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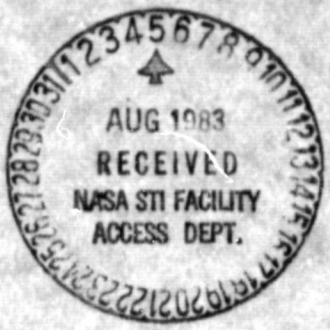

Flux-Based Acceleration of the Euler Equations

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FLUX-BASED ACCELERATION OF THE EULER EQUATIONS

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SUMMARY

A new coarse-grid acceleration scheme for the Euler equations is presented. This flux-based scheme eliminates the need, exhibited by previous accelerators, for computing flux vector Jacobian matrices. The method is derived and implemented in a two-dimensional flow algorithm. Numerical results are presented for both subcritical and shocked, supercritical flow. These results demonstrate that the flux-based accelerator is more efficient than its Jacobian-based counterpart. Generalization to three dimensions is immediate. Construction of flux-based accelerators for the Navier-Stokes equations is also discussed.

INTRODUCTION

One approach to the solution of the steady Euler equations is to cast them in their unsteady form and apply a numerical time-marching procedure to the resultant purely hyperbolic initial-boundary value problem. In this way, the difficulties associated with the iterative solution of the first-order, mixed elliptic-hyperbolic boundary value problem posed by the steady equations of motion may be avoided.

Whereas methods for the time-accurate solution of hyperbolic systems of conservation laws are fairly well established, current research focuses on the development of stratagies for reaching the temporally asymptotic steady state with minimal computational expense.

The unsteady equations of motion may be modified so as to induce nonphysical transient behavior and, by this means, arrive at the physically-correct steady solution more rapidly. Alternatively, a numerical integration scheme with a stability bound in excess of the CFL limit may be employed to advance the unmodified Euler equations over larger time intervals than would otherwise be admissable. Here both implicit and explicit schemes are possible. The implicit schemes are generally unconditionally linearly stable but incur large operations counts, may suffer unacceptable accuracy loss when operating at large time steps or may have stability constraints imposed through the boundary condition implementation. The explicit schemes have low operations counts, lead to very flexible algorithms and are easy to code and to vectorize. However, they are generally conditionally stable.

This report deals with explicit convergence acceleration procedures for the Euler equations, in the spirit introduced by Ni [1] and generalized by Johnson [2] to [4]. In particular, a new flux-based coarse-grid acceleration scheme is introduced. This scheme eliminates the need, exhibited by previous

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accelerators, for computing flux vector Jacobian matrices and, as a consequence, outperforms its Jacobian-based predecessors.

EQUATIONS OF MOTION

The Euler equations may be written in conservation-law form as

$$q_t = -(f_x + g_y) \tag{1}$$

where:

$$q = \begin{bmatrix} e \\ \rho u \\ \rho v \\ E \end{bmatrix} \qquad f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E + p)u \end{bmatrix} \qquad g = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E + p)v \end{bmatrix}$$

Here ρ, u, v, p and E are, respectively, density, velocity components in the x- and y-directions, pressure and total energy per unit volume. This latter quantity may be expressed as

$$E = \rho\left[e + \frac{1}{2}(u^2 + v^2)\right]$$

where the specific internal energy, e, is related to the pressure and density by the simple law of a calorically perfect gas

$$p = (\gamma - 1)\rho e$$

with γ denoting the ratio of specific heats.

Although, for simplicity, the equations of motion are presented here written in Cartesian coordinates, this does not pose any restriction on the generality of the results to be discussed subsequently.

COARSE-GRID ACCELERATION

Given a basic fine grid on which a numerical solution of Eq. (1) is required and an explicit, CFL-limited solution procedure,* the coarse-grid acceleration concept is to construct a sequence of successively coarser grids by means of which the fine grid solution may be rapidly advanced, while respecting the stability limits on all grids.

*Less restrictive hypotheses are possible. See, for example, Stubbs [5].

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Jacobian-Based-Accelerator

Ni [1] introduced a one-step Lax-Wendroff scheme which may be written as

$$
\begin{aligned}
\delta q_{i,j} = \frac{1}{4}\Bigg\{ & \left[\left(I + \frac{\Delta t}{\Delta x} A + \frac{\Delta t}{\Delta y} B\right) \Delta q\right]_{i-\frac{1}{2},\, j-\frac{1}{2}} \\
& + \left[\left(I + \frac{\Delta t}{\Delta x} A - \frac{\Delta t}{\Delta y} B\right) \Delta q\right]_{i-\frac{1}{2},\, j+\frac{1}{2}} \\
& + \left[\left(I - \frac{\Delta t}{\Delta x} A - \frac{\Delta t}{\Delta y} B\right) \Delta q\right]_{i+\frac{1}{2},\, j+\frac{1}{2}} \\
& + \left[\left(I - \frac{\Delta t}{\Delta x} A + \frac{\Delta t}{\Delta y} B\right) \Delta q\right]_{i+\frac{1}{2},\, j-\frac{1}{2}} \Bigg\}
\end{aligned}
\tag{2}
$$

where:

$$
\Delta q_{i+\frac{1}{2},\, j+\frac{1}{2}} = -\Delta t \left(f_x + g_y\right)_{i+\frac{1}{2},\, j+\frac{1}{2}}
$$

$$
A \equiv \frac{\partial f}{\partial q} \qquad B \equiv \frac{\partial g}{\partial q}
$$

Further details on this scheme may be found in [1] and [2]. Attention here is restricted to the observation that Eq. (2), when rewritten in a coarse-grid setting, may be used to accelerate the convergence of the fine-grid solution.

Given the fine-grid corrections, δq, we wish to use successively coarser grids to propagate this information throughout the computational domain, thus accelerating convergence to the steady state while maintaining the accuracy determined by the fine-grid discretization. Given a basic fine grid with the number of points in each direction expressible as $n(2^p) + 1$ for p and n integers such that $p \geq 0$ and $n \geq 2$, where p is the number of grid coarsenings and n is the number of coarsest-grid intervals, let successively coarser grids be defined by successive deletion of every other point in each coordinate direction.

The coarse-grid acceleration scheme replaces the coarse-grid computation of Δq with a restriction of the latest fine-grid value of δq. The effect of this restricted fine-grid correction is then distributed according to a coarse-grid version of Eq. (2) to obtain a coarse-grid correction. This is, in turn, prolonged to the fine grid to become the new fine-grid correction and update the fine-grid solution. One time-cycle of the multiple-grid scheme is composed

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of an application of the fine-grid solution procedure followed by an application of the coarse-grid acceleration scheme to each successively coarser grid.

In the basic integration scheme, a correction at one grid point affects only its nearest neighbors while, in a k-level multiple-grid scheme, the same correction affects all points up to 2^{k-1} mesh spacings distant. Furthermore, since the coarse-grid scheme simply propagates the effects of fine-grid corrections, fine grid accuracy is maintained.

We call the coarse-grid version of Eq. (2) a Jacobian-based accelerator because of the presence of A and B, the flux vector Jacobian matrices. Their computation and storage and multiplications with them may be interpreted as causing inefficiencies in the coarse-grid accelerator. The flux-based accelerator described below is presented as a more efficient alternative.

Flux-Based-Accelerator

A Lax-Wendroff-type coarse-grid scheme may be expressed as

$$\delta q_{coarse} = \Delta t q_t + \frac{\Delta t^2}{2} q_{tt}$$

By introducing Eq. (1), this may be rewritten as

$$\delta q_{coarse} = -\Delta t(f_x + g_y) - \frac{\Delta t^2}{2} \left(f_x + g_y\right)_t$$

Recalling that

$$\Delta q = -\Delta t(f_x + g_y)$$

and temporally differencing the second-order term, we obtain

$$\delta q_{coarse} = \Delta q - \frac{\Delta t}{2} \left\{\left(f_x + g_y\right)^{n+1} - \left(f_x + g_y\right)^n\right\} \qquad (3)$$

This is a flux-based coarse-grid acceleration scheme where Δq is approximated by a restriction of the latest fine-grid value of δq and

$$f^n = f(q^n) \quad , \quad g^n = g(q^n)$$

$$f^{n+1} = f(q^n + \Delta q) \quad , \quad g^{n+1} = g(q^n + \Delta q)$$

In this way the computation and storage of A and B and multiplications with them are eliminated. In place of these operations, we need only compute f^{n+1} and g^{n+1}, which may be immediately obtained from $q^n + \Delta q$. All other quantities are known.

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NUMERICAL RESULTS

In order to facilitate performance comparisons with results obtained previously using the Jacobian-based accelerator, the subcritical and shocked, supercritical flow test cases described in detail in [2] and [4] have been recomputed using the flux-based accelerator.

The physical problem consists of a straight channel section with a 10 percent half-thick circular arc airfoil mounted on its lower wall. The subcritical case has an isentropic inlet Mach number of 0.5 while that for the supercritical case is 0.675. The basic fine grid has 65 nodes in the streamwise direction and 17 in the transverse direction. When grid coarsening by successive deletion of every other grid line in each direction is employed, the 65 x 17 fine grid allows the generation of three coarser grids with the coarsest grid having 9 x 3 nodes.

With the explicit, two-step Lax-Wendroff scheme due to MacCormack [6] used as a fine-grid solution procedure, both the flux-based and the Jacobian-based accelerators produce the same converged solutions. This indicates that, as expected, the new flux-based accelerator does not adversely affect the accuracy of the multiple-grid solutions. The physical details of the solutions are not presented here. The interested reader may find them in [2] and [4].

The main advantage of the flux-based coarse-grid scheme lies in its lower operations count. For example, in computations using a three-grid sequence, the ratio of coarse-grid work to total work was reduced by approximately 40 percent of its previous, Jacobian-based, value. In the same computations, the convergence rate was improved by 7 to 20 percent. Additionally, the flux-based scheme requires less storage than the Jacobian-based scheme.

Comparisons of the flux-based and Jacobian-based results yielding the best work reduction factors for the two test cases are presented in Tables I and II.

TABLE I. - SUBCRITICAL RESULTS

Accelerator	$\frac{\text{Coarse-grid work}}{\text{Total work}}$	Time cycles for convergence	Work reduction factor	Optimal sequence length
Jacobian-based	0.39	1020	5.1	3
Flux-based	0.23	950	6.9	3

TABLE II. - SUPERCRITICAL RESULTS

Accelerator	$\frac{\text{Coarse-grid work}}{\text{Total work}}$	Time cycles for convergence	Work reduction factor	Optimal sequence length
Jacobian-based	0.31	1220	2.1	4
Flux-based	0.14	1130	2.9	3

By work reduction factor we mean: the ratio of the work required to produce a converged solution on a single fine grid to that required on the multiple-grid sequence which minimizes work. Note further that the convergence criterion used here requires the average absolute value of the fine-grid residual in ρu to be less than 10^{-6}. This represents a decrease of approximately five decades from its initial value. This requirement could be relaxed by at least two decades without any significant accuracy deterioration.

Observe that the work reduction factor for the subcritical case has been improved from 5.1 to 6.9 and that for the supercritical case has been raised from 2.1 to 2.9. These represent performance advantages of the flux-based scheme amounting to approximately 35 percent.

GENERALIZATIONS

The derivation of the flux-based accelerator may be extended without difficulty to three dimensions.

The construction of a flux-based accelerator for the Navier-Stokes equations may also be carried out in a fashion analogous to that illustrated here for the Euler equations. This would result in a full coarse-grid scheme, in the sense described in [3]. Such a scheme would, no doubt, be more efficient than a Jacobian-based full coarse-grid scheme. However, the issue becomes complicated by the existence of the Jacobian-based convective coarse-grid scheme.* This scheme, based on the inviscid flux vector Jacobians, is quite efficient. A flux-based analog remains to be formulated.

The successful derivation and implementation of the flux-based coarse-grid scheme as described in this report suggests the possibility of constructing other, perhaps more efficient, schemes in this class.

CONCLUSIONS

A new coarse-grid acceleration scheme for the Euler equations has been presented.

This flux-based scheme is more efficient than its Jacobian-based analog.

Results to substantiate this claim have been presented for two-dimensional subcritical and shocked, supercritical flows. Performance improvements of approximately 35 percent have been obtained.

Generalizations of the flux-based coarse-grid scheme to three dimensions and to the Navier-Stokes equations are possible.

*For details see [3].

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