An Efficient Algorithm for Generating Random Number Pairs Drawn From a Bivariate Normal Distribution
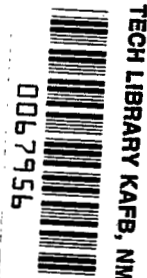
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An Efficient Algorithm for Generating Random Number Pairs Drawn From a Bivariate Normal Distribution

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ACKNOWLEDGMENT

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TECHNICAL PAPER

AN EFFICIENT ALGORITHM FOR GENERATING RANDOM NUMBER PAIRS DRAWN FROM A BIVARIATE NORMAL DISTRIBUTION

INTRODUCTION

An efficient algorithm was devised to generate random numbers drawn from a bivariate normal distribution with any desired values of the two means, two standard deviations, and correlation coefficient. A sequence of pairs of uniformly distributed random numbers are transformed to obtain a sequence of random number pairs which are bivariate normally distributed. The only accuracy limitations are in the quality of the uniform random number generation routine, and in the ability of the computer to perform exact calculations.

The present technique is a special case of a general method described by Scheuer and Stoller [1] who devised a method for generating sequences of random n-vectors drawn from a multivariate normal distribution with any desired covariance matrix. For an n-vector a sequence of n independent normal variates was generated. A linear transformation was applied to each of the n-vectors of the sequence, and the result was a sequence with the desired properties. Since the transformation was linear the transformed sequence was also linear, but the vector components are no longer uncorrelated. By judicious selection of the transformation matrix, the desired covariance matrix was obtained. For the present case, the linear transformation involves multiplication by scale factors and rotation.

MATHEMATICAL DEVELOPMENT

The bivariate normal density function with zero means is given by:

$$f_{x,y}(x,y) = \frac{1}{2\pi \, \sigma_x \, \sigma_y \sqrt{1-r^2}} \, \exp\left\{\frac{1}{2\sqrt{1-r^2}}\left[\frac{x^2}{\sigma_x{}^2} - \frac{2rxy}{\sigma_x \, \sigma_y} + \frac{y^2}{\sigma_y{}^2}\right]\right\} \quad , \tag{1}$$

where

$$\sigma_x{}^2 = E(x^2)$$

$$\sigma_y{}^2 = E(y^2)$$

$$r = E(xy)/\sigma_x \sigma_y$$

and E( ) means the expected value of the variable in parenthesis. The means can be set to zero without loss of generality because transforming to variables with nonzero means involves a simple addition.

The basis for the algorithm is the generation of a sequence of independent normally distributed random pairs (x',y') (being independent the correlation coefficient r is equal to zero) which are transformed to obtain a new sequence (x,y) which is normally distributed with the desired standard deviations and correlation coefficients. For nonzero means, the desired means are then added to each pair (x,y). Because rotation, multiplication by scale factors, and addition constitute a linear transformation, the resulting pairs will be normally distributed. Any desired method of generating (x',y') can be used, but for convenience, a method described by Box and Muller [2] was utilized. Other methods involving the central limit theorem and rational approximation would be acceptable. Some of these techniques are described by Howell and Rheinfurth [3]. Rheinfurth [4] has also pointed out that the Box and Muller method may have some nonzero values of serial correlation possibly resulting from function evaluation errors or other computer arithmetic inaccuracy. In any event any normally accepted method for generating random normal deviates should work for the current algorithm.

The technique of Box and Muller generates a pair of independent normal deviates by means of the following transformation:

$$x'' = (-2 \ln u_1)^{-1/2} \sin 2\pi u_2$$

$$y'' = (-2 \ln u_1)^{-1/2} \cos 2\pi u_2 \quad . \tag{2}$$

u_1 and u_2 are a pair of random variates uniformly distributed between 0 and 1. The inverse transformations are given by:

$$u_1 = \exp[-(x''^2 + y''^2)/2]$$

$$u_2 = \arctan(y''/x'')/(2\pi) \quad . \tag{3}$$

The Jacobian of these inverse transformations is:

$$J[(u_1,u_2)/(x'',y'')] = \frac{-\exp[-(x''^2 + y''^2)/2]}{2\pi} \quad . \tag{4}$$

The joint distribution of x'' and y'' is given by:

$$f_{x'',y''}(x'',y'') = f_{u_1,u_2} J[(u_1,u_2)/(x'',y'')] \tag{5}$$

or

$$f_{x'',y''}(x'',y'') = 1/(\sqrt{2\pi}) \exp(-x''^2/2) \, 1/(\sqrt{2\pi}) \exp(-y''^2/2) \quad . \tag{6}$$

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From the above equation it is obvious that x'' and y'' are independent normally distributed variables with zero means and unit standard deviations.

The transformations in theory are exact. In practice computer errors of arithmetic, function evaluation, etc. create some inaccuracies which are believed to be small.

Equation (2) generates a sequence of normally distributed random number pairs. Transformation to the desired sequence requires three simple operations. The first is the multiplication given in the following equation:

$$x' = \sigma_{x'} x''$$

$$y' = \sigma_{y'} y''$$
(7)

The standard deviations of x' and y' are $\sigma_{x'}$ and $\sigma_{y'}$, respectively. The actual values of $\sigma_{x'}$ and $\sigma_{y'}$ are yet to be determined.

The next transformation is a rotation. Figure 1 illustrates the problem. x and y are given by the rotational transformation:

$$x = x' \cos \theta - y' \sin \theta$$
(8a)

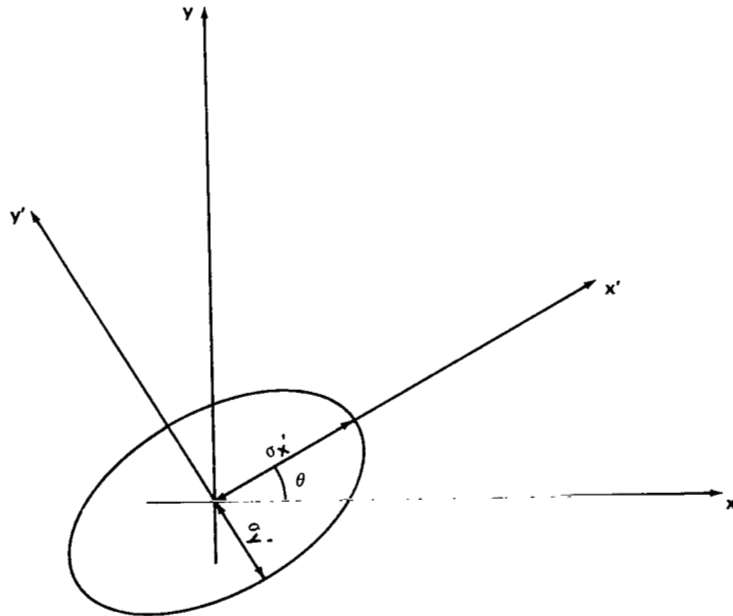$$y = x' \sin \theta + y' \cos \theta \quad .$$
(8b)

Figure 1. Rotation of axes transformation.

The inverse transformation is:

$$x' = x \cos \theta + y \sin \theta \qquad (9a)$$

$$y' = -x \sin \theta + y \cos \theta \quad . \qquad (9b)$$

Multiplying (9a) by (9b) and taking expected values gives:

$$\sigma_{x'} \sigma_{y'} r' = (\sigma_y^2 - \sigma_x^2) \cos \theta \sin \theta + \sigma_x \sigma_y r (\cos \theta - \sin \theta) \quad . \qquad (10)$$

r' is the rotated correlation coefficient and is therefore zero. With the left hand side of (10) set to zero the equation can be solved for θ in terms of the known quantities σ_x, σ_y, and r. The result is:

$$\theta = (1/2) \arctan [2r \sigma_x \sigma_y/(\sigma_x^2 - \sigma_y^2)] \quad . \qquad (11)$$

Now that θ is known, expressions for the unknown quantities $\sigma_{x'}$ and $\sigma_{y'}$ in equation (7) are required. Multiplying equations (9a) and (9b) together and taking expected values yields the following results:

$$\sigma_{x'}^2 = \sigma_x^2 \cos^2 \theta + 2r \sigma_x \sigma_y \cos \theta \sin \theta + \sigma_y^2 \sin^2 \theta \qquad (12a)$$

$$\sigma_{y'}^2 = \sigma_x^2 \sin^2 \theta - 2r \sigma_x \sigma_y \cos \theta \sin \theta + \sigma_y^2 \cos^2 \theta \quad . \qquad (12b)$$

Notice that $\sigma_{x'}$ and $\sigma_{y'}$ are now expressed in terms of known quantities and can be evaluated. Adding (12a) to (12b) gives the following relation:

$$\sigma_{x'}^2 + \sigma_{y'}^2 = \sigma_x^2 + \sigma_y^2 \quad . \qquad (13)$$

If double angle formulae are substituted into (12a) and (12b) and the equations differentiated with respect to θ and the resulting derivatives set to zero, for both (12a) and (12b), equation (11) results. This means that for the standard deviations $\sigma_{x'}$ and $\sigma_{y'}$ of the rotated variates, extrema occur when x' and y' are independent. If x' corresponds to the probability major axis and y' to the minor axis then the value $\sigma_{x'}$ is the maximum standard deviation for any rotation angle and $\sigma_{y'}$ is a minimum.

From equation (11), θ is known, and from two of the three equations (12a), (12b), and (13), $\sigma_{x'}$ and $\sigma_{y'}$ are known. Each of the three parameters $\sigma_{x'}$, $\sigma_{y'}$, and θ are known in terms of the desired parameters σ_x, σ_y, and r. With this information, (x,y) can be calculated.

All required equations are now at hand, and the desired sequence can be generated. The algorithm for generating the desired variate pairs is summarized below.

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1) Input desired value of the two means, two standard deviations, and correlation coefficient to the program.

2) Use these values to calculate the rotation angle, using (11).

3) Calculate $\sigma_{x'}$ and $\sigma_{y'}$ using equations (12a) and (12b).

4) Generate two random numbers (uniformly distributed) u_1, and u_2.

5) Calculate x'', and y'' from equation (2).

6) Calculate x', and y' from equation (7).

7) Calculate x and y from equation (8a) and (8b).

8) Add the desired mean values to x and y.

9) If another random number pair (x,y) is needed return to step 4; otherwise stop.

TEST OF THE ALGORITHM

The previously described algorithm was tested with the computer program described in the Appendix. The code was written for a Hewlett-Packard 1000 F Series computer. Results are summarized in Tables 1 and 2. Table 1 compares the desired values of σ_x, σ_y, and r with those calculated for the sequence. Each sequence consisted of 1000 number pairs. The error is quite small. The purpose of Table 2 is to show that as the size of the sample increases, the deviations from desired values decrease.

TABLE 1. COMPARISON OF STATISTICS OF BIVARIATE PAIRS WITH THE
DESIRED PARAMETER VALUES (N = 1000 FOR EACH CASE)

σ_x	σ_{xm}	σ_y	σ_{ym}	r	r_m	Δr
1.25	1.2379	1.0	1.0002	0.25	0.2538	0.0038
1.25	1.2376	1.0	0.9996	0.50	0.5000	0.0000
1.25	1.2382	1.0	0.9979	0.75	0.7487	-.0013
1.50	1.4871	1.0	0.9998	0.25	0.2562	0.0062
1.50	1.4863	1.0	1.0001	0.50	0.5018	0.0018
1.50	1.4867	1.0	0.9984	0.75	0.7494	-.0006
2.00	1.9845	1.0	0.9993	0.25	0.2576	0.0076
2.00	1.9836	1.0	1.0002	0.50	0.5035	0.0035
2.00	1.9837	1.0	0.9990	0.75	0.7501	0.0001
3.00	2.9782	1.0	0.9990	0.25	0.2582	0.0082
3.00	2.9775	1.0	1.0002	0.50	0.5045	0.0045
3.00	2.9774	1.0	0.9994	0.75	0.7507	0.0007
4.00	3.9714	1.0	0.9989	0.25	0.2584	0.0084
4.00	3.9710	1.0	1.0002	0.50	0.5047	0.0047
4.00	3.9708	1.0	0.9996	0.75	0.7509	0.0009
5.00	4.9646	1.0	0.9988	0.25	0.2584	0.0084
5.00	4.9642	1.0	1.0002	0.50	0.5049	0.0049
5.00	4.9641	1.0	0.9996	0.75	0.7510	0.0010
10.0	9.9300	1.0	0.9988	0.25	0.2585	0.0085
10.0	9.9298	1.0	1.0001	0.50	0.5050	0.0050
10.0	9.9298	1.0	0.9997	0.75	0.7512	0.0012

TABLE 2. CONVERGENCE OF STATISTICS TO DESIRED VALUES FOR INCREASING NUMBERS OF VARIATES

N	$\Delta\sigma_x$	$\Delta\sigma_y$	Δr
10	0.7962	0.7700	0.1787
100	-0.038	0.0794	0.1267
1000	-.0164	0.0002	0.0035
10000	-.0096	0.0044	0.0023

Figures 2 and 3 show interesting trends in the error of the correlation coefficient. In Figure 2 correlation coefficient error Δr, which is the difference in the calculated correlation and the desired correlation, is plotted versus the desired correlation coefficient. The error decreases from a maximum at r = 0 to near zero at r = ±1. This is for the case σ_x = 2, σ_y = 1. Figure 3 plots the correlation coefficient error versus values of the ratio σ_x/σ_y. Each curve corresponds to a different value of the desired correlation coefficient, r. The regularity of the curves depicted in Figures 2 and 3 was somewhat surprising at first glance, but a relatively simple model was developed which explained the qualitative aspects of the plots.

Returning to the transformations (9a) and (9b), the value of the rotated correlation coefficient, r,, is obtained by multiplying the equations together, taking expected values, and manipulating.

$$r' = \frac{\sin 2\theta \ (\sigma_y{}^2 - \sigma_x{}^2)/2 + \sigma_x\sigma_y \cos 2\theta}{\sigma_{x'} \ \sigma_{y'}} \quad . \tag{14}$$

Values of $\sigma_{x'}$ and $\sigma_{y'}$ are obtained from equations (12a) and (12b). The variation of r' with σ_x, σ_y, θ, and r is shown in Figures 4 and 5. The value of the unrotated correlation coefficient, r, for each of the curves is r' evaluated at θ = 0. For each curve four zeroes of r' are seen. The value of θ corresponding to the zero closest to θ = 0 is the value of θ used in transformations (9a) and (9b). For a positive value of the correlation coefficient, r, the closest zero corresponds to a positive value of θ. For a negative value of r, the corresponding value of θ is negative. The four zeroes of r' are expected since the x' axis will correspond to the positive and negative major and minor axes of the probability ellipse for four different values of θ.

Looking at the curves of Figures 4 and 5 and paying particular attention to the zero-crossing behavior of r', one might expect that the error in r may be related to the value of the derivative of r' at the origin. The x'-y' axes are rotated by the angle to obtain the desired values of σ_x, σ_y, and r. Some errors are involved in the transformation and an error in θ results in an error in r; i.e., r = dr'/dθ $\Delta\theta$. In this model Δr varies in magnitude with dr'/dθ evaluated at the origin.

Differentiating equation (14) with respect to θ and evaluating at θ = 0 gives the following result:

$$\frac{dr}{d\theta}\bigg|_{\theta=0} = (\sigma_y/\sigma_x - \sigma_x/\sigma_y) \ (1 - r^2) \quad . \tag{15}$$

If the ratio σ_x/σ_y is held constant, (15) can explain qualitatively the behavior depicted in Figure 2. The maximum error occurs at r = 0, and minimum at r = ±1 as shown.

Figure 3 is not totally explained by the model, however. The sharp decrease of Δr near $\sigma_x/\sigma_y = 0$ is in good agreement with the model, but as $\sigma_x/\sigma_y \to \infty$ the curves seem to level off approaching a constant value. Equation (15) predicts that the error increases linearly with σ_x/σ_y. The reason for the discrepancy is not known, but the actual behavior is better than predicted.
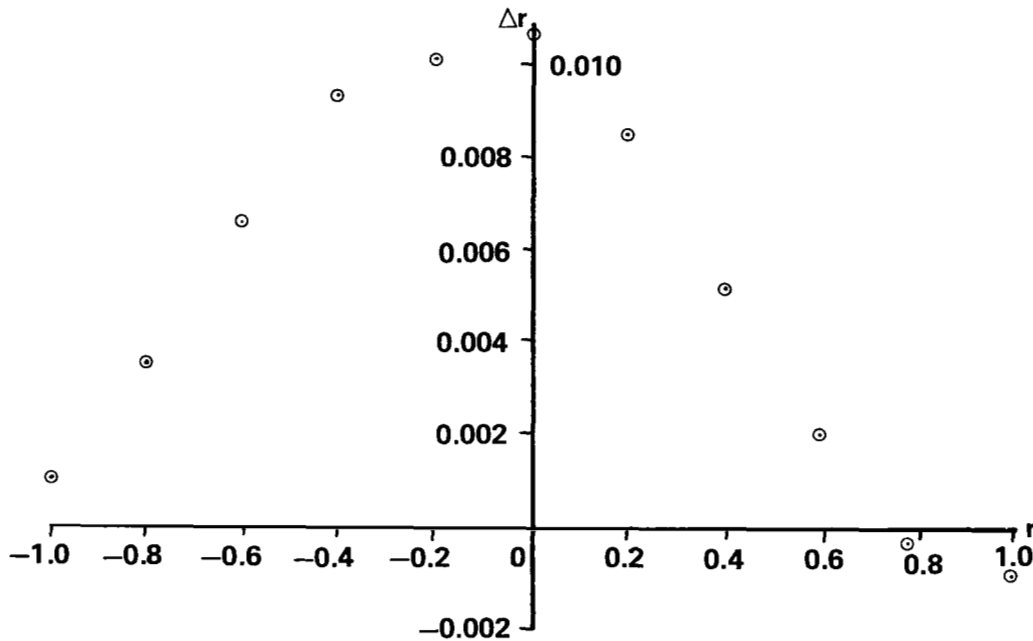
Figure 2. Variation of the absolute error of the correlation coefficient ($\sigma_x = 2$, $\sigma_y = 1$).
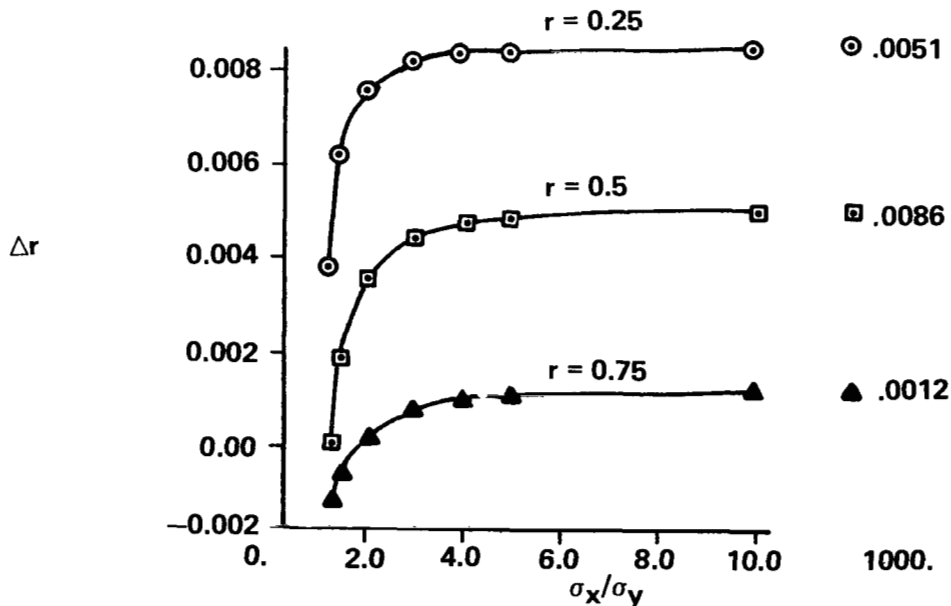
Figure 3. Variation of correlation error with ratio of standard deviations.
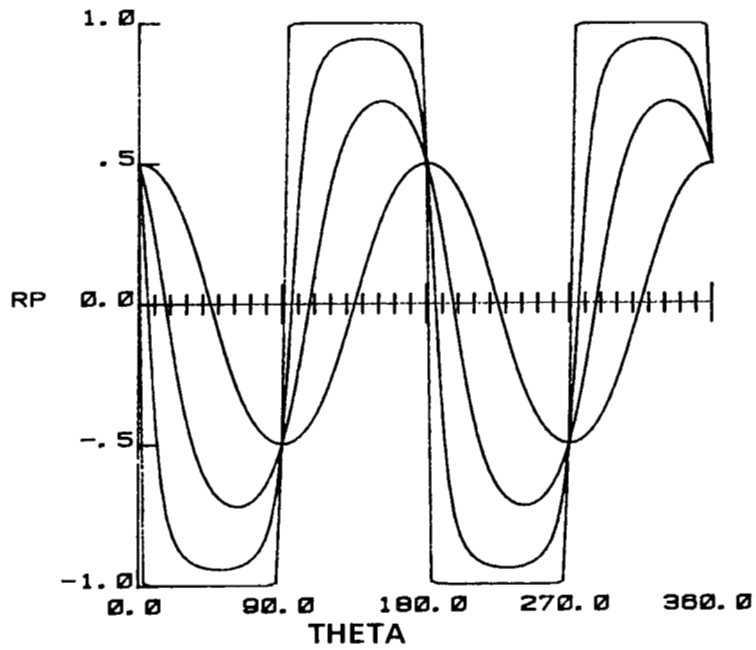
Figure 4. Variation of rotated correlation coefficient with standard deviation ratio $(\sigma_x/\sigma_y = 1, 2, 5, 100; r = 0.5)$.

Figure 5. Variation of unrotated correlation coefficient with rotation angle and unrotated correlation coefficient $(\sigma_x/\sigma_y = 2, r = 0.0, 0.5, 0.9, 1.0)$.

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Qualitatively the increase in Δr with decreasing r as shown in Figure 3 is in agreement with (15). The ratios of corresponding values Δr ($r = 0.25$); Δr ($r = 0.5$); Δr ($r = 0.75$) are not in quantitative agreement, however.

SUMMARY

An algorithm for drawing random number pairs from a bivariate normal distribution with desired means, standard deviations, and correlation coefficien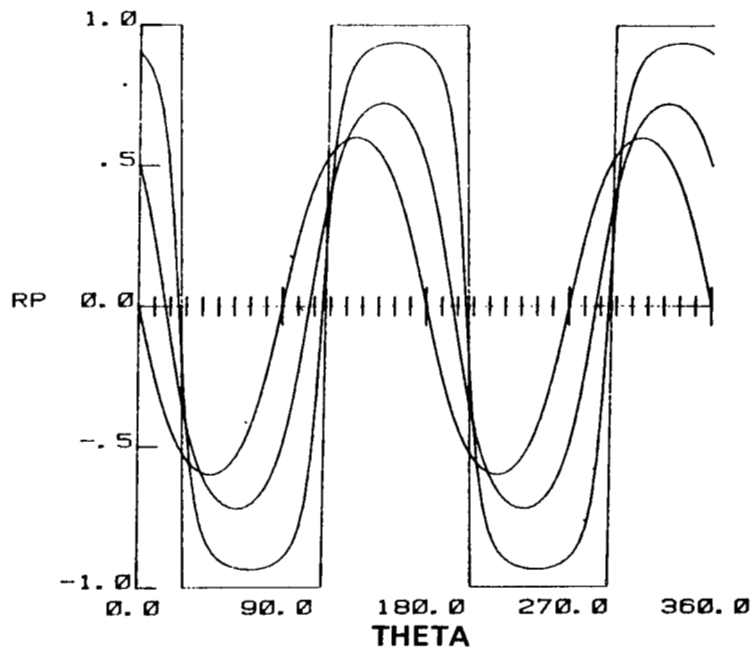t was described. The algorithm is efficient in the sense that a sequence of uniformly distributed random number pairs is converted into a sequence of the same number of bivariate normally distributed random number pairs. In other words, for each pair of random numbers the algorithm generates a pair of bivariate normal random numbers.

A FORTRAN program was written to test the algorithm. The algorithm worked well, with some small errors in the parameters. The regularity of the errors seems to discount the possibility that the errors are entirely statistical in nature.

To better understand the errors, a simple model was devised which qualitatively explained the error in the correlation coefficient. The model was in qualitative agreement with most aspects of the error, and when the model was not in agreement, the actual behavior was better than predicted.

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3. Howell, L. W., and Rheinfurth, M. H.: Generation of Pseudo-Random Numbers. NASA Technical Paper 2105, December 1982.

4. Rheinfurth, M. H. of Marshall Space Flight Center's Systems Dynamics Laboratory. Personal Communication.

5. HP 9206A RTE-IVB Terminal User's Reference Manual, Hewlett-Packard Company, Data Systems Division, Cupertino, California, February 1980.

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APPENDIX

FORTRAN PROGRAM FOR GENERATING BIVARIATE
NORMAL RANDOM NUMBERS

The listing is of a FORTRAN program which generates bivariate normal random numbers. The routine RMPAR is an HP RTE routine which is described in Reference 5. RMPAR passes as many as five integer parameters to the program through the array IPAR. The calling sequence is

RU,BIVAR,IPRT1,IPRT2,NRV

where IPRT1 is the list device where the statistical summary of the random number sequence is sent. IPRT2 is the list device where the random variates are sent, and NRV is the number of variate pairs generated. The program is written for interactive input from a CRT terminal which is logical unit number 1.

System routine URAN generates uniformly distributed random numbers, and is described in Reference 6.

```
FTN4X,L
        PROGRAM BIVAR

*   **  ***********************************************************************
*   **  *                                                                     *
*   **  *   THIS PROGRAM GENERATES BIVARIATE NORMAL RANDOM NUMBERS.           *
*   **  *                                                                     *
*   **  *   PROGRAMMER: WARREN CAMPBELL                                       *
*   **  *                                                                     *
*   **  *   CALLING SEQUENCE: RU,BIVAR,IPRT1,IPRT2,NRV                        *
*   **  *   VARIABLE DEFINITION:                                              *
*   **  *       SIGX = DESIRED X STD DEVIATION                                *
*   **  *       SIGY = DESIRED Y STD DEVIATION                                *
*   **  *       R = DESIRED CORRELATION COEFFICIENT                           *
*   **  *       THETA = ANGLE TO ROTATE AXES                                  *
*   **  *       X = X VALUE OF BIVARIATE NORMAL RANDOM NUMBER                 *
*   **  *       Y = Y VALUE OF BIVARIATE NORMAL RANDOM NUMBER                 *
*   **  *       XPR = ROTATED X VALUE (INDEP OF YPR)                          *
*   **  *       YPR = ROTATED Y VALUE (INDEP OF XPR)                          *
*   **  *       IPRT1 = STATISTICS SUMMARY PRINTER NUMBER                     *
*   **  *       IPRT2 = NORMAL VARIATES PRINTER NUMBER                       *
*   **  *       NRV = NUMBER OF RANDOM VARIATES TO BE GENERATED               *
*   **  *                                                                     *
*   **  ***********************************************************************

        DIMENSION IPAR(5)

        CALL RMPAR(IPAR)
*   **  *********************
        IPRT1 = IPAR(1)
        IPRT2 = IPAR(2)
        NRV = IPAR(3)
*   **  *********************
```

```
       TWOPI = 2.*3.141592654
       SUMX = 0.
       SUMXX = 0.
       SUMY = 0.
       SUMYY = 0.
       SUMXY = 0.
       WRITE(1,9999)
 9999  FORMAT("ENTER DESIRED SIGX,SIGY,R&")
       READ(1,*) SIGX,SIGY,R
       WRITE(IPRT1,9998)
 9998  FORMAT(////)
       WRITE(IPRT1,9990)
 9990  FORMAT(/" ***********************************************************
      +***********************")
       WRITE(IPRT1,9991) NRV
 9991  FORMAT(" NUMBER OF BIVARIATE PAIRS GENERATED IS "IS)
       THETA = 0.5*ATAN(2.*R*SIGX*SIGY/(SIGX**2-SIGY**2))
       A = SIN(THETA)
       B = COS(THETA)
       DTHTA = 360.*THETA/TWOPI
       WRITE(IPRT1,9997) SIGX,SIGY,R,DTHTA
 9997  FORMAT(" SIGX = "F8.3"     SIGY = "F8.3"    R = "F7.4"    THETA = "
      *         F9.4" DEGREES")
       WRITE(IPRT1,9989)
 9989  FORMAT(//)
       SXPR = (B*SIGX)**2 + 2.*R*SIGX*SIGY*A*B + (SIGY*A)**2
       SXPR = SQRT(SXPR)
       SYPR = (A*SIGX)**2 - 2.*R*SIGX*SIGY*A*B + (SIGY*B)**2
       SYPR = SQRT(SYPR)
       WRITE(IPRT1,9996) SXPR,SYPR
 9996  FORMAT(" SXPR = "F9.4"    SYPR = "F9.4)
       WRITE(IPRT1,9989)

       WRITE(IPRT2,9995)
 9995  FORMAT("          I              X              Y              SUMXX
      *        SUMYY          SUMXY")

       DO 100 I=1,NRV
          U1 = URAN(1)
          U2 = URAN(1)*TWOPI
          TEMP = SQRT(-2.*ALOG(U1))
          XPR = TEMP*SIN(U2)*SXPR
          YPR = TEMP*COS(U2)*SYPR
          X = XPR*B - YPR*A
          Y = XPR*A +YPR*B
          SUMX = SUMX + X
          SUMY = SUMY + Y
          SUMXX = SUMXX + X*X
          SUMYY = SUMYY + Y*Y
          SUMXY = SUMXY + X*Y
          WRITE(IPRT2,9994) I,X,Y,SUMXX,SUMYY,SUMXY
 9994     FORMAT(5X,I5,5(5X,F10.5))
 100   CONTINUE
```

```
          XBAR  =  SUMX/NRV
          YBAR  =  SUMY/NRV
          STDX  =  SQRT(SUMXX/(NRV  -  1.))
          STDY  =  SQRT(SUMYY/(NRV  -  1.))
          RM  =  SUMXY/(NRV*STDX*STDY)
          WRITE(IPRT1,9989)
          WRITE(IPRT1,9993) XBAR,YBAR
  9993    FORMAT(" XBAR  =  "F8.4"   YBAR  =  "F8.4)
          WRITE(IPRT1,9992) SIGX,SIGY,R,STDX,STDY,RM
  9992    FORMAT(" SIGX="F9.4"   SIGY="F9.4"   R="F7.4"   STDX="F9.4" STDY="
         *          F9.4"   RM="F7.4)
          WRITE(IPRT1,9990)
          WRITE(IPRT2,9998)
          STOP
          END
          END$
```

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16. ABSTRACT

An efficient algorithm for generating random number pairs from a bivariate normal distribution was developed. Any desired value of the two means, two standard deviations, and correlation coefficient can be selected. Theoretically the technique is exact and in practice its accuracy is limited only by the quality of the uniform distribution random number generator, inaccuracies in computer function evaluation, and arithmetic. A FORTRAN routine was written to check the algorithm and good accuracy was obtained. Some small errors in the correlation coefficient were observed to vary in a surprisingly regular manner. A simple model was developed which explained the qualitative aspects of the errors.

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