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PART II: THE PENALTY METHOD

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PART II: THE PENALTY METHOD

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ABSTRACT

The equilibrium strategy for N-person differential games can be found by studying a min-max problem subject to differential systems constraints [4]. In this paper, we penalize the differential constraints and use finite elements to compute numerical solutions. Convergence proof and error estimates are given. We have also included numerical results and compared them with those obtained by the dual method in [4].

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§0. Introduction

In Part I [4], we first gave a min-max equivalent formulation of equilibrium strategies in N-person differential games, and used the dual and the finite element methods to study and compute them. In this paper, we will study N-person games by another important method - the penalty approach.

The application of the penalty method to optimal control problems, which are just a special case of differential games, has been studied in [3], [5], for example; see also the references therein. Nevertheless, there has not been, to our knowledge, any application of the penalty method to saddle point type problems like differential games. The first main objective of our paper is to investigate this feasibility. Our second objective is to combine penalty with finite elements to compute numerical solutions and to compare them with those in [4] obtained from the dual method.

We inherit some notations from [4] and define some new ones below:

$A(t)$, $B_i(t)$ $(i = 1,2,\ldots,N)$ are, respectively, $n\times n$, $n\times m_i (i = 1,2,\ldots,N)$ time-varying matrices;

$C_i(t)$ $(i = 1,2,\ldots,N)$ are, respectively, $k_i\times n$ time-varying matrices;

$M_i(t)$ $(i = 1,2,\ldots,N)$ are, respectively, symmetric $m_i\times m_i$ time-varying matrices, which induce positive definite linear operators $M_i: L^2_{m_i} \to L^2_{m_i}$;

$z_i(t)$ $(i = 1,2,\ldots,N)$ are, respectively, k_i-vector valued functions;

$$H^k_n \equiv H^k_n(0,T) \equiv \{y: [0,T] \to \mathbb{R}^n \;\big|\; ||y||_{H^k_n} \equiv \sum_{j=0}^{k} ||(\tfrac{d}{dt})^j y||_{L^2_n(0,T)} < \infty\}$$

$$(DE) \equiv \dot{x} - Ax - \sum_{j=1}^{N} B_j u_j - f,\; x \in H^1_n,\; u_j \in L^2_{m_j},\; j = 1,2,\ldots,N,\;\; f \in L^2_n$$

$$(DE)_i \equiv \dot{x}^i - Ax^i - \sum_{j\neq i}^{N} B_j u_j - B_i v_i - f, \quad x^i \in H_n^1,\ u_j \in L_{m_j}^2 (j\neq i)\ ,\ v_i \in L_{m_i}^2$$

$$[DE] \equiv \sum_{i=1}^{N} |(DE)_i|^2$$

$$X \equiv (x^1, x^2, \ldots, x^N)$$

$$H_{0n}^1 \equiv H_n^1 \cap \{y \in H_n^1 \,|\, y(0) = 0\}$$

$$H_{n0}^1 \equiv H_n^1 \cap \{y \in H_n^1 \,|\, y(T) = 0\}$$

$$U \equiv \prod_{i=1}^{N} L_{m_i}^2$$

$$H \equiv H_{0n}^1 \times U \times [H_{0n}^1]^N \times U; \ \bar{H} \equiv L_n^2 \times U \times [L_n^2]^N \times U$$

$$\mathcal{L}: H_{0n}^1 \to L_n^2, \ \mathcal{L}x \equiv \dot{x} - Ax$$

$$\mathcal{L}^*: H_{n0}^1 \to L_n^2, \quad \mathcal{L}^* x \equiv \dot{x} + A^* x.$$

We proceed as follows.

In §1, we present the fundamental penalty theorem. The rate of convergence with respect to the penalty parameters is determined. Our work here extends and generalizes the earlier result of B. T. Polyak [6].

In §2, we specialize to the linear-quadratic case and formulate the finite element variational approach. Error estimates between the computed and the exact solutions with respect to the penalty parameter ϵ and the discretization parameter h are given.

The relationship between the penalty method and the dual method is explored in §3. Their computational advantages and disadvantages are also compared.

Numerical results are presented in §4.

§1. The Penalty Method for N-Person Differential Games. Rate of Convergence.

As in [4], for an N-person game with linear dynamics

$$(1.1)\quad \begin{cases} \dfrac{d}{dt}x(t) = A(t)x(t) + B_1(t)u_1(t) + \dots + B_N(t)u_N(t) + f(t), & 0 \leqq t \leqq T, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

let each player have an associated cost functional $J_i(x,u)$, $1 \leqq i \leqq N$, which is continuous with respect to (x,u) in the $H_n^1 \times U$ norm. Throughout the rest of the paper we assume that we have made the change of variable $x(t) \to x(t) - x_0$ so that $x(0) = 0$. This change of variable results only in minor changes of J_i.

In this section, the costs J_i need not be quadratic.

Following the min-max formulation in [4, §1], we consider

$$(1.2)\quad \inf_{\substack{(x,u)\in H_{0n}^1 \times L_{m_i}^2 \\ (DE)=0}} \; \sup_{\substack{(x^i,v^i)\in H_{0n}^1 \times L_{m_i}^2 \\ (DE)_i=0 \\ 1\leqq i\leqq N}} J(x,u;X,v) \equiv \sum_{i=1}^{N} \left[J_i(x,u) - J_i(x^i,v^i)\right]$$

where $v^i \equiv (u_1,u_2,\dots,u_{i-1},v_i,u_{i+1},\dots,u_N) \in U$. Here, we see that $(DE) = 0$ and $(DE)_i = 0$ $(1 \leqq i \leqq N)$ are $N+1$ equality constraints for the inf-sup problem (1.2). Thus, it appears natural for us to penalize the problem as

$$(1.3)\quad \inf_{(x,u)\in H_{0n}^1 \times U} \; \sup_{(X,v)\in [H_{0n}^1]\times U} J_\varepsilon(x,u;X,v) \equiv J(x,u;X,v)$$

$$+\frac{1}{\varepsilon_0}||(DE)||_{L_n^2}^2 - \sum_{i=1}^{N} \frac{1}{\varepsilon_i} ||(DE)_i||_{L_n^2}^2$$

for some $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N > 0$.

The most important question remains in determining the validity of the above scheme and, if valid, its rate of convergence. Thus, we consider the fundamental theorem of penalty for N-person differential games below.

The following assumptions will be needed:

(B1) $J(x,u;X,v)$ is strictly convex in (x,u) and strictly concave in (X,v);

(B2) $$\inf_{\substack{(x,u)\in H^1_{0n}\times U\\ (DE)=0}} \ \sup_{\substack{(X,v)\in [H^1_{0n}]^N\times U\\ [DE]=0}} J(x,u;X,v)$$ is attained by $(\hat{x},\hat{u};\hat{X},\hat{v}) \in H$;

By (B1), this point $(\hat{x},\hat{u};\hat{X},\hat{v})$ is unique. Also, by [4, Theorem 2.1], there exist Lagrange multipliers $\hat{p}_0$, $\hat{p} = (\hat{p}_1,\ldots,\hat{p}_N)$ such that

(1.4) $$J(\hat{x},\hat{u};\hat{X},\hat{v}) = \min_{\substack{(x,u)\in H^1_{0n}\times U\\ (DE)=0}} \ \max_{\substack{(X,v)\in [H^1_{0n}]^N\times U\\ [DE]=0}} J(x,u;X,v)$$

$$= \max_{p_0\in L^2_n} \ \min_{p\in [L^2_n]^N} \ \min_{(x,u)\in H^1_{0n}\times U} \ \max_{(X,v)\in [H^1_{0n}]^N\times U} [J(x,u;X,v)$$

$$+ \langle p_0, (DE)\rangle + \sum_{i=1}^{N} \langle p_i, (DE)_i\rangle]$$

(B3) The costs $J_i(x,u)$ are of the form

(1.5) $$J_i(x,u) \equiv \int_0^T h_i(x(t),u(t))dt$$

so that $\hat{p}_0 \in H^1_{n0}$, $\hat{p} \in [H^1_{n0}]^N$;

(B4) The first and second derivatives J', J'' exist, and J'' satisfies the global Lipschitz condition

(1.6) $$||J''(x_1,u_1;X_1,v_1) - J''(x_2,u_2;X_2,v_2)|| \le K_1||(x_1-x_2,u_1-u_2;X_1-X_2,v_1-v_2)||_{\bar{H}}$$

for some $K_1 > 0$ uniformly for $(x_1,u_1;X_1,v_1),(x_2,u_2;X_2,v_2)$.

(B5) Let $\mathcal{A}_0 \equiv \partial_x^2 J$, $\mathcal{A}_1 \equiv \partial_X^2 J$, $\mathcal{M}_0 \equiv \partial_u^2 J$, and $\mathcal{M}_1 \equiv \partial_v^2 J$ be second order Fréchet partial derivatives evaluated at $(\hat{x},\hat{u};\hat{X},\hat{v})$. Then $\mathcal{A}_0, \mathcal{M}_0, -\mathcal{A}_1$ and $-\mathcal{M}_1$ are positive definite linear operators on L_n^2, U, $[L_n^2]^N$ and U, respectively. Furthermore, $\mathcal{A}_0 \times \mathcal{A}_1$ maps $H_{0n}^1 \times [H_{0n}^1]^N$ into itself;

(B6) $B_1,\ldots,B_N$ are small relative to $\mathcal{A}_0, \mathcal{M}_0, -\mathcal{A}_1$ and $-\mathcal{M}_1$ (cf.(1.24), (2.10), (2.14))

(B7) The mixed Fréchet partial derivative operators $\partial_x \partial_X J$, $\partial_x \partial_u J,\ldots$, etc., evaluated at $(\hat{x},\hat{u};\hat{X},\hat{v})$ are all 0.

<u>Remark 1.1</u>

(i) In (B3), that the J_i's are assumed to be of the form (1.5) is only for the convenience of discussions.

(ii) Making some other assumptions, one can relax the global Lipschitz condition (1.6) to a local one.

(iii) (B7) is assumed here only for the convenience of discussions, cf. Remark 1.5 later.

<u>Theorem 1.2</u> Under conditions (B1) - (B7), for $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N > 0$ sufficiently small, there exists a unique $(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon) \in H$ satisfying $J'_\varepsilon = 0$ such that

(i) $$||(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon) - (\hat{x},\hat{u};\hat{X},\hat{v})||_{\bar{H}} \leq K_2(\max_{0\leq j\leq N} \varepsilon_j)||(\hat{p}_0,\hat{p})||_{L_n^2\times[L_n^2]^N};$$

(ii) $$||\frac{2}{\varepsilon_0}(\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon - \Sigma B_i\hat{u}_{\varepsilon i} - f) - \hat{p}_0||_{L_n^2} + \sum_{i=1}^N ||\frac{2}{\varepsilon_i}(\dot{\hat{x}}^i - A\hat{x}^i - \sum_{j\neq i} B_j\hat{u}_{\varepsilon j}$$

$$-B_i\hat{v}_{\varepsilon i} - f) - (-\hat{p}_i)||_{L_n^2} \leq K_3(\max_{0\leq j\leq N} \varepsilon_j)||(\hat{p}_0,\hat{p})||_{L_n^2\times[L_n^2]^N},$$

for some $K_2, K_3 > 0$ independent of $\varepsilon_0,\ldots,\varepsilon_N$.

<u>Proof:</u> We introduce the new variables

$$
\begin{aligned}
&\xi_0 = x - \hat{x}\\
&\xi_1 = X - \hat{X}\\
(1.7)\qquad &\eta_0 = u - \hat{u}\\
&\eta_1 = v - \hat{v}\\
&\zeta_0 = \frac{2}{\varepsilon_0}(\dot{x} - Ax - \sum_{i=1}^{N} B_i u_i - f) - \hat{p}_0\\
&\zeta_1^i = -\frac{2}{\varepsilon_i}(\dot{x}^i - Ax^i - \sum_{j\neq i} B_j u_j - B_i v_i - f) - \hat{p}_i \quad , \zeta_1 \equiv (\zeta_1^1, \zeta_1^2, \ldots, \zeta_1^N).
\end{aligned}
$$

In the above, we first choose $x \in H_n^2 \cap H_{0n}^1$, $X \in [H_n^2 \cap H_{0n}^1]^N$,

$u,v \in U \cap \prod_{i=1}^{N} H_{m_i}^1$ and then let $(x,u;X,v)$ tend to an element in H.

We further let

$$\xi = (\xi_0,\xi_1),\ \eta = (\eta_0,\eta_1),\ \zeta = (\zeta_0,\zeta_1).$$

For any $(\delta x,\delta u;\delta X,\delta v) \in H$, we have

$$
\begin{aligned}
(1.8)\qquad J'_\varepsilon(x,u;X,v)\cdot(\delta x,\delta u;\delta X,\delta v) &= J'(x,u;X,v)\cdot(\delta x,\delta u;\delta X,\delta v)\\
&+ \frac{2}{\varepsilon_0} < \dot{x} - Ax - \sum_i B_i u_i - f,\ \delta\dot{x} - A(\delta x) - \sum_i B_i(\delta u_i) >\\
&+ \sum_i \frac{2}{\varepsilon_i} < \dot{x}^i - Ax^i - \sum_{j\neq i} B_j u_j - B_i v_i - f,\ \delta\dot{x}^i - A(\delta x^i)\\
&- \sum_{j\neq i} B_j \delta u_j - B_i \delta v_i >
\end{aligned}
$$

We can use (B4) to write

$$(1.9) \qquad J'(x,u;X,v) = J'(\hat{x},\hat{u};\hat{X},\hat{v}) + J''(\hat{x},\hat{u};\hat{X},\hat{v})(\xi_0,\eta_0;\xi_1,\eta_1) + r(\xi,\eta),$$

where the remainder $r(\xi,\eta)$ (as a linear functional in H) satisfies

$$(1.10) \qquad r(0,0) = 0$$

$$(1.11) \qquad ||r'(\bar{\xi},\bar{\eta}) - r'(\tilde{\xi},\tilde{\eta})|| \le c_1 ||(\bar{\xi} - \tilde{\xi},\bar{\eta} - \tilde{\eta})||_{(L_n^2\times[L_n^2]^N)\times(U\times U)}$$

$$\forall(\bar{\xi},\bar{\eta}),\ (\tilde{\xi},\tilde{\eta}) \in (H_{0n}^1 \times [H_{0n}^1]^N) \times (U\times U).$$

Substituting (1.9) into the first term on the RHS of (1.8) and integrating the remaining terms by parts, we get

$$\begin{aligned}
(1.12) \qquad \text{LHS of } (1.7) = & \left[J'(\hat{x},\hat{u};\hat{X},\hat{v}) + J''(\hat{x},\hat{u};\hat{X},\hat{v})(\xi_0,\eta_0;\xi_1,\eta_1) + r(\xi,\eta)\right]. \\
& (\delta x,\delta u;\delta X,\delta v) - < (\frac{d}{dt} + A^*)\frac{2}{\varepsilon_0}(\dot{x} - Ax - \sum_i B_i u_i - f),\ \delta x > \\
& + \frac{2}{\varepsilon_0} < \dot{x}(T) - A(T)x(T) - \sum_i (B_i u_i)(T) - f(T),\ \delta x(T) >_{\mathbb{R}^n} \\
& - \sum_i < B_i^* \cdot \frac{2}{\varepsilon_0}(\dot{x} - Ax - \sum B_i u_i - f),\ \delta u_i > \\
& + \sum_i < (\frac{d}{dt} + A^*) \cdot \frac{2}{\varepsilon_i}(\dot{x}^i - Ax^i - \sum_{j\neq i} B_j u_j - B_i v_i - f),\ \delta x^i > \\
& - \sum_i \frac{2}{\varepsilon_i} < \dot{x}^i(T) - A(T)x^i(T) - \sum_{j\neq i}(B_j u_j)(T) - (B_i v_i)(T) - f(T),\ \delta x^i(T) >_{\mathbb{R}^n} \\
& + \sum_i \sum_{k\neq i} < B_k^* \cdot \frac{2}{\varepsilon_i}(\dot{x}^i - Ax^i - \sum_{j\neq i} B_j u_j - B_i v_i - f),\ \delta u_k > \\
& + \sum_i < B_i^* \cdot \frac{2}{\varepsilon_i}(\dot{x}^i - Ax^i - \sum_{j\neq i} B_j u_j - B_i v_i - f),\ \delta v_k >
\end{aligned}$$

We now substitute (1.7) into the above and note that

$$(1.13)\qquad J'(\hat{x},\hat{u};\hat{X},\hat{v})\cdot(\delta x,\delta u;\delta X,\delta v) - \langle \mathcal{L}^{*}\hat{p}_0,\delta x\rangle - \sum_i \langle B_i^{*}\hat{p}_0, \delta u_i\rangle$$

$$-\sum_i \langle \mathcal{L}^{*}\hat{p}_i,\delta x^i\rangle - \sum_i \sum_{j\neq i} \langle B_j^{*}\hat{p}_i,\delta u_i\rangle - \sum_i \langle B_i^{*}\hat{p}_i,\delta v_i\rangle = 0,$$

we get that the solution of $J'(x,u;X,v) = 0$ can be found by solving

$$(1.14)\qquad \left[J''(\hat{x},\hat{u};\hat{X},\hat{v})(\xi_0,\eta_0;\xi_1,\eta_1) + r(\xi,\eta)\right]\cdot(\delta x,\delta u;\delta X,\delta v) - \langle \mathcal{L}^{*}\zeta_0, \delta x\rangle$$

$$-\sum_i \langle B_i^{*}\zeta_0,\delta u_i\rangle - \sum_i \langle \mathcal{L}^{*}\zeta_1^i,\delta x^i\rangle - \sum_i \sum_{j\neq i} \langle B_j^{*}\zeta_1^i,\delta u_j\rangle - \sum_i \langle B_i^{*}\zeta_1^i,\delta v_i\rangle =$$

Note that all the $\langle , \rangle_{\mathbb{R}^n}$ terms on the RHS of (1.12) disappear because of the arbitrariness of $\delta x(T)$ and $\delta x^i(T)$. By (B5) and (B7), we have

$$(1.15)\qquad J''(\hat{x},\hat{u};\hat{X},\hat{v})(\xi_0;\eta_0;\xi_1,\eta_1) = \begin{bmatrix} \mathcal{A}_0 & 0 & 0 & 0 \\ 0 & \mathcal{A}_1 & 0 & 0 \\ 0 & 0 & \mathcal{M}_0 & 0 \\ 0 & 0 & 0 & \mathcal{M}_1 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \eta_0 \\ \eta_1 \end{bmatrix}$$

Therefore, from (1.14), we get

$$(1.16)\qquad \begin{cases} \mathcal{A}_0\xi_0 - \mathcal{L}_0^{*}\zeta_0 = -r_1(\xi,\eta) \\ (\mathcal{A}_1\xi_1)i - \mathcal{L}_0^{*}\zeta_1^i = -r_2(\xi,\eta) \\ (\mathcal{M}_0\eta_0)^i - B_i^{*}\zeta_0 - \sum_{j\neq i} B_j^{*}\zeta_1^i = -r_3^i(\xi,\eta) & , 1 \leqq i \leqq N, \\ (\mathcal{M}_1\eta_1)^i - B_i^{*}\zeta_1^i = -r_4(\xi,\eta) & , 1 \leqq i \leqq N, \end{cases}$$

where r_1, r_2, r_3, r_4 are the respective components of $r(\xi,\eta)$ and the superscript i denotes the i-th component.

Combining (1.16) with (1.7.5) and (1.7.6), we get the following nonlinear "matrix" equation

$$(1.17)\qquad \begin{bmatrix} \mathcal{A}_0 & 0 & 0 & 0 & -\mathcal{L}^* & 0 \\ 0 & \mathcal{A}_1 & 0 & 0 & 0 & \begin{bmatrix} -\mathcal{L}^* & & 0 \\ & \ddots & \\ 0 & & -\mathcal{L}^* \end{bmatrix} \\ 0 & 0 & \mathcal{M}_0 & 0 & \mathcal{B}_1^* & \mathcal{B}_2^* \\ 0 & 0 & 0 & \mathcal{M}_1 & 0 & \mathcal{B}_3^* \\ -\mathcal{L} & 0 & \mathcal{B}_1 & 0 & \frac{\varepsilon_0}{2} I & 0 \\ 0 & \begin{bmatrix} -\mathcal{L} & & 0 \\ & \ddots & \\ 0 & & -\mathcal{L} \end{bmatrix} & \mathcal{B}_2 & \mathcal{B}_3 & 0 & \begin{bmatrix} \frac{\varepsilon_1}{2} I & & \\ & \ddots & \\ 0 & & \frac{\varepsilon_N}{2} I \end{bmatrix} \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \eta_0 \\ \eta_1 \\ \zeta_0 \\ \zeta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\varepsilon_0}{2}\hat{p}_0 \\ \begin{bmatrix} -\frac{\varepsilon_1}{2}\hat{p}_1 \\ \vdots \\ -\frac{\varepsilon_N}{2}\hat{p}_N \end{bmatrix} \end{bmatrix} + \begin{bmatrix} -r_1(\xi,\eta) \\ -r_2(\xi,\eta) \\ -r_3(\xi,\eta) \\ -r_4(\xi,\eta) \\ 0 \\ 0 \end{bmatrix}$$

wherein

$$\mathcal{B}_1 \equiv \begin{bmatrix} -B_1 \\ -B_2 \\ \vdots \\ \vdots \\ -B_N \end{bmatrix}, \quad \mathcal{B}_2 \equiv \begin{bmatrix} 0 & -B_2 & \cdots & -B_N \\ -B_1 & 0 & & -B_N \\ \vdots & \vdots & & \vdots \\ -B_1 & -B_2 & \cdots & 0 \end{bmatrix}, \quad \mathcal{B}_3 \equiv \begin{bmatrix} -B_1 & & & \\ & -B_2 & & 0 \\ & & \ddots & \\ & & & -B_N \end{bmatrix}.$$

We further abbreviate (1.17) as

$$(1.18)\qquad D_\varepsilon \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \equiv \begin{bmatrix} \mathcal{A} & 0 & -\tilde{\mathcal{L}}^* \\ 0 & \mathcal{M} & \mathcal{B}^* \\ -\tilde{\mathcal{L}} & \mathcal{B} & I_\varepsilon \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{p}_\varepsilon \end{bmatrix} + \begin{bmatrix} -\tilde{r}_1(\xi,\eta) \\ -\tilde{r}_2(\xi,\eta) \\ 0 \end{bmatrix}$$

where

$$\mathcal{A} \equiv \begin{bmatrix} \mathcal{A}_0 & 0 \\ 0 & \mathcal{A}_1 \end{bmatrix}, \quad \mathcal{M} \equiv \begin{bmatrix} \mathcal{M}_0 & 0 \\ 0 & \mathcal{M}_1 \end{bmatrix}, \quad \mathcal{B} \equiv \begin{bmatrix} \mathcal{B}_1 & 0 \\ \mathcal{B}_2 & \mathcal{B}_3 \end{bmatrix}$$

$$\tilde{L} \equiv \begin{bmatrix} L & & & 0 \\ & L & & \\ & & \ddots & \\ 0 & & & L \end{bmatrix}_{(N+1)\times(N+1)}, \quad \tilde{L}^* \equiv \begin{bmatrix} L^* & & & 0 \\ & L^* & & \\ & & \ddots & \\ 0 & & & L^* \end{bmatrix}_{(N+1)\times(N+1)}$$

$$I_\varepsilon \equiv \begin{bmatrix} \frac{\varepsilon_0}{2} I & & & 0 \\ & -\frac{\varepsilon_1}{2} I & & \\ & & \ddots & \\ 0 & & & -\frac{\varepsilon_N}{2} I \end{bmatrix}_{(N+1)\times(N+1)}$$

$$\tilde{r}_1(\xi,\eta) \equiv \begin{bmatrix} r_1(\xi,\eta) \\ \\ r_2(\xi,\eta) \end{bmatrix}, \quad \tilde{r}_2(\xi,\eta) \equiv \begin{bmatrix} r_3(\xi,\eta) \\ \\ r_4(\xi,\eta) \end{bmatrix}, \quad \tilde{p}_\varepsilon = \begin{bmatrix} -\frac{\varepsilon_0}{2}\hat{p}_0 \\ \frac{\varepsilon_1}{2}\hat{p}_1 \\ \vdots \\ \frac{\varepsilon_N}{2}\hat{p}_N \end{bmatrix}.$$

By (A5), D_ε is a closed linear operator on $(L_n^2 \times [L_n^2]^N) \times (U\times U) \times (L_n^2 \times [L_n^2]^N)$ with domain $\operatorname{dom}(D_\varepsilon) = (H_{0n}^1 \times [H_{0n}^1]^N) \times (U\times U) \times (H_{n0}^1 \times [H_{n0}^1]^N)$.

<u>Lemma 1.3</u> Under conditions (A5), (A6) and (A7), for all $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N > 0$ sufficiently small, the operator D_ε introduced above has an inverse and

(1.19) $\qquad ||D_\varepsilon^{-1}|| \leq K_4$

for some $K_4 > 0$ independent of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N$.

<u>Proof:</u> For an arbitrarily given $(\alpha,\beta,\gamma) \in (L^2_n \times [L^2_n]^N) \times (U\times U) \times (L^2_n \times [L^2_n]^N)$, we wish to find some $(\bar\xi,\bar\eta,\bar\zeta) \in (H^1_{0n} \times [H^1_{0n}]^N) \times (U\times U) \times (H^1_{n0} \times [H^1_{n0}]^N)$ such that

$$(1.20) \qquad D_\varepsilon \begin{bmatrix} \bar\xi \\ \bar\eta \\ \bar\zeta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix},$$

or, in detail,

$$(1.21) \qquad \begin{cases} \mathcal{A}\bar\xi - \tilde{\mathcal{L}}^*\zeta = \alpha \\ \mathcal{M}\bar\eta + \mathcal{B}^*\bar\zeta = \beta \\ -\tilde{\mathcal{L}}\bar\xi + \mathcal{B}\bar\eta + I_\varepsilon\bar\zeta = \gamma. \end{cases}$$

Let $\Phi(t,s)$ be the fundamental $n \times n$ matrix solution satisfying

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,s) = A(t)\Phi(t,s) & , \ 0 \leqq s \leqq t \leqq T \\ \Phi(s,s) = I_{n\times n}. \end{cases}$$

It is easy to see that $\tilde{\mathcal{L}}$ is invertible with inverse

$$(\tilde{\mathcal{L}})^{-1}\lambda = \int_0^t \begin{bmatrix} \Phi(t,s) & & 0 \\ & \ddots & \\ 0 & & \Phi(t,s) \end{bmatrix}_{(N+1)\times(N\times 1)} \lambda(s)ds.$$

Thus, we have from (1.21.3),

$$(1.22) \qquad \bar\xi = (\tilde{\mathcal{L}})^{-1}(\mathcal{B}\bar\eta + I_\varepsilon\bar\zeta - \gamma).$$

Substituting (1.22) into (1.21.1), we get

$$\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\bar{\eta} + (\mathcal{A}\tilde{\mathcal{L}}^{-1}I_\varepsilon - \tilde{\mathcal{L}}^*)\bar{\zeta} = \alpha + \mathcal{A}\tilde{\mathcal{L}}^{-1}\gamma.$$

The integrodifferential operator $\tilde{\mathcal{L}}^* - \mathcal{A}\tilde{\mathcal{L}}^{-1}I_\varepsilon$ is easily seen to be invertible for $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N)$ sufficiently small, thus we have

$$(1.23) \qquad \bar{\zeta} = (\tilde{\mathcal{L}}^* - \mathcal{A}\tilde{\mathcal{L}}^{-1}I_\varepsilon)^{-1}\,[\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\bar{\eta} - (\alpha + \mathcal{A}\tilde{\mathcal{L}}^{-1}\gamma)].$$

Substituting (1.23) into (1.21.2), we get

$$[\mathcal{M} + \mathcal{B}^*(\tilde{\mathcal{L}}^* - \mathcal{A}\tilde{\mathcal{L}}^{-1}I_\varepsilon)^{-1}\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}]\bar{\eta} = \beta + \mathcal{B}^*(\tilde{\mathcal{L}}^* - \mathcal{A}\tilde{\mathcal{L}}^{-1}I_\varepsilon)^{-1}(\alpha + \mathcal{A}\tilde{\mathcal{L}}^{-1}\gamma).$$

Now we invoke (B6): since $\mathcal{M}$ is invertible, if $\mathcal{B}$ is relatively smaller than $\mathcal{M}$ such that

$$(1.24) \qquad \mathcal{M} + \mathcal{B}^*(\tilde{\mathcal{L}}^* - \mathcal{A}\tilde{\mathcal{L}}^{-1}I_\varepsilon)^{-1}\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B} \qquad \text{is invertible}$$

(for ε sufficiently small), we have

$$(1.25) \qquad \bar{\eta} = \mathcal{J}_\varepsilon^{-1}\,[\beta + \mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1}(\alpha + \mathcal{A}\tilde{\mathcal{L}}^{-1}\gamma)].$$

where

$$(1.26) \qquad \mathcal{J}_\varepsilon \equiv \mathcal{M} + \mathcal{B}^*(\tilde{\mathcal{L}}^* - \mathcal{A}\tilde{\mathcal{L}}^{-1}I_\varepsilon)^{-1}\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}, \quad \tilde{\mathcal{L}}_\varepsilon \equiv \tilde{\mathcal{L}}^* - \mathcal{A}\tilde{\mathcal{L}}^{-1}I_\varepsilon.$$

Using (1.25) in (1.22) and (1.23), we obtain

$$\bar{\xi} = \tilde{\mathcal{L}}^{-1}\{[\mathcal{B}\mathcal{J}_\varepsilon^{-1}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1} + I_\varepsilon\tilde{\mathcal{L}}_\varepsilon^{-1}(\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1} - I)]\alpha + [\mathcal{B}\mathcal{J}_\varepsilon^{-1} + I_\varepsilon\tilde{\mathcal{L}}_\varepsilon^{-1}\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{J}_\varepsilon^{-1}]\beta$$

$$+ [\mathcal{B}\mathcal{J}_\varepsilon^{-1}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1}\mathcal{A}\tilde{\mathcal{L}}^{-1} - I + I_\varepsilon\tilde{\mathcal{L}}_\varepsilon^{-1}(\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1} - I)\mathcal{A}\tilde{\mathcal{L}}^{-1}]\gamma\}$$

$$\tilde{\zeta} = \tilde{\mathcal{L}}_\varepsilon^{-1}\{(\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1} - I)\alpha + \mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{J}_\varepsilon^{-1}\beta + (\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1} - I)\mathcal{A}\tilde{\mathcal{L}}^{-1}\gamma\}.$$

Therefore, D_ε is invertible, with

$$D_\varepsilon^{-1} = \begin{bmatrix} \begin{array}{l}\tilde{\mathcal{L}}^{-1}[\mathcal{B}\mathcal{J}_\varepsilon^{-1}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1} \\ + I_\varepsilon\tilde{\mathcal{L}}_\varepsilon^{-1}(\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon - I)]\end{array} & \begin{array}{l}\tilde{\mathcal{L}}^{-1}[\mathcal{B}\mathcal{J}_\varepsilon^{-1} + I_\varepsilon\tilde{\mathcal{L}}_\varepsilon^{-1}\cdot \\ \mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{J}_\varepsilon^{-1}]\end{array} & \begin{array}{l}\tilde{\mathcal{L}}^{-1}\{\mathcal{B}\mathcal{J}_\varepsilon^{-1}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1}\mathcal{A}\tilde{\mathcal{L}} - I + I_\varepsilon\tilde{\mathcal{L}}_\varepsilon^{-1}\cdot \\ \cdot(\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{B}^*\tilde{\mathcal{L}}^{-1} - I)\mathcal{A}\tilde{\mathcal{L}}^{-1}]\end{array} \\ \mathcal{J}_\varepsilon^{-1}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1} & \mathcal{J}_\varepsilon^{-1} & \mathcal{J}_\varepsilon^{-1}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1}\mathcal{A}\tilde{\mathcal{L}}^{-1} \\ \tilde{\mathcal{L}}_\varepsilon^{-1}(\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1} - I) & \mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{J}_\varepsilon^{-1} & (\mathcal{A}\tilde{\mathcal{L}}^{-1}\mathcal{B}\mathcal{B}^*\tilde{\mathcal{L}}_\varepsilon^{-1} - I)\mathcal{A}\tilde{\mathcal{L}}^{-1} \end{bmatrix}$$

Since each entry of the matrix D_ε^{-1} is bounded, we have proved that D_ε^{-1} is bounded for ε sufficiently small.

We will need the following lemma from [6]:

<u>Lemma 1.4</u> Let $\mathcal{H}$ be a given Hilbert space and T be a densely defined closed linear operator from $\mathrm{dom}(T) \subseteq \mathcal{H}$ onto $\mathcal{H}$ with a bounded inverse $\|T^{-1}\| \le c_1$, and let $r(x)$ be a nonlinear (Fréchet) differentiable operator on $\mathcal{H}$ such that $r(0) = 0$, $\|r'(x)\| \le c_2\|x\|$ for all $x \in \mathcal{H}$. Then for any $a \in \mathcal{H}$, $\|a\| \le \dfrac{1}{4c_1^2c_2}$, the equation

$$Tx = a + r(x)$$

has in the sphere $\|x\| < 4c_1\|a\|$ a unique solution $\hat{x} \in \mathrm{dom}(T)$ satisfying

$$\|\hat{x}\| \le \frac{c_1}{2}\|a\|. \qquad \square$$

We note that although in [6, p.6, Lemma 2], it is assumed that T be <u>bounded</u>, a careful examination of the proof shows that that assumption is redundant.

Using $T = D_\varepsilon$, $c_1 = K_4$ and $c_2 = K_1$ in Lemma 1.4 and applying it to (1.18), we obtain that for

$$\|\tilde{p}_\varepsilon\|_{L_n^2\times[L_n^2]^N} \leq \frac{1}{4K_1K_4^2} ,$$

which is clearly satisfied if

$$\min_{0\leq j\leq N} \frac{2}{\varepsilon_j} \geq 4K_1K_4^2 \|(-\hat{p}_0,\hat{p})\|_{L_n^2\times[L_n^2]^N} ,$$

(1.20) has a solution $(\hat{\xi}_\varepsilon,\hat{\eta}_\varepsilon,\hat{\zeta}_\varepsilon) \in [H_{0n}^1]^{N+1}\times U^2 \times [H_{n0}^1]^{N+1}$ satisfying

$$\|(\hat{\xi}_\varepsilon,\hat{\eta}_\varepsilon,\hat{\zeta}_\varepsilon)\|_{[L_n^2]^{N+1}\times U^2\times[L_n^2]^{N+1}} \leq \frac{K_4}{4} (\max_{0\leq j\leq N} \varepsilon_j) \|(-\hat{p}_0,\hat{p})\|_{[L_n^2]^{N+1}} .$$

From (1.7), writing

$$\hat{x}_\varepsilon = \hat{x} + \hat{\xi}_{\varepsilon,0} \quad , \hat{X}_\varepsilon = \hat{X} + \hat{\xi}_{\varepsilon,1} ,$$

$$\hat{u}_\varepsilon = \hat{u} + \hat{\eta}_{\varepsilon,0} \quad , \hat{v}_\varepsilon = \hat{v} + \hat{\eta}_{\varepsilon,1} ,$$

$$\frac{2}{\varepsilon_0}(\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon - \sum_i B_i\hat{u}_{\varepsilon,i} - f) = \hat{\zeta}_{\varepsilon,0} + \hat{p}_0$$

$$\frac{2}{\varepsilon_i} (\dot{\hat{x}}_\varepsilon^i - A\hat{x}_\varepsilon^i - \sum_{j\neq i} B_j\hat{u}_{\varepsilon,j} - B_i\hat{v}_{\varepsilon,i} - f) = -\hat{\zeta}_{\varepsilon,1}^i - \hat{p}_i, \quad 1 \leq i \leq N,$$

we obtain that for

$$\max_{0\leq j\leq N} \varepsilon_j \leq [2K_1K_4^2 \|(\hat{p}_0,\hat{p})\|_{[L_n^2]^{N+1}}]^{-1} \qquad (\| -\hat{p}_0,\hat{p})\| = \|(\hat{p}_0,\hat{p})\|),$$

a point $(\hat{x}_\varepsilon, \hat{u}_\varepsilon; \hat{X}_\varepsilon, \hat{v}_\varepsilon) \in H^1_{0n} \times U \times [H^1_{0n}]^N \times U$ has been found for which

$J'_\varepsilon(\hat{x}_\varepsilon, \hat{u}_\varepsilon; \hat{X}_\varepsilon, \hat{v}_\varepsilon) = 0$ and

$$\| \hat{x}_\varepsilon - \hat{x} \|_{L^2_n} = \| \hat{\xi}_{\varepsilon,0} \|_{L^2_n} \le \| (\hat{\xi}_\varepsilon, \hat{\eta}_\varepsilon, \hat{\zeta}_\varepsilon) \|_{[L^2_n]^{N+1} \times U^2 \times [L^2_n]^{N+1}}$$

$$\le \frac{K_4}{4} \left(\max_{0 \le j \le N} \varepsilon_j \right) \| (\hat{p}_0, \hat{p}) \|_{[L^2_n]^{N+1}} ;$$

similarly,

$$\| \hat{u}_\varepsilon - \hat{u} \|_U \le \frac{K_4}{4} \left(\max_{0 \le j \le N} \varepsilon_j \right) \| (\hat{p}_0, \hat{p}) \|_{[L^2_n]^{N+1}} ,$$

$$\| \hat{X}_\varepsilon - \hat{X} \|_{[L^2_n]^N} \le \frac{K_4}{4} \left(\max_{0 \le j \le N} \varepsilon_j \right) \| (\hat{p}_0, \hat{p}) \|_{[L^2_n]^{N+1}} ,$$

$$\| \hat{v}_\varepsilon - \hat{v} \|_U \le \frac{K_4}{4} \left(\max_{0 \le j \le N} \varepsilon_j \right) \| (\hat{p}_0, \hat{p}) \|_{[L^2_n]^{N+1}} ,$$

$$\left\| \frac{2}{\varepsilon_0} \left(\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon - \sum_i B_i \hat{u}_{\varepsilon,i} - f \right) - \hat{p}_0 \right\|_{L^2_n} \le \frac{K_4}{4} \left(\max_{0 \le j \le N} \varepsilon_j \right) \| (\hat{p}_0, \hat{p}) \|_{[L^2_n]^{N+1}}$$

$$\left\| \frac{2}{\varepsilon_i} \left(\dot{\hat{x}}^i_\varepsilon - A\hat{x}^i_\varepsilon - \sum_{j \ne i} B_j \hat{u}_{\varepsilon,j} - B_i \hat{v}_{\varepsilon,i} - f \right) - (-\hat{p}_i) \right\|_{L^2_n}$$

$$\le \frac{K_4}{4} \left(\max_{0 \le j \le N} \varepsilon_j \right) \| (\hat{p}_0, \hat{p}) \|_{[L^2_n]^{N+1}} .$$

$\square$

The proof of Theorem 1.2 is complete.

Remark 1.5 From the proof given above, we see that assumption (B7) can be relaxed; we need only require that the mixed partial derivative operators $\partial_x\partial_X J, \partial_x\partial_u J, \ldots$, etc., be dominated by $\partial_x^2 J$, $\partial_u^2 J$, $\partial_X^2 J$, $\partial_v^2 J$ at $(\hat{x},\hat{u};\hat{X},\hat{v})$. □

Remark 1.6 Although $J_\varepsilon(x,u;X,v)$ is concave in (X,v) for all $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_N$, in general it is not necessarily true that $J_\varepsilon(x,u;X,v)$ is convex in (x,u). Thus $(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)$ need not be a saddle point for J_ε. Compare Lemma 2.2 later. □

Corollary 1.7 Under the conditions of Theorem 1.2, assume, in addition, that $J(x,u;X,v)$ is quadratic in the sense that

$$J(\tilde{x},\tilde{u};\tilde{X},\tilde{v}) = J(x,u;X,v) + 2J'(x,u;X,v)\cdot(\tilde{x}-x,\ \tilde{u}-u;\ \tilde{X}-X,\ \tilde{v}-v)$$
$$+ < J''(x,u;X,v)\cdot(\tilde{x}-x,\ \tilde{u}-u;\ \tilde{X}-X,\ \tilde{v}-v),\ (\tilde{x}-x,\ \tilde{u}-u;\ \tilde{X}-X,\ \tilde{v}-v) >$$

holds for all $(x,u;X,v)$, $(\tilde{x},\tilde{u};\tilde{X},\tilde{v}) \in H^1_{Cn} \times U \times [H^1_{0n}]^N \times U$. Then Theorem 1.2(i) can be strengthened to

$$(1.27)\qquad \| (\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon) - (\hat{x},\hat{u};\hat{X},\hat{v})\|_H \le K_2' \,(\max_{0\le j\le N} \varepsilon_j)\,\| (\hat{p}_0,\hat{p})\|_{[L^2_n]^{N+1}}$$

for all $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_N$ sufficiently small.

Proof: Since J is quadratic, so is J_ε. Therefore $r(\xi,\eta) = 0$ in the proof of Theorem 1.2. By (1.18), we have

$$(1.28)\qquad D_\varepsilon \begin{bmatrix} \hat{\xi}_\varepsilon \\ \hat{\eta}_\varepsilon \\ \hat{\zeta}_\varepsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{p}_\varepsilon \end{bmatrix}.$$

Thus

$$\| D_\varepsilon(\hat{\xi}_\varepsilon,\hat{\eta}_\varepsilon,\hat{\zeta}_\varepsilon)\|^2_{[L^2_n]^{N+1}\times U^2\times[L^2_n]^{N+1}} \le \|\tilde{p}_\varepsilon\|^2_{[L^2_n]^{N+1}}.$$

For $\varepsilon_0,\ldots,\varepsilon_N$ sufficiently small, it is easily seen that there exist K_5, $K_6 > 0$ such that

$$K_5 \|(\xi,\eta,\zeta)\|^2_{[L^2_n]^{N+1}\times U^2\times[L^2_n]^{N+1}} + \|D_\varepsilon(\xi,\eta,\zeta)\|^2_{[L^2_n]^{N+1}\times U^2\times[L^2_n]^{N+1}} \tag{1.29}$$

$$\geq K_6\|(\xi,\eta,\zeta)\|^2_{[H^1_{0n}]^{N+1}\times U^2\times[H^1_{n0}]^{N+1}}$$

for all $(\xi,\eta,\zeta) \in [H^1_{0n}]^{N+1}\times U^2\times[H^1_{n0}]^{N+1}$, thanks to the coercivity

$$\|\tilde{\mathcal{L}}\xi\|^2_{[H^1_{0n}]^{N+1}} \geq K_7\|\xi\|^2_{[H^1_{0n}]^{N+1}}$$

$$\|\tilde{\mathcal{L}}^*\zeta\|^2_{[H^1_{n0}]^{N+1}} \geq K_7\|\zeta\|_{[H^1_{n0}]^{N+1}} \quad .$$

Combining (1.28) and (1.29) with Theorem 1.2(i), we conclude (1.27). □

§2. Penalty for Linear Quadratic Differential Games. Finite Element Error Analysis.

For each (the i-th) player, we let his cost functional be of the same form as in [4]:

$$J_i(x,u) \equiv \tfrac{1}{2}\int_0^T \left[|C_i(t)x(t) - z_i(t)|^2 + \langle M_i(t)u_i(t),\, u_i(t)\rangle\right] dt.$$

By (1.3), we have

$$(2.1)\quad J_\varepsilon(x,u;X,v) = \frac{1}{2}\sum_{i=1}^{N}\int_0^T \Big[|C_i(t)x(t) - z_i(t)|^2 + < M_i(t)u_i(t), u_i(t) >$$

$$-|C_i(t)x^i(t) - z_i(t)|^2 - < M_i(t)v_i(t), v_i(t) >\Big]$$

$$+ \frac{1}{\varepsilon_0} \| \dot{x} - Ax - \sum_i B_iu_i - f \|^2 - \sum_i \frac{1}{\varepsilon_i} \| \dot{x}^i - Ax^i - \sum_{j\neq i} B_ju_j$$

$$- B_iv_i - f\|^2 .$$

Consider

$$(2.2)\qquad \min_{(x,u)\in H^1_{0n}\times U} \ \max_{(X,v)\times [H^1_{0n}]^N\times U} J_\varepsilon(x,u;X,v).$$

Using the notations in §1, we have

$$\mathcal{A}_0 = \partial_x^2 J = C_1^*C_1 + \ldots + C_N^*C_N$$

$$\mathcal{A}_1 = \partial_{\mathbf{x}}^2 J = -\begin{bmatrix} C_1^*C_1 & & 0 \\ & \ddots & \\ 0 & & C_N^*C_N \end{bmatrix}$$

$$\mathcal{M}_0 = \partial_u^2 J = \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_N \end{bmatrix} \quad , \quad \mathcal{M}_1 = \partial_v^2 J = -\begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_N \end{bmatrix}.$$

From now on, assume that the operators $C_i^*C_i$, $1 \le i \le N$, are all invertible. Then $\mathcal{A}_0$, $-\mathcal{A}_1$, $\mathcal{M}_0$ and $-\mathcal{M}_1$ are positive definite, and (B5) will be met.

For any given $(x,u) \in H^1_{0n} \times U$, define

$$(2.3)\qquad \bar{J}_\varepsilon(x,u) = \max_{(X,v)\in [H^1_{0n}]^N\times U} J_\varepsilon(x,u;X,v)$$

if the maximum is attained.

<u>Lemma 2.1</u> (i) $\bar{J}_\varepsilon(x,u)$ in (2.3) is well-defined.

(ii) If $B_1,\ldots,B_N$ are relatively smaller than $M_1,\ldots,M_N$, then $\bar{J}_\varepsilon(x,u)$ is strictly convex in (x,u), and

$$(2.4) \qquad \lim_{\|(x,u)\|_{H^1_{0n}\times U}\to\infty} \bar{J}_\varepsilon(x,u) = +\infty.$$

Consequently, $\min\limits_{(x,u)\in H^1_{0n}\times U} \bar{J}_\varepsilon(x,u)$ has a unique solution.

<u>Proof:</u> Since $J_\varepsilon(x,u;X,v)$ is strictly concave in (X,v), negatively coercive, i.e.,

$$\lim_{\|(X,v)\|\to\infty} J_\varepsilon(x,u;X,v) = -\infty,$$

we see that for any given (x,u), $\max\limits_{(X,v)} J_\varepsilon(x,u;X,v)$ is uniquely attained at some $(\hat{X}_\varepsilon(x,u),\hat{v}_\varepsilon(x,u))$. Solving $\max\limits_{(X,v)} J_\varepsilon(x,u;X,v)$ is equivalent to solving

$$(2.5) \qquad \max_{(X,v)\in[H^1_{0n}]^N\times U} -\sum_i \Big[J_i(x^i,u_1,\ldots u_{i-1},v_i,u_{i+1},\ldots,u_N) + \frac{1}{\varepsilon_i}\Big\|\dot{x}^i - Ax^i - \sum_{j\neq i} B_j u_j - B_i v_i - f\Big\|^2\Big].$$

For any given $u_1,\ldots,u_N$, we choose $\tilde{x}^1,\ldots,\tilde{x}^N,\tilde{v}_1, \tilde{v}_2,\ldots,\tilde{v}_{N-1}$ and $\tilde{v}_N$ such that

$$(2.6) \qquad \dot{\tilde{x}}^i - A\tilde{x}^i - \sum_{j\neq i} B_j u_j - B_i\tilde{v}_i - f = 0, \quad \tilde{x}^i(0) = 0, \ i = 1,2,\ldots,N.$$

Then

$$(2.5) \ \geq\ -\sum_{i=1}^N \Big[J_i(\tilde{x}^i,u_1,\ldots,u_{i-1},\tilde{v}_i,u_{i+1},\ldots,u_N) + \frac{1}{\varepsilon_i}\Big\|\dot{\tilde{x}}^i - A\tilde{x}^i - \sum_{j\neq i} B_j u_j - B_i\tilde{v}_i - f\Big\|^2\Big]$$

$$= -\sum_i J_i(\tilde{x}^i, u_1, \ldots, u_{i-1}, \tilde{v}_i, u_{i+1}, \ldots, u_N) .$$

From (2.6), we have

$$\tilde{x}^i(t) = \int_0^t \Phi(t,s) \Big[\sum_{j\neq i} B_j(s)u_j(s)\Big]ds + \int_0^t \Phi(t,s)\big[B_i(s)\tilde{v}_i(s) + f(s)\big]ds.$$

Thus

$$(2.5) \geq -\sum_i J_i(\tilde{x}^i, u_1, \ldots, u_{i-1}, \tilde{v}_i, u_{i+1}, \ldots, u_N)$$

$$= -\sum_i \Big\{\Big\| C_i\Big(\int_0^t \Phi(t,s) \Big[\sum_{j\neq i} B_j(s)u_j(s)\Big]ds + \int_0^t \Phi(t,s)\big[B_i(s)\tilde{v}_i(s) + f(s)\big]ds\Big) - z_i(t)\Big\|^2 + < M_i v_i, v_i >\Big\} .$$

Hence

$$\bar{J}_\varepsilon(x,u) = \sum_i J_i(x,u) + \frac{1}{\varepsilon_0} \| \dot{x} - Ax - \sum_i B_i u_i - f\|^2 + (2.5)$$

$$\geq \Big\{ \sum_i \Big[\|C_i x - z_i\|^2 + < M_i u_i, u_i >\Big] + \frac{1}{\varepsilon_0} \| \dot{x} - Ax - \sum_i B_i u_i - f\|^2$$

$$- \Big\| C_i \int_0^t \Phi(t,s) \Big[\sum_{j\neq i} B_j(s)u_j(s)\Big]ds \Big\|^2\Big\}$$

$$- 2\Big\{ \sum_i < C_i \int_0^t \Phi(t,s) \sum_{j\neq i} B_j(s)u_j(s)ds, \int_0^t \Phi(t,s)\big[B_i(s)\tilde{v}_i(s) + f(s)\big]ds>\Big\}$$

$$+ \text{ remaining terms involving only } \tilde{v}_i \text{ and } f.$$

As $\|(x,u)\|_{H^1_{0n}\times U} \to \infty$, the first parenthesized term, which is quadratic, dominates the second. Since we have assumed that $M_1,\ldots,M_N$ are positive definite and sufficiently larger than $B_1,\ldots,B_N$, we see that the first parenthesized term is positive definite in $u_1,\ldots,u_N$. Hence (2.4) is proved. □

Lemma 2.2 If $B_1,\ldots,B_N$ are relatively smaller than $M_1,\ldots,M_N$ such that $J_\varepsilon(x,u;X,v)$ is strictly convex and coerive in (x,u) for each given v, then the saddle point property

$$(2.7)\qquad \min_{(x,u)\in H^1_{0n}\times U}\ \max_{(X,v)\in[H^1_{0n}]^N\times U} J_\varepsilon(x,u;X,v)$$

$$= \max_{(X,v)\in[H^1_{0n}]^N\times U}\ \min_{(x,u)\in H^1_{0n}\times U} J_\varepsilon(x,u;X,v)$$

holds for all ε.

Proof: We know that $J_\varepsilon(x,u;X,v)$ is always strictly concave and negatively coercive in (X,v). From the proof of Lemma 2.1, we easily see that when $B_1,\ldots,B_N$ are relatively smaller than $M_1,\ldots,M_N$, $J_\varepsilon(x,u;X,v)$ is strictly convex and coercive in (x,u) for each given (X,v).

The saddle point property (2.7) follows in the some manner as the proof of Theorem 4.4 in [4]. □

Now it is not hard to see that all of the assumptions of Theorem 1.1 are met, and by Lemmas 2.1 and 2.2 we see that the saddle point $(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)$ is determined by solving

$$\partial_x J_\varepsilon\Big|_{(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)} = 0\ ,$$

$$\partial_u J_\varepsilon\Big|_{(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)} = 0\ ,$$

$$\partial_X J_\varepsilon\Big|_{(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)} = 0\ ,$$

$$\partial_v J_\varepsilon\Big|_{(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)} = 0\ .$$

Thus we can make a direct variational analysis on J_ε and obtain

$$J'_\varepsilon(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)\cdot(\delta x,\delta u;\delta X,\delta v)=0$$

$$= \sum_{i=1}^{N}\Big[<C_i\hat{x}_\varepsilon - z_i, C_i(\delta x)> + <M_i\hat{u}_{\varepsilon,i},\delta u_i> - <C_i\hat{x}^i_\varepsilon - z_i, C_i(\delta x^i)>$$

$$- <M_i\hat{v}_{\varepsilon,i},\delta v_i>\Big] + \frac{2}{\varepsilon_0}<\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon - \sum_i B_i\hat{u}_{\varepsilon,i} - f, \dot{\delta x} - A(\delta x) - \Sigma B_i(\delta u_i)\ >$$

$$- \sum_{i=1}^{N}\ \frac{2}{\varepsilon_i}<\dot{\hat{x}}^i_\varepsilon - A\hat{x}^i_\varepsilon - \sum_{j\neq i} B_j\hat{u}_{\varepsilon,j} - B_i\hat{v}_{\varepsilon,i} - f, \delta\dot{x}^i - A(\delta x^i)$$

$$- \sum_{j\neq i} B_j(\delta u_j) - B_i(\delta v_i) >$$

for all $(\delta x,\delta u;\delta X,\delta v)\in H^1_{0n}\times U\times [H^1_{0n}]^N\times U$. This gives the following variational equation

$$\text{(2.8)}\qquad a_\varepsilon\left(\begin{bmatrix}\hat{x}_\varepsilon\\ \hat{u}_\varepsilon\\ \hat{X}_\varepsilon\\ \hat{v}_\varepsilon\end{bmatrix},\begin{bmatrix}\delta x\\ \delta u\\ \delta X\\ \delta v\end{bmatrix}\right)=\theta_\varepsilon\left(\begin{bmatrix}\delta x\\ \delta u\\ \delta X\\ \delta v\end{bmatrix}\right),$$

where a_ε is a bilinear form defined by

$$a_\varepsilon\left(\begin{bmatrix}\xi_1\\ \mu_1\\ \Xi_1\\ \nu_1\end{bmatrix},\begin{bmatrix}\xi_2\\ \mu_2\\ \Xi_2\\ \nu_2\end{bmatrix}\right)\equiv\sum_{i=1}^{N}\Big[<C_i\xi_1,C_i\xi_2> + <M_i\mu_{1,i},\ \mu_{2,i}> - <C_i\Xi^i_1,\ C_i\Xi^i_2>$$

$$- <M_1\nu_{1,i},\ \nu_{2,i}>\Big] + \frac{2}{\varepsilon_0}<\dot{\xi}_1 - A\xi_1 - \sum_i B_i\mu_{1,i}, \dot{\xi}_2 - A\xi_2 - \sum_i B_i\mu_{2,i}>$$

$$- \sum_i\frac{2}{\varepsilon_i}<\dot{\Xi}^i_1 - A\Xi^i_1 - \sum_{j\neq i} B_j\mu_{1,j} - B_i\nu_{1,i}, \dot{\Xi}^i_2 - A\Xi^i_2 - \sum_{j\neq i} B_j\mu_{2,j} - B_i\nu_{2,i}>$$

and θ_ε is a linear form defined by

$$\theta_\varepsilon \left(\begin{bmatrix}\xi\\ \mu\\ \Xi\\ \nu\end{bmatrix}\right) = \sum_i \left[< z_i, C_i\xi> - <z_i, C_i\Xi^i>\right] + \frac{2}{\varepsilon_0} < f, \dot{\xi} - A\xi - \sum B_i\mu_i >$$

$$- \sum_i \frac{2}{\varepsilon_i} < f, \dot{\Xi}^i - A\Xi^i - \sum_{j\neq i} B_j\mu_j - B_i\nu_i >,$$

for $(\xi,\mu;\Xi,\nu)$, $(\xi_1,\mu_1;\Xi_1,\nu_1)$ and $(\xi_2,\mu_2;\Xi_2,\nu_2) \in H^1_{0n}\times U\times[H^1_{0n}]^N\times U$.

We assume that $\exists\Gamma > 0$ such that for all ε sufficiently small, a_ε satisfies

$$\text{(H1)} \qquad \inf_{\left\|\begin{bmatrix}\xi_2\\ \mu_2\\ \Xi_2\\ \nu_2\end{bmatrix}\right\|_H=1} \quad \sup_{\left\|\begin{bmatrix}\xi_1\\ \mu_1\\ \Xi_1\\ \nu_1\end{bmatrix}\right\|_H=1} \quad a_\varepsilon \left(\begin{bmatrix}\xi_1\\ \mu_1\\ \Xi_1\\ \nu_1\end{bmatrix}, \begin{bmatrix}\xi_2\\ \mu_2\\ \Xi_2\\ \nu_2\end{bmatrix}\right) \geq \Gamma > 0.$$

How realistic is the above assumption? This is partly answered in

<u>Proposition 2.3</u> If $B_1,\ldots,B_N$ are comparatively smaller than $M_1,\ldots,M_N$, then (H1) is valid.

<u>Proof:</u> For any given $(\xi_1,\mu_1;\Xi_1,\nu_1) \in H$ with unit norm, if we choose $\tilde{\Xi}_1 \in [H^1_{0n}]^N$ such that

$$\text{(2.9)} \qquad \dot{\tilde{\Xi}}^i_1 - A\tilde{\Xi}^i_1 - \sum_{j\neq i} B_j\mu_{1,j} - B_i\nu_{1,i} = -\dot{\Xi}^i_1 + A\Xi^i_1 + \sum_{j\neq i} B_j\mu_{1,j} + B_1\nu_{1,i},$$

thus

$$\tilde{\Xi}^i_1(t) = -\Xi^i_1(t) + 2\int_0^t \Phi(t,s)\left[\sum_{j\neq i} B_j\mu_{1,j} + B_i\nu_{1,i}\right]ds,$$

then

$$(2.10)\qquad \sup_{\left\|\begin{bmatrix}\xi_2\\ \mu_2\\ \Xi_2\\ \nu_2\end{bmatrix}\right\|_H=1} a_\varepsilon\left(\begin{bmatrix}\xi_1\\ \mu_1\\ \Xi_1\\ \nu_1\end{bmatrix},\begin{bmatrix}\xi_2\\ \mu_2\\ \Xi_2\\ \nu_2\end{bmatrix}\right) \geq \frac{1}{\|(\xi_1,\mu_1;\tilde{\Xi}_1,-\nu_1)\|_H} a_\varepsilon\left(\begin{bmatrix}\xi_1\\ \mu_1\\ \Xi_1\\ \nu_1\end{bmatrix},\begin{bmatrix}\xi_1\\ \mu_1\\ \Xi_1\\ \nu_1\end{bmatrix}\right)$$

$$= \frac{1}{\|(\xi_1,\mu_1;\tilde{\Xi}_1,-\nu_1)\|_H}\Big\{\sum_{i=1}^{N}\Big[< C_i\xi_1,\ C_i\xi_1 > + < M_i\mu_{1,i}\ ,\ \mu_{1,i} >$$

$$+ < C_i\Xi_1^i,\ C_i\Big[\Xi_1^i - 2\int_0^t \Phi(t,s)\ \big(\sum_{j\neq i} B_j\mu_{1,j} + B_1\nu_{1,i}\big)ds\Big]> + < M_i\nu_{1,i},\ \nu_{1,i}>\Big]$$

$$+ \frac{2}{\varepsilon_0}\ \|\ \dot{\xi}_1 - A\xi_1 - \sum_i\ B_i\mu_{1,i}\|^2\ + \sum_i \frac{2}{\varepsilon_i}\ \|\ \dot{\Xi}_1^i - A\Xi_1^i - \sum_{j\neq i} B_j\mu_{1,j} - B_i\nu_{1,i}\|^2 .$$

But

$$\sum_i < C_i\Xi_1^i,\ C_i\Xi_1^i - 2C_i\int_0^t \Phi(t,s)\ \big(\sum_{j\neq i} B_j\mu_{1,j} + B_i\nu_{1,i}\big)ds >$$

$$\geq \sum_i\Big[\|\ C_i\Xi_1^i\|^2 - \tfrac{1}{2}\|\ C_i\Xi_1^i\|^2 - 2\|\ C_i\int_0^t \Phi(t,s)\ \big(\sum_{j\neq i} B_j\mu_{1,j} + B_i\nu_{1,i}\big)ds\|^2\ \Big]$$

$$= \tfrac{1}{2}\sum_i\ \|\ C_i\Xi_1^i\|^2 - 2\ \sum_i\|\ C_i\int_0^t \Phi(t,s)\ \big(\sum_{j\neq i} B_j\mu_{1,j} + B_i\nu_{1,i}\big)ds\|^2 .$$

The second term above can be absorbed into a fraction of $\sum_i\ \big[<M_i\mu_{1,i},\ \mu_{1,i} > + < M_i\nu_{1,i},\ \nu_{1,i}>\big]$ provided that $M_1,\ldots,M_N$ are comparatively larger than $B_1,\ldots,B_N$, i.e., we have

$$(2.11)\qquad \text{LHS of (2.14)} \geq \frac{1}{\|(\xi_1,\mu_1;\tilde{\Xi}_1,-\nu_1)\|_H}\cdot\alpha\Big\{\sum_{i=1}^{N}\ \big[<C_i\xi_1,\ C_i\xi_1 >$$

$$+ < M_i\mu_{1,i},\ \mu_{1,i} > + < C_i\Xi_1^i,\ C_i\Xi_1^i > + < M_i\nu_{1,i},\ \nu_{1,i} >\big] + \frac{2}{\varepsilon_0}\ \|\ \dot{\xi}_1 - A\xi_1$$

$$- \sum_i B_i\mu_{1,i}\|^2 + \sum_i \frac{2}{\varepsilon_i}\ \|\ \dot{\Xi}_1^i - A\Xi_1^i - \sum_{j\neq i} B_j\mu_{1,j} - B_i\nu_{1,i}\|^2\Big\}\ ,$$

for some α: $0 < \alpha < \tfrac{1}{2}$.

Let $\beta \in (0,1)$ be fixed. For the terms on the RHS of (2.11), if $(\xi_1, \mu_1; \Xi_1, \nu_1)$ satisfies

$$\frac{1}{\varepsilon_0} \|\dot{\xi}_1 - A\xi_1 - \sum_i B_i \mu_{1,i}\|^2 \geq \beta \|\dot{\xi}_1 - A\xi_1\|^2 ,$$

(2.12)

$$\frac{1}{\varepsilon_i} \|\dot{\Xi}^i_1 - A\Xi^i_1 - \sum_{j \neq i} B_j \mu_{1,j} - B_i \nu_{1,i}\|^2 \geq \beta \|\dot{\Xi}^i_1 - A\Xi^i_1\|^2 , \quad i = 1,\ldots,N,$$

then we easily observe that

$$\text{(RHS) of (2.11)} \geq \Gamma > 0 , \quad \text{for some } \Gamma \text{ independent of } (\xi_1, \mu_1; \Xi_1, \nu_1),$$

is satisfied.

If, on the contrary, say, for $i = 1$, we have

$$\text{(2.13)} \qquad \frac{1}{\varepsilon_1} \|\dot{\Xi}^1_1 - A\Xi^1_1 - \sum_{j=2}^{N} B_j \mu_{1,j} - B_1 \nu_{1,1}\|^2 \leq \beta \|\dot{\Xi}^1_1 - A\Xi^1_1\|^2,$$

while the rest of (2.12) remains valid, then (2.13) gives

$$\text{(2.14)} \qquad \|\dot{\Xi}^1_1 - A\Xi^1_1\|^2 + \|\sum_{j=2}^{N} B_j \mu_{1,j} + B_1 \nu_{1,1}\|^2 \leq \beta\varepsilon_1 \|\dot{\Xi}^1_1 - A\Xi^1_1\|^2 + 2 < \dot{\Xi}^1_1 - A\Xi^1_1, \sum_{j=2}^{N} B_j \mu_{1,j} + B_1 \nu_{1,1} >$$

$$(1-\beta\varepsilon_1)\|\dot{\Xi}^1_1 - A\Xi^1_1\|^2 + \|\sum_{j=2}^{N} B_j \mu_{1,j} + B_1 \nu_{1,1}\|^2 \leq \tfrac{1}{4}\|\dot{\Xi}^1_1 - A\Xi^1_1\|^2 + 4\|\sum_{j=2}^{N} B_j \mu_{1,j} + B_1 \nu_{1,1}\|^2$$

$$(1 - \beta\varepsilon_1 - \tfrac{1}{4}) \|\dot{\Xi}^1 - A\Xi^1\|^2 \leq 3 \|\sum_{j=2}^{N} B_j \mu_{1,j} + B_1 \nu_{1,1}\|^2.$$

Hence

$$\|\sum_{j=2}^{N} B_j \mu_{1,j} + B_1 \nu_{1,1}\|^2 \geq \frac{1}{3} (1 - \beta\varepsilon_1 - \frac{1}{4}) \|\dot{\Xi}^1_1 - A\Xi^1_1\|^2 .$$

Because M_i, $1 \leq i \leq N$ are positive definite operators considerably larger than B_i, $1 \leq i \leq N$, we have

$$(2.15) \qquad \alpha' \sum_i \left[\langle M_i\mu_{1,i}, \mu_{1,i}\rangle + \langle M_i\nu_{1,i},\nu_{1,i}\rangle\right] \geq \left\| \sum_{j=2}^{N} B_j\mu_{1,j} + B_1\nu_{1,1}\right\|^2$$

for some α';: $0 < \alpha' < 1$. Therefore, for some $\alpha'' < \alpha$, $\alpha'' > 0$, we have

$$\text{(LHS) of (2.11)} \geq \frac{1}{\|(\xi_1,\mu_1;\Xi_1,-\nu_1)\|_H} \alpha''\{\sum_i \left[\langle C_i\xi_1, C_i\xi_1\rangle\right.$$

$$+ \langle M_i\mu_{1,i}, \mu_{1,i}\rangle + \langle C_i\Xi_1^i, C_i\Xi_1^i\rangle + \left.\langle M_i\nu_{1,i},\nu_{1,i}\rangle\right]$$

$$+\frac{1}{\varepsilon_0} \| \dot{\xi}_1 - A\xi_1 - \sum_i B_i\mu_{1,i}\|^2 + \sum_{i=2}^{N} \beta\| \dot{\Xi}_1^i - A\Xi_1^i\|^2$$

$$+\frac{1}{3} (1 - \beta\varepsilon_1 - \frac{1}{4})\|\dot{\Xi}_1^1 - A\Xi_1^1\|^2\} .$$

Hence the LHS of (2.11) is again $\geq \Gamma > 0$ for some Γ, independent of $(\xi_1,\mu_1;\Xi_1, \nu_1)$ and $\varepsilon_0,\ldots,\varepsilon_N$. Therefore (H1) is realistic. □

We now let $S_h^0 \subset H_n^1(0,T)$ be a $(\tau_0,1)$-system ([1], [4]) and let $S_h^i \subset L^2_{m_i}(0,T)$, $i = 1,2,\ldots,N$ be $(\tau,0)$-systems, and denote

$$(2.16) \qquad S_h = S_h^0 \times (\prod_{i=1}^{N} S_h^i) \times (\prod_{i=1}^{N} S_h^0) \times (\prod_{i=1}^{N} S_h^i)$$

We assume, furthermore, that

$$\text{(H2)} \quad \inf_{\left\|\begin{bmatrix}\xi_2^h\\ \mu_2^h\\ \Xi_2^h\\ \nu_2^h\end{bmatrix}\right\|_H=1} \ \sup_{\left\|\begin{bmatrix}\xi_1^h\\ \mu_1^h\\ \Xi_1^h\\ \nu_1^h\end{bmatrix}\right\|_H=1} a_\varepsilon\left(\begin{bmatrix}\xi_1^h\\ \mu_1^h\\ \Xi_1^h\\ \nu_1^h\end{bmatrix}, \begin{bmatrix}\xi_2^h\\ \mu_2^h\\ \Xi_2^h\\ \nu_2^h\end{bmatrix}\right) \geqslant \Gamma_h \geqslant \Gamma > 0, \quad \forall h,$$

wherein $(\xi_1^h,\mu_1^h;\Xi_1^h,\nu_1^h)$, $(\xi_2^h,\mu_2^h;\Xi_2^h,\nu_2^h) \in S_h$. It should be noted that if

$B_1,\ldots,B_N$ are sufficiently small compared with $M_1,\ldots,M_N$, then (H2) is also valid.

We consider

$$(2.17)\qquad \min_{(x,u)\in S_h^0\times(\prod_{i=1}^N S_h^i)} \ \max_{(X,v)\in(\prod_{i=1}^N S_h^0)\times(\prod_{i=1}^N S_h^i)} J_\varepsilon(x,u;X,v).$$

Arguing in the same manner as in the early part of this section, we see that (2.17) leads to finding the solution $(\hat{x}^h_\varepsilon,\hat{u}^h_\varepsilon;\hat{X}^h_\varepsilon,\hat{v}^h_\varepsilon)$ of the variational equation

$$(2.18)\quad a_\varepsilon\left(\begin{bmatrix}\hat{x}^h_\varepsilon\\ \hat{u}^h_\varepsilon\\ \hat{X}^h_\varepsilon\\ \hat{v}^h_\varepsilon\end{bmatrix},\begin{bmatrix}\delta x^h\\ \delta u^h\\ \delta X^h\\ \delta v^h\end{bmatrix}\right)=\theta_\varepsilon\left(\begin{bmatrix}\delta x^h\\ \delta u^h\\ \delta X^h\\ \delta v^h\end{bmatrix}\right),\qquad (\delta x^h,\delta u^h,\delta X^h,\delta v^h)\in S_h.$$

Let $\{\phi_0^{i_0}\}_{i_0=1}^{I_0}\times\{\phi_1^{i_1}\}_{i_1=1}^{I_1}\times\ldots\times\{\phi_N^{i_N}\}_{i_N=1}^{I_N}$ be a basis for $S_h^0\times S_h^1\times\ldots\times S_h^N$.

Then (2.18) is a matrix equation $\bar{M}^h_\varepsilon\,\bar{q}_h=\bar{\theta}^h_\varepsilon$, where the matrix $\bar{M}^h_\varepsilon$ and the vector $\bar{\theta}^h_\varepsilon$ have entries

$$[\bar{M}^h_\varepsilon]_{ij}=a_\varepsilon(\psi_i,\psi_j)\qquad ;\ \psi_i,\psi_j\in S_h,$$

$$\psi_i=(\psi_0^{i_0},(\psi_1^{i_1},\ldots,\psi_N^{i_N}),(\bar{\psi}_0^{\bar{i}_1},\bar{\psi}_0^{\bar{i}_2},\ldots,\bar{\psi}_0^{\bar{i}_N}),(\tilde{\psi}_1^{\tilde{i}_1},\tilde{\psi}_2^{\tilde{i}_2},\ldots,\tilde{\psi}_N^{\tilde{i}_N}))$$

$$\psi_j=(\psi_0^{j_0},(\psi_1^{j_1},\ldots,\psi_N^{j_N}),(\bar{\psi}_0^{\bar{j}_1},\bar{\psi}_0^{\bar{j}_2},\ldots,\bar{\psi}_0^{\bar{j}_N}),(\tilde{\psi}_1^{\tilde{j}_1},\tilde{\psi}_2^{\tilde{j}_2},\ldots,\tilde{\psi}_N^{\tilde{j}_N}))$$

$$(\bar{\theta}^h_\varepsilon)_i=\theta_\varepsilon(\psi_i);\qquad \psi_i\in S_h,$$

for all i, j: $1\le i, j\le (N+1)I_0+2\sum_{j=1}^N I_j$, where in the above,

$$\psi_0^{i_0},\quad \psi_0^{j_0},\quad \bar{\psi}_0^{\bar{i}_1},\ldots,\ \bar{\psi}_0^{\bar{i}_N} \in \{\psi_0^{i_0}\}_{i_0=1}^{I_0};\ \psi_1^{i_1},\ \tilde{\psi}_1^{\tilde{i}_1},\ \psi_1^{j_1},\ \tilde{\psi}_1^{\tilde{j}_1} \in \{\phi_1^{i_1}\}_{i_1=1}^{I_1};\ldots;$$

$$\psi_N^{i_N},\quad \tilde{\psi}_N^{\tilde{i}_N},\quad \psi_N^{j_N},\quad \tilde{\psi}_N^{\tilde{j}_N} \in \{\phi_N^{i_N}\}_{i_n=1}^{I_N}.$$

Blockwise, we can write

$$\bar{M}_\varepsilon^h = \begin{bmatrix} \bar{M}_\varepsilon^h(1,1) & \bar{M}_\varepsilon^h(1,2) & \bar{M}_\varepsilon^h(1,3) & \bar{M}_\varepsilon^h(1,4) \\ \bar{M}_\varepsilon^h(2.1) & \bar{M}_\varepsilon^h(2,2) & \bar{M}_\varepsilon^h(2,3) & \bar{M}_\varepsilon^h(2,4) \\ \bar{M}_\varepsilon^h(3,1) & \bar{M}_\varepsilon^h(3,2) & \bar{M}_\varepsilon^h(3,3) & \bar{M}_\varepsilon^h(3,4) \\ \bar{M}_\varepsilon^h(4,1) & \bar{M}_\varepsilon^h(4,2) & \bar{M}_\varepsilon^h(4,3) & \bar{M}_\varepsilon^h(4,4) \end{bmatrix}$$

$$\bar{\theta}_\varepsilon^h = \begin{bmatrix} \bar{\theta}_\varepsilon^h(1) \\ \bar{\theta}_\varepsilon^h(2) \\ \bar{\theta}_\varepsilon^h(3) \\ \bar{\theta}_\varepsilon^h(4) \end{bmatrix},$$

wherein

$$\left[\bar{M}_\varepsilon^h(1,1)\right]_{i_0 j_0} = \sum_{k=1}^{N}\left[< C_k\psi_0^{i_0},\ C_k\psi_0^{j_0} > + \frac{2}{\varepsilon_0} <\dot{\psi}_0^{i_0} - A\psi_0^{i_0},\ \dot{\psi}_0^{j_0} - A\psi^{j_0} >\right] \quad ;$$

$$1 \le i_0, j_0 \le I_0,$$

$$\left[\bar{M}_\varepsilon^h(2,1)\right]_{i_0\left[\sum_{j=1}^{\max(1,\ell-1)} I_j\cdot \mathrm{sgn}(j-1)+j_\ell\right]} = -\frac{2}{\varepsilon_0} <\dot{\psi}_0^{i_0} - A\psi_0^{i_0},\ B_\ell\psi_\ell^{j_\ell}>; \quad \ell = 1,2,\ldots,N;$$

$$1 \le i_0 \le I_0; \qquad 1 \le j_\ell \le I_\ell; \qquad \mathrm{sgn}\ a \equiv \begin{cases} \dfrac{a}{|a|} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0; \end{cases}$$

$\bar{M}_\varepsilon^h(3,1) \equiv 0;$

$\bar{M}_\varepsilon^h(4,1) \equiv 0;$

$[\bar{M}_\varepsilon^h(2,2)]$

$$\Big(\sum_{j=1}^{\max(1,\ell-1)} I_j \cdot \mathrm{sng}(j-1)+i_\ell\Big)\Big(\sum_{k=1}^{\max(1,p-1)} I_k \cdot \mathrm{sgn}(k-1)+i_p\Big)$$

$$= \delta_{p\ell} < M_\ell \psi_\ell^{i_\ell},\ \psi_p^{i_p} > + \frac{2}{\varepsilon_0} < B_\ell \psi_\ell^{i_\ell},\ B_p \psi_p^{i_p} > - \sum_{i=1}^{N} (1-\delta_{i\ell})\ (1-\delta_{ip}) \frac{2}{\varepsilon_i}$$

$$\cdot < B_\ell \psi_\ell^{i_\ell},\ B_p \psi_p^{i_p} >,$$

$1 \le p,\ \ell \le N; \quad 1 \le i_\ell \le I_\ell; \quad 1 \le i_p \le I_p;$

$[\bar{M}_\varepsilon^h(3,2)]$

$$\Big(\sum_{j=1}^{\max(1,\ell-1)} I_j \cdot \mathrm{sgn}(j-1)+i_\ell\Big)(\mathrm{sgn}(p-1)\cdot(p-1)I_0+i_p)$$

$$= (1-\delta_{p\ell}) \frac{2}{\varepsilon_\ell} < B_\ell \psi_\ell^{i_\ell},\ \dot{\psi}_0^{i_p} - A\psi_0^{i_p} >;$$

$1 \le p,\ell \le N; \quad 1 \le i_\ell \le I_\ell; \quad 1 \le i_p \le I_0;$

$[M_\varepsilon^{-h}(4,2)]$

$$\Big(\sum_{j=1}^{\max(1,\ell-1)} I_j \cdot \mathrm{sgn}(j-1)+i_\ell\Big)\Big(\sum_{k=1}^{\max(1,p-1)} I_k \cdot \mathrm{sgn}(k-1)+i_p\Big)$$

$$= -(1-\delta_{p\ell}) \frac{2}{\varepsilon_\ell} < B_\ell \psi_\ell^{i_\ell}, B_p \psi_p^{i_p} >;$$

$1 \le p,\ell \le N; \quad 1 \le i_\ell \le I_0; \quad 1 \le i_p \le I_p.$

$[\bar{M}^h(3,3)]$

$$(\mathrm{sgn}(\ell-1)\cdot(\ell-1)I_0+i_\ell)(\mathrm{sgn}(p-1)\cdot(p-1)I_0+i_p)$$

$$= -\sum_j < C_j \psi_0^{i_\ell},\ C_j \psi_0^{i_p} > - \frac{2}{\varepsilon_\ell} < \dot{\psi}_0^{i_\ell} - A\psi_0^{i_\ell},\ \dot{\psi}_0^{i_p} - A\psi_0^{i_p} >;$$

$1 \le p,\ell \le N; \quad 1 \le i_\ell \le I_0; \quad 1 \le i_p \le I_p;$

$[\bar{M}^h_\varepsilon(4,3)]$

$$(\mathrm{sgn}(\ell-1)\cdot(\ell-1)I_0+i_\ell)(\sum_{k=1}^{\max(1,p-1)} I_k\cdot\mathrm{sgn}(k-1)+i_p)$$

$$= \delta_{p\ell}\cdot\frac{2}{\varepsilon_\ell} < \dot{\psi}_0^{i_\ell}-A\psi_0^{i_\ell}, \dot{\psi}_p^{i_p}-A\psi_p^{i_p} > \quad ;$$

$$1 \le p,\ell \le N; \quad 1 \le i_\ell \le I_0; \quad 1 \le i_p \le I_p;$$

$[\bar{M}^h_\varepsilon(4,4)]$

$$(\sum_{j=1}^{\max(1,p-1)} I_j\cdot\mathrm{sgn}(j-1)+i_\ell)(\sum_{j=1}^{\max(1,p-1)} I_j\cdot\mathrm{sgn}(j-1)+i_p)$$

$$= -\delta_{\ell p}[\frac{2}{\varepsilon_\ell} <B_\ell\psi_\ell^{i_\ell}, B_p\psi_p^{i_p}> + <M_\ell\psi_\ell^{i_\ell};\psi_p^{i_p}>] \quad ;$$

$$1 \le p,\ell \le N; \quad 1 \le i_\ell \le I_\ell; \quad 1 \le i_p \le I_p;$$

and $\bar{M}^h_\varepsilon(q,r) = \bar{M}^{h^*}_\varepsilon(r,q)$ for $q < r$, $1 \le q,r \le 4$;

$$[\bar{\theta}^h_\varepsilon(1)]_{i_0} = \sum_{i=1}^{N} < z_i, C_i\psi_0^{i_0} + \frac{2}{\varepsilon_0} < f, \dot{\psi}_0^{i_0} - A\psi_0^{i_0}>; \quad 1 \le i_0 \le I_0;$$

$$[\bar{\theta}^h_\varepsilon(2)]_{(\sum_{j=1}^{\max(1,\ell-1)} I_j\cdot\mathrm{sgn}(j-1)+i_\ell)} = -\frac{2}{\varepsilon_0} < f, B_\ell\psi_\ell^{i_\ell} > \quad ;$$

$$1 \le \ell \le N; \quad 1 \le i_\ell \le I_\ell;$$

$$[\bar{\theta}^h_\varepsilon(3)]_{((\ell-1)\cdot\mathrm{sgn}(\ell-1)\cdot I_0+i_\ell)} = -<z_\ell, C_\ell\psi_0^{i_\ell} > - \frac{2}{\varepsilon_\ell} <f, \dot{\psi}_0^{i_\ell} - A\psi_0^{i_\ell} >;$$

$$1 \le \ell \le N; \quad 1 \le i_\ell \le I_0;$$

$$[\bar{\theta}^h(4)]_{(\sum_{j=1}^{\max(1,\ell-1)} I_j\cdot\mathrm{sgn}(j-1)+i_\ell)} = \frac{2}{\varepsilon_\ell} < f, B_\ell\psi_\ell^{i_\ell}>;$$

$$1 \le \ell \le N; \quad 1 \le i_\ell \le I_\ell.$$

<u>Theorem 2.4</u> Let $\{S_h\}$ be a one-parameter family of finite element spaces as mentioned in (2.16). Let $(\hat{x}^h_\varepsilon, \hat{u}^h_\varepsilon; \hat{X}^h_\varepsilon, \hat{u}^h_\varepsilon) \in S_h$ be the solution of (2.18) Assume that $(\hat{x}_\varepsilon, \hat{u}_\varepsilon; \hat{X}_\varepsilon, \hat{v}_\varepsilon)$, the solution of (2.8) (or (2.2)), belongs to $H^{s_1}_n \times (\prod_{i=1}^{N} H^{s_2}_{m_i}) \times ([H^{s_1}_n]^N) \times (\prod_{i=1}^{N} H^{s_2}_{m_i})$. Let $(\hat{x}, \hat{u}; \hat{X}, \hat{v})$ be the solution of (1.2) Under (H1) and (H2), we have

$$(2.19)\quad \|\hat{x}^h_\varepsilon - \hat{x}\|_{H^1_n} + \|\hat{u}^h_\varepsilon - \hat{u}\|_U + \|\hat{X}^h_\varepsilon - \hat{X}\|_{[H^1_n]^N} + \|\hat{v}^h_\varepsilon - \hat{v}\|_U$$

$$\leq \left(1 + \frac{1}{\Gamma}\frac{K_8}{\min\limits_{0\leq i\leq N} \varepsilon_i}\right) h^\mu \|(\hat{x}_\varepsilon, \hat{v}_\varepsilon; \hat{X}_\varepsilon, \hat{v}_\varepsilon)\|_{H^{s_1}_n \times (\prod_{i=1}^{N} H^{s_2}_{m_i}) \times [H^{s_1}_n]^N \times (\prod_{i=1}^{N} H^{s_2}_{m_i})}$$

$$+ K_2' \left(\max_{0\leq j\leq N} \varepsilon_j\right) \|(\hat{p}_0, \hat{p})\|_{[L^2_n]^{N+1}}$$

for some constant $K_8 > 0$ independent of h, ε and $(\hat{x}_\varepsilon, \hat{u}_\varepsilon; \hat{X}_\varepsilon, \hat{v}_\varepsilon)$, where $\mu = \min(\tau_0 - 1, \tau, s_1 - 1, s_2)$, $(\hat{p}_0, \hat{p})$ is the dual multiplier, and K_2' is the same constant as in (1.27).

<u>Proof:</u> We use the triangle inequality

$$(2.20)\qquad \|(\hat{x}^h_\varepsilon, \hat{u}^h_\varepsilon; \hat{X}^h_\varepsilon, \hat{v}^h_\varepsilon) - (\hat{x}, \hat{u}; \hat{X}, \hat{v})\|_H$$

$$\leq \|(\hat{x}^h_\varepsilon, \hat{u}^h_\varepsilon; \hat{X}^h_\varepsilon, \hat{v}^h_\varepsilon) - (\hat{x}_\varepsilon, \hat{u}_\varepsilon; \hat{X}_\varepsilon, \hat{v}_\varepsilon)\|_H + \|(\hat{x}_\varepsilon, \hat{v}_\varepsilon; \hat{X}_\varepsilon, \hat{v}_\varepsilon) - (\hat{x}, \hat{u}; \hat{X}, \hat{v})\|_H.$$

Since a_ε satisfies

$$|a_\varepsilon(\phi,\psi)| \leq \frac{K_8}{\min\limits_{0\leq i\leq N} \varepsilon_i} \|\phi\|_H \|\psi\|_H$$

for some $K_8 > 0$ for all $\phi, \psi \in H$, by assumptions (H1) and (H2), and [1,p.186], we have

$$(2.21)\quad \|(\hat{x}^h_\varepsilon,\hat{u}^h_\varepsilon;\hat{X}^h_\varepsilon,\hat{v}^h_\varepsilon) - (\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)\|_H \leq (1 + \frac{1}{\Gamma}\frac{K_8}{\min\limits_{0\leq i\leq N} \varepsilon_i})h^\mu$$

$$\cdot\|(\hat{x}_\varepsilon,\hat{v}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)\|_{H^{s_1}_n\times(\prod_{i=1}^N H^{s_2}_{m_i})\times[H^{s_1}_n]^N\times(\prod_{i=1}^N H^{s_2}_{m_i})}.$$

with $\mu = \min((\tau_0-1),\tau,s_1-1,s_2)$.

Combining (2.20), (2.21) and Corollary 1.7, we conclude (2.19). □

From Theorem 2.4, we see that if

$$\lim_{\varepsilon_0,\ldots,\varepsilon_N\downarrow 0} \|(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)\|_{H^{s_1}_n\times(\prod_{i=1}^N H^{s_2}_{m_i})\times([H^{s_1}_n]^N)\times(\prod_{i=1}^N H^{s_2}_{m_i})} < \infty,$$

then the error estimate is of the order of magnitude

$$(2.22)\qquad \|(\hat{x}^h_\varepsilon,\hat{u}^h_\varepsilon;\hat{X}^h_\varepsilon,\hat{v}^h_\varepsilon) - (\hat{x},\hat{u};\hat{X},\hat{v})\|_H = (\frac{h^\mu}{\min\limits_{0\leq i\leq N} \varepsilon_i} + \max_{0\leq i\leq N} \varepsilon_i),\ \forall\ \varepsilon,\ \forall\ h.$$

Thus if we choose $\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_N \equiv \bar{\varepsilon}$ and $\bar{\varepsilon} = \mathcal{O}(h^{\mu/2})$, the RHS of (2.22) is optimal and we have

$$\|(\hat{x}^h_\varepsilon,\hat{u}^h_\varepsilon;\hat{X}^h_\varepsilon,\hat{v}^h_\varepsilon) - (\hat{x},\hat{u};\hat{X},\hat{v})\|_H = \mathcal{O}(h^{\mu/2}).$$

§3. Duality and Penalty

The relationship between penalty has already been indicated in Theorem 1.2: we see that the Lagrange multipliers $\hat{p}_0,\ldots,\hat{p}_N$ are actually the strong limits of (some scalar multiples of) the penalized differential equation, and the rate of convergence is $\mathcal{O}(\varepsilon)$.

Let us explore this relationship a little further here. Consider, as in (2.2),

$$(3.1) \quad \min_{(x,u)\in H^1_{0n}\times U} \; \max_{(X,v)\in[H^1_{0n}]^N\times U} J_\varepsilon(x,u;X,v)$$

$$\equiv \frac{1}{2}\sum_{i=1}^{N}\Big[\| C_i x - z_i \|^2 + \langle M_i u_i, u_i\rangle - \| C_i x^i - z_i\|^2 - \langle M_i v_i, v_i\rangle$$

$$- \frac{2}{\varepsilon_i}\|\dot{x}^i - Ax^i - \sum_{j\neq i} B_j u_j - B_i v_i - f\|^2\Big] + \frac{1}{\varepsilon_0}\|\dot{x} - Ax - \sum_i B_i u_i - f\|^2 .$$

We can regard the above as a <u>primal</u> min-max problem subject to constraints $\dot{x} = \frac{d}{dt}x$, $\dot{x}^1 = \frac{d}{dt}x^1,\ldots,\dot{x}^N = \frac{d}{dt}x^N$. Thus, formally, we introduce Lagrange multipliers $p_0,p_1,\ldots,p_N$ and consider

$$(3.2) \quad \max_{p_0\in L^2_n} \; \min_{p\in[L^2_n]^N} \; \min_{(x,u)\in H^1_{0n}\times U} \; \max_{(X,v)\in[H^1_{0n}]^N\times U} L_\varepsilon(p_0,p;x,u;X,v)$$

where

$$(3.3) \quad L_\varepsilon(p_0,p;x,u;X,v) \equiv J_\varepsilon(x,u;X,v) + \langle p_0, \frac{d}{dt}x - \dot{x}\rangle + \sum_{i=1}^{N}\langle p_i, \frac{d}{dt}x^i - \dot{x}^i\rangle ,$$

$$p = (p_1,\ldots,p_N).$$

For given $p_0,p_1,\ldots,p_N \in H^1_{n0}$, proceeding formal variational analysis as in [4], we get

$$\sum_{i=1}^{N}\{<C_i^*(C_i\hat{x}_\varepsilon - z_i),\delta x> + <M_i\hat{u}_{\varepsilon i}, \delta u_i> - <C_i^*(C_i\hat{x}_\varepsilon^i - z_i), \delta x^i> - <M_i\hat{v}_{\varepsilon i}, \delta v_i>$$

$$- \frac{2}{\varepsilon_i}\Big[<\dot{\hat{x}}_\varepsilon^i - A\hat{x}_\varepsilon^i - \sum_{j\neq i} B_j\hat{u}_{\varepsilon j} - B_i\hat{v}_{\varepsilon i} - f, \delta x^i> - <A^*(\dot{\hat{x}}_\varepsilon^i - A\hat{x}_\varepsilon^i - \sum_{j\neq i} B_j\hat{u}_{\varepsilon j} - f), \delta x^i$$

$$- \sum_{j\neq i} <B_j^*(\dot{\hat{x}}_\varepsilon^i - A\hat{x}_\varepsilon^i - \sum_{k\neq i} B_k\hat{u}_{\varepsilon k} - B_i\hat{v}_{\varepsilon i} - f), \delta u_j> - <B_i^*(\dot{\hat{x}}_\varepsilon^i - A\hat{x}_\varepsilon^i - \sum_{k\neq i} B_k\hat{u}_{\varepsilon k}$$

$$- B_i\hat{v}_{\varepsilon i} - f), \delta v_i>\Big]\} + \frac{2}{\varepsilon_0}\Big[<\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon - \sum_{j=1}^{N} B_j\hat{u}_{\varepsilon j} - f, \delta\dot{x}> - <A^*(\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon$$

$$- \sum_j B_j\hat{u}_{\varepsilon j} - f), \delta x> - \sum_{j=1}^{N} <B_j^*(\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon - \sum_k B_k\hat{u}_{\varepsilon k} - f), \delta u_j>\Big] - <p_0, \delta\dot{x}>$$

$$- <\frac{d}{dt}p_0, \delta x> - \sum_{i=1}^{N}\Big[<p_i, \delta\dot{x}^i> + <\frac{d}{dt}p_i, \delta x^i>\Big] = 0$$

Thus, we get, for $i = 1,2,\ldots,N$,

(3.4) $$\sum_j C_j^*(C_j\hat{x}_\varepsilon - z_j) - \frac{2}{\varepsilon_0} A^*(\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon - \sum_j B_j\hat{u}_{\varepsilon j} - f) - \frac{d}{dt} p = 0$$

(3.5) $$-C_i^*(C_i\hat{x}_\varepsilon^i - z_i) + \frac{2}{\varepsilon_i} A^*(\dot{\hat{x}}_\varepsilon^i - A\hat{x}_\varepsilon^i - \sum_{j\neq i} B_j\hat{u}_{\varepsilon j} - B_i\hat{v}_{\varepsilon i} - f) - \frac{d}{dt} p_i = 0$$

(3.6) $$M_i\hat{u}_{\varepsilon i} - \frac{2}{\varepsilon_0} B_i^*(\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon - \sum_j B_j\hat{u}_{\varepsilon j} - f) + \sum_{j\neq i}\frac{1}{\varepsilon_i}[B_j^*(\dot{\hat{x}}_\varepsilon^i - A\hat{x}_\varepsilon^i - \sum_{k\neq i} B_k\hat{u}_{\varepsilon k} - B_i\hat{v}_{\varepsilon i} - f)] = 0$$

(3.7) $$-M_i\hat{v}_{\varepsilon i} + \frac{2}{\varepsilon_i} B_i^*(\dot{\hat{x}}_\varepsilon^i - A\hat{x}_\varepsilon^i - \sum_{k\neq i} B_k\hat{u}_{\varepsilon k} - B_i\hat{v}_{\varepsilon i} - f) = 0$$

(3.8) $$-p_0 + \frac{2}{\varepsilon_0}(\dot{\hat{x}}_\varepsilon - A\hat{x}_\varepsilon - \sum_j B_j\hat{u}_{\varepsilon j} - f) = 0$$

(3.9) $$-p_i - \frac{2}{\varepsilon_i}(\dot{\hat{x}}_\varepsilon^i - A\hat{x}_\varepsilon^i - \sum_{j\neq i} B_j\hat{u}_{\varepsilon j} - B_i\hat{v}_{\varepsilon i} - f) = 0 \quad , 1 \le i \le N.$$

Substituting $p_0, p_1, \ldots, p_N$ from (3.8), (3.9) into (3.4) - (3.7), we get

$$(3.10) \qquad \frac{d}{dt} p_0 = -A^* p_0 + \sum_{i=1}^{N} C_i^* (C_i \hat{x}_\varepsilon - z_i)$$

and for $i = 1, 2, \ldots, N$,

$$(3.11) \qquad \frac{d}{dt} p_i = -A^* p_i - C_i^* (C_i \hat{x}_\varepsilon^i - z_i)$$

$$(3.12) \qquad \hat{u}_{\varepsilon,i} = M_i^{-1} B_i^* (p_0 + \sum_{j \neq i} p_j)$$

$$(3.13) \qquad \hat{v}_{\varepsilon,i} = -M_i^{-1} B_i^* p_i,$$

wherein for p_i, $i = 0, 1, \ldots, N$, the terminal conditions $p_i(T) = 0$ has been imposed.

Comparing (3.10), (3.11), (3.12) and (3.13), respectively, with [4, (3.9), (3.6), (3.10) and (3.7)], we find that they are correspondingly identical. If we assume as in [4, (A1)] that

(3.14) $\quad C_i^* C_i$ $(1 \leq i \leq N)$ are positive definite,

then we have

$$(3.15) \qquad \hat{x}_\varepsilon = \mathbb{C}_0^{-1}(\dot{p} + A^* p + \sum_i C_i z_i); \qquad \mathbb{C}_0 \equiv \sum_i C_i^* C_i;$$

$$(3.16) \qquad \dot{\hat{x}}_\varepsilon = A\left[\mathbb{C}_0^{-1}(\dot{p} + A^* p + \sum_i C_i z_i)\right] + \sum_i B_i\left[M_i^{-1} B_i^* (p_0 + \sum_{j=i} p_j)\right] + f + \frac{\varepsilon_0}{2} p_0$$

$$(3.17) \qquad \hat{x}_\varepsilon^i = -\mathbb{C}_i^{-1}(\dot{p}_i + A^* p_i - C_i^* z_i); \qquad \mathbb{C}_i \equiv C_i^* C_i, \quad 1 \leq i \leq N;$$

$$(3.18) \qquad \dot{\hat{x}}_\varepsilon^i = A\left[-\mathbb{C}_i^{-1}(\dot{p}_i + A^* p_i - C_i^* z_i)\right] + \sum_{j \neq i} B_j\left[M_j^{-1} B_j^* (p_0 + \sum_{k \neq i} p_k)\right] + f$$

$$- B_i M_i^{-1} B_i^* p_i - \frac{\varepsilon_i}{2} p_i, \qquad 1 \leq i \leq N.$$

Integrating by parts for the last terms $\langle p_0, \frac{d}{dt}x - \dot{x}\rangle + \sum_i \langle p_i, \frac{d}{dt}x^i - \dot{x}^i\rangle$ in (3.3) and using (3.15) - (3.18) to substitute p_i, $\dot{p}_i$, $0 \le i \le N$ for $\hat{x}_\varepsilon$, $\dot{\hat{x}}_\varepsilon$, $\hat{x}^1_\varepsilon$, $\dot{\hat{x}}^1_\varepsilon, \ldots,$ $\hat{x}^N_\varepsilon$, $\dot{\hat{x}}^N_\varepsilon$, $\hat{u}_{\varepsilon 1}, \ldots, \hat{u}_{\varepsilon N}$, $\hat{v}_{\varepsilon 1}, \ldots, \hat{v}_{\varepsilon N}$, we get

$$\bar{L}_\varepsilon(p_0,p) \equiv \min_{(x,u)\in H^1_{0n}\times U} \; \max_{(X,v)\in[H^1_{0n}]^N\times U} \; L_\varepsilon(p_0,p;x,u;X,v)$$

$$= -\frac{1}{2}\langle \dot{p}_0 + A^*p_0, \mathbb{C}_0^{-1}(\dot{p}_0 + A^*p_0)\rangle + \frac{1}{2}\sum_i\langle \dot{p}_i + A^*p_i, \mathbb{C}_i^{-1}(\dot{p}_i + A^*p_i)\rangle$$

$$-\frac{1}{2}\langle p_0 + \sum_i p_i, S(p_0 + \sum_i p_i)\rangle + \langle p_0 + \sum_i p_i, \sum_i B_iM_i^{-1}B_i^*p_i\rangle$$

$$-\langle p_0 + A^*p_0, \mathbb{C}_0^{-1}\sum_{i=1}^N C_i^*z_i\rangle - \sum_i\langle \dot{p}_i + A^*p_i, \mathbb{C}_i^{-1}C_i^*z_i\rangle$$

$$-\langle p_0 + \sum_i p_i, f\rangle - \frac{1}{2}\langle \mathbb{C}_0^{-1}(\sum_i C_i^*z_i), \sum_i C_i^*z_i\rangle + \frac{1}{2}\sum_i \|z_i\|^2$$

$$-\frac{1}{2}\varepsilon_0\|p_0\|^2 + \frac{1}{2}\sum_i \varepsilon_i\|p_i\|^2,$$

which differs from [4, (4.3)] only by $-\frac{1}{2}\varepsilon_0\|p_0\|^2 + \frac{1}{2}\sum_i \varepsilon_i\|p_i\|^2$. (The term $\langle p_0(0) + \sum_i p_i(0), x_0\rangle$ vanishes because $x_0 = 0$). These two terms do not affect the convexity of p_0 and the concavity of p in $\bar{L}_\varepsilon(p_0,p)$. Thus we conclude that <u>the dual of the penalized problem is just an ε-perturbation of the dual problem</u>.

We compare briefly the amount of computing involved in the dual and the penalty methods. Assume that in the error estimate (2.19) s_1 and s_2 are sufficiently large and that

$$\lim_{\varepsilon\to 0}\|(\hat{x}_\varepsilon,\hat{u}_\varepsilon;\hat{X}_\varepsilon,\hat{v}_\varepsilon)\|_{H_n^{s_1}\times(\prod_{i=1}^N H_{m_i}^{s_2})\times[H_n^{s_1}]^N\times(\prod_{i=1}^N H_{m_i}^{s_2})} < \infty.$$

For simiplicity, we only consider $n = m_1 = \ldots = m_N = 1$. In order that $\hat{x}_h^{\varepsilon}$, the penalty solution, converges to $\hat{x}$ with the same rate as $\|\hat{x}_h - \hat{x}\|$ in [4,(6.21)](where $\hat{x}_h$ is the duality solution), we must choose

$$\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_N = \mathcal{O}(h^{\bar{\mu}/2})$$

, with $\mu = \bar{\mu}$ in (2.19) and $\bar{\mu} = 2\mu$, where μ is the same μ in [4, (6.21)]. This implies that

$$\bar{\tau}_0 - 1 = \bar{\tau} = \bar{\mu} = 2\mu.$$

Assume also that in [4,Theorem 6.2] that ℓ is sufficiently large so that $\mu = \tau - 1$. Thus, using the same $h = T/M$ in both approaches by dividing the interval $[0,T]$ into M equal parts, the finite element space S_h (in [4] Theorem 6.2) has $(N + 1) \cdot (M\mu + 1)$ basis elements, while the finite element space S_h in (2.16) has $(N + 1) \cdot (2M\mu + 1) + 2N[(2\mu - 1) M + 1]$ basis elements, assuming that $S_h^1 = S_h^2 = \ldots = S_h^N = $ a $(\bar{\mu},0)$-system. Thus the corresponding matrix equations

$$\bar{M}_h \bar{q}_h = \bar{\theta}_h \quad \text{(cf. [4,(6.10)])} \tag{3.19}$$

$$\bar{M}_h^{\varepsilon} \bar{q}_h^{\varepsilon} = \bar{\theta}_h^{\varepsilon} \quad (2.18) \tag{3.20}$$

have respective sizes

$$\bar{M}_h: \ [(N+1) \cdot (M\mu+1)]^2$$

$$\bar{M}_h^{\varepsilon}: \ [(N+1) \cdot (2M\mu+1) + 2N(2\mu-1)M+2N]^2$$

Thus the ratio of computing time between (3.19) and (3.20) is

$$\left[\frac{(N+1)(M\mu+1)}{(N+1)(2M\mu+1)+2N(2\mu-1)M+2N}\right]^3 \tag{3.21}$$

It appears to us that even after we take into account the sparseness and block structures of the matrices $\overline{M}_h$ and $\overline{M}_h^{\varepsilon}$, the above ratio is still valid asymptotically. Therefore we see that the dual method is much more efficient than the penalty method, especially when the number of players N is large.

Nevertheless, the dual method is feasible only under assumption (A1) in [4], which requires the invertibility of $\mathbb{C}_1,\ldots,\mathbb{C}_N$ and is therefore quite restrictive. Computationally, the penalty method is not restricted by such a condition.

4. Numerical Results

Example 1 We consider the very same example as in [4,§7,Example 1]

$$
(4.1)\begin{cases}
\begin{cases}\dot{x}(t) = x(t) + u_1(t) + 2u_2(t) + 1, & 0 \le t \le T, \quad T = \pi/4 \\ x(0) = 0 \end{cases} \\
J_1(x,u) = \int_0^T \left[\left| x(t) - (\cos t + \frac{1}{2})\right|^2 + \frac{1}{2}\left|u_1(t)\right|^2\right] dt \\
J_2(x,u) = \int_0^T \left[\left|x(t) - \sin t\right|^2 + 2\left|u_2(t)\right|^2\right] dt,
\end{cases}
$$

which is a 2-person non zero-sum game and is known to have a unique equilibrium strategy for all $T > 0$.

$J_\varepsilon(x,u;X,v)$ is given as in (2.1). We choose for S_h^0 a $(\tau_0,1) = (3,1)$ system of quadratic splines as approximation spaces for x, x^1 and x^2, and for S_h^1, S_h^2 a $(\tau,0) = (2,0)$ system of piecewise linear finite elements as approximation spaces for u_1,u_2,v_1 and v_2.

It is not difficult to see that conditions (B1) - (B5) and (B7) in §1 are all satisfied. We are, however, unable to verify (B6); similarly, nor are we able to verify the validity of (H1) and (H2) in §2.

Our numerical results are plotted in the following figures. We use $h = \frac{\pi}{4}/32$ and $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 \equiv \varepsilon$.

In the first three figures, we use $\varepsilon = 10^{-1}$, 10^{-2}, 10^{-3}, 10^{-4} and 10^{-5}, respectively.

Figure 1 contains graphs of u_1, versus time t, for various values of ε.

Figure 2 contains graphs of u_2.

Figure 3 contains graphs of x.

The trajectories of u_1, u_2 and x versus various values of ε cluster closer and closer as ε becomes small.

Figure 4 shows two graphs of u_1. The solid line represents numerical data obtained from duality in [4], with (4,1)-cubics and $h = \frac{\pi}{4}/32$. The broken line represents data obtained from penalty, with (3,1)-quadratics for state and (2,0) piecewise continuous linear elements for strategies also with $h = \frac{\pi}{4}/32$, and $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 10^{-3}$.

Figure 5 shows two graphs of u_2, obtained in the same fashion as u_1. Figure 6 contains two graphs of x.

From Figures 4 and 5, we see that numerical results for u_1 and u_2 obtained from duality and penalty show remarkable agreement. In Figure 6, we see that the two graphs of x agree very well everywhere except at the initial and terminal time 0 and T, where the duality graph is rougher and less accurate.

We list values of $u_1, u_2, v_1, v_2, x, x^1, x^2$ at selected points in Table 1.

All of our calculations were carried out with double precision.

The values of J_ε are obtained as follows:

$J_\varepsilon = 0.5159708038688868 \times 10^{-1}$, $\qquad \varepsilon = 10^{-1}$;

$J_\varepsilon = 0.5698840889975811 \times 10^{-1}$, $\qquad \varepsilon = 10^{-2}$;

$J_\varepsilon = 0.8583978319547797 \times 10^{-3}$, $\qquad \varepsilon = 10^{-3}$;

$J_\varepsilon = 0.3287468594749820 \times 10^{-3}$, $\qquad \varepsilon = 10^{-4}$;

$J_\varepsilon = -\ 0.1245174244833003 \times 10^{-3}$, $\qquad \varepsilon = 10^{-5}$.

In the following table, we use $h = \frac{\pi}{4}/32$ and use the following to denote

P_1: penalty solution with $\varepsilon_0=\varepsilon_1=\varepsilon_2=10^{-3}$

P_2: penalty solution with $\varepsilon_0=\varepsilon_1=\varepsilon_2=10^{-5}$

D: duality solution

		$t = \frac{1}{4} \cdot \frac{\pi}{4}$	$t = \frac{1}{2} \cdot \frac{\pi}{4}$	$t = \frac{3}{4} \cdot \frac{\pi}{4}$	$t = \frac{\pi}{4} = T$
u_1	P_1	-2.077473	-1.238577	-0.562432	0.000086
	P_2	-2.078433	-1.239262	-0.562789	-0.004539
	D	-2.064450	-1.229223	-0.556116	0.0
u_2	P_1	0.440848	0.285103	0.131847	-0.000053
	P_2	0.441103	0.285264	0.131923	-0.002366
	D	0.436094	0.281693	0.129746	0.0
x	P_1	-0.125946	-0.136808	-0.053823	0.118535
	P_2	-0.125870	-0.136707	-0.053713	0.118661
	D	-0.126924	-0.137191	-0.053709	-0.250000

Table 1

Remark: The above are rounded-off figures with 6 decimal place accuracy.

Example 2 (The Primal-Finite Difference Method)

We return to the primal approach in Part I [4]. Consider the same example as in Example 1:

$$(4.2) \qquad \min_{(x,u)} \ \max_{(X,v)} \ J(x,u;X,v)$$

where $(x,u;X,v)$ is subject to the differential constraints $(DE) = 0$, $[DE] = 0$, $x(0) = 0$, $X(0) = 0$.

We discretize the differential constraints by the crude Euler finite difference scheme. For example, $(DE) = 0$ is discretized as

$$(4.3) \qquad \frac{x(t_{i+1}) - x(t_i)}{h} = A(t_i)x(t_i) + \sum_{j=1}^{N} B_j(t_i)u_j(t_i) + f(t_i) \ , \quad i = 0,\ldots,M-1.$$

Substituting (4.3) (and others) into (4.2), we proceed to solve the min-max problem.

We use $M = 32$ and $h = \frac{\pi}{4}/32$ and the primal approach to compute Example 1. The following values are obtained at selected points:

	$t = \frac{1}{4} \cdot \frac{\pi}{4}$	$t = \frac{1}{2} \cdot \frac{\pi}{4}$	$t = \frac{3}{4} \cdot \frac{\pi}{4}$	$t = \frac{\pi}{4} = T$
u_1	-2.0416074	-1.2191050	-0.5541710	0.0
u_2	0.4394483	0.2837087	0.1311784	0.0
x	-0.1244138	-0.1363107	-0.0565919	0.1097617

Table 2

The reader may compare the values in Table 2 with those in Table 1.

Example 3: Consider again the following 2-person nonzero-sum game

$$\begin{cases} \begin{cases} \dot{x}(t) = x(t) + \cos t \cdot u_1(t) + \sin t \cdot u_2(t) + 1, \quad 0 \le t \le T, \\ x(0) = 0, \end{cases} \\ J_1(x,u) \equiv \int_0^T [\,|x(t) - d_1(\cos t + \frac{1}{2})|^2 + \frac{1}{3} u_1^2(t)]dt, \\ J_2(x,u) \equiv \int_0^T [\,|x(t) - d_2 \sin t|^2 + \frac{1}{2} u_2^2(t)]dt, \end{cases}$$

where $T = 2\pi$ and $(d_1,d_2) = (-1,0.9)$, as in [4,§7,Example 3]. It is not clear to us whether the assumptions in [4] or in this paper are satisfied by this problem. However, as noted in [4,§7,Example 3], the values of L seem to be divergent.

Let us manage to compute the numerical solutions in a straightforward manner, using $h = 2\pi/32$. In Figures 7-9, the graphs of u_1,u_2 and x are plotted. The solid lines always represent the duality solutions of u_1,u_2 and x, while the broken lines represent the penalty solutions of u_1, u_2 and x, using $\varepsilon = \varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 10^{-3}$ and 10^{-4}, respectively.

It can be seen from these graphs that smaller values of ε cause futher deviations between the penalty and duality solutions, if h is not adjusted according to ε. This offers partial evidence that ε and h are coupled in the error bounds (2.19). Compare the results in [3].

We have also plotted the graphs of u_1, u_2 and x, respectively, versus t with $\varepsilon = 10^{-1}$, 10^{-2}, 10^{-3}, 10^{-4} and 10^{-5}, in Figures 10, 11 and 12. The computed values of J_ε are

$$J_\varepsilon = 0.4582465783358920, \quad \varepsilon = 10^{-1};$$
$$J_\varepsilon = 0.2730100141206180, \quad \varepsilon = 10^{-2};$$
$$J_\varepsilon = 0.1231759476555612, \quad \varepsilon = 10^{-3};$$
$$J_\varepsilon = 0.9989590297527213, \quad \varepsilon = 10^{-4};$$
$$J_\varepsilon = 0.3011451125450394\times10^{3}, \ \varepsilon = 10^{-5}.$$

We see that when $\varepsilon = 10^{-5}$, the "numerical solutions" become completely meaningless. □

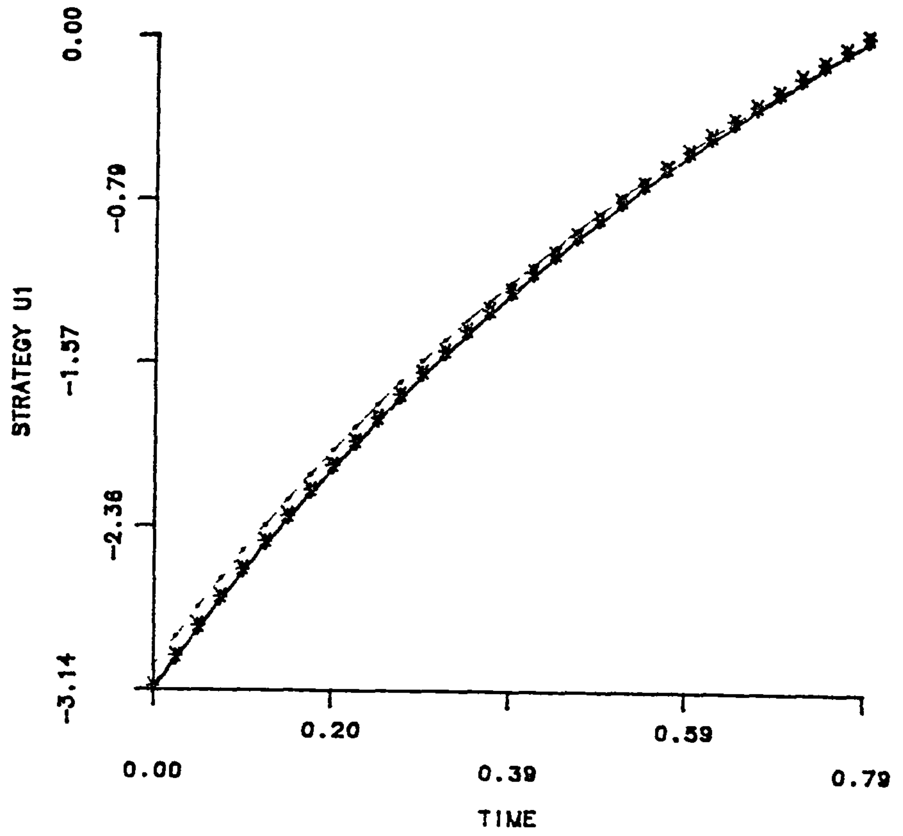


Figure 1: The Penalty Solution of u_1

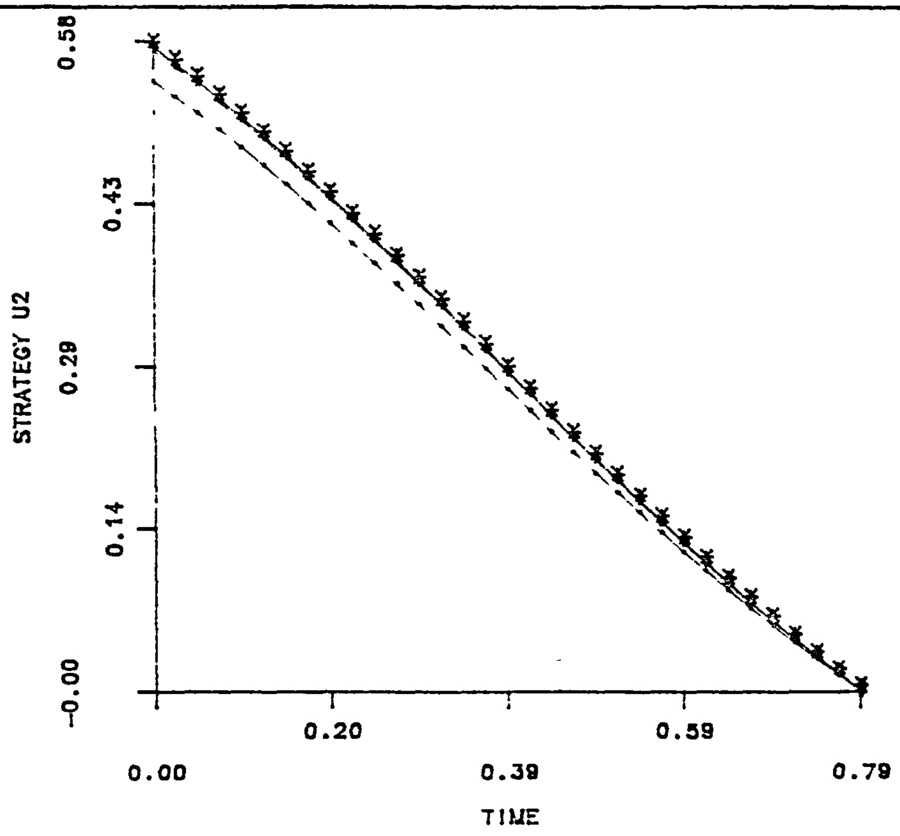


Figure 2: The Penalty Solution of u_2

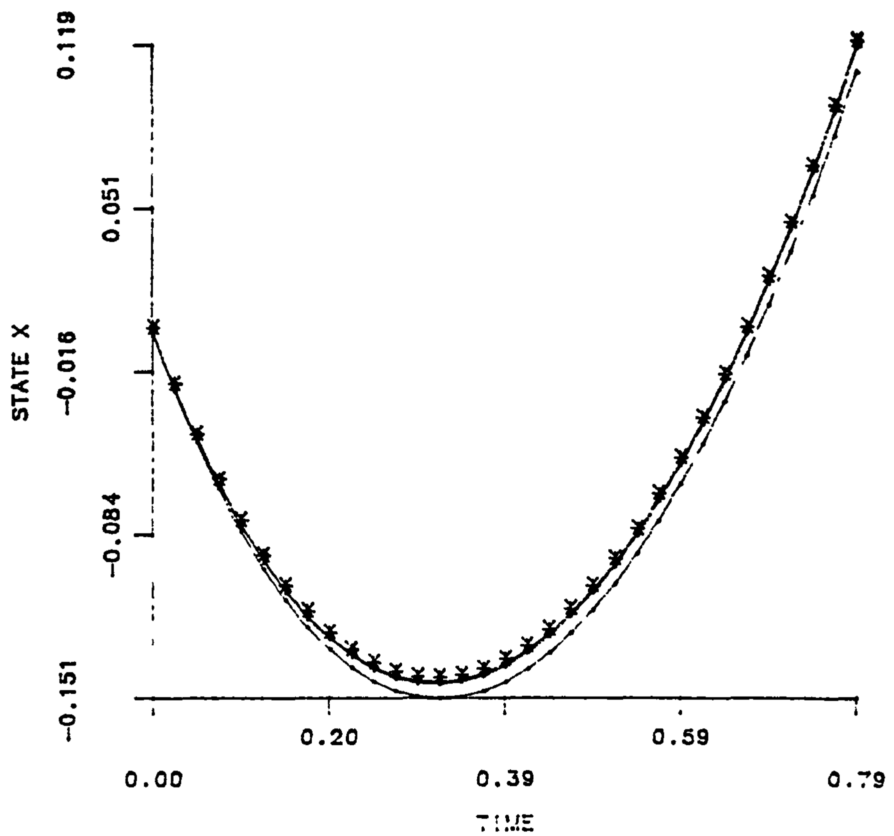


Figure 3: The Penalty Solution of the State x

Throughout Figures 1, 2 and 3, we use the following legend:

EPS1 ... EPS2 *** EPS3 +++ EPS4 XXX EPS5 ,,,

where $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 =$ EPSI, I = 1, 2, 3, 4, 5 and

EPS1 = 10^{-1}, EPS2 = 10^{-2}, EPS3 = 10^{-3}, EPS4 = 10^{-4}, EPS5 = 10^{-5}.

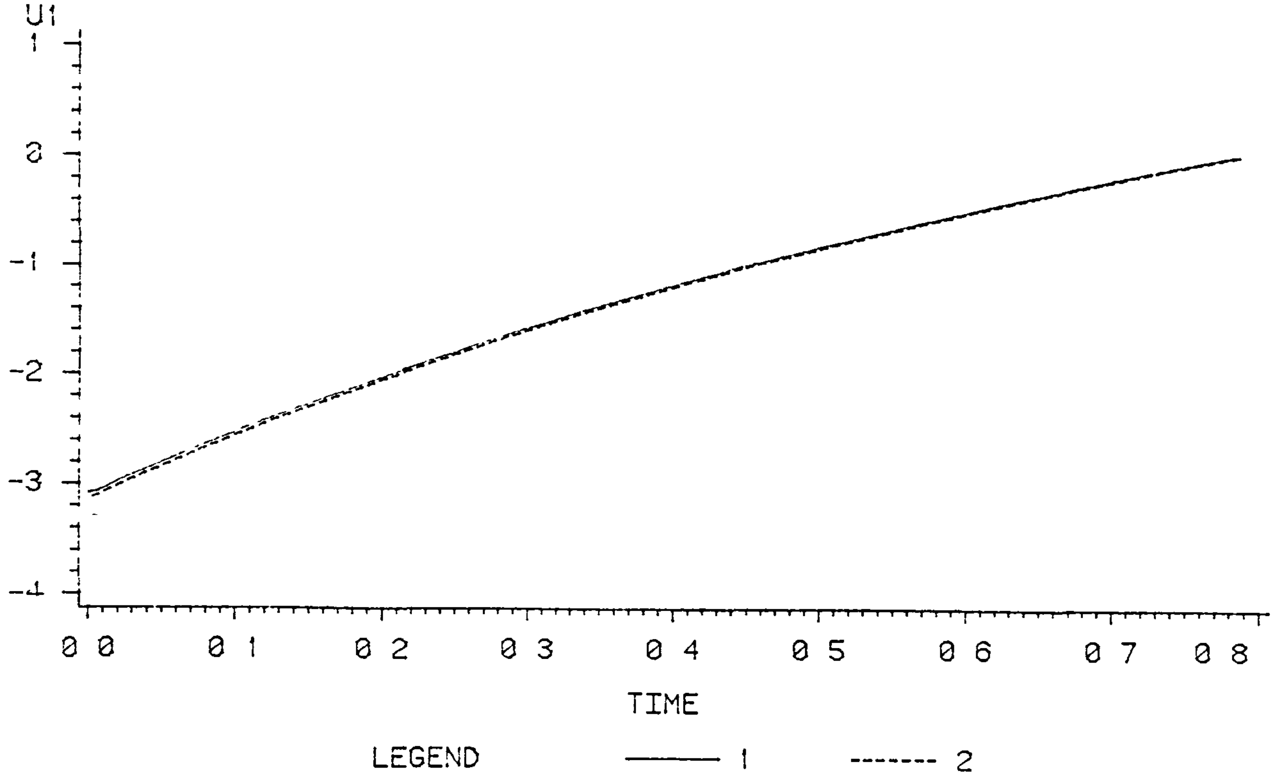


Figure 4: Comparison of the Penalty and Duality Solutions of u_1

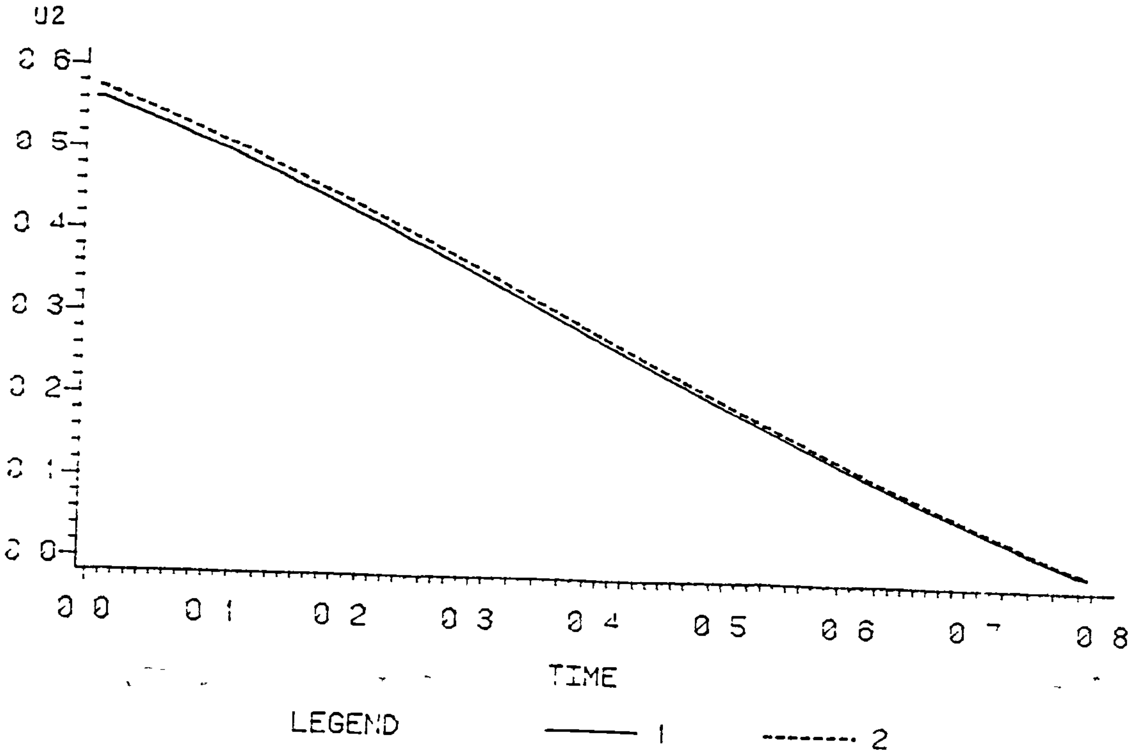


Figure 5: Comparison of the Penalty and Duality Solutions of u_2

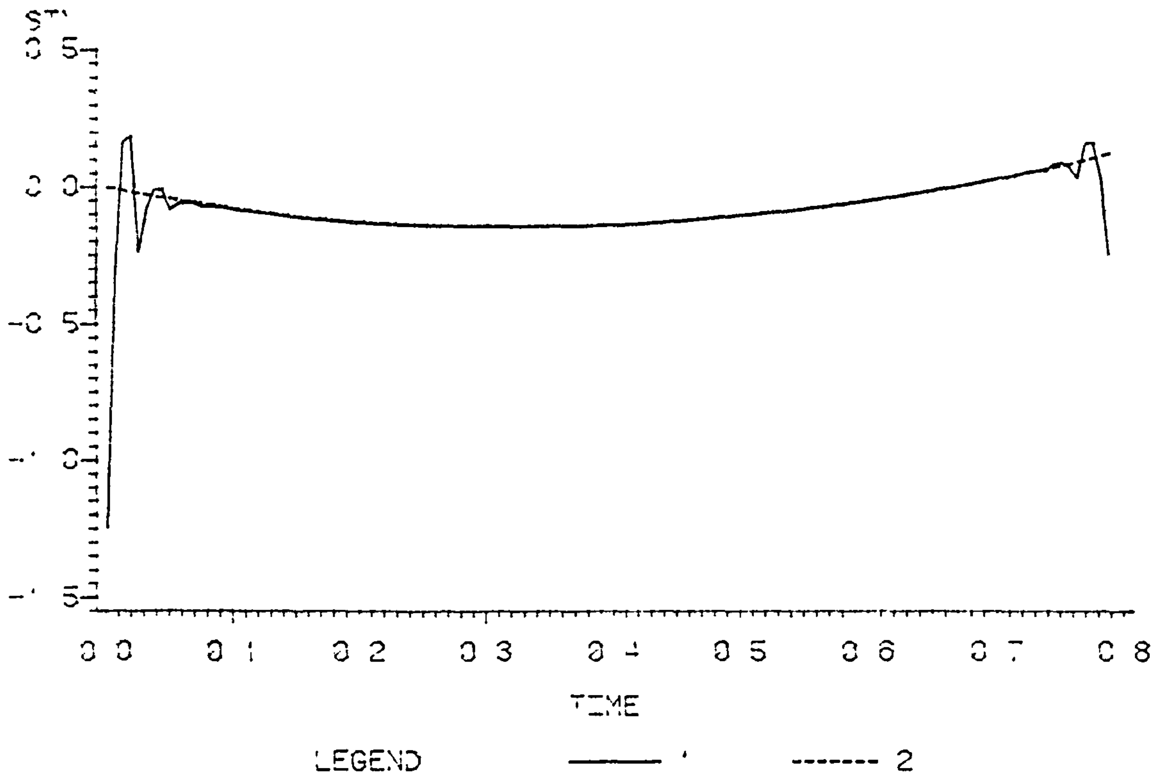


Figure 6: Comparison of the Penalty and Duality Solution of x

Throughout Figures 4, 5 and 6, the solid curve (1) represents the duality solution while the broken curve (2) represents the penalty solution, with $\varepsilon_0=\varepsilon_1=\varepsilon_2= 10^{-3}$, both use $h = \frac{\pi}{4}/32$.

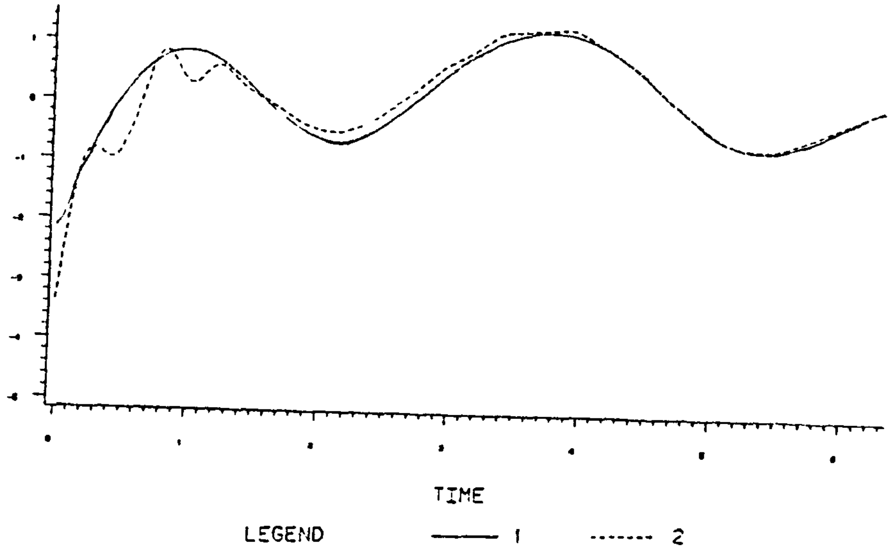


Figure 7.1 Comparison Between the Penalty and Duality Solutions of u_1, $\varepsilon = 10^{-3}$.

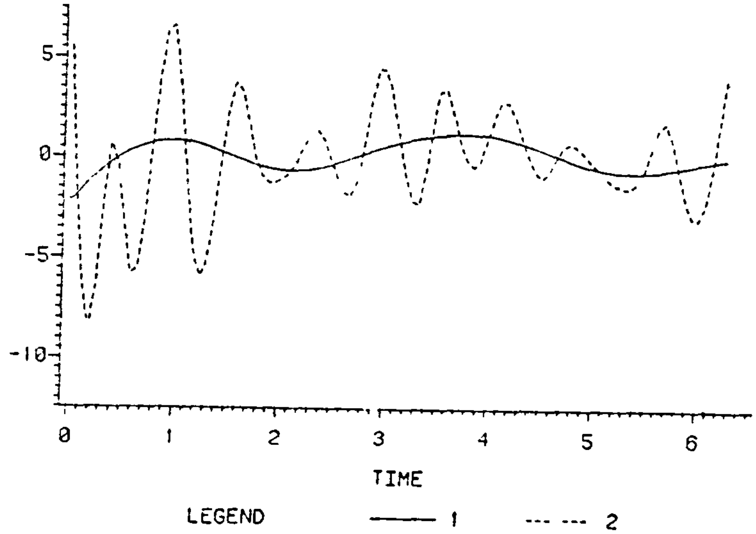


Figure 7.2 Comparison Between the Penalty and Duality Solutions of u_1, $\varepsilon = 10^{-4}$.

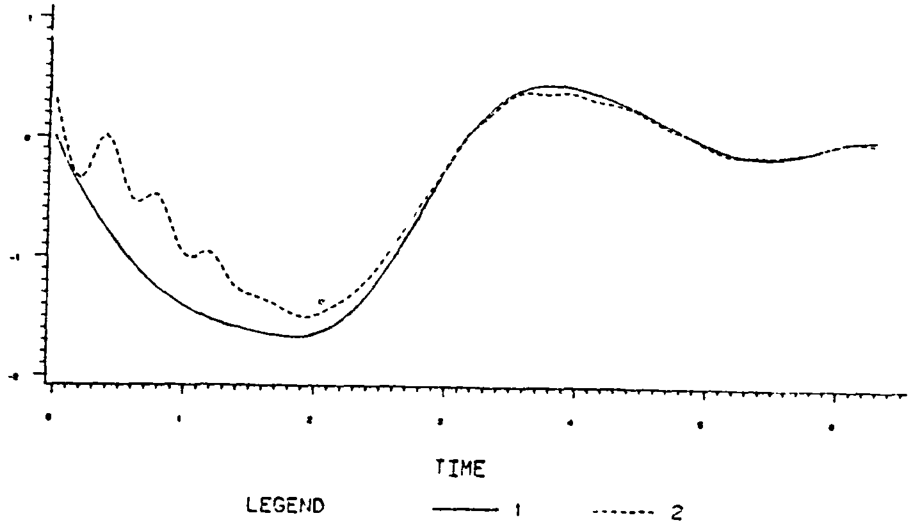
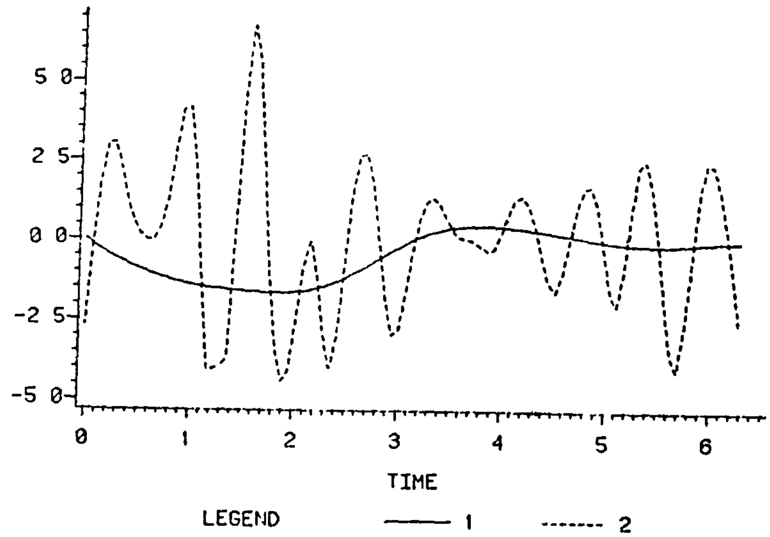
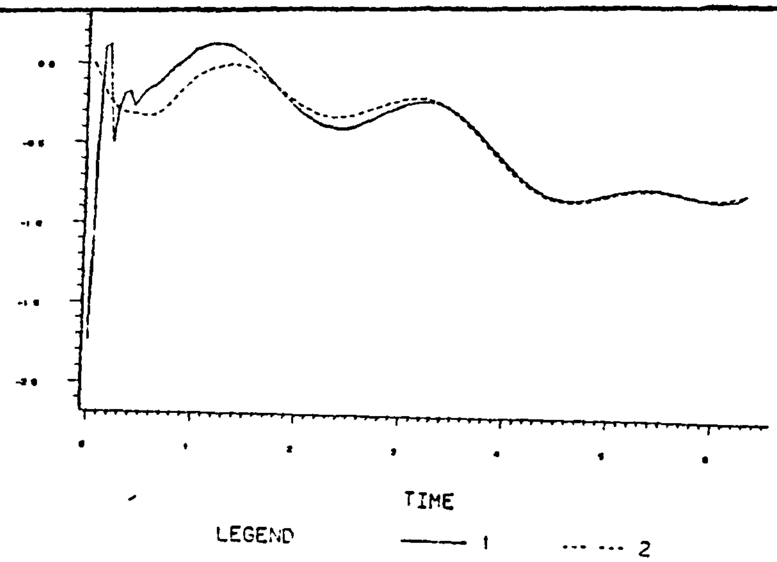


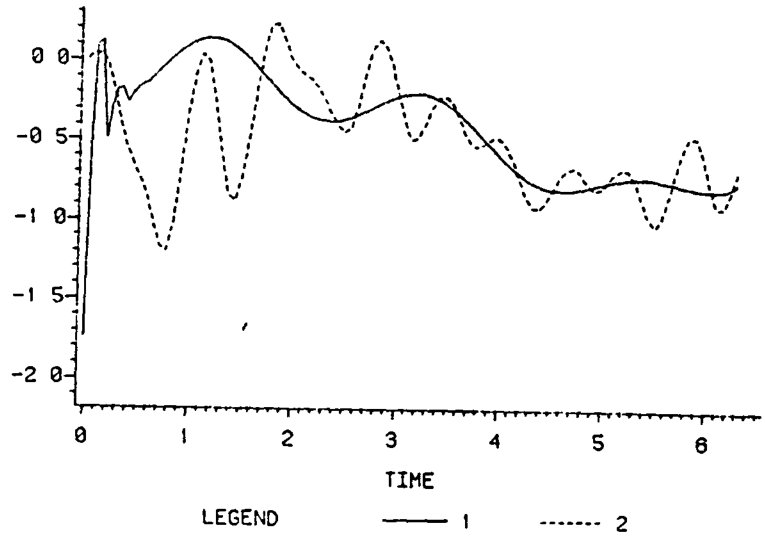
Figure 8.1 Comparison Between the Penalty and Duality Solutions of u_2, $\varepsilon = 10^{-3}$.



<u>Figure 8.2</u> Comparison Between the Penalty and Duality Solutions of u_2, $\varepsilon = 10^{-4}$.



<u>Figure 9.1</u> Comparison Between the Penalty and Duality Solutions of x, $\varepsilon = 10^{-3}$.



<u>Figure 9.2</u> Comparison Between the Penalty and Duality Solutions of x, $\varepsilon = 10^{-4}$.

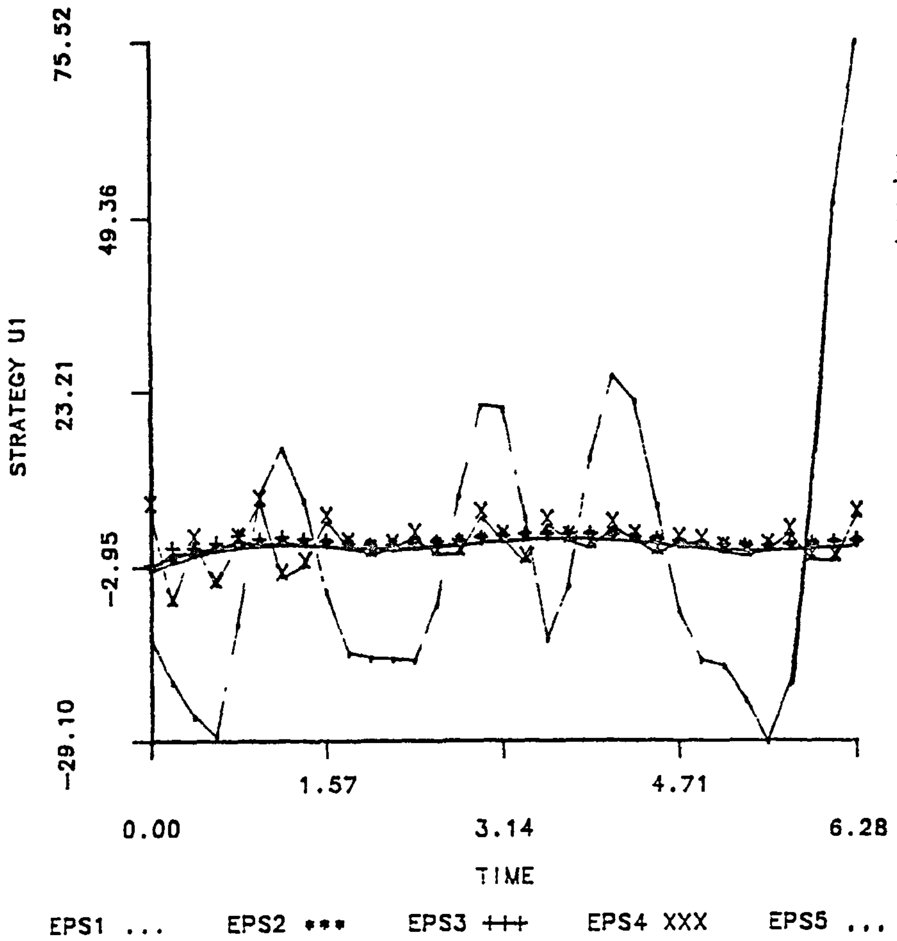


Figure 10 u_1 in Example 3 for various values of ε.

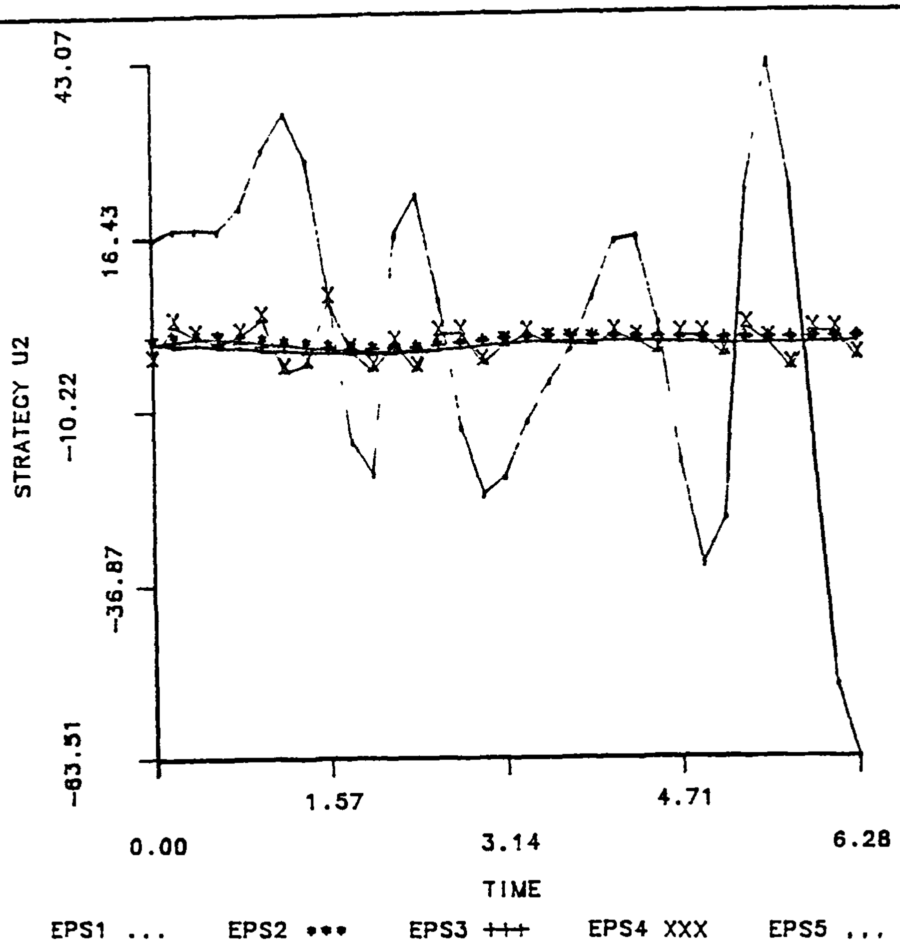
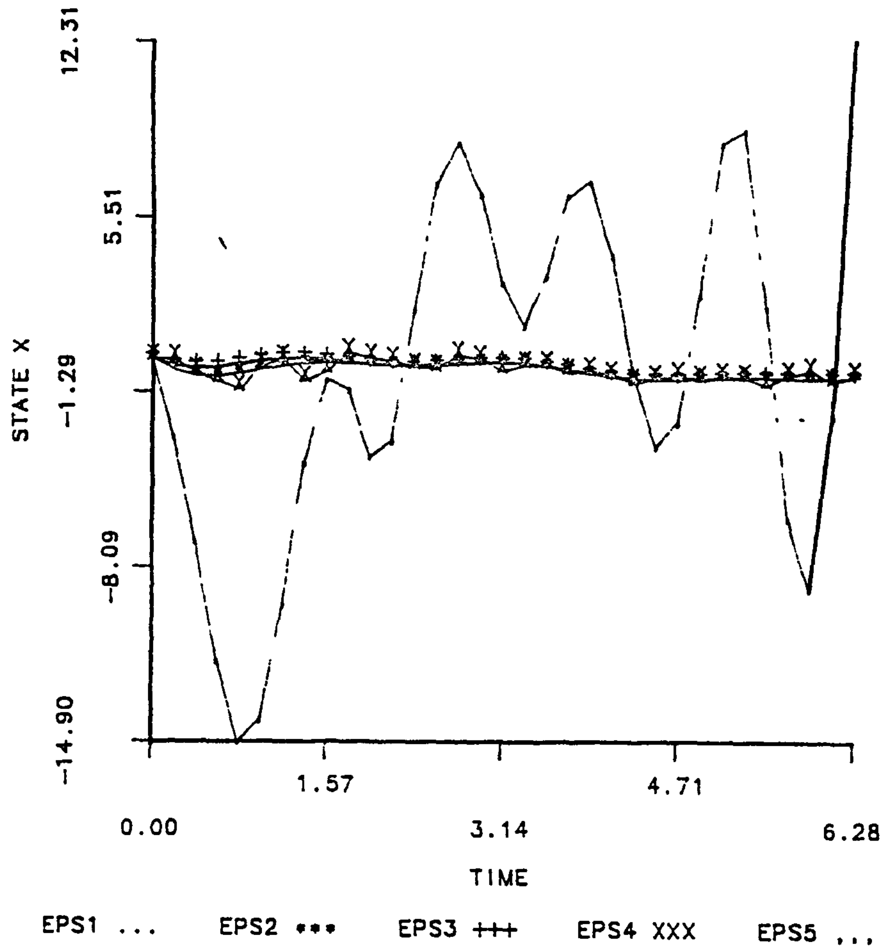


Figure 11 u_2 in Example 3 for various values of ε.



<u>Figure 12</u> x in Example 3 for various values of ε.

Throughout Figures 10,11 and 12, we use

$$EPS1 = 10^{-1},\ EPS2 = 10^{-2},\ EPS3 = 10^{-3},\ EPS4 = 10^{-4} \text{ and } EPS5 = 10^{-5}.$$

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16 Abstract
The equilibrium strategy for N-person differential games can be found by studying a min-max problem subject to differential systems constraints [4]. In this paper, we penalize the differential constraints and use finite elements to compute numerical solutions. Convergence proof and error estimates are given. We have also included numerical results and compared them with those obtained by the dual method in [4].

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