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A simple derivation of the formula to calculate synthetic long-period seismograms in a heterogeneous Earth by normal mode summation

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Summary

A simple modification of Gilbert's (1970) formula to account for slight lateral heterogeneity of the Earth leads to a convenient formula to calculate synthetic long-period seismograms. Partial derivatives are easily calculated, thus the formula is suitable for direct inversion of seismograms for lateral heterogeneity of the Earth.

Derivation

Gilbert(1970), using the classic results due to Rayleigh and Routh, derived a convenient formula to calculate synthetic seismograms by normal mode summation. We show that a simple modification of his approach to account for slight lateral heterogeneity is possible.

The equation of motion is given by

$$(\rho_o + \delta\rho)\partial_t^2 u = (H_o + H)u + f(t, x_s), \tag{1}$$

where ρ_o is density and H_o is an operator appropriate in a spherically symmetric Earth, while $\delta\rho$ and H are their perturbations. The source is described by $f(t, x_s)$. The Laplace transformation with respect to time yields

$$(\rho_o + \delta\rho)p^2\bar{u} = (H_o + H)\bar{u} + \bar{f}(p, x_s), \tag{2}$$

where we used the bar to denote the transformed quantities.

Now we use the eigenfunctions of the spherically symmetric Earth, $\bar{u}_n^{(0)}$ ($n=1,2,...$), to expand $\bar{u}$, where $\bar{u}_n^{(0)}$ satisfies

$$-\rho_o \omega_n^2 \bar{u}_n^{(0)} = H_o \bar{u}_n^{(0)}. \tag{3}$$

The normalization of $\bar{u}_n^{(0)}$ is done by

$$\int_E \rho_o \bar{u}_m^{(0)*} \bar{u}_n^{(0)} dV = \delta_{mn}, \tag{4}$$

where $*$ means complex conjugation. We express $\bar{u}$ by

$$\bar{u} = \sum_n a_n \bar{u}_n^{(0)}. \tag{5}$$

Substitute (5) in (2), multiply $\bar{u}_n^{(0)*}$ and integrate over the whole volume of the Earth. We obtain

$$(p^2+\omega_n^2)a_n = \sum_s a_s (<n|H|s> - p^2<n|\delta\rho|s>) + F_n. \tag{6}$$

where

$$<n|\delta\rho|s> = \int_E \delta\rho \bar{u}_n^{(0)*} \bar{u}_s^{(0)} dV,$$

$$<n|H|s> = \int_E \bar{u}_n^{(0)*} H \bar{u}_s^{(0)} dV, \tag{7}$$

$$F_n = \int_E \bar{u}_n^{(0)*} \bar{f}(p,x_s) dV.$$

Considering that the first two terms on the right-hand side of (6) are small, the first approximation of a_n is given by

$$a_n = \frac{F_n}{(p^2+\omega_n^2)}. \tag{8}$$

Then the next higher approximation is obviously

$$a_n = \sum_s (<n|H|s>-p^2<n|\delta\rho|s>)\frac{F_s}{(p^2+\omega_s^2)(p^2+\omega_n^2)} + \frac{F_n}{(p^2+\omega_n^2)}. \tag{9}$$

We use in this order of approximation in the following derivation.

Formula for the step function type source

If we assume the step function type source, F_n should be replaced by F_n/p.

$$a_n = \sum_s (<n|H|s>-p^2<n|\delta\rho|s>)\frac{F_s}{p(p^2+\omega_s^2)(p^2+\omega_n^2)} + \frac{F_n}{p(p^2+\omega_n^2)}. \tag{10}$$

For a point source described by a second order tensor M_{ij}, F_n is expressed by

$$F_n = \sum_{ij} M_{ij}\varepsilon_{ij},$$

where ε_{ij} is the strain of a mode at the source, as is well known.

Substituting (10) in (5) and inverting to the time domain, we obtain

$$u = \sum_n a_n^{(0)}\left[F_n\frac{(1-\cos\omega_n t)}{\omega_n^2}\right.$$

$$+ \sum_{s \atop \omega_n=\omega_s} F_s\left\{\frac{(1-\cos\omega_n t)}{\omega_n^4}<n|H|s> - \frac{t}{2\omega_n^3}\sin\omega_n t\,(<n|H|s>+\omega_n^2<n|\delta\rho|s>)\right\}$$

$$+ \sum_{s \atop \omega_n\neq\omega_s} F_s(\frac{1}{\omega_n^2\omega_s^2}<n|H|s>+\frac{\cos\omega_n t}{(\omega_n^2-\omega_s^2)\omega_n^2}(<n|H|s>+\omega_n^2<n|\delta\rho|s>)$$

$$\left. -\frac{\cos\omega_s t}{(\omega_n^2-\omega_s^2)\omega_s^2}(<n|H|s>+\omega_s^2<n|\delta\rho|s>))\right] \tag{11}$$

Note that the first term is equivalent to Gilbert's (1970) formula and the rest are corrections due to the heterogeneity of the Earth.

Simplification

A few simplifications to the formula (11) are possible. First of all, static terms are not necessary in many applications, since they are usually filtered out in the real data. Secondly, if we assume that $\omega_n = \omega_s$ occurs only within the same multiplet, then the sum over s is replaced by the sum over the azimuthal order number m only. Also for a short time approximation, the secular term can be absorbed in $\cos \bar{\omega}_n t$ as

$$\cos \bar{\omega}_n t = \cos \omega_n t + \frac{t}{2\omega_n} \sin \omega_n t \frac{\sum_{m'} \sum_{m} \bar{u}_{m'}^{(0)}(<m'|H|m>+\omega_n^2<m'|\delta\rho|m>)F_m}{\sum_{m} \bar{u}_{m}^{(0)} F_m} \tag{.12}$$

where

$$\bar{\omega}_n^2 = \omega_n^2 - \frac{\sum_{m'} \sum_{m} \bar{u}_{m'}^{(0)}(<m'|H|m>+\omega_n^2<m'|\delta\rho|m>)F_m}{\sum_{m} \bar{u}_{m}^{(0)} F_m} \tag{13}$$

and it is understood that the sum over m and m' are taken within the multiplet specified by n. Note that the last term in (13) devided by $2\omega_n$, is the multiplet location parameter (Jordan, 1978, Woodhouse and Girnius, 1981). Thirdly, the last term in (11) can be exchanged with the last term in the case of $\bar{u}_s^{(0)}$ for computational convenience. Then we have

$$u = -\sum_{n} \frac{\cos \bar{\omega}_n t}{\omega_n^2} (F_n \bar{u}_n^{(0)} + \sum_{m'} \sum_{m} \frac{\bar{u}_{m'}^{(0)}<m'|H|m>F_m}{\omega_n^2})$$

$$-\sum_{n} \frac{\cos \bar{\omega}_n t}{\omega_n^2} \sum_{\substack{s \\ \omega_n \neq \omega_s}} \frac{1}{(\omega_n^2 - \omega_s^2)} \sum_{m'} \sum_{m} [F_s^m \bar{u}_n^{(0)m'} (<nm'|H|sm>+\omega_n^2<nm'|\delta\rho|sm>)$$

$$+ F_n^{m} \overline{u}_s{}^{(0) \, m} (<sm \, | \, H \, | \, nm' > + \omega_n^2 <sm \, | \, \delta\rho \, | \, nm' >)] \, , \qquad (14)$$

to the accuracy of the above assumptions. s and n specify multiplets and m and m' refer to certain singlets within each multiplet respectively. The definitions of the new symbols should be clear in the context.

It is straight-forward to use (14), since some relevant formulae for $<n \, | \, H \, | \, s>$ and $<n \, | \, \delta\rho \, | \, s>$ are available in the literature (e.g. Woodhouse and Dahlen, 1978) and others are not so difficult to derive. Also it is possible to express $<n \, | \, H \, | \, s>$ in terms of elastic constants, $\delta\lambda$ and $\delta\mu$, thus (14) is useful to invert the data for $\delta\rho$, $\delta\lambda$ and $\delta\mu$.

Care must be taken in (14), because (14) breaks down if ω_s is very close to ω_n. This is because (14) assumes that the terms from summation over s are small perturbations. Such a case can be treated in (11), however, by incorporating those modes in the case for $\omega_n = \omega_s$.

Lastly, the equation (14) seems to be equivalent to the formula in Woodhouse(1983), although the two derivations are very different.

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