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CONSTITUTIVE RELATIONSHIPS FOR ANISOTROPIC HIGH-TEMPERATURE ALLOYS

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SUMMARY

A constitutive theory is presented for representing the anisotropic viscoplastic behavior of high-temperature alloys that possess directional properties resulting from controlled grain growth or solidification. The theory is an extension of a viscoplastic model that has been applied in structural analyses involving isotropic metals. Anisotropy is introduced through the definition of a vector field that identifies a preferential (solidification) direction at each material point. Following the development of a full multiaxial theory, application is made to homogeneously stressed elements in pure shear and to a uniaxially stressed rectangular block in plane stress with the stress direction oriented at an arbitrary angle with the material direction. It is shown that an additional material parameter introduced to characterize the degree of anisotropy can be determined on the basis of simple creep tests.

INTRODUCTION

The need for greater efficiency in aircraft engines places increasing demands on the high-temperature structural alloys used for engine components. As higher operating temperatures are sought, advanced materials are being developed to meet these increased demands. Good examples are the single crystal (SC) and directionally solidified (DS) polycrystalline materials finding application as turbine airfoil components. An advantage of these materials over conventionally cast alloys is their increased strength (e.g., creep and creep-rupture strength, yield strength, etc.) in a preferential (grain growth) direction, which in the case of a turbine blade can be advantageously oriented radially (centrifugally). Improved creep and creep-fatigue properties result as well as reduced susceptibility to grain boundary corrosion and oxidation.

The directional properties of SC or DS metals render them highly anisotropic relative to conventional alloys. This introduces additional complexity in understanding and mathematically representing their mechanical behavior over and above the already enormous complexities associated with elevated temperature.

Here, the unified constitutive model of Robinson (refs. 1 and 2) that has found application in representing important behavioral features of high-temperature isotropic metals is extended to account for the effects of anisotropy. Each material point is taken to have a uniquely identifiable direction designated by a vector. An extended material body is thus treated as being locally transversely anisotropic although the preferential direction may vary from point to point as represented by a vector field. It is believed that this relatively simple model captures the essence of anisotropy as induced by directional grain growth and solidification without undue complication.

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The isotropic, isothermal form of the Robinson viscoplastic theory is first discussed with emphasis on its derivability from a potential function. Full isotropy is treated by taking the applied and internal stress dependence of the potential function in terms of the principal invariants of the stress tensors. The extension to anisotropy is made by replacing the principal invariants with another set of stress invariants that reflect the appropriate material symmetry.

Following the development of the full multiaxial theory, application is made to simple states of shear stress oriented transverse to and along the preferential material direction. A final application is made to a uniaxially stressed rectangular block of material in plane stress with a uniformly oriented material direction taken at an arbitrary angle with the direction of stress.

SYMBOLS

a_{ij}	components of deviatoric internal stress
d_i	components of unit vector
F	scalar function of stress
f	material function
G	scalar function of stress
g	material function
g_h	hardening function
g_r	recovery function
H	material constant
h	hardening function
I_i	invariants of effective stress
$\mathcal{I}_i$	invariants of internal stress
J_i	principal invariants of effective stress
$\mathcal{J}_i$	principal invariants of internal stress
K	threshold transverse shear stress
K_d	threshold longitudinal shear stress
m	material constant
n	material constant
R	material constant
s	internal stress in transverse shear
$\bar{s}$	internal stress in longitudinal shear
S_{ij}	components of applied deviatoric stress
x_i	coordinate directions
α	uniaxial internal stress

β material constant

$\dot{\gamma}$ shear strain rate

δ_{ij} Kronecker delta

$\dot{\varepsilon}$ uniaxial inelastic strain rate

$\dot{\varepsilon}_{ij}$ components of inelastic strain rate

Ω potential function

σ uniaxial normal stress

σ_{ij} components of applied stress

τ shear stress

Σ_{ij} components of effective stress

ϕ angle between x_1 axis and stress direction

μ material constant

THE ISOTROPIC VISCOPLASTIC MODEL

The flow and evolutionary equations in the Robinson model are taken to be derivable from a potential function Ω of the applied and internal stress. The components of these stress tensors are denoted by σ_{ij} and α_{ij}, respectively. Thus, we have

$$\Omega = \Omega(\sigma_{ij}, \alpha_{ij}). \tag{1}$$

For the sake of simplicity, the present development is restricted to isothermal conditions. Extension to nonisothermal conditions follows the development presented in references 1 and 2.

As moderate hydrostatic stress is known to have essentially no effect on inelastic behavior, the stress dependence is taken in terms of the deviatoric components of the applied stress

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{2}$$

and of the internal stress

$$a_{ij} = \alpha_{ij} - \frac{1}{3}\alpha_{kk}\delta_{ij} \tag{3}$$

where the symbol δ_{ij} denotes the usual Kronecker delta. We further identify

$$\Sigma_{ij} = S_{ij} - a_{ij} \tag{4}$$

as the effective stress. The potential nature of Ω is expressed by

$$\dot{\varepsilon}_{ij} = \frac{\partial\Omega}{\partial\sigma_{ij}} \tag{5}$$

and

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$$-\frac{\dot{a}_{ij}}{h}=\frac{\partial\Omega}{\partial a_{ij}} \tag{6}$$

where $\dot{\varepsilon}_{ij}$ represents the inelastic strain rate and h is a scalar function of the internal stress. Equation (5) is termed the flow law. Equation (6) is termed the evolutionary law.

The appropriateness of equations (5) and (6) has been discussed on physical and thermodynamical grounds by several authors including Rice (ref. 3), Ponter and Leckie (ref. 4), Valanis (ref. 5), and Robinson (ref. 6). Equations (5) and (6) are shown in reference 4 to hold exactly for an individual slip system in a polycrystalline metal deforming at high temperature where a_{ij} is interpreted as the local internal flow stress on the slip plane. Equation (5) remains valid (ref. 3) for a polycrystalline metal where σ_{ij} and $\dot{\varepsilon}_{ij}$ are interpreted as the average stress and inelastic strain rate over a volume of material that is large compared to the crystal size. The derivative in equation (6), however, as applied to an individual slip system, requires constant local stress, whereas in the present context it implies constant average stress. As pointed out in reference 4, it is not generally possible to assume that constant average stress implies constant local stress and, consequently, that equation (6) remains exactly true for a polycrystalline material. Nevertheless, constitutive relationships have been derived from equations (5) and (6) that are consistent generalizations of well established classical equations (refs. 3 and 4) and that are capable of accurately representing important features of high temperature behavior of metals including rate-dependent plasticity, creep, recovery and their interactions (refs. 1 and 2).

The function Ω in the Robinson model can be written as

$$\Omega=K^2\left(\int\frac{1}{2\mu}f(F)dF+\int\frac{R}{H}g(G)dG\right) \tag{7}$$

where the stress dependence enters through the scalar functions

$$F(\Sigma_{ij}) \text{ and } G(a_{ij}) \tag{8}$$

taken as depending on the effective and internal stress, respectively. The functions f and g and the material parameters K, μ, R, and H are assumed known for present purposes; they are determined as described in earlier writings (refs. 1 and 2).

Under full isotropy, the functions F and G can be taken to depend on the principal stress invariants

$$\left.\begin{aligned} J_2&=\frac{1}{2}\Sigma_{ij}\Sigma_{ji}\\ J_3&=\frac{1}{3}\Sigma_{ij}\Sigma_{jk}\Sigma_{ki}\end{aligned}\right\} \tag{9}$$

and

$$\left.\begin{aligned} \mathcal{J}_2 &= \frac{1}{2} a_{ij}a_{ji} \\ \mathcal{J}_3 &= \frac{1}{3} a_{ij}a_{jk}a_{ki} \end{aligned}\right\} \tag{10}$$

In the spirit of von Mises, we retain only J_2 and $\mathcal{J}_2$, quadratic in stress, and take

$$F = \frac{J_2}{K^2} - 1 \tag{11}$$

and

$$G = \frac{\mathcal{J}_2}{K^2} \tag{12}$$

Equation (11) plays the role of a (Bingham) yield condition with K denoting the magnitude of the threshold shear stress below which inelastic deformation does not occur - inelastic strain occurs only for $F > 0$. For our purposes, we treat K as a constant; more generally it is considered a scalar state variable.

The flow and evolutionary equations are determined directly from equations (5) and (6) making use of equations (7) to (12) and taking

$$h = \frac{H}{g_h(G)} \tag{13}$$

in equation (6). The details of this development are given in appendix A. Here we state just the result, i.e., the flow law

$$2\mu\dot{\varepsilon}_{ij} = f(F)\Sigma_{ij} \tag{14}$$

and the evolutionary law

$$\dot{a}_{ij} = \frac{H}{g_h(G)}\,\dot{\varepsilon}_{ij} - Rg_r(G)a_{ij} \tag{15}$$

in which

$$g_r(G) = \frac{g(G)}{g_h(G)} \tag{16}$$

This is essentially the form of the Robinson model for a fully isotropic material and for isothermal conditions. Some important features of the model, such as the accompanying inequalities (refs. 1 and 2), are not expressed or discussed here as they do not pertain directly to the extension to anisotropy.

The evolutionary law (eq. (15)) is of the widely accepted Bailey-Orowan type, which presumes that high-temperature deformation occurs under the action of two simultaneously competing mechanisms, a hardening process proceeding with accumulated deformation (characterized by the first term in eq. (15)) and a recovery term proceeding with time (characterized by the second term). Steady state then corresponds to the condition where the two competing mechanisms balance and $\dot{a}_{ij} = 0$.

In most applications of the theory to date, the function f has been chosen as

$$\left.\begin{aligned} f(F) &= F^n \\ &\text{or} \\ f(F) &= (\sinh F)^n \end{aligned}\right\} \tag{17}$$

and, as suggested by the experimental results of Mitra and McLean (ref. 7),

$$g(G) = G^m \tag{18}$$

$$g_h(G) = G^{\beta} \tag{19}$$

so that

$$g_r(G) = G^{m-\beta} \tag{20}$$

where n, m, and β are constants.

EXTENSION TO ANISOTROPY

The direction of grain growth or solidification at each point in a SC or DS solid can be characterized by a field of unit vectors $d_i(x_k)$. The mechanical behavior at each point must then depend not only on the stress and deformation history at the point but also on the local preferential direction. This requires that dependence on d_i be included in F and G in equation (8). However, as the sense of d_i is immaterial, the dependence is properly taken in terms of the product $d_i d_j$. Thus, we replace equation (8) with

$$F(\Sigma_{ij}, d_i d_j) \text{ and } G(a_{ij}, d_i d_j) \tag{21}$$

As indicated in appendix B, the theory of tensorial invariants (refs. 8 to 10) requires that, for form-invariance under arbitrary rigid-body rotations, F and G must be expressible in terms of the principal invariants of their respective tensorial arguments and invariants involving various products of these tensors. Here, as argued in appendix B, we take the functions F and G as depending on the subset of these invariants,

$$\left.\begin{aligned} I_1 &= \frac{1}{2}\,\Sigma_{ij}\Sigma_{ji} \\ I_2 &= d_i d_j \Sigma_{jk}\Sigma_{ki} \\ I_3 &= \frac{1}{2}\,d_i d_j \Sigma_{ji} \end{aligned}\right\} \tag{22}$$

for F, and

$$\left.\begin{aligned} \mathcal{I}_1 &= \frac{1}{2}\,a_{ij}a_{ji} \\ \mathcal{I}_2 &= d_i d_j a_{jk}a_{ki} \\ \mathcal{I}_3 &= \frac{1}{2}\,d_i d_j a_{ji} \end{aligned}\right\} \tag{23}$$

for G. Thus, we take

$$F = \frac{I_1}{K^2} + \left(\frac{1}{K_d^2} - \frac{1}{K^2}\right)\left(I_2 - I_3^2\right) - 1 \tag{24}$$

$$G = \frac{\mathcal{I}_1}{K^2} + \left(\frac{1}{K_d^2} - \frac{1}{K^2}\right)\left(\mathcal{I}_2 - \mathcal{I}_3^2\right) \tag{25}$$

As in the fully isotropic development we have sought generalizations of a von Mises type theory and have, therefore, restricted our choice of invariant expressions to those quadratic in stress. Analogous to the earlier development, K denotes the threshold (Bingham) shear stress transverse to the preferential material direction and K_d denotes the same for shear along the material direction (fig. 1). For $K = K_d$, indicating no difference in shear strength across and along the direction d_i, equations (24) and (25) reduce to their isotropic counterparts (eqs. (11) and (12)).

As before, the flow and evolutionary equations are obtained from equations (5) and (6), this time by making use of equations (7), (13), and (22 to 25). Again, the details are reserved for appendix A. The resulting flow law is given by

$$2\mu\dot{\varepsilon}_{ij} = f(F)\left[\Sigma_{ij} + \left(\frac{K^2}{K_d^2} - 1\right)\left(d_j d_k \Sigma_{ki} + d_k d_i \Sigma_{jk} - \frac{1}{2}\,d_\ell d_k \Sigma_{k\ell}(\delta_{ij} + d_i d_j)\right)\right] \tag{26}$$

and the evolutionary law by

$$\dot{a}_{ij} = \frac{H}{g_h(G)}\dot{\varepsilon}_{ij} - Rg_r(G)\left[a_{ij} + \left(\frac{K^2}{K_d^2} - 1\right)\left(d_j d_k a_{ki} + d_k d_i a_{jk} - \frac{1}{2} d_\ell d_k a_{k\ell}(\delta_{ij} + d_i d_j)\right)\right] \qquad (27)$$

Note that $\dot{\varepsilon}_{ii} = 0$ and $\dot{a}_{ii} = 0$, the former indicating incompressibility of the inelastic deformation and the latter confirming the deviatoric nature of a_{ij}. As before, when $K = K_d$ equations (26) and (27) reduce to those of the isotropic case (eqs. (14) and (15)).

Recall that the functions f, g and g_h and the material constants K, μ, R, and H are determined just as in the isotropic case. Determination of K_d or alternately the ratio K/K_d is discussed in the following section.

APPLICATIONS

We first consider applications of the foregoing theory to the cases of homogeneously stressed elements in pure shear, transverse (fig. 1(a)) and longitudinal (fig. 1(b)) to the preferential material direction. In each case, the x_1 coordinate direction is aligned with the material direction, i.e., $\underset{\sim}{d} = (1, 0, 0)$. For creep in transverse shear (fig 1(a)), we have

$$\left.\begin{aligned} \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = 0 \\ \sigma_{23} = \tau = \text{const.} \end{aligned}\right\} \qquad (28)$$

and

$$\left.\begin{aligned} \alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha_{12} = \alpha_{13} = 0 \\ \alpha_{23} = s \neq 0 \end{aligned}\right\} \qquad (29)$$

From equation (24) we obtain for F

$$F = \frac{(\tau - s)^2}{K^2} - 1 \qquad (30)$$

which depends only on the parameter K. From equation (26), the flow law for the shear rate component $\dot{\gamma}_{23} \equiv \dot{\gamma}_{tr}$ is

$$\mu\dot{\gamma}_{tr} = \left[\frac{(\tau - s)^2}{K^2} - 1\right]^n (\tau - s) \qquad (31)$$

where we have made use of the first of equations (17). Now for creep under longitudinal shear (fig. 1(b)), we write

$$\left.\begin{aligned} \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \\ \\ \sigma_{12} = \tau = \text{const.} \end{aligned}\right\} \tag{32}$$

and

$$\left.\begin{aligned} \alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha_{13} = \alpha_{23} = 0 \\ \\ \alpha_{12} = \bar{s} \neq 0 \end{aligned}\right\} \tag{33}$$

This time F from equation (24) is

$$F = \frac{(\tau - \bar{s})^2}{K_d^2} - 1 \tag{34}$$

which now depends only on K_d. The flow law for $\dot{\gamma}_{12} \equiv \dot{\gamma}_{\ell o}$ is likewise obtained from equation (26) and is

$$\mu\dot{\gamma}_{\ell o} = \left[\frac{(\tau - \bar{s})^2}{K_d^2} - 1\right]^n \left(\frac{K^2}{K_d^2}\right)(\tau - \bar{s}) \tag{35}$$

With $F >> 0$ and $s \approx \bar{s} \approx 0$ (corresponding to the initial stage of a creep test) we obtain by dividing equation (35) by equation (31) and solving for K^2/K_d^2,

$$\frac{K^2}{K_d^2} = \left(\frac{\dot{\gamma}_{\ell o}}{\dot{\gamma}_{tr}}\right)^{(1/n+1)} \tag{36}$$

As the parameter n is assumed known, the ratio K/K_d is defined by equation (36) in terms of the ratio of creep strain rates along and transverse to the preferential material direction. Simple shear tests of this type can, in principle, be used to determine the ratio K/K_d; however, a more practical method on the basis of uniaxial tests will be suggested in the following paragraphs.

Next consider an application of the theory to that of uniaxial plane stress as depicted in figure 2. The plane stress element lies in the plane $x_3 = 0$ with the tensile stress applied along x_1. The preferential material direction is constant throughout the body and, in this case, makes an angle ϕ with the x_1 axis (fig. 2). The stress components are

$$\left.\begin{aligned} &\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0 \\ &\text{with} \\ &\sigma_{11} = \sigma \neq 0 \end{aligned}\right\} \tag{37}$$

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The unit vector d denoting the material direction is

$$d = (\cos\phi, \sin\phi, 0) \tag{38}$$

F and G from equations (24) and (25) become

$$F = \frac{(\sigma - \alpha)^2}{3K^2}\left[1 + \frac{1}{3}\left(\frac{K^2}{K_d^2} - 1\right)\left(4\cos^2\phi + \sin^2\phi - \frac{1}{4}(2\cos^2\phi - \sin^2\phi)^2\right)\right] - 1 \tag{39}$$

and

$$G = \frac{\alpha^2}{3K^2}\left[1 + \frac{1}{3}\left(\frac{K^2}{K_d^2} - 1\right)\left(4\cos^2\phi + \sin^2\phi - \frac{1}{4}(2\cos^2\phi - \sin^2\phi)^2\right)\right] \tag{40}$$

where $\alpha = \alpha_{11}$ is the uniaxial component of the internal stress α_{ij}. The governing equations for the extensional strain rate $\dot{\epsilon} = \dot{\epsilon}_{11}$ are given by equations (26) and (27) as:

$$\dot{\epsilon} = \frac{1}{3\mu} f(F)(\sigma - \alpha)\left[1 + \frac{1}{4}\left(\frac{K^2}{K_d^2} - 1\right)(4\cos^2\phi + 3\cos^2\phi\sin^2\phi + \sin^2\phi)\right] \tag{41}$$

$$\dot{\alpha} = \frac{3H}{2g_h(G)}\dot{\epsilon} - Rg_r(G)\alpha\left[1 + \frac{1}{4}\left(\frac{K^2}{K_d^2} - 1\right)(4\cos^2\phi + 3\cos^2\phi\sin^2\phi + \sin^2\phi)\right] \tag{42}$$

As expected, the in-plane shear strain rate $\dot{\gamma} = \dot{\gamma}_{12}$ is not generally zero and is given by:

$$\dot{\gamma} = \frac{1}{4\mu} f(F)(\sigma - \alpha)\left(\frac{K^2}{K_d^2} - 1\right) \sin^2\varphi \sin 2\varphi \tag{43}$$

The shear strain rate is zero when the preferential material direction is $\varphi = 0^\circ$ or $\varphi = 90^\circ$ and when the material is isotropic, i.e., $K/K_d = 1$. With $\varphi = 0^\circ$ and $f(F) = F^n$, equations (39) and (41) become

$$F_0 = \frac{(\sigma - \alpha)^2}{3K_d^2} - 1 \tag{44}$$

$$\dot{\epsilon}_0 = \frac{1}{3\mu} F_0^n \left(\frac{K^2}{K_d^2}\right)(\sigma - \alpha) \tag{45}$$

With $\varphi = 90^\circ$, equations (39) and (41) give

$$F_{90} = \frac{(\sigma - \alpha)^2}{3K^2}\left[1 + \frac{1}{4}\left(\frac{K^2}{K_d^2} - 1\right)\right] - 1 \tag{46}$$

$$\dot{\epsilon}_{90} = \frac{1}{3\mu} F_{90}^n \left[1 + \frac{1}{4}\left(\frac{K^2}{K_d^2} - 1\right)\right](\sigma - \alpha) \tag{47}$$

Under constant stress creep conditions with $F \gg 0$, the ratio of <u>initial</u> creep rates corresponding to $\varphi = 0^\circ$ and $\varphi = 90^\circ$ is given by

$$\left(\frac{\dot{\epsilon}_0}{\dot{\epsilon}_{90}}\right) = \left[\frac{\dfrac{K^2}{K_d^2}}{1 + \dfrac{1}{4}\left(\dfrac{K^2}{K_d^2} - 1\right)}\right]^{n+1} \tag{48}$$

solving for K^2/K_d^2 we get

$$\frac{K^2}{K_d^2} = \left[\frac{3\left(\dfrac{\dot{\epsilon}_0}{\dot{\epsilon}_{90}}\right)^{(1/n+1)}}{4 - \left(\dfrac{\dot{\epsilon}_0}{\dot{\epsilon}_{90}}\right)^{(1/n+1)}}\right] \tag{49}$$

As the material parameter n is presumed known, equation (49) allows the determination of the ratio K/K_d from uniaxial tests with stress directed along and transverse to the grain growth or solidification direction. Uniaxial tests conducted with the applied stress at arbitrary angles to the preferential material direction will provide information on which assessments of the present theory can be made.

For a complete elastic-viscoplastic theory, needed for structural analysis, a compatible anisotropic elasticity formulation must be coupled with the present model. This will not be dealt with here but will constitute a topic of subsequent research.

SUMMARY AND CONCLUSIONS

A constitutive theory has been presented for representing the anisotropic viscoplastic behavior of high-temperature alloys that have directional properties resulting from controlled grain growth or solidification. The theory is constructed by defining a vector field that identifies the preferential direction at each material point. This results in a locally transversely anisotropic model with allowance for spatially varying directional properties. The anisotropic theory is based on the isotropic viscoplastic model of Robinson that has already been successfully applied in elevated temperature structural analysis.

Application of the anisotropic theory is made to homogeneously stressed elements in pure shear with the shear direction taken transverse to and along the preferential material direction. These simple applications help to illustrate the physical origin of the pertinent material parameters K and K_d. Application is also made to a uniaxially stressed rectangular block in a state of plane stress with the spatially constant material direction making an arbitrary angle with the stress direction. As expected, the results indicate that shear strain generally develops in the absence of shear stress. In other words, the principal axes of stress and strain are not in alignment as is generally true under conditions of anisotropy.

It is shown that the critical material parameter K_d (or alternately the ratio K/K_d) can be determined on the basis of uniaxial creep tests with the uniaxial stress direction along and transverse to the preferential material direction (grain growth or solidification direction).

For a complete elasto-viscoplastic model, an appropriate elasticity formulation must be coupled with the present model.

APPENDIX A

DERIVATION OF FLOW AND EVOLUTIONARY EQUATIONS

Isotropic Case

We first present the derivation of the flow and evolutionary laws for the fully isotropic case, i.e., leading to equations (14) and (15). From equation (5) we write:

$$\dot{\varepsilon}_{ij} = \frac{\partial\Omega}{\partial\sigma_{ij}} = \frac{\partial\Omega}{\partial F}\frac{dF}{dJ_2}\frac{\partial J_2}{\partial S_{k\ell}}\frac{\partial S_{k\ell}}{\partial\sigma_{ij}} \tag{1A}$$

Making use of equations (7) to (12), we have

$$\frac{\partial\Omega}{\partial F} = \frac{K^2}{2\mu} f(F) \tag{2A}$$

$$\frac{dF}{dJ_2} = \frac{1}{K^2} \tag{3A}$$

$$\frac{\partial J_2}{\partial S_{ij}} = \Sigma_{ij} \tag{4A}$$

and

$$\frac{\partial S_{k\ell}}{\partial\sigma_{ij}} = \delta_{ki}\delta_{\ell j} - \frac{1}{3}\delta_{ij}\delta_{k\ell} \tag{5A}$$

Substitution of equations (2A) to (5A) into equation (1A) leads directly to equation (14),

$$2\mu\dot{\varepsilon}_{ij} = f(F)\Sigma_{ij} \tag{6A}$$

Next, from equation (6) we write:

$$\dot{a}_{ij} = -h\frac{\partial\Omega}{\partial\alpha_{ij}} = -h\left[\frac{\partial\Omega}{\partial F}\frac{dF}{dJ_2}\frac{\partial J_2}{\partial a_{k\ell}} + \frac{\partial\Omega}{\partial G}\frac{dG}{d\mathcal{J}_2}\frac{\partial\mathcal{J}_2}{\partial a_{k\ell}}\right]\frac{\partial a_{k\ell}}{\partial\alpha_{ij}} \tag{7A}$$

where in addition to equations (2A) to (5A), we have

$$\frac{\partial\Omega}{\partial G} = K^2\left(\frac{R}{H}\right)g(G) \tag{8A}$$

$$\frac{dG}{d\mathcal{J}_2} = \frac{1}{K^2} \tag{9A}$$

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$$\frac{\partial J_2}{\partial a_{ij}} = -\Sigma_{ij} \tag{10A}$$

$$\frac{\partial \mathcal{J}_2}{\partial a_{ij}} = a_{ij} \tag{11A}$$

and

$$\frac{\partial a_{k\ell}}{\partial \alpha_{ij}} = \delta_{ki}\delta_{\ell j} - \frac{1}{3}\,\delta_{ij}\delta_{k\ell} \tag{12A}$$

Combining these and using equation (13) gives equation (15),

$$\dot{a}_{ij} = \frac{H}{g_h(G)}\,\dot{\varepsilon}_{ij} - Rg_r(G)a_{ij} \tag{13A}$$

Anisotropic Case

The derivation of equations (26) and (27) from equations (5) and (6) will now be outlined. From equation (5) we write:

$$\dot{\varepsilon}_{ij} = \frac{\partial \Omega}{\partial \sigma_{ij}} = \frac{\partial \Omega}{\partial F}\left[\frac{\partial F}{\partial I_1}\frac{\partial I_1}{\partial S_{k\ell}} + \frac{\partial F}{\partial I_2}\frac{\partial I_2}{\partial S_{k\ell}} + \frac{\partial F}{\partial I_3}\frac{\partial I_3}{\partial S_{k\ell}}\right]\frac{\partial S_{k\ell}}{\partial \sigma_{ij}} \tag{14A}$$

Using equation (7) together with equations (22) to (25) we have (without repeating terms already included in equations (2A) to (12A))

$$\frac{\partial F}{\partial I_1} = \frac{1}{K^2} \tag{15A}$$

$$\frac{\partial I_1}{\partial S_{ij}} = \Sigma_{ij} \tag{16A}$$

$$\frac{\partial F}{\partial I_2} = \left(\frac{1}{K_d^2} - \frac{1}{K^2}\right) \tag{17A}$$

$$\frac{\partial I_2}{\partial S_{ij}} = d_j d_k \Sigma_{ki} + d_k d_i \Sigma_{jk} \tag{18A}$$

$$\frac{\partial F}{\partial I_3} = -\left(\frac{1}{K_d^2} - \frac{1}{K^2}\right) d_i d_j \Sigma_{ji} \tag{19A}$$

$$\frac{\partial I_3}{\partial S_{ij}} = \frac{1}{2} d_j d_i \tag{20A}$$

$$\frac{\partial S_{k\ell}}{\partial \sigma_{ij}} = \delta_{ki}\delta_{\ell j} - \frac{1}{3}\delta_{ij}\delta_{k\ell} \tag{21A}$$

Substituting the appropriate terms from equations (2A) to (12A) and equations (15A) to (21A) into equation (14A) we get equation (26),

$$2\mu\dot{\varepsilon}_{ij} = f(F)\left[\Sigma_{ij} + \left(\frac{K^2}{K_d^2} - 1\right)\left(d_j d_k \Sigma_{ki} + d_k d_i \Sigma_{jk} - \frac{1}{2} d_\ell d_k \Sigma_{k\ell}(\delta_{ij} + d_i d_j)\right)\right] \tag{22A}$$

Finally, we write from equation (6)

$$\dot{a}_{ij} = -h\frac{\partial\Omega}{\partial F}\left[\frac{\partial F}{\partial I_1}\frac{\partial I_1}{\partial a_{k\ell}} + \frac{\partial F}{\partial I_2}\frac{\partial I_2}{\partial a_{k\ell}} + \frac{\partial F}{\partial I_3}\frac{\partial I_3}{\partial a_{k\ell}}\right]\frac{\partial a_{k\ell}}{\partial \alpha_{ij}}$$

$$- h\frac{\partial\Omega}{\partial G}\left[\frac{\partial G}{\partial \mathcal{I}_1}\frac{\partial \mathcal{I}_1}{\partial a_{k\ell}} + \frac{\partial G}{\partial \mathcal{I}_2}\frac{\partial \mathcal{I}_2}{\partial a_{k\ell}} + \frac{\partial G}{\partial \mathcal{I}_3}\frac{\partial \mathcal{I}_3}{\partial a_{k\ell}}\right]\frac{\partial a_{k\ell}}{\partial \alpha_{ij}} \tag{23A}$$

The terms in equation (23A) not previously stated are

$$\frac{\partial I_1}{\partial a_{ij}} = -\Sigma_{ij} \tag{24A}$$

$$\frac{\partial I_2}{\partial a_{ij}} = -\left[d_j d_k \Sigma_{ki} + d_k d_i \Sigma_{jk}\right] \tag{25A}$$

$$\frac{\partial a_{k\ell}}{\partial \alpha_{ij}} = \delta_{ki}\delta_{\ell j} - \frac{1}{3}\delta_{ij}\delta_{k\ell} \tag{26A}$$

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$$\frac{\partial G}{\partial \mathcal{I}_1} = \frac{1}{K^2} \tag{27A}$$

$$\frac{\partial \mathcal{I}_1}{\partial a_{ij}} = a_{ij} \tag{28A}$$

$$\frac{\partial G}{\partial \mathcal{I}_2} = \left(\frac{1}{K_d^2} - \frac{1}{K^2}\right) \tag{29A}$$

$$\frac{\partial \mathcal{I}_2}{\partial a_{ij}} = d_j d_k a_{ki} - d_k d_i a_{jk} \tag{30A}$$

$$\frac{\partial G}{\partial \mathcal{I}_3} = -\left(\frac{1}{K_d^2} - \frac{1}{K^2}\right) d_i d_j a_{ji} \tag{31A}$$

$$\frac{\partial \mathcal{I}_3}{\partial a_{ij}} = \frac{1}{2} d_j d_i \tag{32A}$$

Combining the appropriate terms in equation (23A) and again making use of equation (13) we get equation (27),

$$\dot{a}_{ij} = \frac{H}{g_h(G)} \dot{\epsilon}_{ij} - Rg_r(G) \left[a_{ij} + \left(\frac{K^2}{K_d^2} - 1\right)\left(d_j d_k a_{ki} + d_k d_i a_{jk} - \frac{1}{2} d_\ell d_k a_{k\ell}(\delta_{ij} + d_i d_j)\right)\right] \tag{33A}$$

APPENDIX B

BASIS FOR SELECTION OF INVARIANTS

It follows from the theory of algebraic invariants (refs. 8 to 10) that the scalar function F, specified in equation (21) as a function of the two symmetric tensors

$$\boldsymbol{\Sigma} = [\Sigma_{ij}] \tag{1B}$$

and

$$\mathbf{D} = [d_i d_j] \tag{2B}$$

is form-invariant under arbitrary rigid body rotations if expressed in terms of the invariants

$$\left.\begin{aligned} &\mathrm{tr}\,\mathbf{D} = \mathrm{tr}\,\mathbf{D}^2 = \mathrm{tr}\,\mathbf{D}^3 = 1 \\ &\mathrm{tr}\,\boldsymbol{\Sigma} = 0,\ \mathrm{tr}\,\boldsymbol{\Sigma}^2,\ \mathrm{tr}\,\boldsymbol{\Sigma}^3 \\ &\mathrm{tr}\,\mathbf{D}\boldsymbol{\Sigma} = \mathrm{tr}\,\mathbf{D}^2\boldsymbol{\Sigma},\ \mathrm{tr}\,\mathbf{D}\boldsymbol{\Sigma}^2 \end{aligned}\right\} \tag{3B}$$

A set of nontrivial invariants extracted from equations (3B) is

$$\mathrm{tr}\,\boldsymbol{\Sigma}^2,\ \mathrm{tr}\,\boldsymbol{\Sigma}^3,\ \mathrm{tr}\,\mathbf{D}\boldsymbol{\Sigma} \quad \text{and} \quad \mathrm{tr}\,\mathbf{D}\boldsymbol{\Sigma}^2 \tag{4B}$$

Seeking generalizations of a von Mises type, we limit the dependence of F on combinations of the invariants in equations (4B) that are quadratic in stress, i.e.,

$$\left.\begin{aligned} I_1 &= \frac{1}{2}\,\mathrm{tr}\,\boldsymbol{\Sigma}^2 = \frac{1}{2}\,\Sigma_{ij}\Sigma_{ji} \\ I_2 &= \mathrm{tr}\,\mathbf{D}\boldsymbol{\Sigma}^2 = d_i d_j \Sigma_{jk}\Sigma_{ki} \\ I_3^2 &= \left(\frac{1}{2}\,\mathrm{tr}\,\mathbf{D}\boldsymbol{\Sigma}\right)^2 = \left(\frac{1}{2}\,d_i d_j \Sigma_{ji}\right)^2 \end{aligned}\right\} \tag{5B}$$

In particular, we take

$$F = \frac{I_1}{K^2} + \left(\frac{1}{K_d^2} - \frac{1}{K^2}\right)\left(I_2 - I_3^2\right) - 1 \tag{6B}$$

as expressed in equation (24). A parallel argument applies to the function G. According the equations (21) G depends on the symmetric tensors

$$\mathbf{A} = [a_{ij}] \tag{7B}$$

and

$$\mathbf{D} = [d_i d_j] \tag{8B}$$

and we are led, using arguments similar to the above, to express G in terms of the invariants

$$\left.\begin{aligned} \mathcal{I}_1 &= \frac{1}{2}\,\mathrm{tr}\,\mathbf{A}^2 = \frac{1}{2}\, a_{ij}a_{ji} \\ \mathcal{I}_2 &= \mathrm{tr}\,\mathbf{DA}^2 = d_i d_j a_{jk} a_{ki} \\ \mathcal{I}_3^2 &= \left(\frac{1}{2}\,\mathrm{tr}\mathbf{DA}\right)^2 = \left(\frac{1}{2}\, d_i d_j a_{ji}\right)^2 \end{aligned}\right\} \tag{9B}$$

Specifically, we write

$$G = \frac{\mathcal{I}_1}{K^2} + \left(\frac{1}{K_d^2} - \frac{1}{K^2}\right)\left(\mathcal{I}_2 - \mathcal{I}_3^2\right) \tag{10B}$$

as given by equation (25).

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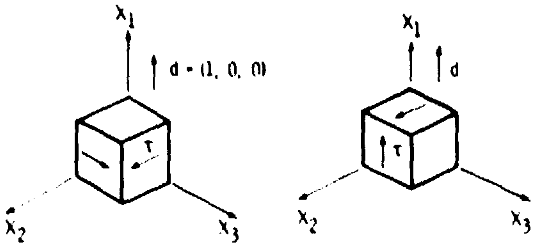

(a) Transverse shear state. (b) Longitudinal shear state.

Figure 1. - Homogeneously stressed elements in pure shear.

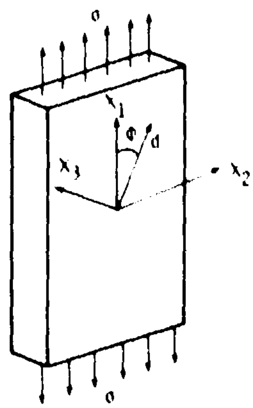

Figure 2. Rectangular uniaxial element in plane stress. Material direction is oriented at angle ϕ with x_1 axis.