate an approximation $\underset{\sim}{E}(\underset{\sim}{X}_0^{(p)}(t),t)$ to $\underset{\sim}{E}(\underset{\sim}{X}(t),t)$ for $t = 0$ and 1 as the Schur decomposition of $\underset{\sim}{G}(\underset{\sim}{X}^{(p)}(t),t)$. The examples of Section 4 were calculated using analytical formulas for $\underset{\sim}{E}$ rather than the numerical procedures of Golub and Wilkinson [13], Ruhe [26], or Bjork [5]. Finally, $\underset{\sim}{L}^{(p)}$ and $\underset{\sim}{R}^{(p)}$ are obtained by using Gaussian Elimination to row reduce $\underset{\sim}{A}(\underset{\sim}{X}_0^{(p)}(0),0)\underset{\sim}{E}(\underset{\sim}{X}_0^{(p)}(0),0)$ and $\underset{\sim}{B}(\underset{\sim}{X}_0^{(p)}(1),1)\underset{\sim}{E}(\underset{\sim}{X}_0^{(p)}(1),1)$, respectively.

When this procedure converges to $(\underset{\sim}{X}_0(t),\underset{\sim}{Y}_0(t))$, we calculate boundary layer corrections $\underset{\sim}{\mu}_0(\tau)$ and $\underset{\sim}{\nu}_0(\sigma)$, for a given value of ε, using Eqs. (2.9). (3.6a), (2.14), and (3.11a), and add these to the reduced solution in order to get the $0(\varepsilon)$ asymptotic approximation (1.4). For moderately small values of ε, this approximation may not provide a sufficiently accurate representation of the solution and, in this case, we use it as an initial guess to COLSYS and solve the complete problem (1.1, 2). However, this procedure may fail unless we also provide COLSYS with an initial nonuniform partition

$$\pi := \{0 = t_0 < t_1 < \cdots < t_N = 1\} \tag{3.12}$$

that is appropriately graded within the boundary layers. Following Ascher, Christiansen, and Russell [1], we seek to find π such that the error on each subinterval satisfies

$$||\underset{\sim}{e}||_i \leq \delta(1 + ||\underset{\sim}{u}||_i), \qquad i = 1, 2, \ldots, N, \tag{3.13}$$

where δ is a prescribed tolerance,

$$\underset{\sim}{u}^T = [\underset{\sim}{x}^T, \underset{\sim}{y}^T] , \tag{3.14}$$

$\underset{\sim}{e}(t)$ is the difference between $\underset{\sim}{u}(t)$ and its collocation approximation,

$$||\underset{\sim}{u}||_i = \max_{t_{i-1} \leq t \leq t_i} |\underset{\sim}{u}(t)| , \quad \text{and} \quad |\underset{\sim}{u}(t)| = \max_{1 \leq j \leq m+n} |u_j(t)| . \tag{3.15}$$

We assume that the final partition selected by COLSYS to solve the reduced problem satisfies (3.13) outside of boundary layer regions and we seek to refine it within the boundary layers. We further assume that derivatives of $\underset{\sim}{u}$ can adequately be approximated by either $\underset{\sim 0}{\mu}(\tau)$ or $\underset{\sim 0}{\nu}(\sigma)$ in the left or right boundary layer, respectively.

It is known (cf. Russell and Christiansen [27]) that if the solution of (1.1,2) is smooth

$$||\underset{\sim}{e}||_1 = c_j ||\underset{\sim}{u}^{(j+1)}||_1 h_1^{j+1} + O(h^{j+2}) \tag{3.16}$$

for collocation at the image of j Gauss-Legendre points per subinterval. Here c_j is a known constant,

$$h_1 = t_1 - t_{1-1} , \quad \text{and} \quad h = \max_{1 \leq i \leq N} h_i . \tag{3.17}$$

In the left boundary layer we approximate $\underset{\sim}{u}$ in (3.16) by $\underset{\sim 0}{\mu}$ using (2.9) and attempt to find a partition that satisfies

$$c_j h_1^{j+1} ||\underset{\sim 0}{\mu}^{(j+1)}(t/\varepsilon)||_1 \approx \delta(1 + ||u||_1) . \tag{3.18}$$

Finally, we use (2.9) and (3.1) to approximate $\underset{\sim 0}{\mu}$ and the subinterval lengths as

$$t_i - t_{i-1} \approx \frac{\varepsilon}{\alpha}\left[\frac{\delta(1 + \|\underset{\sim}{u}\|_1)}{c_j |\underset{\sim 0}{\mu}(t_{i-1}/\varepsilon)|}\right]^{1/(j+1)}, \qquad (3.19)$$

where α is the magnitude of the largest diagonal element of $\underset{\sim\sim}{T}(\underset{\sim 0}{X}(0),0)$. A similar formula can be obtained for selecting subintervals in the right boundary layer.

Starting with $i = 1$, we use Eq. (3.19) to generate a partition until we either reach $t = 1/2$ or a point where a subinterval length selected by Eq. (3.19) is larger than that used locally by COLSYS to solve the reduced problem. We then repeat the procedure in the right boundary layer.

We have written a computer code called SPCOL that implements the algorithms that are described in this section; thus, it (i) uses COLSYS to solve the reduced problem, (ii) calculates and adds appropriate boundary layer corrections to the reduced problem, and (iii) (optionally) suggests a mesh that can be used by COLSYS to solve the complete problem.

4. Examples.

In order to appraise the performance of SPCOL, we have applied it to a problem involving the deformation of a thin nonlinear elastic beam (Example 1) and a third order model problem that has multiple solutions (Example 2).

Example 1. We consider problems involving the deformation of a nonlinear elastic beam that is resting on an elastic foundation with unit spring constant and is subjected to the combined action of a horizontal end thrust P and a unit uniform lateral load. This problem is discussed in detail in Flaherty and O'Malley [11] and herein we only present the governing differential equations, which in dimensionless form are

$$\dot{x}_1 = \cos x_3, \quad \dot{x}_2 = \sin x_3, \quad \dot{x}_3 = y_1, \tag{4.1a,b,c}$$

$$\varepsilon\dot{y}_1 = -y_2, \quad \varepsilon\dot{y}_2 = (x_2 - 1)\cos x_3 - Ty_1, \tag{4.1d,e}$$

where

$$T = \sec x_3 + \varepsilon y_2 \tan x_3 . \tag{4.1f}$$

The slow variables (x_1, x_2) and x_3 represent the Cartesian coordinates and the tangent angle of a material particle on the centerline of the beam that was at the Cartesian location $(t,0)$ in the undeformed state. The fast variables y_1 and y_2 are the internal bending moment and transverse shear force, respectively. Finally, the small parameter is

$$\varepsilon^2 = EI/PL^2, \tag{4.2}$$

where EI is the flexural rigidity and L is the length of the beam; thus, our beam is much stronger in extension than it is in bending.

This example does not precisely fit our hypotheses since the axial force T is a function of the fast variable y_2 and, thus, $g_{\underset{\sim}{y}}$ also depends on $\underset{\sim}{y}$. However, our theory and methods will still apply as long as $\underset{\sim}{y}$ remains bounded and $|x_3| < \pi/2$ as $\varepsilon \to 0$. Flaherty and O'Malley [11] show that unbounded solutions can occur when certain types of boundary conditions are prescribed for Eq. (4.1). In this paper we present results for the following three sets of boundary conditions:

(i). $$x_1(0,\varepsilon) = x_2(0,\varepsilon) = y_1(0,\varepsilon) = x_2(1,\varepsilon) = y_1(1,\varepsilon) = 0 \ , \tag{4.2a}$$

(ii). $$x_1(0,\varepsilon) = 0, \ -10x_2(0,\varepsilon) + y_2(0,\varepsilon) = 0, \ -x_3(0,\varepsilon)+10y_1(0,\varepsilon) = 0$$
$$10x_2(1,\varepsilon) + y_2(1,\varepsilon) = 0, \ 10x_3(1,\varepsilon)+y_1(1,\varepsilon) = 0 \ , \tag{4.2b}$$

(iii). $$x_1(0,\varepsilon) = x_2(0,\varepsilon) = x_3(0,\varepsilon) = x_2(1,\varepsilon) = x_3(1,\varepsilon) = 0 \ . \tag{4.2c}$$

Equations (4.2a) correspond to "simple supports", Eqs. (4.2c) correspond to "clamped supports", and Eqs. (4.2b) correspond to elastic supports that are almost simply supported at t = 0 and almost clamped at t = 1. Conditions (4.2b) could arise because, say, friction introduces some coupling between lateral and rotational effects at the supports. As we shall see, $\underset{\sim}{y}$ remains bounded for conditions (4.2a,b), but becomes unbounded as $\varepsilon \to 0$ when conditions (4.2c) are applied. The problem is that the boundary conditions for the clamped beam only involve the slow variables and the slow vector $\underset{\sim}{x}$ cannot generally satisfy all five of them without having boundary layers. This in turn forces the fast vector $\underset{\sim}{y}$ to become unbounded like $O(1/\varepsilon)$ at

the endpoints. Thus, the solution cannot have an asymptotic expansion of the form of Eq. (2.1); however, an appropriate asymptotic expansion was obtained by Flaherty and O'Malley [11]. We do not repeat those results here, but in order to emphasize the diverse behavior that can occur with nonlinear singularly perturbed problems, we present solutions for x_2, x_3, and y_3 corresponding to each of the boundary conditions (4.2a), (4.2b), and (4.2c) in Figures 1, 2, and 3, respectively.

Our methods apply to problems having boundary conditions (4.2a) and (4.2b) and, in these cases, the orthogonal matrix

$$\underset{\sim}{E}(\underset{\sim}{x}(0),0) = (1+\alpha^2)^{-1/2} \begin{bmatrix} 1 & -|\alpha| \\ |\alpha| & 1 \end{bmatrix} \tag{4.3a}$$

where

$$\alpha^2 = \sec x_3(0) \tag{4.3b}$$

reduces

$$\underset{\sim}{G}(\underset{\sim}{x}(0),0) = \begin{bmatrix} 0 & -1 \\ -\alpha^2 & 0 \end{bmatrix} \tag{4.4}$$

to the Schur form given by (3.1) at $t = 0$ while $\underset{\sim}{E}^T(\underset{\sim}{x}(1),1)$ will reduce $\underset{\sim}{G}(\underset{\sim}{x}(1),1)$ to the form given by (3.7) at $t = 1$.

We solved Eq. (4.1) with conditions (4.2a) and (4.2b) in two ways: (1) using COLSYS to solve the complete problem with continuation from a large

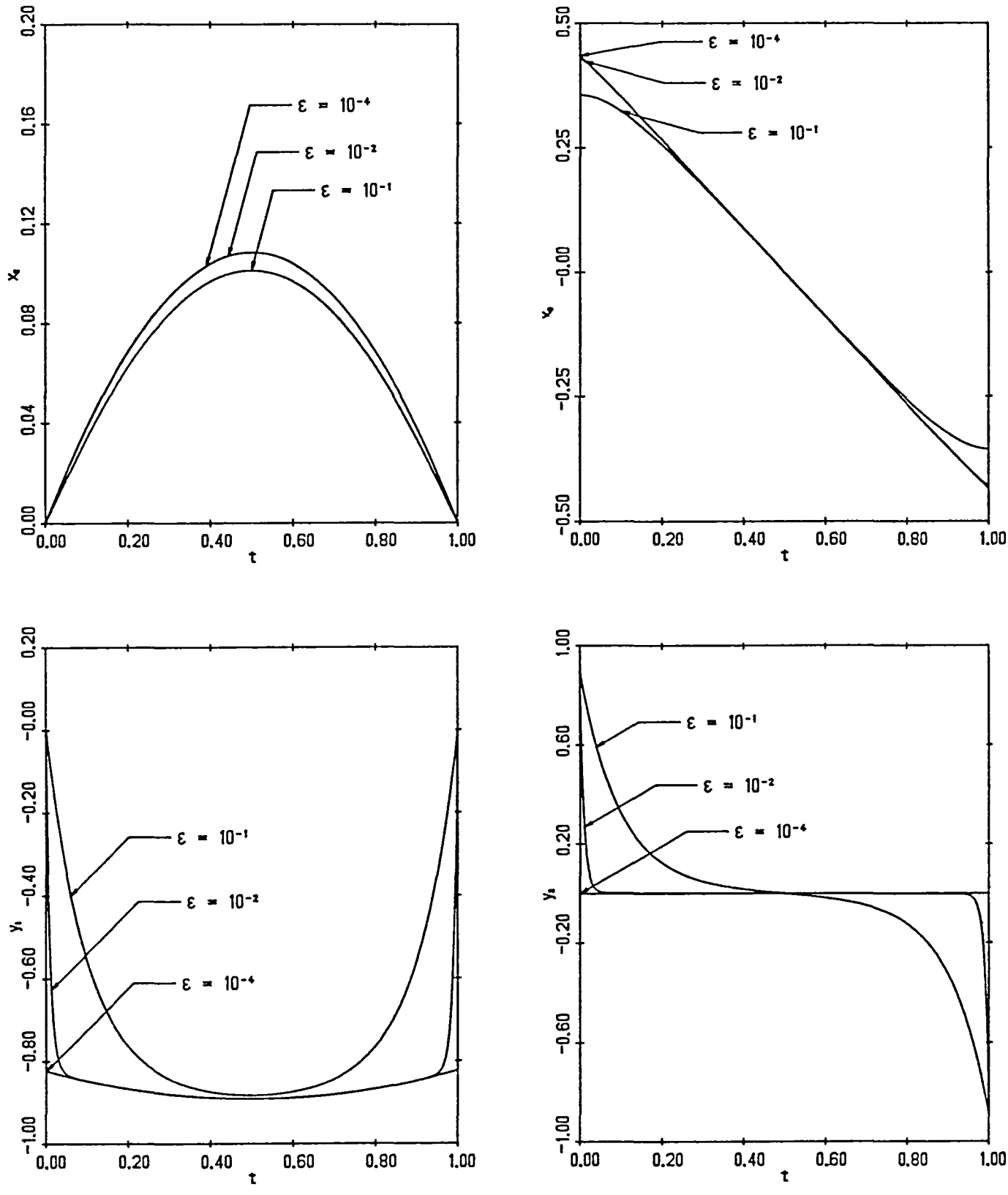

Figure 1. Numerical solution of simply supported beam, Example 1 with boundary conditions given by equation (4.2a).

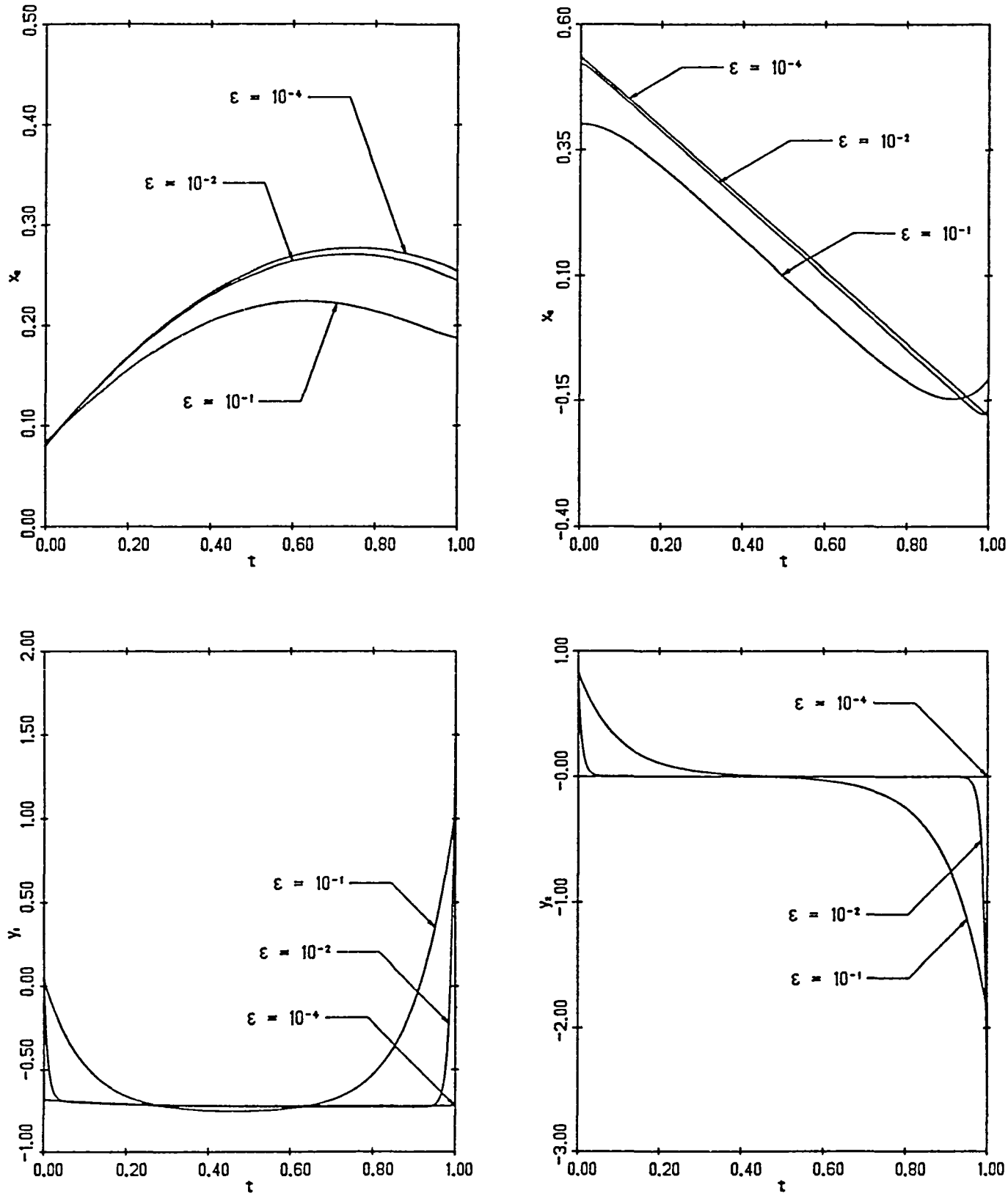

Figure 2. Numerical solution of elastically supported beam, Example 1 with boundary conditions given by equation (4.2b).

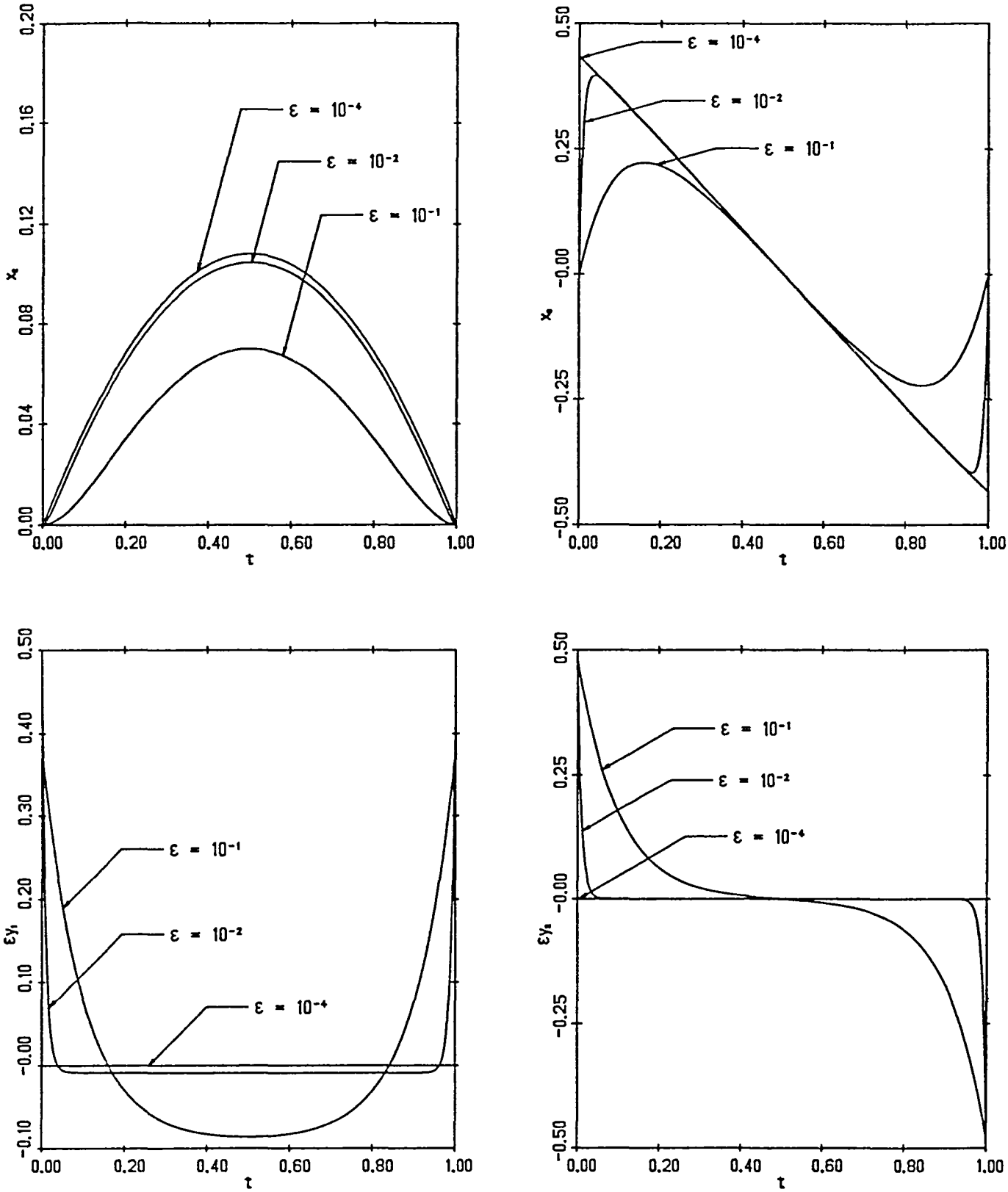

Figure 3. Numerical solution of clamped beam, Example 1 with boundary conditions given by equation (4.2c). Note that y_1 and y_2 are multiplied by ε.

to a small value of ε and (ii) using our code SPCOL to compute an initial asymptotic approximation and to recommend a nonuniform mesh and using this with COLSYS to calculate an improved solution. All calculations were performed using double precision arithmetic on an IBM 3033 computer, two collocation points per subinterval, and an error tolerance δ (cf. Eq. (3.19)) of 10^{-6} for slow variables and 10^{-3} for fast variables.

Our results for the normalized CP times and the number of subintervals (NSUB) that are either used by COLSYS or recommended by SPCOL are shown in Tables 1 and 2 for the simply supported beam and in Tables 3 and 4 for the elastically supported beam. Tables 1 and 3 contain the continuation results and Tables 2 and 4 contain the SPCOL results with COLSYS improvement. The CP times (for all examples) were normalized with respect to the ε sequence in Table 1. Differences between our initial asymptotic approximation and the final solution obtained by COLSYS are shown for $x_2(1/2,\varepsilon)$ and $y_2(0,\varepsilon)$ for the simply supported beam in Table 5 and for $x_3(0,\varepsilon)$ and $y_2(0,\varepsilon)$ for the elastically supported beam in Table 6. All of the differences are decreasing like $O(\varepsilon)$ as expected. Differences that are recorded as zero are less than 10^{-8}.

The results reported in these Tables need some additional explanation. The number of subintervals and CP times used with continuation depended heavily on the ε sequence that was used. The results in Tables 1 and 3 are about the best insofar as they gave the smallest total CP time for the sequence. We see in almost every instance that the COLSYS correction is using about twice the number of subintervals that were suggested by SPCOL. This mesh doubling strategy is often used in COLSYS to estimate errors or when the Newton iteration has convergence difficulties. Thus, in some sense our mesh

ε	NSUB	CP	TOTAL CP
10^{-1}	80	6.1	6.1
10^{-2}	72	6.3	12.5
10^{-4}	112	18.4	30.9
10^{-6}	158	27.2	58.1
10^{-8}	254	41.9	100.0

TABLE 1. EXAMPLE 1 WITH SIMPLE SUPPORTS. NUMBER OF SUB-INTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY COLSYS WITH CONTINUATION IN ε. TOTAL CP IS THE ACCUMULATED TIME FOR THE ε SEQUENCE.

ε	SPCOL		COLSYS CORRECTION		TOTAL CP
	NSUB	CP	NSUB	CP	
10^{-1}	20	1.3	80	5.7	7.0
10^{-2}	28	1.3	112	8.7	10.0
10^{-4}	34	1.3	136	9.0	10.3
10^{-8}	35	1.3	92	9.3	10.6

TABLE 2. EXAMPLE 1 WITH SIMPLE SUPPORTS. NUMBER OF SUB-INTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY SPCOL AND TO IMPROVE IT BY COLSYS. THE CP TIMES INCLUDE THE TIME TO CALCULATE THE REDUCED SOLUTION, WHICH WAS 1.3 TIME UNITS. TOTAL CP IS THE SUM OF THE SPCOL CP AND THE COLSYS CP.

ε	NSUB	CP	TOTAL CP
10^{-1}	80	6.9	6.9
10^{-2}	78	6.3	14.6
10^{-4}	78	16.8	31.4
10^{-6}	156	38.3	69.7
10^{-8}	100	16.4	86.1

TABLE 3. EXAMPLE 1 WITH ELASTIC SUPPORTS. NUMBER OF SUB-INTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY COLSYS WITH CONTINUATION IN ε. TOTAL CP IS THE ACCUMULATED TIME FOR THE ε SEQUENCE.

ε	SPCOL		COLSYS CORRECTION		TOTAL CP
	NSUB	CP	NSUB	CP	
10^{-1}	40	3.9	100	10.2	14.1
10^{-2}	47	3.9	94	10.5	14.4
10^{-4}	56	3.9	112	12.8	16.7
10^{-8}	57	3.9	134	16.8	20.7

TABLE 4. EXAMPLE 1 WITH ELASTIC SUPPORTS. NUMBER OF SUB-INTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY SPCOL AND TO IMPROVE IT BY COLSYS. THE CP TIMES INCLUDE THE TIME TO CALCULATE THE REDUCED SOLUTION, WHICH WAS 3.8 TIME UNITS. TOTAL CP IS THE SUM OF THE SPCOL CP AND THE COLSYS CP.

ε	$\Delta x_2(1/2,\varepsilon)$	$\Delta y_2(0,\varepsilon)$
10^{-1}	7.1×10^{-3}	3.2×10^{-2}
10^{-2}	6.7×10^{-5}	3.6×10^{-3}
10^{-4}	0	3.6×10^{-5}
10^{-8}	0	0

TABLE 5. EXAMPLE 1 WITH SIMPLE SUPPORTS. DIFFERENCES BETWEEN SPCOL AND COLSYS SOLUTIONS, i.e., $\Delta(\ ) := |(\ )_{SPCOL} - (\ )_{COLSYS}|$.

ε	$\Delta x_3(0,\varepsilon)$	$\Delta y_2(0,\varepsilon)$
10^{-1}	1.3×10^{-1}	4.2×10^{-2}
10^{-2}	1.4×10^{-2}	5.2×10^{-3}
10^{-4}	1.5×10^{-4}	5.4×10^{-5}
10^{-8}	0	0

TABLE 6. EXAMPLE 1 WITH ELASTIC SUPPORTS. DIFFERENCES BETWEEN SPCOL AND COLSYS SOLUTIONS, i.e., $\Delta(\ ) := |(\ )_{SPCOL} - (\ )_{COLSYS}|$.

strategy is doing about as well as can be expected; however, it seems that fewer points should be necessary. We tried placing the subintervals according to a pointwise error criteria, as suggested by Ascher and Weiss [2, 3, 4], rather than the global criteria used in Eq. (3.19), but this gave very similar results (cf. Flaherty and O'Malley [12]). We also tried suggesting an initial mesh to COLSYS that consisted of every other point in the mesh suggested by SPCOL. This is clearly a risky strategy, since collocation at the Gauss-Legendre points is known to produce oscillations unless the mesh is appropriately fine in the boundary layers (cf. Ascher and Weiss [2]). Nevertheless, this did give some improvement for values of $\varepsilon > 10^{-8}$ (cf. Flaherty and O'Malley [12]). Perhaps the results could be improved further by using higher order collocation and/or collocation at the Gauss-Lobatto points as suggested by Ascher and Weiss [2, 3, 4].

We see from Tables 1 to 4 that for $\varepsilon = 10^{-8}$ the SPCOL solution can be computed in less than 5% of the time of the continuation solution and the COLSYS improvement with the SPCOL solution as an initial guess can be computed in less than 24% of the time of the continuation solution for both simple and elastic supports.

Example 2. We consider the third order model problem

$$\dot{x} = 1 - x , \qquad \varepsilon\dot{y}_1 = y_2 , \qquad \varepsilon\dot{y}_2 = \alpha^2(x)y_1 + 8x(1-x) \qquad (4.5a,b,c)$$

with

$$\alpha(x) = 1 + 2x \qquad (4.5d)$$

and the linear boundary conditions

$$x(0,\varepsilon) + y_1(0,\varepsilon) = 0, \quad -\gamma x(0,\varepsilon) + y_2(0,\varepsilon) = 0, \quad x(1,\varepsilon) + y_1(1,\varepsilon) = 0. \quad (4.6)$$

The matrix $\underset{\sim}{G}(x,t)$ for this example is the negative of that given by (4.4) for Example 1 with α now being given by (4.5d). Thus, $\underset{\sim}{G}$ has one negative and one positive eigenvalue provided that $\alpha(x)$ is nonzero and $\underset{\sim}{G}$ may be reduced to Schur form at $t = 0$ using the orthogonal matrix $\underset{\sim}{E}^T(\underset{\sim}{x}(0),0)$ and at $t = 1$ using $\underset{\sim}{E}(\underset{\sim}{x}(1),1)$ (with $\underset{\sim}{E}(\underset{\sim}{x},t)$ given by Eq. (4.3a)).

Flaherty and O'Malley [10] studied this problem and showed that the reduced system is

$$\dot{X}_0 = 1 - X_0, \quad Y_{20} = 0, \quad \alpha^2(X_0)Y_{10} + 8X_0(1-X_0) = 0 \quad (4.7)$$

with the initial condition

$$|\alpha(X_0(0))|[X_0(0) + Y_{10}(0)] - \gamma X_0(0) = 0. \quad (4.8)$$

They show that there are three solutions of (4.7,8) for each value of the constant γ provided that there are no "turning points", i.e., provided that there are no values of $x(t)$ for which $\alpha(x) = 0$ on $0 \le t \le 1$. The three solutions can be characterized by their value of $X_0(0)$ which is determined as

$$X_0(0) = 0, \ \frac{1}{4}[\gamma s - 6 \pm \sqrt{(\gamma s-4)^2 + 48}], \quad s = \operatorname{sgn}(\alpha(X_0(0))). \quad (4.9)$$

For $\gamma = 2$ the three values of $X_0(0)$ are 0, 0.803, and -4.29 and the three corresponding solutions for $y_1(t,\varepsilon)$ are shown in Figure 4. For $X_0(0) = 0$, the initial layer correction $\mu_0(\tau)$ is trivial; however, the other two solutions have initial layer jumps.

It can be easily verified that $\alpha(X_0(t))$ has a zero on $0 \leq t \leq 1$ when $(-3.08 \approx) -3e/2 + 1 \leq X_0(0) \leq -1/2$. In this case (4.5) has a turning point and Y_{10} becomes unbounded. Our theory and methods do not apply in this case; however, if ε is not too small, the solution of (4.5) can be calculated using COLSYS. In order to contrast solutions with and without turning points, we illustrate $y_1(t,\varepsilon)$ for $\gamma = -2$ and $X_0(0) = -2.80$ in Figure 5.

Solutions obtained using SPCOL and the corresponding COLSYS corrections are shown for $\gamma = 2$ and $X_0(0) = 0$, 0.803, and -4.29 in Tables 7, 9, and 11, respectively. The COLSYS correction failed to converge for $\varepsilon \leq 10^{-6}$ when $X_0(0) = 0$ and -4.29. We have no explanation as to why the solution with $X_0(0) = 0.803$ was so much easier to calculate. The relative difference between the SPCOL and COLSYS solutions for $x(1,\varepsilon)$ and $y_2(1,\varepsilon)$ are shown in Table 13 for $\gamma = 2$ and $X_0(0) = -4.29$. These results are typical of those that we obtained for all three solutions.

Using COLSYS with continuation in ε and the default initial guess can find at most one solution, and, for this example, it found the $X_0(0) = 0$ solution. The results of this calculation are shown in Table 8 for $\gamma = 2$. Although several ε sequences were tried, we were unable to obtain convergence for $\varepsilon \leq 10^{-6}$. Again, this situation could possibly be remedied by using collocation at the Gauss-Lobatto points as in Ascher and Weiss [2, 3, 4]. The other two solutions when $\gamma = 2$ can also be calculated using continuation in ε provided that we use a suitable initial guess. Results for the solutions corresponding

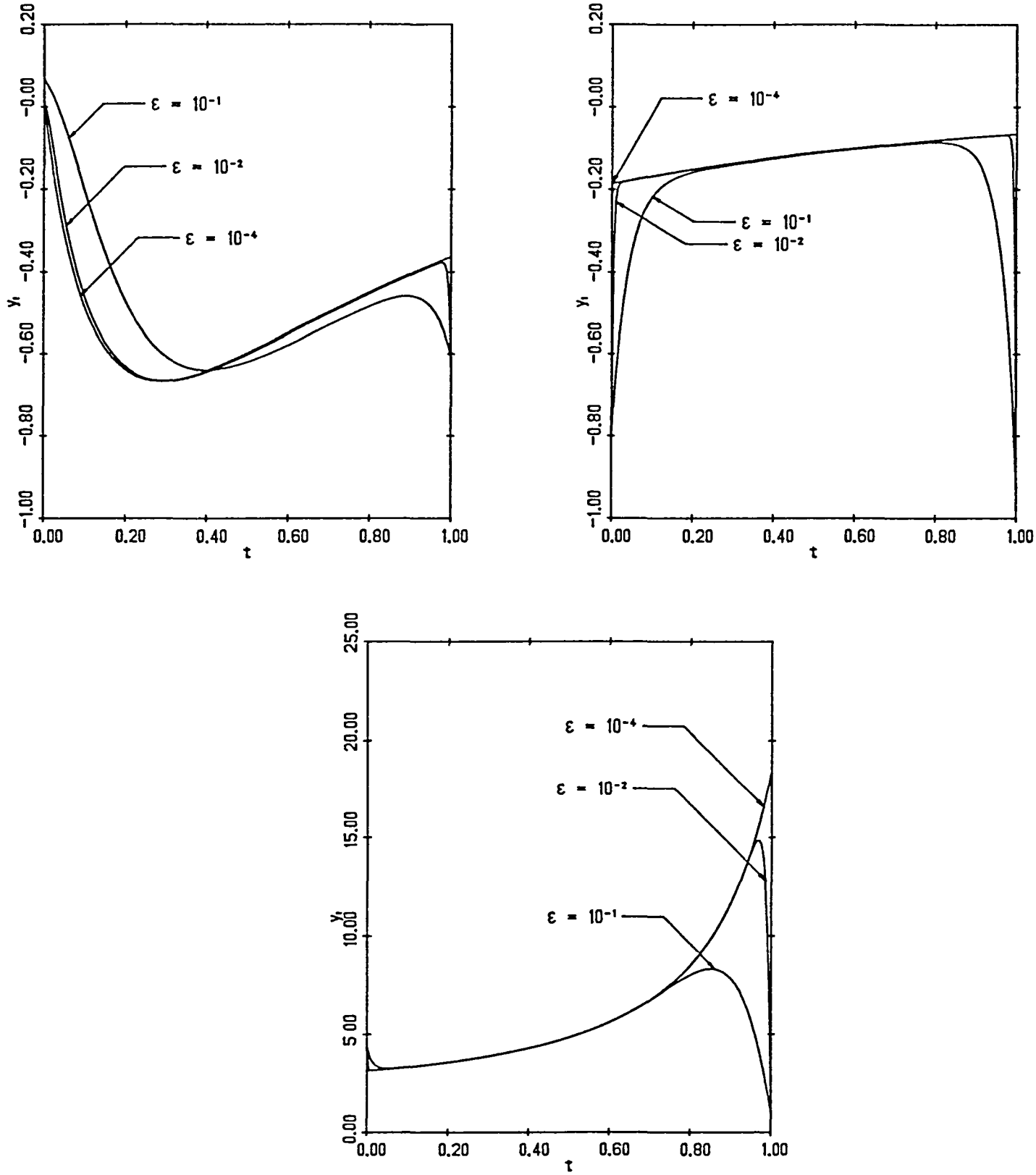

Figure 4. Numerical solution for $y_1(t)$ of Example 2 with $\gamma = 2$ and $X_0(0) = 0$, 0.803, and -4.29.

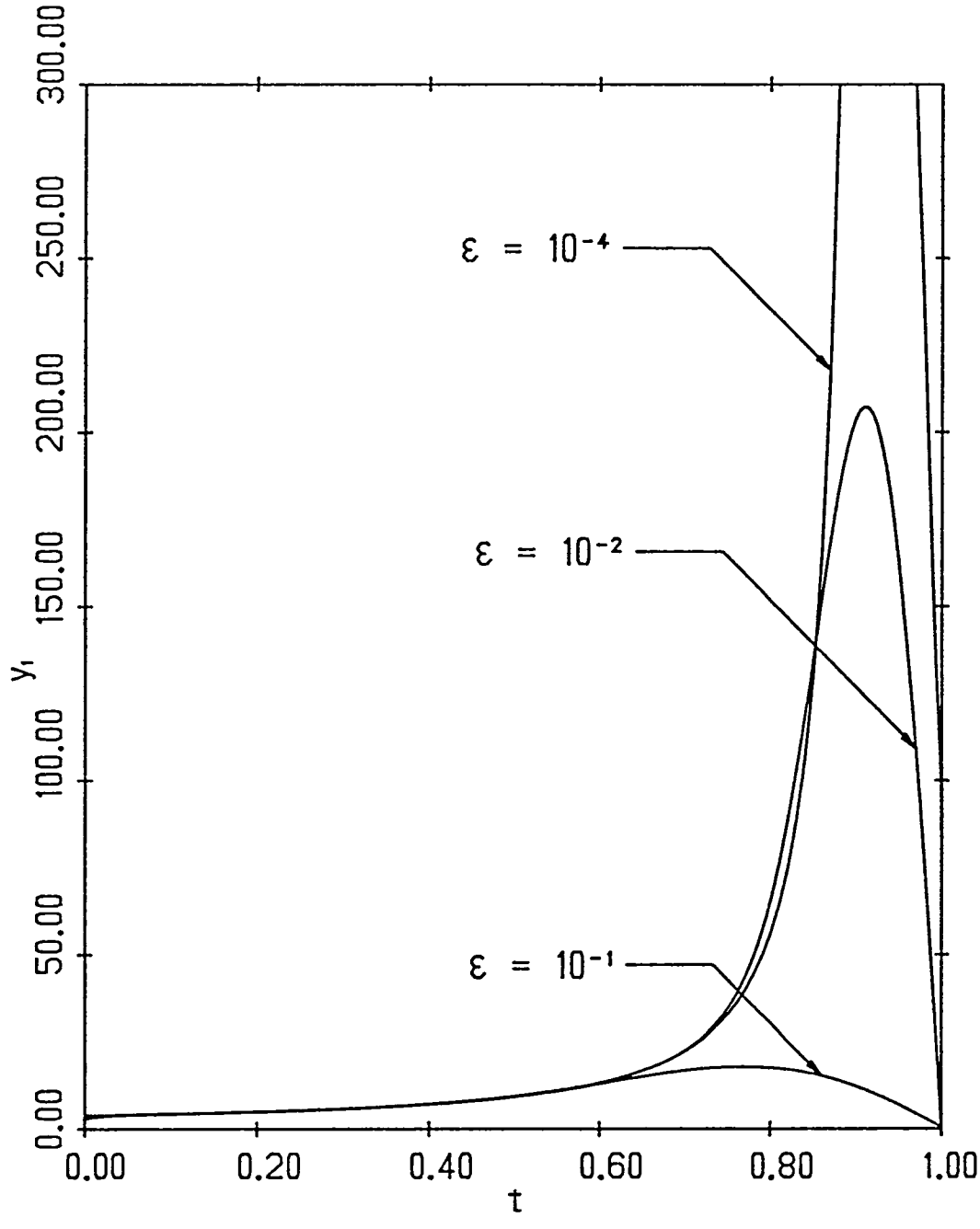

Figure 5. Numerical solution for $y_1(t)$ of Example 2 with $\gamma = 2$ and $X_0(0) = -2.80$.

to $X_0(0) = 0.803$ and -4.29 are presented in Tables 10 and 12, respectively, using continuation with SPCOL furnishing an initial guess. These results seem to point to the possibility of using a combination of asymptotics and continuation to solve singular perturbation problems.

5. Discussion.

We have obtained asymptotic approximations for a restricted class of nonlinear singularly perturbed two-point boundary value problems and have shown how to construct approximate solutions numerically and use them to suggest a nonuniform mesh that may be used as input to a two-point boundary value code in order to calculate improved solutions. Clearly this approach offers some advantages over the more standard technique of continuation in ε steps; however, the picture is far from clear and several questions still remain as to how best to use asymptotic analysis in conjunction with numerical analysis.

In Example 2 of Section 4 we have shown that asymptotic methods may be used to distinguish different solutions in problems having multiple solutions. These asymptotic approximations may be used to provide initial guesses to a two-point boundary value code.

In Example 1 of Section 4 we have shown that unbounded solutions can result from seemingly minor changes in the boundary conditions of singularly perturbed boundary value problems. Other very diverse behaviors can occur when turning point problems are considered (cf., e.g., Kevorkian and Cole [18] or O'Malley [20]). Since phenomena cannot easily be predicted, a sensible

ε	SPCOL		COLSYS CORRECTION		
	NSUB	CP	NSUB	CP	TOTAL CP
10^{-1}	40	0.6	88	6.3	6.9
10^{-2}	44	0.6	88	6.2	6.8
10^{-4}	47	0.6	192	18.2	18.8
10^{-6}	47	0.6	failed		

TABLE 7. EXAMPLE 2 WITH $\gamma = 2$ AND $X_0(0) = 0$. NUMBER OF SUBINTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY SPCOL AND TO IMPROVE IT BY COLSYS. THE CP TIMES INCLUDE THE TIME TO CALCULATE THE REDUCED SOLUTION, WHICH WAS 0.5 TIME UNITS. TOTAL CP IS THE SUM OF THE SPCOL CP AND THE COLSYS CP.

ε	NSUB	CP	TOTAL CP
10^{-1}	40	1.8	1.8
10^{-2}	44	3.3	5.2
10^{-4}	264	13.4	18.6
10^{-5}	372	20.2	38.7
10^{-6}	failed		

TABLE 8. EXAMPLE 2 WITH $\gamma = 2$ and $X_0(0) = 0$. NUMBER OF SUBINTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY COLSYS WITH CONTINUATION IN ε FROM $\varepsilon = 10^{-1}$. THE DEFAULT INITIAL GUESS THAT IS PROVIDED IN COLSYS WAS USED TO START THE CONTINUATION SEQUENCE. TOTAL CP IS THE ACCUMULATED TIME FOR THE SEQUENCE.

ε	SPCOL		COLSYS CORRECTION		TOTAL CP
	NSUB	CP	NSUB	CP	
10^{-1}	42	1.5	42	3.0	4.5
10^{-2}	52	1.6	52	3.0	4.6
10^{-4}	57	1.6	58	2.6	4.2
10^{-6}	57	1.6	114	10.9	12.5

TABLE 9. EXAMPLE 2 WITH $\gamma = 2$ AND $X_0 = 0.803$. NUMBER OF SUBINTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY SPCOL AND TO IMPROVE IT BY COLSYS. THE CP TIMES INCLUDE THE TIME TO CALCULATE THE REDUCED SOLUTION, WHICH WAS 1.5 TIME UNITS. TOTAL CP IS THE SUM OF THE SPCOL CP AND THE COLSYS CP.

ε	NSUB	CP	TOTAL CP
10^{-4}	58	2.6	2.6
10^{-5}	58	2.4	5.0
10^{-6}	70	4.3	9.3

TABLE 10. EXAMPLE 2 WITH $\gamma = 2$ AND $X_0(0) = 0.803$. NUMBER OF SUBINTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY COLSYS WITH CONTINUATION IN ε FROM $\varepsilon = 10^{-4}$. TOTAL CP IS THE ACCUMULATED TIME FOR THE SEQUENCE.

ε	SPCOL		COLSYS CORRECTION		
	NSUB	CP	NSUB	CP	TOTAL CP
10^{-1}	44	0.9	62	3.6	4.5
10^{-2}	52	0.9	84	3.8	4.7
10^{-4}	59	0.9	232	15.3	16.2
10^{-6}	59	0.9	failed		

TABLE 11. EXAMPLE 2 WITH $\gamma = 2$ AND $X_0(0) = -4.29$. NUMBER OF SUBINTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY SPCOL AND TO IMPROVE IT BY COLSYS. THE CP TIMES INCLUDE THE TIME TO CALCULATE THE REDUCED SOLUTION, WHICH WAS 0.9 TIME UNITS. TOTAL CP IS THE SUM OF THE SPCOL CP AND THE COLSYS CP.

ε	NSUB	CP	TOTAL CP
10^{-2}	84	3.8	3.8
10^{-4}	168	21.3	25.1
10^{-6}	322	40.8	65.9

TABLE 12. EXAMPLE 2 WITH $\gamma = 2$ AND $X_0(0) = -4.29$. NUMBER OF SUBINTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY COLSYS WITH CONTINUATION IN ε FROM $\varepsilon = 10^{-2}$. TOTAL CP IS THE ACCUMULATED TIME FOR THE SEQUENCE.

ε	$\frac{\Delta x(1,\varepsilon)}{\lvert x(1,\varepsilon)_{COLSYS}\rvert}$	$\frac{\Delta y_2(1,\varepsilon)}{\lvert y_2(1,\varepsilon)_{COLSYS}\rvert}$
10^{-1}	9.7×10^{-3}	2.4×10^{-1}
10^{-2}	9.6×10^{-4}	3.9×10^{-2}
10^{-4}	9.6×10^{-6}	4.3×10^{-4}
10^{-6}	1.0×10^{-7}	4.5×10^{-6}

TABLE 13. EXAMPLE 2 WITH $\gamma = 2$ and $X_0(0) = -4.29$. RELATIVE DIFFERENCE BETWEEN SPCOL AND COLSYS SOLUTIONS WITH $\Delta(\ ) := \lvert(\ )_{SPCOL} - (\ )_{COLSYS}\rvert$.

course to follow is perhaps to use asymptotic and numerical methods in tandem. For example, a rough numerical solution could be obtained for several values of ε which could then be used to suggest the form of an asymptotic solution. The asymptotic approximation could then be used to refine the numerical solution, and so on. It is also possible that singular perturbation theory could be used to construct special methods that are appropriate for specific problems as e.g., in Flaherty and Mathon [9] and Ascher and Weiss [2, 3, 4].

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16 Abstract

We develop numerical procedures for constructing asymptotic solutions of certain nonlinear singularly perturbed vector two-point boundary value problems having boundary layers at one or both endpoints. he asymptotic approximations are generated numerically and can either be used as is or to furnish a general purpose two-point boundary value code with an initial approximation and the nonuniform computational mesh needed for such problems. The procedures are applied to a model problem that has multiple solutions and to problems describing the deformation of a thin nonlinear elastic beam that is resting on an elastic foundation.

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