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NASA

Technical Memorandum 85008

On the Joint Bimodality of Temperature and Moisture Near Stratocumulus Cloud Tops

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March 1983

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Near Stratocumulus Cloud Tops

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ABSTRACT

The observed distributions of the thermodynamic variables near stratocumulus top are highly bimodal. We examine two simple models of sub-grid fractional cloudiness motivated by this observed bimodality. In both models, certain low-order moments of two independent, moist-conservative thermodynamic variables are assumed to be known.

The first model is based on the assumption of two <u>discrete</u> populations of parcels: a warm-dry population and a cool-moist population. If only the first and second moments are assumed to be known, the number of unknowns exceeds the number of independent equations. If the third moments are assumed to be known as well, the number of independent equations exceeds the number of unknowns.

The second model is based on the assumption of a <u>continuous</u> joint bimodal distribution of parcels, obtained as the weighted sum of two binormal distributions. For this model, the third moments are used to obtain 9 independent non-linear algebraic equations in 11 unknowns. Two additional equations are needed to determine the covariances within the two subpopulations. In case these two internal covariances vanish, the system of equations can be solved analytically.

1. Introduction

Stratocumulus sheets are one of the most widespread cloud forms over the globe. For this reason, and in view of the importance of cloudiness as a forecast product for numerical weather prediction, and as a pivotal component of the climate system (JOC, 1975), major efforts have been mounted to simulate stratocumulus clouds in general circulation models (Randall, 1976, 1982; Slingo, 1980; Ramanathan and Dickinson, 1981; Suarez et. al., 1983). These efforts have been limited by a number of simplifying assumptions, notably including the assumption that at a given level the fractional cloudiness is either zero or one.

Fractional cloudiness enters the problem in two different ways. First, as discussed by Sundqvist (1978, 1981), cloud decks are rarely unbroken over an area the size of a general circulation model grid box (several hundred kilometers on a side). Sundqvist has proposed methods to determine the macroscale fractional cloudiness within a grid box, and this will undoubtedly be an active area of research in coming years.

However, fractional cloudiness also enters into the micro-scale dynamics governing the local structure of the clouds. It is this aspect of the fractional cloudiness problem that is addressed in the current proposal. Radiation and the buoyancy flux, which strongly control the rate of cloud-top entrainment (Randall, 1980a, b), depend sensitively on the fractional cloudiness at each level. Near cloud top, there is a layer within which the fractional cloudiness decreases with height from near 100% to zero.

The upper boundary of a layer of stratocumulus clouds is inevitably somewhat irregular; turrets can be pushed up here and there by convective turbulence, and shearing instabilities can give rise to wave-like features. The bumpiness of

cloud top complicates the theoretical description of such important cloud-top processes as radiative cooling and turbulent entrainment, since at a given level the fractional cloudiness is greater than zero but less than one. The clear air near cloud top is typically very warm and dry compared to the cool, moist cloudy air lying nearby at the same height (Fig. 1).

Up to now, only the high-resolution three-dimensional model of Deardorff (1980) has explicitly simulated the effects of partial cloudiness near cloud top; Deardorff remarked that even in his model the vertical resolution (50 m) appeared to be inadequate to accurately represent processes near cloud-top. He advocated the use of a parameterization of sub-grid fractional cloudiness as an aid in the simulation of cloud-top processes.

Such parameterizations have been proposed and discussed by Deardorff and Someria (1977), Mellor (1977), Oliver et. al. (1978), and Bougeault (1981a, b). The first two studies dealt with fractional cloudiness in air characterized by binormal probability distributions of a pair of moist-conservative thermodynamic variables, such as liquid-water static energy and total mixing ratio. Oliver et. al. proposed a simpler model, based on ad hoc assumptions. Bougeault generalized the Deardorff-Someria-Mellor model to allow skewed distributions, at the expense of additional tunable parameters.

However, the observed distributions of the thermodynamic variables near stratocumulus cloud-top are not well-represented by gaussian or even skewed-gaussian distributions, because of the very sharp changes of temperature and moisture with height in the capping inversion (Lumley, 1979). The observed distributions are stongly bimodal; there are a lot of warm-dry parcels from the inversion above, and a lot of cool-moist parcels from the cloud below, but relatively few warm-moist or cool-dry parcels. An example given by Mahrt and

Paumier (1982) is shown in Fig. 2. A parameterization of the subgrid fractional cloudiness near stratocumulus top must take this bimodality into account.

The purpose of this paper is to report on efforts to develop a new parameterization of subgrid-fractional cloudiness, in which joint bimodality of the thermodynamic variables is permitted. The parameterization is intended to represent the subgrid-fractional cloudiness near the top of a stratocumulus deck; however, it may have wider applications. The results of this study should make possible improved parameterizations of stratiform cloudiness for use in climate models. They may also provide some guidance for parameterizations of macro-scale fractional cloudiness for climate models.

We begin by investigating a highly idealized model for which bimodality is the whole story. Consider a mass of air consisting entirely of two kinds of parcels. Parcels of Type 1 have liquid water static energy s_1 and total mixing ratio (vapor plus liquid) q_1; parcels of Type 2 have liquid water static energy s_2 and total mixing ratio q_2. Since (s,q) can assume only the discrete values (s_1, q_1) and (s_2, q_2), we refer to this as the "discrete model." At a given height, the fraction of Type 1 air is f_1, and the fraction of Type 2 air is $(1 - f_1)$. The physical picture is illustrated in Fig. 3. The joint probability density function is zero except at a pair of points, where it is infinite (Fig. 4).

The first and second moments of s and q are given by

$$\bar{s} = s_1 f_1 + s_2 (1 - f_1) \quad , \tag{2.1}$$

$$\bar{q} = q_1 f_1 + q_2 (1 - f_1) \quad , \tag{2.2}$$

$$\overline{s'^2} = (s_1 - \bar{s})^2 f_1 + (s_2 - \bar{s})^2 (1 - f_1) \quad , \tag{2.3}$$

$$\overline{q'^2} = (q_1 - \bar{q})^2 f_1 + (q_2 - \bar{q})^2 (1 - f_1) \quad , \tag{2.4}$$

$$\overline{s'q'} = (s_1 - \bar{s})(q_1 - \bar{q}) f_1 + (s_2 - \bar{s})(q_2 - \bar{q})(1 - f_1). \tag{2.5}$$

We assume that these moments are known, as for example in a second-order closure model. Then (2.1-2.5) appear to represent 5 equations which can be solved for the 5 unknowns s_1, s_2, q_1, q_2, and f_1.

From (2.1-2.4) we can solve for s_1, s_2, q_1, and q_2 in terms of f_1, as follows: From (2.1), we obtain

$$s_1 - \overline{s} = (s_1 - s_2)(1 - f_1) \quad , \tag{2.6}$$

$$s_2 - \overline{s} = (s_2 - s_1)f_1 \quad . \tag{2.7}$$

Substituting into (2.3), we find that

$$(s_2 - s_1)^2 = \overline{s'^2}/[f_1(1 - f_1)]. \tag{2.8}$$

Going back to (2.6), we square both sides and substitute from (2.8) to obtain

$$s_1 = \overline{s} \pm \{[(1 - f_1)/f_1]\,\overline{s'^2}\}^{1/2} \quad . \tag{2.9}$$

Similarly, from (2.7) and (2.8), we find that

$$s_2 = \overline{s} \pm \{[f_1/(1 - f_1)]\,\overline{s'^2}\}^{1/2} \quad . \tag{2.10}$$

By analogy, q_1 and q_2 are given by

$$q_1 = \overline{q} \pm \{[1 - f_1)/f_1]\,\overline{q'^2}\}^{1/2} \quad , \tag{2.11}$$

$$q_2 = \overline{q} \pm \{[f_1/(1 - f_1)]\,\overline{q'^2}\}^{1/2} \quad . \tag{2.12}$$

If we choose the + in (2.9), then $s_1 > \overline{s}$ necessarily, and so s_2 must be less than $\overline{s}$, i.e., we must choose the minus in (2.10). A similar argument holds for (2.11) and (2.12). Suppose we choose the (+) in (2.9), so that $s_1 > \overline{s}$. Then from (2.5) we see that choice of the (+) in (2.4), implying $q_1 > \overline{q}$, would guarantee $\overline{s'q'} > 0$. Thus if the sign of $\overline{s'q'}$ is known, the choice of signs in

(2.9) and (2.11) cannot be made independently. Then let's proceed as follows.
Define

$$M \equiv \begin{cases} 1, & \overline{s'q'} \geq 0 \\ -1, & \overline{s'q'} < 0. \end{cases} \qquad (2.13)$$

We arbitrarily choose the + in (2.9), and all else follows:

$$s_1 = \overline{s} + \{[(1 - f_1)/f_1] \overline{s'^2}\}^{1/2} \quad , \qquad (2.14)$$

$$s_2 = \overline{s} - \{[f_1/(1 - f_1)] \overline{s'^2}\}^{1/2} \quad , \qquad (2.15)$$

$$q_1 = \overline{q} + M \{[(1 - f_1)/f_1] \overline{q'^2}\}^{1/2} \quad , \qquad (2.16)$$

$$q_2 = \overline{q} - M \{[f_1/(1 - f_1)] \overline{q'^2}\}^{1/2} \quad . \qquad (2.17)$$

Note that now by definition Type 1 air has a higher liquid water static energy than Type 2 air.

From (2.14-2.17) we can obtain s_1, s_2, q_1, and q_2 if f_1 is known. Thus we rely on (2.5) to determine f_1. Substituting (2.14-2.17) into (2.5), we obtain

$$\overline{s'q'} = M \cdot (\overline{s'^2} \; \overline{q'^2})^{1/2} \quad . \qquad (2.18)$$

Since f_1 has dropped out of (2.18), knowledge of the covariance does not help us to determine f_1. We have more unknowns than we have equations. Some new information is needed.

One possibility is to assume that the predicted variances are explained by the smallest possible differences between the two types of air, i.e. assume

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that $(s_2 - s_1)^2(q_2 - q_1)^2$ is minimized. However, this leads to $f_1 = 1/2$ in every case, so the assumption is not useful.

Another possibility is to introduce the third moments: $\overline{s'^3}$, $\overline{q'^3}$, $\overline{s'^2 q'}$, and $\overline{s' q'^2}$. These we assume to be known, as for example in a third-order closure model. We find that

$$\overline{s'^3} = (s_1 - \overline{s})^3 f_1 + (s_2 - \overline{s})^3 (1 - f_1)$$

$$= \{[(1 - f_1)/f_1] \, \overline{s'^2}\}^{3/2} \cdot f_1 - \{[f_1/(1 - f_1)] \, \overline{s'^2}\}^{3/2}(1 - f_1).$$

$$(2.19)$$

From (2.19) we can derive a quadratic equation for f_1:

$$f_1^2[4 + (\overline{s'^3})^2(\overline{s'^2})^{-3}] - f_1[4 + (\overline{s'^3})^2(\overline{s'^2})^{-3}] + 1 = 0; \qquad (2.20)$$

the solution is

$$f_1 = \frac{1}{2} \left\{[1 - \overline{s'^3} \, [4(\overline{s'^2})^3 + (\overline{s'^3})^2 \,]^{-1/2}\right\} . \qquad (2.21)$$

In (2.21), the sign before the radical has been chosen by the following considerations. Regardless of the choice of sign, we have $f_1 = 1/2$ for $\overline{s'^3} = 0$. Fig. 5 shows a section along which $f_1 < 1/2$. Suppose for simplicity that $s_1 > 0$ and $s_2 = -s_1$. (This is consistent with our choice $s_1 > s_2$.) It is clear from the figure that $\overline{s'^3} > 0$, since along the section there is a small amount of air with large positive s', and a large amount with small negative s'. We conclude that $f_1 < 1/2$ for $\overline{s'^3} > 0$, and $f_1 < 1/2$ for $\overline{s'^3} > 0$. This explains the choice of the minus sign in (2.21).

Since we need $0 \leq f_1 \leq 1$, we must require that

$$-1 \leq \overline{s'^3} \, [4(\overline{s'^2})^3 + (\overline{s'^3})^2]^{-1/2} \leq 1. \qquad (2.22)$$

This will be satisfied as long as $\overline{s'^2} \geq 0$.

Of course, we can also determine f_1 from $\overline{q'^2}$ and $\overline{q'^3}$:

$$f_1 = \{1 - M\overline{q'^3} \ [4(\overline{q'^2})^3 + (\overline{q'^3})^{-1/2}\}/2 \qquad (2.23)$$

The problem is that there is no guarantee that (2.21) and (2.23) will agree. We can show that the mixed triple moments $\overline{s'^2q'}$ and $\overline{s'q'^2}$ provide two more independent formulae for f_1. With only the first and second moments we had too many unknowns and not enough equations. Now with the addition of the third moments we have too many equations and not enough unknowns. We have to introduce some additonal unknowns by revising our fundamental assumptions, and admitting that the variances are not entirely due to a mixture of just two types of air.

This we could do by introducing Type 3 air, with properties (s_3, q_3) and fraction f_3. However, this approach is not consistent with the observed bimodal distribution of properties at cloud top. A more promising approach is to allow continuous if highly bimodal distributions of s and q.

3. The Continuous Model

Again we consider a mass of air containing two populations of parcels, but this time we assume that within each population s and q have joint normal distributions about their mean values within the population (Fig. 6), i.e.

$$p(s,q) = f_1 G(s - s_1, \sigma_{s_1}, q - q_1, \sigma_{q_1}, r_{sq_1})$$

$$+ (1 - f_1) \, G \, (s - s_2, \sigma_{s_2}, q - q_2, \sigma_{q_2}, r_{sq_2}) \quad , \qquad (3.1)$$

where G is the joint-normal distribution:

$$G \, (x, \sigma_x, y, \sigma_y, r_{xy}) \equiv [2\pi\sigma_x\sigma_y(1 - r_{xy}^2)^{1/2}]^{-1}$$

$$\exp\left\{\frac{-1}{2(1 - r_{xy}^2)} \left[\frac{x^2}{\sigma_x^2} - \frac{2r_{xy}xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right]\right\} \quad , \qquad (3.2)$$

and $r_{xy} \equiv \overline{x'y'}/(\sigma_x\sigma_y)$ is the correlation between x and y within each population. As before, we have assigned weight f_1 to population 1, and weight $(1 - f_1)$ to population 2. Of course,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(s,q)dsdq = 1 \quad . \qquad (3.3)$$

Since s and q are continuously distributed, we refer to this as the "continuous model." In case the two populations are the same, the model reduces to the gaussian model of Deardorff and Sommeria (1977), and Mellor (1977).

Using the properties of the joint-normal distribution G, we can derive the following relations:

$$\bar{s} = f_1 s_1 + (1 - f_1)s_2 \quad , \tag{3.4}$$

$$\bar{q} = f_1 q_1 + (1 - f_1)q_2 \quad , \tag{3.5}$$

$$\overline{s'^2} = f_1[\sigma_{s_1}^2 + (s_1 - \bar{s})^2] + (1 - f_1)[\sigma_{s_2}^2 + (s_2 - \bar{s})^2] \quad , \tag{3.6}$$

$$\overline{q'^2} = f_1[\sigma_{q_1}^2 + (q_1 - \bar{q})^2] + (1 - f_1)[\sigma_{q_2}^2 + (q_2 - \bar{q})^2] \quad , \tag{3.7}$$

$$\overline{s'q'} = f_1[(s_1 - \bar{s})(q_1 - \bar{q}) + (\overline{s'q'})_1] + (1 - f_1)[(s_2 - \bar{s})(q_2 - \bar{q}) + (\overline{s'q'})_2] \quad , \tag{3.8}$$

$$\overline{s'^3} = f_1(s_1 - \bar{s})[3\sigma_{s_1}^2 + (s_1 - \bar{s})^2] + (1 - f_1)(s_2 - \bar{s})[3\sigma_{s_2}^2 + (s_2 - \bar{s})^2] \quad , \tag{3.9}$$

$$\overline{q'^3} = f_1(q_1 - \bar{q})[3\sigma_{q_1}^2 + (q_1 - \bar{q})^2] + (1 - f_1)(q_2 - \bar{q})[3\sigma_{q_2}^2 + (q_2 - \bar{q})^2] \quad , \tag{3.10}$$

$$\overline{s'^2q'} = f_1\{[\sigma_{s_1}^2 + (s_1 - \bar{s})^2](q_1 - \bar{q}) + 2(s_1 - \bar{s})(\overline{s'q'})_1\} ,$$
$$+ (1 - f_1)\{[\sigma_{s_2}^2 + (s_2 - \bar{s})^2](q_2 - \bar{q}) + 2(s_2 - \bar{s})(\overline{s'q'})_2\} , \tag{3.11}$$

$$\overline{s'q'^2} = f_1\{(s_1 - \bar{s})[\sigma_{q_1}^2 + (q_1 - \bar{q})^2] + 2(q_1 - \bar{q})(\overline{s'q'})_1\}$$
$$+ (1 - f_1)\{(s_2 - \bar{s})[\sigma_{q_2}^2 + (q_2 - \bar{q})^2] + 2(q_2 - \bar{q})(\overline{s'q'})_2\} . \tag{3.12}$$

In (3.8), (3.11), and (3.12), $(\overline{s'q'})_1$ and $(\overline{s'q'})_2$ are the covariances associated with populations 1 and 2, respectively. The 9 equations (3.4 - 3.12) involve the 11 unknowns s_1, s_2, q_1, q_2, $\sigma_{s_1}^2$, $\sigma_{s_2}^2$, $\sigma_{q_1}^2$, $\sigma_{q_2}^2$, f_1, $(\overline{s'q'})_1$, and $(\overline{s'q'})_2$. Two additional equations will be required for solution of the system. These additional equations essentially determine $(\overline{s'q'})_1$ and $(\overline{s'q'})_2$.

If any of the triple moments are non-zero, we will necessarily find that $s_1 \neq s_2 \neq \bar{s}$, or $q_1 \neq q_2 \neq \bar{q}$, or both. This follows from the fact that, for Gaussian distributions, all odd moments about the mean vanish.

From (3.4) and (3.6), we can derive

$$s_1 = \bar{s} + [(\overline{s'^2} - \overline{\sigma_s^2})(1 - f_1)/f_1]^{1/2} \quad , \tag{3.13}$$

$$s_2 = \bar{s} - [(\overline{s'^2} - \overline{\sigma_s^2})f_1/(1 - f_1)]^{1/2} \quad , \tag{3.14}$$

where

$$\overline{\sigma_s}^2 \equiv f_1 \sigma_{s_1}^2 + (1 - f_1) \sigma_{s_2}^2 . \tag{3.15}$$

Similarly, from (3.5) and (3.7), we find that

$$q_1 = \bar{q} + M [(\overline{q'^2} - \overline{\sigma_q^2})(1 - f_1)/f_1]^{1/2} \quad , \tag{3.16}$$

$$q_2 = \bar{q} - M [(\overline{q'^2} - \overline{\sigma_q^2})f_1/(1 - f_1)]^{1/2} \quad , \tag{3.17}$$

where

$$\overline{\sigma_q}^2 \equiv f_1 \sigma_{q_1}^2 + (1 - f_1)\sigma_{q_2}^2 . \tag{3.18}$$

From (3.16-3.17), we recognize the rather obvious requirements

$$\overline{\sigma_s}^2 \leq \overline{s'^2} \quad , \tag{3.19}$$

$$\overline{\sigma_q}^2 \leq \overline{q'^2} \quad . \tag{3.20}$$

We now use (3.4) and (3.5) to eliminate s_2 and q_2 from (3.6) - (3.12):

$$\overline{s'^2} = \overline{\sigma_s}^2 + (s_1 - \bar{s})^2 f_1/(1 - f_1) \quad , \tag{3.21}$$

$$\overline{q'^2} = \overline{\sigma_q}^2 + (q_1 - \bar{q})^2 f_1/(1 - f_1) \quad , \tag{3.22}$$

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$$\overline{s'q'} = (s_1 - \bar{s})(q_1 - \bar{q})f_1/(1 - f_1) + f_1(\overline{s'q'})_1 + (1 - f_1)(\overline{s'q'})_2 \,, \tag{3.23}$$

$$\overline{s'^3} = f_1(s_1 - \bar{s})[3\delta\sigma_s^2 + (s_1 - \bar{s})^2(1 - 2f_1)/(1 - f_1)^2] \,, \tag{3.24}$$

$$\overline{q'^3} = f_1(q_1 - \bar{q})[3\delta\sigma_q^2 + (q_1 - \bar{q})^2(1 - 2f_1)/(1 - f_1)^2] \,, \tag{3.25}$$

$$\overline{s'^2q'} = f_1\{(q_1 - \bar{q})[\delta\sigma_s^2 + (s_1 - \bar{s})^2(1-2f_1)/(1-f_1)^2] + 2(s_1 - \bar{s})\delta(\overline{s'q'})\} \,, \tag{3.26}$$

$$\overline{s'q'^2} = f_1\{(s_1 - \bar{s})[\delta\sigma_q^2 + (q_1 - \bar{q})^2(1-2f_1)/(1-f_1)^2] + 2(q_1 - \bar{q})\delta(\overline{s'q'})\} \,. \tag{3.27}$$

Here we have used the convenient definitions

$$\delta\sigma_s^2 \equiv \sigma_{s_1}^2 - \sigma_{s_2}^2 \,, \tag{3.28}$$

$$\delta\sigma_q^2 \equiv \sigma_{q_1}^2 - \sigma_{q_2}^2 \,, \tag{3.29}$$

$$\delta(\overline{s'q'}) \equiv (\overline{s'q'})_1 - (\overline{s'q'})_2 \,. \tag{3.30}$$

Since $\bar{\sigma}_s^2$ and $\bar{\sigma}_q^2$ appear only in (3.21) and (3.22), respectively, the remaining five equations (3.23-3.27) can be used to determine the five unknowns $(s_1 - s)$, $(q_1 - q)$, $\delta\sigma_s^2$, $\delta\sigma_q^2$, and f_1, as functions of $(\overline{s'q'})_1$ and $(\overline{s'q'})_2$. Inspection of (3.24-27) shows that f_1 cannot be zero unless all of the triple moments vanish.

We can eliminate $\delta\sigma_s^2$ between (3.24) and (3.26), obtaining

$$\overline{s'^3}(q_1 - \bar{q}) - 3\overline{s'^2q'}(s_1 - \bar{s}) = -2f_1(s_1 - \bar{s})^3(q_1 - \bar{q})(1 - 2f_1)/(1 - f_1)^2$$
$$- 6f_1(s_1 - \bar{s})^2\delta(\overline{s'q'}) \,. \tag{3.31}$$

This can simplified using (3.23):

$$\overline{s'^3}(q_1 - \bar{q}) - 3\overline{s'^2q'}(s_1 - \bar{s}) = -(s_1 - \bar{s})S, \tag{3.32}$$

where

$$S \equiv (s_1 - \bar{s})\{F(f_1)[\overline{s'q'} - f_1(\overline{s'q'})_1 - (1 - f_1)(\overline{s'q'})_2] + 6f_1\delta(\overline{s'q'})\} \,, \tag{3.33}$$

and

$$F(f_1) \equiv 2(1 - 2f_1)/(1 - f_1) \quad . \tag{3.34}$$

Similarly, we can use (3.23), (3.25), (3.27), and (3.33) to obtain

$$\overline{q'^3}(s_1 - \overline{s}) - 3\overline{s'q'^2}(q_1 - \overline{q}) = -(q_1 - \overline{q})^2 S/(s_1 - \overline{s}) \quad . \tag{3.35}$$

From (3.32) and (3.35), we obtain a cubic equation for S:

$$S^3 - S^2 \, 6 \, \overline{s'^2 q'} + S \, 3[\overline{s'q'^2} \cdot \overline{s'^3} + 3(\overline{s'^2 q'})^2]$$

$$+ \overline{s'^3}(\overline{s'^3} \, \overline{q'^3} - 9 \, \overline{s'^2 q'} \, \overline{s'q'^2}) = 0 \quad . \tag{3.36}$$

We can also show that

$$Q^3 - Q^2 6 \, \overline{s'q'^2} + Q 3[\overline{s'^2 q'} \cdot \overline{q'^3} + 3(\overline{s'q'^2})^2]$$

$$+ \overline{q'^3}(\overline{s'^3} \, \overline{q'^3} - 9 \, \overline{s'^2 q'} \, \overline{s'q'^2}) = 0, \tag{3.37}$$

where

$$Q \equiv (q_1 - \overline{q}) \left\{ F(f_1)[\overline{s'q'} - f_1(\overline{s'q'})_1 - (1 - f_1)(\overline{s'q'})_2] + 6f_1 \delta(\overline{s'q'}) \right\}. \tag{3.38}$$

The cubic equations (3.36) and (3.37) can be solved by standard methods (e.g., Abramowitz and Segun, 1968); we choose the roots which are always real. In this way, S and Q can be obtained directly from the known moments.

In the remainder of this Section, we consider the special case in which $(\overline{s'q'})_1 = (\overline{s'q'})_2 = 0$. For this case, we can use (3.33) and (3.38) in (3.23) to show that f_1 satisfies

$$(16 + X)f_1^2 - (16 + X)f_1 + 4 = 0, \tag{3.39}$$

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where

$$X \equiv SQ/(\overline{s'q'})^3 . \qquad (3.40)$$

Since SQ and $\overline{s'q'}$ always have the same sign, X cannot be negative. The solution to (3.39) is

$$f_1 = \{1 \pm [X/(16 + X)]^{1/2}\}/2 . \qquad (3.41)$$

For $X = 0$, we have $f_1 = 1/2$. From the form of (3.41), it is clear that $0 \leq f_1 \leq 1$ will be satisfied.

How can we choose the sign in (3.41)? For very large X, f_1 approaches either 1 or 0, depending on the choice of sign. From (3.34), we see that $F \geq 0$ for $0 \leq f_1 \leq 1/2$ (choice of the minus), and $F \leq 0$ for $1/2 \leq f_1 \leq 1$ (choice of the plus). Since, by convention, $(s_1 - \overline{s}) \geq 0$, (3.33) implies that F must be positive when $\overline{s'q'}$ and S have the same sign; otherwise, F must be negative. Therefore, when S and $\overline{s'q'}$ have the same sign, we take the minus in (3.41); otherwise, we take the plus.

If S and Q both turn out to be non-zero, completing the solution is straightforward. After solving for S and Q using (3.36) and (3.37), we obtain f_1 from (3.41). Next, $s-s_1$ and $q-q_1$ are obtained from (3.33) and (3.38), respectively. Of course, $s_2-\overline{s}$ and $q_2-\overline{q}$ are determined from (3.4) and (3.5). Finally, $\overline{\sigma}_s^2$, $\overline{\sigma}_q^2$, $\delta\sigma_s^2$, and $\delta\sigma_q^2$ are obtained from (3.21), (3.22), (3.26), and (3.27). It is then easy to obtain $\sigma_{s_1}^2$, $\sigma_{s_2}^2$, $\sigma_{q_1}^2$, and $\sigma_{q_2}^2$.

On the other hand, suppose that $S = 0$. Then from (3.40) and (3.41) we have $f_1 = 1/2$, which implies $F = 0$. It follows that $Q = 0$. Thus, S and Q are always zero or non-zero together; and when they are zero, $f_1 = 1/2$. Inspection of (3.36) and (3.37) shows that this happens for

$$\overline{s'^3}\ \overline{q'^3} - 9\ \overline{s'^2q'}\ \overline{s'q'^2} = 0. \qquad (3.42)$$

In this case, we cannot use (3.33) and (3.38) to solve for $s_1 - \bar{s}$ and $q_1 - \bar{q}$. Instead, we return to (3.23) – (3.27), which are now simplified to

$$\overline{s'q'} = (s_1 - \bar{s})(q_1 - \bar{q}) \, , \tag{3.43}$$

$$(2/3)\ \overline{s'^3} = (s_1 - \bar{s})\ \delta\sigma_s^2 \quad , \tag{3.44}$$

$$(2/3)\ \overline{q'^3} = (q_1 - \bar{q})\ \delta\sigma_q^2 \quad , \tag{3.45}$$

$$2\ \overline{s'^2 q'} = (q_1 - \bar{q})\ \delta\sigma_s^2 \quad , \tag{3.46}$$

$$2\ \overline{s'q'^2} = (s_1 - \bar{s})\ \delta\sigma_q^2 \quad . \tag{3.47}$$

From (3.45) and (3.47), we obtain

$$3\ \overline{s'q'^2}\ (q_1 - \bar{q}) = \overline{q'^3}\ (s_1 - \bar{s}). \tag{3.48}$$

Multiplying (3.48) through by $(s_1 - \bar{s})$, and using (3.43), we find that

$$s_1 - \bar{s} = (3\ \overline{s'q'^2}\ \overline{s'q'}\ /\ \overline{q'^3})^{1/2} \quad . \tag{3.49}$$

Here we have followed our convention that $s_1 > \bar{s}$. Similarly, we have

$$q_1 - \bar{q} = M\ (3\ \overline{s'^2 q'}\ \overline{s'q'}\ /\ \overline{s'^3})^{1/2} \, . \tag{3.50}$$

From (3.46) and (3.47), we now find that

$$\delta\sigma_s^2 = 2M\ (\overline{s'^2 q'}\ \overline{s'^3}\ /\ \overline{s'q'})^{1/2} \quad , \tag{3.51}$$

$$\delta\sigma_q^2 = 2(\overline{s'q'^2}\ \overline{q'^3}\ /\ \overline{s'q'})^{1/2} \quad . \tag{3.52}$$

If we multiply (3.49) and (3.50) together, we get

$$(s_1 - \bar{s})(q_1 - \bar{q}) - M\ \overline{s'q'}\ [9\ \overline{s'^2 q'}\ \overline{s'q'^2}\ /\ (\overline{s'^3}\ \overline{q'^3})]^{1/2} \, ; \tag{3.53}$$

use of (3.42) now allows us to recover (3.43).

Obviously, each quantity that appears under a square root in (3.49)-(3.52) must be non-negative. There is no guarantee that this will be the case; if it is not, then the given (or predicted) moments simply cannot be described by the model. This must be accepted as a possible outcome. If such a failure of the model occurs, it is analogous to the failure of a higher-order closure model that predicts an unrealizable set of moments.

4. Discussion and conclusion

We have solved the continous model for the case $\overline{(s'q')}_2 = 0$. Is this sufficient? Such a solution will have $s_1 \neq s_2$ and $q_1 \neq q_2$ whenever $\overline{s'q'} \neq 0$; otherwise (3.8) cannot be satisfied. Thus we will almost always have two distinct populations. However, the joint distribution of s and q need not necessarily have two <u>maxima</u>. Perhaps the additional generality provided by non-zero values of $\overline{(s'q')}_1$ and $\overline{(s'q')}_2$ is not needed. Further study is needed to resolve this issue.

To apply the theory in a higher order closure model, it is necessary to consider the joint distributions of s and q with the vertical velocity w, in order to determine the buoyancy flux. Can we assume that (s,w) and (q,w) are jointly bimodal? This would imply that at a given level the thermodynamic properties of rising and sinking air have been determined by different processes- a very plausible idea. Further studies are in progress.

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ACKNOWLEDGEMENTS

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FIGURE LEGENDS

Figure 1:   Schematic diagram illustrating the irregularities in cloud top
            height.

Figure 2:   Observed joint probability distributions of potential temperature
            and specific humidity for an aircraft flight-leg during AMTEX, as
            reported by Mahrt and Paumier (1982). Darkest shading indicates
            $\geq$ 200 observations, double-hatched $\geq$ 100, single-hatched $\geq$ 50, and
            outlined $\geq$ 25.

Figure 3:   Schematic diagram illustrating the spatial distributions of Type 1
            and Type 2 air in the discrete model.

Figure 4:   Sketch showing the assumed joint probability distributions of
            liquid water static energy and total mixing ratio for the discrete
            model.

Figure 5:   Diagram used to relate changes in $\overline{s'^3}$ to changes in f_1, for the
            discrete model. Along the dashed line, $f_1 < 1/2$.

Figure 6:   Sketch showing the assumed joint probability distributions of liquid
            water static energy and total mixing ratio for the continuous model.
            For illustrative purposes, it is assumed that the cool-wet population
            has an internal correlation consistent with the Clausius-Clapeyron
            relation, and relatively large standard deviations, while the warm-
            dry population has no internal correlation, and relatively small
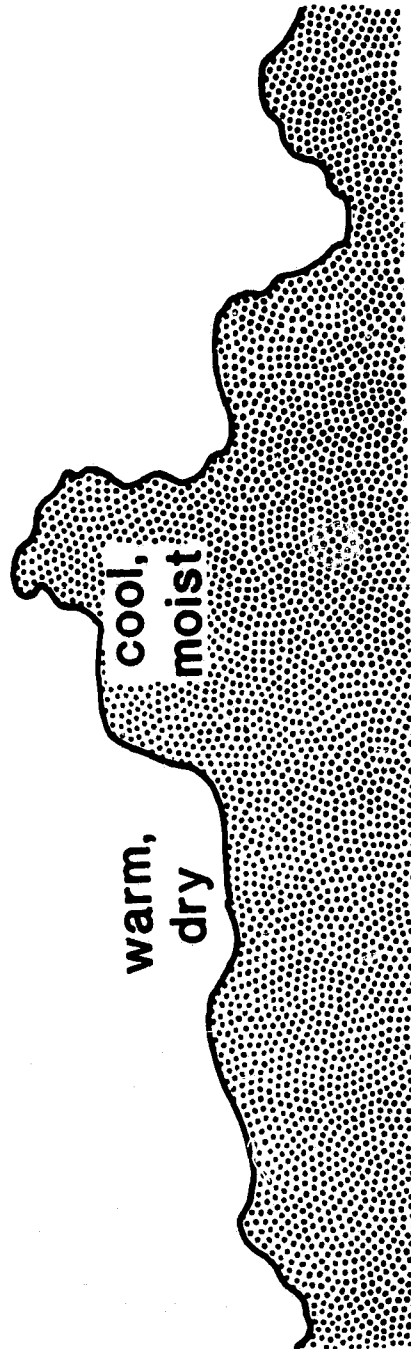   standard deviations.

cool, moist

warm, dry

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6