NASA

Technical Memorandum 85028

ON EXCITATION OF EARTH'S FREE WOBBLE AND REFERENCE FRAMES

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JUN 1983
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MAY 1983

National Aeronautics and
Space Administration

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TM-85028

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May 1983

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ABSTRACT

In this paper, we study the excitation of the Earth's polar motion in connection with problems that are associated with the diversity of reference frames involved in observations and theoretical computations. Thus, following the dynamics of the Earth's polar motion, the kinematics that relates observations from different reference frames is developed. The conventional procedures of studying the seismic excitation of polar motion are then re-examined accordingly—subject constantly to the question: relative to what reference frame? It is concluded that an inconsistency in reference frames has prevailed in the literature. While this inconsistency is indeed far from trivial, the resultant discrepancy, however, is small for all practical purposes.

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ON EXCITATION OF EARTH'S FREE WOBBLE AND REFERENCE FRAMES

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1. INTRODUCTION

As any geodetic observations made on the surface of the Earth, the polar motion is a relative quantity; and the specification of reference frames is a common requirement for all investigations of the subject. As a matter of fact, now that the precision of measurement is being greatly improved, the incorporation of various astronomical/geodetic reference frames has become a subject of great interest and significance, as exemplified by two IAU colloquia (No. 26, 1975 and No. 56, 1980). However, in the study of seismic excitation of the Earth's polar motion, there has been a lack of consistency in the reference frames used (not as much among different investigators as in each individual study, see Section 5). For example, theoretical computations are made with respect to reference frames that are defined dynamically while having no real-world counterpart from the standpoint of observations. The present paper is an effort to resolve this inconsistency and to assess its implications.

2. DYNAMICS OF THE EARTH'S FREE WOBBLE

We should point out at the outset that in this paper the term 'polar motion' is used synonymously with the free (Chandler) wobble, with no regard to any forced annual wobble which is outside the scope of this study. The basic physical principle governing the Earth's free rotation, in the absence of any external torques, is the conservation of the angular momentum vector $\vec{H}$. In a reference frame which is under a rotation at angular velocity $\vec{\omega}$, it can be expressed as

$$\frac{d}{dt}\vec{H} + \vec{\omega} \times \vec{H} = \vec{0} \qquad (1a)$$

where $\vec{H}$ can be resolved into two parts:

$$\vec{H} = (\text{Moment of inertia tensor}) \cdot \vec{\omega} + (\text{Relative angular momentum}). \qquad (1b)$$

Equation (1) is known as the Liouville equation. In reality, the departure of the Earth's rotational motion from the state of a constant rotation (the latter corresponds to zero polar motion and zero length-of-day variation) is $\lesssim 0(10^{-6})$ (where 0 reads 'on the order of'). As a result, a set of two disjoint, linearized equations of motion for the Earth's free rotation can be derived based on the Liouville equation by means of a first-order perturbation scheme (Munk & MacDonald 1960). They can be written in the following form:

$$(\dot{m} - i\sigma m) + (\dot{c} + i\Omega c) + (\dot{h} + i\Omega h) = 0 \tag{2}$$

$$\left(1 + \frac{\sigma}{\Omega}\right)\dot{m}_3 + \dot{c}_{33} + \dot{h}_3 = 0, \tag{3}$$

where

$\Omega = 2\pi/(1 \text{ day})$

$\sigma = \Omega(C-A)/A$

A,A,C = three principal moments of inertia of the unperturbed (axial-symmetric) Earth, C>A

$< m_1, m_2, 1+m_3 > = \vec{\omega}/\Omega$

$c = c_{13} + ic_{23}$ (4)

$$\begin{pmatrix} 1+c_{11} & c_{12} & c_{13} \\ c_{12} & 1+c_{22} & c_{23} \\ c_{13} & c_{23} & 1+\frac{\sigma}{\Omega}+c_{33} \end{pmatrix} = (\text{moment of inertia})/A$$

$h = h_1 + ih_2$,

$< h_1, h_2, h_3 > = (\text{relative angular momentum})/A\Omega$,

overdot denotes time derivative, and $< >$ denotes Cartesian vector. Equation (2) governs the polar motion m, and equation (3) the length-of-day variation m_3. Note that m, m_3 (the 'm-terms'), c, c_{33} (the 'c-terms'), and h, h_3 (the 'h-terms') are all dimensionless functions of time with infinitesimal magnitudes, $\lesssim 0(10^{-6})$. As usual, we consider the c- and h-terms as the geophysical 'sources' that excite the m-terms which can be observed astronomically. We shall not take into account the 'feedback part' in the c- and h-terms from the m-terms due to the elastic yielding effect. The latter

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effect is responsible for lengthening the Chandler period from 10 months to 14 months, but has no bearing on our forthcoming studies.

3. KINEMATICS OF THE EARTH'S FREE WOBBLE

Equations (2) and (3), as pointed out by Munk & MacDonald (1960), are valid in any (non-inertial) reference frame with respect to which the m-, c-, and h-terms remain infinitesimal. We shall, loosely, call such a reference frame a 'terrestrial frame' (and use the term in a stricter sense later, see Section 4); and we shall always let the origin of our terrestrial frame be coincident with the center of mass of the Earth so that translational motions will not enter into our discussion.

It is obvious that the quantities m-, c-, and h-terms are all frame-dependent, that is, they will be given different quantitative descriptions by observers from different terrestrial frames. Let us now study the kinematic relations between these quantities as viewed from one terrestrial frame (Frame (1)) and that from another terrestrial frame (Frame (2)). Let Frame (2) be related to Frame (1) by the set of three Eulerian angles (ϕ, θ, ψ), as depicted in Figure 1. In general, these Eulerian angles are functions of time, and $\theta(t)$ stays infinitesimal throughout the motion. Now the transformation law (see e.g., Goldstein 1958) for the angular momentum vector $\vec{H}$ is, to first order in θ,

$$\vec{H}^{(2)} = \begin{pmatrix} \cos\Delta & \sin\Delta & \theta\sin\psi \\ -\sin\Delta & \cos\Delta & \theta\cos\psi \\ \theta\sin\phi & -\theta\cos\phi & 1 \end{pmatrix} \vec{H}^{(1)} \tag{5}$$

where the superscript (1) or (2) indicates reference frame, and

$$\Delta = \phi + \psi . \tag{6}$$

Substituting equations (1b) and (4) into (5) and retaining only the first-order terms, we obtain the following basic kinematic relations that relates observations from two different terrestrial frames:

$$m^{(2)} + c^{(2)} + h^{(2)} = \exp(-i\Delta)\,(m^{(1)} + c^{(1)} + h^{(1)}) + i\left(1 + \frac{\sigma}{\Omega}\right)\exp(-i\psi)\,\theta \tag{7}$$

$$\left(1 + \frac{\sigma}{\Omega}\right) m_3^{(2)} + c_{33}^{(2)} + h_3^{(2)} = \left(1 + \frac{\sigma}{\Omega}\right) m_3^{(1)} + c_{33}^{(1)} + h_3^{(1)} . \tag{8}$$

We see that the quantity in equation (8) is frame-independent, or, an invariant. This is not surprising because the actual frame-dependency is of second order. That the quantity is in fact also *time*-independent can be seen readily from equation (3). We shall leave equations (3) and (8), and hence the length-of-day problem, as such without further discussions. Equation (7), on the other hand, is of central importance to forthcoming discussions. Note that, to first order in θ, Δ is simply the rotation angle between the two x-y planes belonging to Frames (1) and (2) (see Figure 1). Hence the first term on the right side of equation (7) represents nothing but the rotation transformation about the z-axis through an angle Δ. It is the second term that has non-trivial implications, as we shall see.

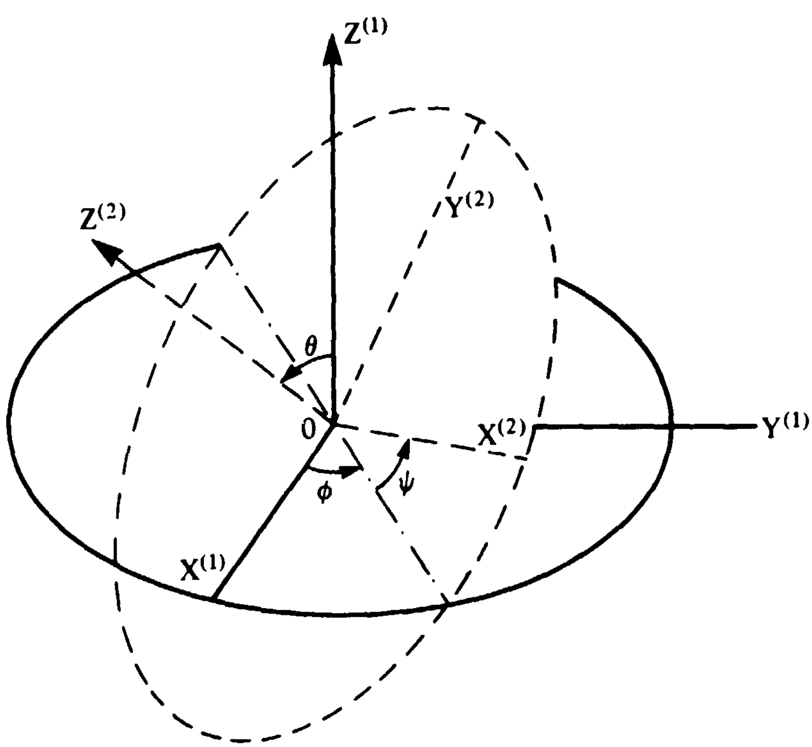

Figure 1. The Eulerian angles (ϕ, θ, ψ) between two reference frames, Frame (1) and Frame (2).

4. EXCITATION OF EARTH'S FREE WOBBLE AND REFERENCE FRAMES

The solutions to the polar motion equation (2) is

$$m(t) = -\exp(i\sigma t)\int_{-\infty}^{t} [i\Omega(c+h) + (\dot{c}+\dot{h})] \exp(-i\sigma\tau)\, d\tau. \quad (9)$$

For simplicity, let us assume that prior to time $t = 0$ all the m-, c-, h-terms as well as their time derivatives are identically zero, and the Earth simply rotates in space at the constant angular velocity $\vec{\Omega} = \Omega\hat{z}$ about its figure axis.

Suppose that at time $t = 0$ the (formerly unperturbed) Earth undergoes a sudden internal redistribution of mass that can be described by an infinitesimal displacement field and that after the moment $t = 0$ the Earth 'freezes' into its perturbed configuration. This event inevitably induces a free wobble, known as the Chandler wobble. Now let us restrict ourselves to a particular subset of terrestrial frames, namely those 'evolve' at $t = 0$ and end up as a body frame of the Earth, so that the event can be described by

$$c(t) = c_0 H(t), \qquad h(t) = h_0 \delta(t) \tag{10}$$

where $H(t)$ is the Heaviside step function and $\delta(t)$ the Dirac delta function. We shall call such an event an H/δ event. Note that the Eulerian angles (ϕ, θ, ψ) between any two terrestrial frames are now time-independent and that the dimension of h_0 is $|\text{time}|$.

Substitute equation (10) into (9), we obtain the polar motion excited by an H/δ event:

$$m(t) = -\left(1 + \frac{\Omega}{\sigma}\right)(c_0 + i\sigma h_0)\exp(i\sigma t) + \frac{\Omega}{\sigma}c_0, \qquad t > 0. \tag{11}$$

Thus, in any given terrestrial frame, after the H/δ event that occurred at $t = 0$, the pole undergoes a prograde, circular motion at the angular rate σ and amplitude $\left(1 + \frac{\Omega}{\sigma}\right)|c_0 + i\sigma h_0|$ about the 'mean pole position' $(\Omega/\sigma)c_0$, starting from the point $-c_0 - \left(1 + \frac{\Omega}{\sigma}\right) i\sigma h_0$.

Now, combining equations (7) and (11), we obtain the following conditions that relate observations from two given terrestrial frames with respect to the H/δ event, true to first order:

$$c_0^{(2)} = \exp(-i\Delta)\, c_0^{(1)} + i\frac{\sigma}{\Omega}\exp(-i\psi)\,\theta \tag{12a}$$

$$h_0^{(2)} = \exp(-i\Delta)\, h_0^{(1)} - \frac{1}{\Omega}\exp(-i\psi)\,\theta. \tag{12b}$$

It is an easy exercise to see that the two (complex) equations (12a, b) are equivalent to the following four (real) expressions:

$$\theta = \Omega \mid c_0^{(2)} h_0^{(1)} - c_0^{(1)} h_0^{(2)} \mid / \mid c_0^{(1)} + i\sigma h_0^{(1)} \mid \quad (13a)$$

$$\psi = \mathrm{Arg}(c_0^{(1)} + i\sigma h_0^{(1)}) - \mathrm{Arg}(c_0^{(2)} h_0^{(1)} - c_0^{(1)} h_0^{(2)}) \quad (13b)$$

$$\phi = \mathrm{Arg}(c_0^{(2)} + i\sigma h_0^{(2)}) + \mathrm{Arg}(c_0^{(2)} h_0^{(1)} - c_0^{(1)} h_0^{(2)}) \quad (13c)$$

$$\mid c_0^{(1)} + i\sigma h_0^{(1)} \mid = \mid c_0^{(2)} + i\sigma h_0^{(2)} \mid, \quad (13d)$$

where Arg denotes the argument of a complex quantity. Equations (13a-c) give the Eulerian angles between two terrestrial frames (1) and (2) induced by an H/δ event in terms of their respective c_0 and h_0. Note that it can be readily shown from equations (12a, b) that the quantity $c_0 + i\sigma h_0$ acts as if it were a vector under the 2-dimensional rotation through the angle Δ. This leads directly to equation (13d), which states that the (real) quantity $|c_0 + i\sigma h_0|$ associated with an H/δ event is an invariant and, hence, so is $\left(1 + \frac{\Omega}{\sigma}\right) |c_0 + i\sigma h_0|$, the amplitude of the polar motion induced by an H/δ event. Thus we see that while observers from two different terrestrial frames do not agree on the direction and magnitude of the static shift of the mean pole, nor on the direction of the instantaneous displacement of the pole that occurs at t = 0, they certainly agree on the amplitude of the polar motion. It implies that, in any terrestrial frame, c_0 and h_0 are not entirely independent quantities. The restriction is imposed by our physical requirement that the Earth 'freezes' after the event.

Now let us study the implication of equations (13a-d) with respect to some specific terrestrial frames. A trivial, but instructive, special case ensues when $\theta = 0$, i.e., when the two frames share the same z-axis. Then equations (12a, b) reduce to the kinematic transformation law for c_0 and h_0, respectively, under the two-dimensional rotation Δ in the x-y plane (see Figure 1).

Two other special cases of terrestrial frame are of interest, namely the principal axes and the Tisserand's (mean) axes (Munk & MacDonald, 1960):

(1) The principal axes (henceforth called the P-frame) correspond to the terrestrial frame in which c = 0. The polar motion induced by the H/δ event (equation 11) reduces to

$$m^{(P)}(t) = -i(\Omega + \sigma)\, h_0^{(P)} \exp(i\sigma t). \tag{14}$$

The z-axis of the P-frame, about which the pole rotates, is the figure axis by definition.

(2) Tisserand's axes (henceforth called the T-frame) are defined as the terrestrial frame in which $h = h_3 = 0$. The polar motion in this case becomes

$$m^{(T)}(t) = \frac{\Omega}{\sigma} c_0^{(T)} - \left(1 + \frac{\Omega}{\sigma}\right) c_0^{(T)} \exp(i\sigma t). \tag{15}$$

We point out here that for the P- and T-frames, equations (13a) and (13d) reduce to

$$\theta_{PT} = \Omega/\sigma\, |c_0^{(T)}| = \Omega\, |h_0^{(P)}|, \tag{16}$$

a relation that will be used in the next section. Figure 2 illustrates, among other things, the polar motion with respect to both reference frames.

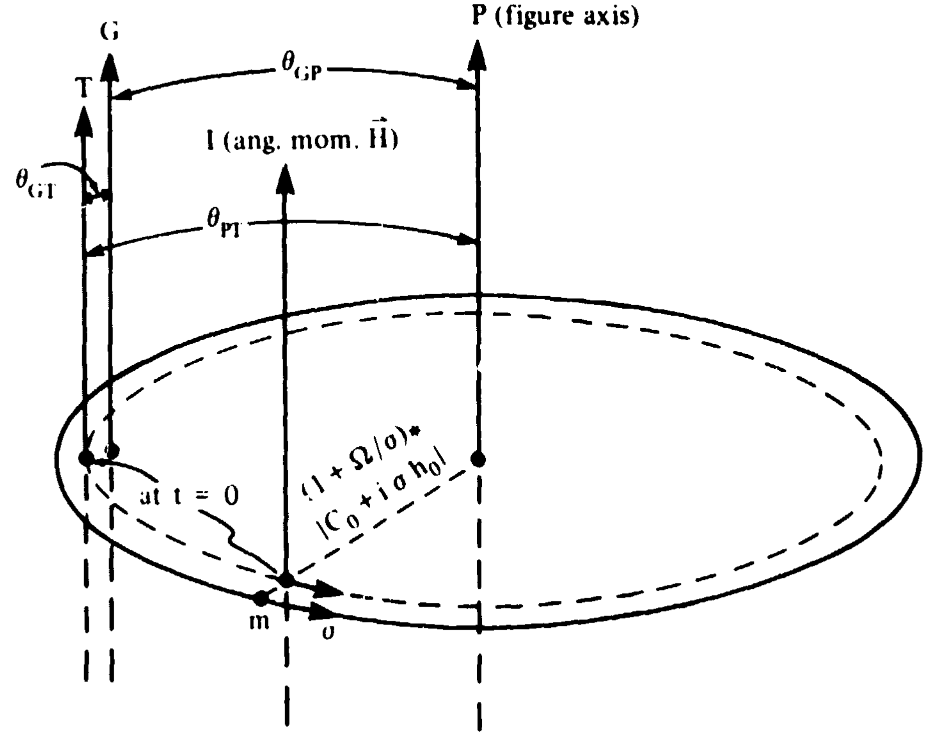

Figure 2. The z-axes of various reference frames and their relationship with the polar motion m. P: principal frame, T: Tisserand's Frame, G: geographic frame, I: invariant frame.

5. ARE COMPUTATIONS AND OBSERVATIONS COMPATIBLE?

The foregoing mathematical treatment is valid with respect to the Earth to an accuracy within 10^{-6}; and an H/δ event is a convenient representation of an earthquake whose duration is generally much shorter than the Chandler period. In fact, in the study of seismic excitation of the polar motion, the conventional procedure connecting changes in the polar motion with the occurrence of major earthquakes has been as follows: (1) compute according to some theoretical formulae the c-term, the change in the product of inertia of the Earth, accompanying a given earthquake faulting with observed fault geometry; (2) neglect the h-term, use the computed c-term and the T-frame equation (15) to obtain the polar motion m; and (3) compare the resultant m with the observed polar motion. This procedure has been used by various investigators; among them, Mansinha & Smylie (1967), Ben-Menahem & Israel (1970), Smylie & Mansinha (1971), Dahlen (1971, 1973), Rice & Chinnery (1972), O'Connell & Dziewonski (1976), Mansinha et. al. (1979). However, no strong conclusions have been drawn to date. While there are indeed aspects that remain uncertain (for example, whether we know enough about the seismic source, or whether the available Earth models are adequate), we shall here examine a fundamental problem associated with the above procedure, namely the compatibility between the polar motion observed astronomically and that computed according to geophysical observations of the seismic source. This problem arises from the diversity of reference frames involved. In particular, here we should bring in the so-called 'geographical frame' (henceforth the G-frame) (Munk & MacDonald 1960). A G-frame is a terrestrial frame defined with respect to a number of 'fixed' points on the surface of the Earth, from which observations are made. Thus, corresponding to the said procedure (with steps (1) and (2) in reverse order), three levels of questions should be raised:

(i) When is the h-term negligible from equation (11)?

(ii) Given a fault geometry, what reference frame is used for the computation of the c-term (and the h-term if it is to be taken into account)?

(iii) Are the two G-frames defined respectively by the seismic network and the astronomical stations compatible?

For question (i), what we should really be concerned with is: In the presence of the c-term, is the h-term negligible *in the G-frame*? Indeed, the neglect of the h-term is certainly valid in the T-frame where σh_0 is zero and the only contribution comes from c_0, while being evidently absurd in the P-frame where the converse is true. (Note also that we have asserted in equation (16) that the magnitudes of $c_0^{(T)}$ and $\sigma h_0^{(P)}$ are equal.) Now, let θ_{GT} and θ_{GP} be, respectively, the departure angle between (the z-axis of) the G-frame and the T-frame, and that between the G-frame and the P-frame (see Figure 2). From equation (13a) we have

$$\theta_{GT} = \Omega \, |h_0^{(G)}| \tag{17a}$$

$$\theta_{GP} = (\Omega/\sigma) \, |c_0^{(G)}| \, . \tag{17b}$$

Therefore $\theta_{GT}/\theta_{GP} = \sigma \, |h_0^{(G)}| / |c_0^{(G)}|$, and we conclude that $\sigma \, |h_0^{(G)}|$ is negligible compared with $|c_0^{(G)}|$ only if

$$\theta_{GT} \ll \theta_{GP} \, , \tag{18}$$

that is, only if the z-axis departure of the G-frame from the T-frame is much smaller than that from the P-frame. In principle, there is no *a priori* reason why this is true because the P- and T-frames are defined dynamically whereas the G-frame has a purely empirical definition. However, quantitatively, as long as the G-frame is not 'ill-defined', it may be argued that the P-frame appears to be very sensitive to any mass redistribution in the Earth in the sense that θ_{GP} has been 'magnified' by the factor Ω/σ (equation 17b), which, in the case of the Earth because of its nearly spherical configuration, is $\sim$0(300). Therefore, for all practical purposes, equation (18) indeed holds (see Figure 2); and it is legitimate to use the T-frame formula (15). Notice that, for the instantaneous displacement $-c_0 - \left(1 + \frac{\Omega}{\sigma}\right) i\sigma h_0$ at $t = 0$, the neglect of the h-term evidently yields totally incorrect results. Fortunately, this quantity is itself $\sim$0(1/300) compared with the polar motion amplitude (see equation 11), and hence no serious error will be introduced in the polar motion.

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Question (ii) is one of strictly theoretical nature. Thus, given a fault geometry, earlier investigations, despite differences in the actual approaches, have all computed the c-term (as presumably would for the h-term ›n a hypothetical non-rotating earth model. The reference frame used is thus the (non-rotating) inertial frame, constrained by the vanishing of the total angular momentum $\vec{H}$, which in the present case is identical to the relative angular momentum h-terms. (The latter fact can be easily seen by letting $\vec{\omega} = \vec{0}$ in equation 1.) In the case of rotating reference frames, the above constraint corresponds to the T-frame, by definition.

It is interesting to point out here a totally different procedure for computing the polar motion m due to a given earthquake, namely the normal-mode approach of Smith (1977) who used the 'invariant frame' (call it the I-frame), a frame that continues to rotate in space at the constant angular velocity $\vec{\Omega}$ regardless of what happens to the Earth. The I-frame is, again, defined dynamically; its z-axis coincides with the constant angular momentum vector $\vec{H}$. But unlike the two previously defined dynamical frames (the P- and T-frames), it is not a (body-fixed) terrestrial frame. Yet, contrary to what we might expect from Section 4, the computed mean pole shift by Smith (1977), relative to the I-frame, are in good agreement with corresponding results by other investigators using the T-frame (see above). This, of course, is not a coincidence. The reason is that, at t = 0, the I-frame is in fact coincident with the T-frame. This can be shown easily by, for example, comparing equation (15) with the Poinsot representation of a rigid body rotation (see e.g., Lambeck 1980). After t = 0, as seen in a terrestrial frame, the I-frame rotates about the figure axis, always keeping pace with the pole position m. Figure 2 summarizes the relation among various reference frames.

Thus, in conclusion, we see that all the computed c-terms, and hence the resultant polar motion m, in the literature to date are in fact what would have been observed in the T-frame. Although they do not in any way correspond to actual observations (from a G-frame), we have asserted above that the difference is small, ~O(1/300), compared with the polar motion m itself. Note further that, in the T-frame under the constraint of vanishing h-terms, it is the relative displacement (rather than the absolute displacement) at the fault, or the corresponding seismic moment, that enters the computation of the c-term. Therefore, in principle, even the small difference between the T- and G-frames can be accounted for provided we can resolve the ambiguity in the absolute displacement as seen from a G-frame. This presumably can be achieved by means of geodetic techniques such as the San Andreas Fault Experiment (SAFE, see Smith et al. 1979) that uses satellite laser ranging to tie fault movements to a network of observatories that defines a G-frame.

The philosophical question (iii) arises because even if we reduced all quantities to a geophysical G-frame, the latter can still, in principle, be different from an astronomical G-frame. This, however, is a much milder problem and can be answered simply by stating that, by increasing the geographical coverage and density of both seismic and astronomical networks, the two thus defined G-frames will in general approach coincidence in a statistical sense.

ACKNOWLEDGMENTS

This work was completed during my tenure at the Goddard Space Flight Center as an NAS/NRC Research Associate.

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