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16. Abstract

In this report, the author presents a literature survey on the earth's magnetic field, citing the works of Gauss, Erman-Petersen, Quintus Icilius and Neumayer. The general formulas for the representation of the potential and components of the earth's magnetic force are presented. An analytical representation of magnetic condition of the earth based on observations is also made.

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No. 3

Mathematical Developments Regarding the General Theory of the Earth Magnetism

by Adolf Schmidt, Gotha

More than half a century ago Gauss crowned his work whose most important basis he had created six years earlier in the treatise "Intensitas vis magneticae ad mensuram absolutam revocata", with his "General Theory of the Earth Magnetism". He had already formulated the most essential part of this work, the theory itself, sometime ago; however, the application to the actual magnetic condition of the earth proved to be impossible at the beginning because of the insufficient knowledge of just this condition. Only in the year 1838 did it become possible, by the publication of Sabine's chart of the total intensity, to complete the empirical basis of the theory to such an extent that Gauss could start the calculations which he had intended to perform for quite some time and which he had started several times unsuccessfully. But he himself called this undertaking only an attempt from which one should expect no more than a rough approximation. Indeed the observation material which he could use was still quite insufficient. For extensive regions of the earth's surface there were none, for others only insufficient measurements were available which often contradicted themselves. The usable observations were obtained to an overwhelming degree still with older means which Gauss and Weber had only replaced a short time ago with more complete methods and instruments, a fact which reduced the accuracy considerably especially for intensity measurements. To this was added the fact that the measurements referred in no way to one and the same point in time without the possibility to reduce them to a given epoch. Thus, the influence of the secular variations could not be taken into account; naturally, a way to remove the periodic and irregular fluctuations associated with the measured results was even more unthinkable. All these

* Numbers in the margin indicate pagination of foreign text.

deficiencies distort not only the charts of the magnetic curves on which Gauss based his calculations, they also disturb the comparison between the directly observed values of the magnetic elements and those calculated by the theory which he eventually conducted to check his theory. In view of these facts, the end results of that comparison must be considered as quite favorable even though the largest deviations remain behind occasional differences which were found between several observations at one and the same location.

If in this way the first application of the theory founded on the consideration of the potential led to a satisfactory result, the expectation seemed justified that a repetition of the calculation based on a more complete and exact material should lead to a higher degree for the approximation. This expectation was not fulfilled.

The Gaussian calculation has already been repeated several times. In the years 1846 to 1848, H.Petersen made an improved calculation* with the exclusive utilization of the magnetic observations made by A. Erman on his trip around the earth (1828 - 1830). However, he obtained, in this way, that the expression of the potential thus found presented those observations on the average twice as accurately as the expression calculated by Gauss. As was to be expected, however, greater deviations were found at points which were far removed from Erman's line of travel. Nevertheless, this work is of great interest because it establishes in the strictest sense a uniform observation material which is referenced to almost the same epoch.

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* The same has been published in the "Report of the eighteenth meeting of the British association held in 1848". The above statements are taken from the paper cited in the next footnote.

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Almost 30 years later, 1874, A. Erman and H. Petersen published, as the result of an extraordinarily laborious and careful work*, the determination of the earth's magnetic potential for the year 1829. They based this determination,which in the real sense can be considered as an improved repetition of that conducted by Gauss, referenced to approximately the same period of time, as far as possible on all observations made till 1870 after they had reduced them to the selected standard epoch. The careful and well-planned conduct of this reduction constitutes a particularly to be praised merit of the Erman-Petersen work. However, with regard to their results they exhibited a better agreement with the observations as those calculated by Gauss. However, the progress made was generally only minor; a noticeable improvement was made only in such areas in which Gauss had almost no observation available to him. The average deviation of the calculated and the observed values of the components was found to be equal to 0.005 to 0.010 $cm^{-\frac{1}{2}}g^{\frac{1}{2}}s^{-1}$, thus equal to a value which considerably exceeds the uncertainty of careful observations.

A new potential determination, referenced to the timepoint 1880,0 was undertaken by Quintus Icilius**. He based his calculation on the charts of the magnetic elements*** published by the German Naval Observatory. Without presenting the differences between observation and calculation found by him, he still pointed out that the latter were not substantially smaller than those for Gauss.

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* The bases of the Gaussian theory and the appearances of the earth's magnetism in the year 1829. Calculated by taking into account the secular variations from all available observations and presented by A. Erman and H. Petersen. Published under commission of the Emperor's Admiralty. Berlin 1874.

** The magnetic state of the earth calculated in accordance with the magnetic charts published by the German Naval Observatory for 1880,0 by G.V. Quintus Icilius. - From the Archives of the German Naval Observatory, IV. annual issue, 1881. No. 2.

*** These appeared, in addition to a detailed source listing, in the annals of Hydrography and Maritime Meteorology, 1880, H. XII.

For the epoch 1885,0 the present director of the German Naval Observatory finally derived the earth's magnetic potential, supported in the calculation part of the effort by the untiring researcher H. Petersen already mentioned several times, and compared the results obtained therefrom with the results of experimentation. Several charts which present the distribution of the magnetic elements and the potential values calculated therefrom on the surface of the earth, have recently appeared in the new edition of the "Physical Atlas" by Berghaus; except for this the bases and the results of the calculation have not yet been published. However, a clearer picture of the same and a development of the most essential conclusions from them are contained in the presentation* given by Mr. Neumayer in the opening session in the VIII Conference of German Geographers. From the information presented therein by the lecturer as well as from the extraordinarily numerous charts prepared with painstaking care presented to the conference, it is clear that none of the potential determinations till now are based on an amount of observation material as extensive and as reliable as this latest one. This impression already obtained from a cursory inspection which is only reinforced by a detailed inspection of the charts such as was made available to the writer by the kindness of the author, justifies the assumption that in the subject potential determination one obtains essentially that close contact between theory and experiment which can be attained at all by following the path pursuit till now. However - and this was made very clear by the lecturer who, because of his official position and because of his scientific activity is more familiar with the totality of the research material available at present than anybody else - our empirical knowledge of the magnetic condition of the surface of the earth still has many gaps.

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* Regarding the presently available material for the earth- and world magnetic research. Presentation given at the VIII. Conference of German Geographers in Berlin by Geheimrath Dr. G. Neumayer, Hamburg. Proceedings of the VIII. Conference of German Geographers, Berlin 1889.

Of large areas especially of the southern hemisphere we have still very little information, of the antarctic zone practically none. In /3 particular we are still missing much with regard to an even only approximate reliable knowledge of the secular changes. However, these deficiencies can be made harmless to a large extent by a purposeful critical evaluation of that which is known with certainty. The uncertainty regarding the value of the secular variations has little effect if one utilizes only such observations which lie close to the standard epoch. However, one cannot always practice such continence; in areas for which no newer determinations are available one will still have to evaluate the older measurements after careful inspection. However, one can now take into account the requirement for possible simultaneousness of the observations to be used to a much higher degree than this was possible in earlier times. And, in the second place, with regard to the large gaps of the observation network, the potential theory in particular presents the most reliable means to fill these gaps by interpolation, and for that reason this theory in particular is disturbed less by such gaps as those of all other observation methods. One will find this statement to be reasonable without detailed justification if one remembers that, strictly speaking, knowledge of the magnetic elements at a small number of points distributed over the surface of the earth is already sufficient to calculate the potential. The differences between the observed and the calculated values, after filling in the gaps still existing today, would probably be noticeably reduced only in the areas occupied by them but would be changed only little in the regions already sufficiently investigated at present and will even appear to be somewhat enlarged in certain locations. (A comparison of the Gaussian and the Erman-Petersen calculation provides a good confirmation of this assumption resulting from mathematical reasons). From the subject considerations it follows that the Neumayer potential determination, despite many present unavoidable imperfections, undoubtedly offers to the empirical basis a generally true picture of the degree with which the theory approaches experimental experience. A further confirmation for this is furnished by the fact that the unexplained deviations do not appear randomly, but exhibit in their distribution over the surface of the earth the clearly pronounced, not excessively complicated regularity. The difference charts for the individual elements prepared by Mr. Neumayer, who was the first one to follow the suggestions made by Gauss, as the keystone of his work, which in a simplified version are also attached to the preprint of his lecture, clearly show these regular arrangements of the differences between observation and calculation. It is probably that in this regard the Neumayer potential calculation will present a noticeable step forward compared to the earlier ones based on an incomplete foundation, especially as compared to those of Gauss. On the other hand, no such progress is made with regard to the magnitude of the deviations; a noticeable reduction of these deviations is not being attained. But these same deviations which could be considered in the first tests made by Gauss as a consequence of the then present inaccuracies and gaps of the empirical data, exceed today, by far, the considerably narrowed limits which include the potential contribution of these disturbing circumstances. And thus, the repeated efforts to present a satisfactory picture of the earth-magnetic phenomena based on the theory based by Gauss, lead with increasing certainty to the conclusion that this is not possible if one follows the paths used almost exclusively until now and often deviating from the one prescribed by the strict theory.

It is the merit of Mr. Neumayer to have clearly recognized this and to have expressed this in definite terms. He considers this to be the principle purpose of the previously mentioned lecture as well as of a complimentary lecture* given at the meeting of natural scientists in Heidelberg to direct the attention to the unsatisfactory nature of the present experiments and in this way to stimulate further formulation of the theoretical foundations for the same.

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* The results of a new calculation of the earth-magnetic constants. Lecture given by Mr. G. Neumayer, Hamburg, in The Physical Section of the Meeting of German Natural Scientists and Physicians in Heidelberg, 1889.

Just what should constitute this further formulation can be easily seen. There must be only few cases for the application of mathematical theory to natural phenomena in which one would not be initially satisfied with a simplified and abbreviated picture resulting from the introduction of numerous omissions. As long as the errors resulting therefrom do not reach the uncertainty of the observations, such a method is not only allowable, but should be preferred to a /4 more rigorous one since the latter, even if it can be carried out initially, requires increased work expenditures uselessly, or even at the expense of clarity and distinctness. Only whenever the sharpness of the observations increases, the theoretical treatment must abandon the simplifying but less rigorous assumptions. This is also the case here. When Gauss developed his theory, he could only apply to it a relatively deficient amount of observations and was therefore forced to introduce several simplifications. Thus, he chose to forego above all to take into account the flattening (ellipticity) of the earth. Furthermore, he was satisfied to retain from the infinite potential series only the terms of the first four orders. Finally, he derived the potential by means of a calculation encompassing all three components of the earth-magnetic force since a separate treatment of the horizontal and vertical partial force, as prescribed by rigorous theory, would have split up the available material too much. Now in all these relations the later calculations, despite the fact that they were spaced on a more extensive and accurate knowledge of the earth-magnetic phenomena, have followed the example given by Gauss. This is lamentable; however, this still leads to the advantage that final experiences were gathered concerning the results to be attained over the simplified path. Thus, one will be able to later confirm with surety those results which are obtained by a more rigorous theory free of omissions.

Thus, the path which must be followed in the future has been clearly prescribed. The omissions previously employed are no longer acceptable. The deviation of the earth from a spherical shape must be taken into account; the calculation must be set up in such a way that an extension of this series to terms of a higher order easily possible; the combination of the three components possible only with the aid of though probable, but dispensable, hypotheses must be abandoned. With regard to the second item, there is available already some experience, although of an unfavorable type. Mr. Neumayer expanded the calculation during the last few months to the terms of the fifth order without containing in this way a satisfactory connection to the results of the observations. However, a definite conclusion cannot be derived from this as long as the other two improvements have not been carried out. In a certain sense one could even interpret this experience as favorable in that it allows one to suspect that one will also in the future be able to dispense with a substantial increase of the expansion of a potential series.

About three years ago I touched upon the subject problem in a memorial paper presented to the International Polar Commission and pointed out at that time that one could take into account the flattening of the earth without large effort and without any essential changes in the form of the calculations. For a simple consideration shows that the most pronounced deviation of the calculation referenced to a sphere is affected by the appearance of certain factors which depend only on the magnitude of the ellipticity and which for that reason can be calculated in advance once and forever. At that time I started this, although rather laborious, but as previously mentioned, calculation to be made only once with the purpose to publish the results obtained therefrom. Another very time consuming work which I started soon thereafter and which I did not finish until the spring of the current year, forced me, however, to interrupt that calculation soon after I started. Now I have resumed the latter, stimulated to a considerable degree by the presentation of Mr. Neumayer, and have concluded it. The results of this calculation listed in several tables constitute, in addition to the developments leading thereto, the first section of the following communication. In the second section I then attempt to describe the path which a theoretical investigation of the phenomena of the earth's magnetism must follow which is rigorous-free of unnecessary assumptions.

Development of the General Formulas for the Representation of the Potential and the Components of the Earth-magnetic Force.

In order to simplify the understanding of the developments referring to the earth as an ellipsoid and in order to allow at the same time to more clearly stand out the effect which the taking into account of the flattening of the earth has on the potential determination, I precede these developments with a summary of the known formulas which apply when we consider the earth to be a sphere. This is /5
also important for the reason that I deviate in the establishing of the algebraic signs at two points from the previously conventional usage going back to Gauss. For I am basing, as briefly noted for explanation and justification, my calculation on those stipulations which in theoretical physics, which also includes the science of the earth's magnetism, have taken on a dominating role and which at the present time, are being used almost exclusively. (One should compare in this regard a dissertation by Budde in the February issue of the "Annalen der Physik und Chemie" (Annals of Physics and Chemistry) from the year 1887). Even if we are dealing here with a quite unimportant triviality for the theory, nevertheless, in a practical application, the carrying out of a given generally accepted designation method is valuable particularly in potential science and should be aspired.

In order to express the position of a point in its relation to the earth in terms of numbers, let us first assume an orthogonal coordinate system whose axes intersect at the center of the earth. Let the positive semi-axis of the ξ be directed toward the North Pole while that of the η intersects the equator at the meridian of Greenwich, that of the ζ intersects the same at a point displaced further eastward by 90°. An eye which looks along the first axis from negative to positive, thus has the third semi-axis to the right of the second one of the same name, following the usual arrangement. In an arrangement agreeing therewith let us establish at each point of the surface of the earth a coordinate system with the x-axis toward the north, the y-axis toward the east, and the z-axis vertical downward. (One can proceed similarly at any arbitrary point. One need only put, in place of the surface of the earth, the spherical surface concentric with it and passing through that point.) The components of the earth-magnetic total force T form in the three designated directions I will call X,Y,Z. For the second the algebraic sign is opposite to the one introduced by Gauss. In conjunction therewith, the declination which Gauss counts as positive for a considerable deviation of the north magnetic pole must assume here, for an eastward deviation, the positive sign. Additionally, the latter counting method is recommended also because of its agreement with the conventional azimuth calculation method.

In the following consideration I first start with the assumption that the earth's magnetic force derives from a distribution of equal quantities of free positive and negative magnetism. The results obtained with this assumption generally also retain their validity if that force is based partly or wholly on electrical currents. Only for such volume regions through which the currents themselves flow, certain changes are necessary which I shall discuss in the next section.

The potential of a magnetic mass whose density in the volume element dk is equal to μ, has, at every point at a distance of R from that element the value

$$(1) \ldots\ldots\ldots\ldots \quad V = \int \frac{\mu dk}{R}$$

if the integration is extended over the entire volume filled by the magnetism. The force acting at any points according to the direction of a line element dp is

$$(2) \ldots\ldots \quad P = -\frac{dV}{dp}, \quad X = -\frac{dV}{dx}, \quad Y = -\frac{dV}{dy}, \quad Z = -\frac{dV}{dz}.$$

Thus, it is positive in the direction from higher to lower initial values. Gauss designates the value of the integral given in (1) by -V and thus obtains the force acting in any direction at the same, not as here, as the differential quotient of the potential V taken in the opposite direction.

The foundation for the evaluation of the above integral is now formed in every case by the representation of the reciprocal distance between two points in a version simplifying the integration as much as possible. The choice of that form depends on the shape of that surface for whose points the potential, in addition to its differential quotient, must be determined as a first priority, thus here on the surface of the earth*. If we consider this to be spherical, then one must perform the known development with spherical functions. For this purpose, one introduces the spherical polar coordinates r, u, λ. defined by the equations /6

$$(3) \ldots\ldots\ldots\ldots\ldots\ldots \quad \xi = r\cos u \qquad \eta = r\sin u\cos\lambda \qquad \zeta = r\sin u\sin\lambda$$

The reciprocal value of the distance R between two points $(r_1\ u_1\ \lambda_1)$ and $(r_2 u_2 \lambda_2)$ is thus represented by the sum of a convergent infinite series. It is

$$(4) \ldots\ldots \quad \frac{1}{R} = \frac{1}{r_2}\sum_{n=0}^{n=\infty}\sum_{m=0}^{m=n} a_n^m\left(\frac{r_1}{r_2}\right)^n P_n^m(\cos u_1)\, P_n^m(\cos u_2)\cos m(\lambda_1-\lambda_2) \qquad \text{for } r_1<r_2$$

The spherical functions appearing here are defined by the equation

$$(5) \ldots P_n^m(x) = (1-x^2)^{\frac{m}{2}} x^{n-m}\left[1-\frac{(n-m)(n-m-1)}{2(2n-1)}x^{-2}+\frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2.4.(2n-1)(2n-3)}x^{-4}-\ldots\right] \quad (m\leq n)$$

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* If a regular distribution of the acting masses is given, one may under certain circumstances allow this to be the determining factor for the shape of the plot.

and we obtain

$$(6)\ldots\ldots\ldots\ldots a_n^0 = \frac{(1.3.5\ldots(2n-1))^2}{n!\,n!} \qquad a_n^m = 2.\frac{(1.3.5\ldots(2n-1))^2}{(n-m)!\,(n+m)!} \qquad m>0$$

If one now wishes to determine with the aid of this development the value of the earth-magnegnetic potential at any arbitrary point $(r\ \mu\ \lambda)$, then one must break it up into two parts to be calculated separately. The first derives from magnetic masses at such points which are closer to the center of the earth than that point for which $r_1 < r$; the second, on the other hand, derives from such farther removed points for which thus the relation $r_2 > r$ applies. By taking into account this fact, one finds for the potential the following expression which, by the introduction of the radius of the earth r_0, is brought into such a form that its coefficients c, s, γ, σ possess the dimension of the intensity of a magnetic field ($L^{-\frac{1}{2}}\ M^{\frac{1}{2}}\ T^{-1}$).

$$(7)\ldots V = r_0 \sum_{n=0}^{n=\infty} \sum_{m=0}^{m=n} P_n^m(\cos u)\left[\left(\frac{r}{r_0}\right)^{-n-1}(c_n^m \cos m\lambda + s_n^m \sin m\lambda) + \left(\frac{r}{r_0}\right)^{n}(\gamma_n^m \cos m\lambda + \sigma_n^m \sin m\lambda)\right]$$

The just named coefficients appearing herein depend on the distribution of the magnetic masses. They are expressed by the following definite integrals in which μ_1 and μ_2 denote the magnetic density in the volume elements $r_1^2 \sin u_1 dr_1 du_1 d\lambda_1$ and $r_2^2 \sin u_2 dr_2 du_2 d\lambda_2$:

$$(8)\ldots\ldots\ldots\ldots \begin{cases} c_n^m = a_n^m r_0^{-n-2}\int_0^{2\pi}\cos m\lambda_1\, d\lambda_1\int_0^{\pi}P_n^m(\cos u_1)\sin u_1\, du_1\int_0^{r}\mu_1 r_1^{n+2}\, dr_1 \\ s_n^m = a_n^m r_0^{-n-2}\int_0^{2\pi}\sin m\lambda_1\, d\lambda_1\int_0^{\pi}P_n^m(\cos u_1)\sin u_1\, du_1\int_0^{r}\mu_1 r_1^{n+2}\, dr_1 \\ \gamma_n^m = a_n^m r_0^{n-1}\int_0^{2\pi}\cos m\lambda_2\, d\lambda_2\int_0^{\pi}P_n^m(\cos u_2)\sin u_2\, du_2\int_r^{\infty}\mu_2 r_2^{-n+1}\, dr_2 \\ \sigma_n^m = a_n^m r_0^{n-1}\int_0^{2\pi}\sin m\lambda_2\, d\lambda_2\int_0^{\pi}P_n^m(\cos u_2)\sin u_2\, du_2\int_r^{\infty}\mu_2 r_2^{-n+1}\, dr_2 \end{cases}$$

After the potential for any arbitrary point has been set up, one obtains the same for the points of the surface of the earth if one puts the radius of the earth r_0 in place of r. λ and μ then become the geographical longitude and the complement of the geographical latitude. The potential at the surface of the earth, knowledge of which is particularly important, thus appears without any additional transformation as a function of the two just given geographical coordinates. This is a consequence of the fact that the equation of that surface in the coordinate system on which it is based takes on the simple shape r = constant.

From the potential we now obtain the components of the earth's magnetic force in accordance with (2) as a result of the equations /7

$$(9)\quad X = -\frac{dV}{dx} = \frac{1}{r_0}\frac{dV}{d\mu} \qquad Y = -\frac{dV}{dy} = -\frac{1}{r_0 \sin\mu}\frac{dV}{d\lambda} \qquad Z = -\frac{dV}{dz} = \frac{dV}{dr} \qquad (r = r_0)$$

The position of the axes to which these components are referenced depends on the shape of the surface of the earth. If therefore their deviation from the spherical shape must be taken into account, the subject equations (9) (in case the earth is considered to be a body of rotation, only the first and the third) must be subjected to only certain, easily stated changes. However, all other developments will retain, since they represent only the general equations of the potential theory, their validity even then when the flattening of the earth is taken into account. Nevertheless it would serve no purpose to make use of them in this case for the reason that the equation of the ellipsoid-shaped surface of the earth in spherical coordinates takes on no simple shape so that the transition to these from the geographical coordinates makes necessary involved transformations. Of even greater importance is the fact that the two parts of the potential which appear in equation (7) separately, lose their simple significance to represent the effect of the magnetic masses found within and outside of the surface of the earth separately.

By the introduction of elliptical polar coordinates the disadvantages described can be avoided. These coordinates are defined by the equations

$$(10) \ldots\ldots\ldots \xi = r \cos v \qquad \eta = \sqrt{r^2+e^2} \sin v \cos \lambda \qquad \zeta = \sqrt{r^2+e^2} \sin v \sin \lambda$$

in which e represents a constant. All points for which r takes on the same value fill the surface of an ellipsoid revolution whose polar radius r and whose equatorial radius is $\sqrt{r^2 + e^2}$. Thus the surfaces belonging to different values of r are all confocal. They also include the surface of the earth which, if its axis of rotation is 2 b, is characterized by the simple equation r = b. For the points of this surface λ and v go over to the geographical longitude and the complement of the so-called reduced latitude whichhas a simple relation to the geographical latitude. If one again designates this, as for the sphere, by (90°- u), and if the equatorial diameter which according to the statements just made is equal to $\sqrt{b^2 + e^2}$, is called a , then we obtain

$$(11) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a \operatorname{tg} u = b \operatorname{tg} v$$

Naturally the coordinates of every other point also make possible a very similar geometrical interpretation; in place of the surface of the earth we put the ellipsoid drawn through every point and confocal with the latter. The equation (11) is thus transformed into the following

$$(12) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \sqrt{r^2+e^2} \operatorname{tg} u = r \operatorname{tg} v$$

From the magnetic potential expressed in elliptical polar coordinates we obtain the three force components X,Y,Z by means of equation (2) by taking into account the known properties of the ellipsoid of revolution. (Because of this as well as for the justification of (11) we refer to the textbooks of geodesy, e.g. to "Helmert: The Mathematical and Physical Theories of Higher Geodesy". These interesting formulas (13) are also found in "Heine: Handbook of Spherical Functions" on page 328.) On the ellipsoid, whose axis of rotation is 2r and whose second axis is $2\sqrt{r^2+e^2}$, the parallel circle part of the to be reduced pole distance v, has the radius $\sin v\sqrt{r^2+e^2}$ and the element of the meridian corresponding to the change in the latitude dv has the value $dv\sqrt{r^2+e^2\cos v^2}$ for the same polar distance. Exactly there, the normal distance of the ellipsoid from the one infinitely next to it whose polar radius is (r+dr), is equal to $dr\sqrt{r^2+e^2\cos v^2} : \sqrt{r^2+e^2}$. Thus one obtains

$$dx = -dv\sqrt{r^2+e^2\cos v^2} \qquad dy = d\lambda \sin v\sqrt{r^2+e^2} \qquad dz = -dr\sqrt{r^2+e^2\cos v^2} : \sqrt{r^2+e^2}$$

and accordingly for the component of the force one obtains

$$(18)\ldots\ldots\; X = \frac{1}{\sqrt{r^2+e^2\cos v^2}}\frac{dV}{dv} \qquad Y = -\frac{1}{\sin v\sqrt{r^2+e^2}}\frac{dV}{d\lambda} \qquad Z = \frac{\sqrt{r^2+e^2}}{\sqrt{r^2+e^2\cos v^2}}\frac{dV}{dr}$$

By introducing the special value b in place of r one obtains herefrom the values valid for the surface of the earth.

Now I turn to the evaluation of the potential of a definite magnetic mass coverage. The mathematical developments which lead to the solution not only of these, but also of several other problems of theoretical physics equivalent to it, have already been presented by the work of Lamé, Heine, and F. Neumann quite some time ago. Here it is only necessary to bring the result derived by these research scientists into a version as suitable as possible for numerical calculations. /8

I start with the formula which Heine has found in his "Handbook of Spherical Function" for the reciprocal value of the distance R between two points. (also in the above page 320.) In the designation used by him this formula is:

$$\frac{1}{R} = \frac{1}{e}\sum_{n=0}^{n=\infty}\sum_{m=0}^{m=n}(-1)^m a_n^m P_n^m(\cos\theta_1) P_n^m(\cos\theta_2) P_n^m(\rho_1) Q_n^m(\rho_2)\cos m(\psi_1-\psi_2) \qquad (\rho_1<\rho_2)$$

The function which is called here P_n^m does not agree completely with the just designated version, defined by equation (5); rather it is equal to the product from it and the factor $(\sqrt{-1})^m$. The so-called spherical function of second type Q_n^m is determined by the following equation:

$$(14)\ldots Q_n^m(x) = (x^2-1)^{\frac{m}{2}} x^{-n-m-1}\left[1+\frac{(n+m+1)(n+m+2)}{2(2n+3)}x^{-2}+\frac{(n+m+1)(n+m+2)(n+m+3)(n+m+4)}{2.4(2n+3)(2n+5)}x^{-4}+\ldots\right]$$

(Mod $x>1$)

In addition it should be remarked that this series results if one places $-(n+1)$ in place of n in the expression valid for P_n^m (in accordance with Heine's designation). However, an important difference of both series rests in the fact that the one serving to represent Q_n^m is infinite while the other one breaks off after a finite number of terms.

I now rewrite Heine's formula in the here conventional manner. For $\theta_1, \theta_2, \psi_1, \psi_2$ one must write v_1, v_2, λ_1, λ_2. In place of e we put $\pm ei$, that is $\pm e\sqrt{-1}$, since Heine denotes $\sqrt{b^2-a^2}$ by e. I choose the lower algebraic sign so that I might put i:e for 1:e. Therefore one must write for ρ, which represents the quotient r:e, now ir:e. The factor $(-1)^m$ finally disappears by the introduction of the definition of P_n^m given in equation (5). Thus the reciprocal distance of two points $(r_1 v_1 \lambda_1)$ and $(r_2 v_2 \lambda_2)$ appears in the form

$$(15)\ldots\ldots \frac{1}{E} = \frac{i}{e}\sum_{n=0}^{n=\infty}\sum_{m=0}^{m=n} a_n^m P_n^m(\cos v_1) P_n^m(\cos v_2) P_n^m\left(\frac{ir_1}{e}\right) Q_n^m\left(\frac{ir_2}{e}\right)\cos m(\lambda_1-\lambda_2) \quad \text{for } r_1<r_2$$

In order to make this expression more convenient for the numerical calculation, I alter the designation somewhat. I place

$$(16)\begin{cases} P_n^m(x) = x^n \cdot p_n^m(x) \\ Q_n^m(x) = x^{-n-1} \cdot q_n^m(x) \end{cases}$$ and furthermore write for shortening: $$p_n^m\left(\frac{ir}{e}\right) = p_n^m[r] \qquad q_n^m\left(\frac{ir}{e}\right) = q_n^m[r]$$

An inspection of equations (5) and (14) allows one to recognize the significance of the hereby introduced functions p_n^m and q_n^m and shows at the same time that these are real for purely imaginary arguments and are equal to 1 for infinitely large values. Since for the application to the earth ellipsoid e becomes approximately equal to b:17, one can furthermore easily see that, except for the hardly to be considered points near the center of the earth, the values of the arguments take on everywhere large numbers and the values of the function p_n^m and q_n^m are therefore nowhere considerably different from 1.

If one observes that according to equation (16) /9

$$\frac{i}{e} P_n^m\left(\frac{ir_1}{e}\right) Q_n^m\left(\frac{ir_2}{e}\right) = \frac{i}{e}\left(\frac{ir_1}{e}\right)^n \left(\frac{ir_2}{e}\right)^{-n-1} p_n^m\left(\frac{ir_1}{e}\right) q_n^m\left(\frac{ir_2}{e}\right) = \frac{r_1^n}{r_2^{n+1}} p_n^m[r_1]\, q_n^m[r_2]$$

then one finds by substitution into (15)

$$(17)\ldots\ldots \frac{1}{E} = \frac{1}{r_2} \sum_{n=0}^{n=\infty} \sum_{m=0}^{m=n} a_n^m\, p_n^m[r_1]\, q_n^m[r_2] \left(\frac{r_1}{r_2}\right)^n P_n^m(\cos v_1)\, P_n^m(\cos v_2) \cos m(\lambda_1 - \lambda_2) \qquad \text{for } r_1 < r_2$$

One readily recognizes the great analogy of this formula with the one given in (4) which is referred to the sphere and which is derived from it without any additional transformations if we place e = 0.

Let there now be again present a magnetic mass distribution with the density μ in point (r v λ). In a similar manner as previously for the sphere (compare 7 and 3) we then obtain for the value of the potential at point (r v λ)

$$(18)\ldots V = b \sum_{n=0}^{n=\infty} \sum_{m=0}^{m=n} P_n^m(\cos v)\left[q_n^m[r]\left(\frac{r}{b}\right)^{-n-1}(c_n^m \cos m\lambda + s_n^m \sin m\lambda) + p_n^m[r]\left(\frac{r}{b}\right)^n(\gamma_n^m \cos m\lambda + \sigma_n^m \sin m\lambda)\right]$$

The coefficients c, s, γ, σ are determined by the following formulas for whose explanation one must remark that the volume element is expressed in the application of elliptical polar coordinates by $(r^2+e^2 \cos v^2) \sin v \, dr \, dv \, d\lambda$.

$$
(19)\ldots\ldots \begin{cases}
c_n^m = a_n^m b^{-n-2} \int_0^{2\pi} \cos m\lambda_1 \, d\lambda_1 \int_0^{\pi} P_n^m(\cos v_1) \sin v_1 \, dv_1 \int_0^{r} \mu_1 \, p_n^m[r_1] \, r_1^n (r_1^2+e^2 \cos v_1^2) \, dr_1 \\
s_n^m = a_n^m b^{-n-2} \int_0^{2\pi} \sin m\lambda_1 \, d\lambda_1 \int_0^{\pi} P_n^m(\cos v_1) \sin v_1 \, dv_1 \int_0^{r} \mu_1 \, p_n^m[r_1] \, r_1^n (r_1^2+e^2 \cos v_1^2) \, dr_1 \\
\gamma_n^m = a_n^m b^{n-1} \int_0^{2\pi} \cos m\lambda_2 \, d\lambda_2 \int_0^{\pi} P_n^m(\cos v_2) \sin v_2 \, dv_2 \int_r^{\infty} \mu_2 \, q_n^m[r_2] \, r_2^{-n-1} (r_2^2+e^2 \cos v_2^2) \, dr_2 \\
\sigma_n^m = a_n^m b^{n-1} \int_0^{2\pi} \sin m\lambda_2 \, d\lambda_2 \int_0^{\pi} P_n^m(\cos v_2) \sin v_2 \, dv_2 \int^{\infty} \mu_2 \, q_n^m[r_2] \, r_2^{-n-1} (r_2^2+e^2 \cos v_2^2) \, dr_2
\end{cases}
$$

From the expression obtained for V we obtained the potential at the surface of the earth by placing b in the place of r. The three components of the earth-magnetic force obtained from the same expression by the application of the already derived formulas (13) and the subsequent carrying out of the same substitution. The differentiation of V with respect to v and λ and thus the calculation of X and Y offers no reason for additional remarks. For the determination of Z, however, it makes sense to carry out a small transformation. In order to simplify the numerical calculation of the differential quotient appearing in Z of $q_n^m[r] r^{-n-1}$ and $p_n^m[r] r^n$ with respect to r, I place

$$
(20)\ldots\ldots\ldots \begin{cases}
\dfrac{dP_n^m(x)}{dx} = n x^{n-1} \pi_n^m(x) \\
\dfrac{dQ_n^m(x)}{dx} = -(n+1) x^{-n-2} \kappa_n^m(x)
\end{cases}
$$

and for abbreviation: $\pi_n^m\left(\frac{ir}{e}\right) = \pi_n^m[r]$, $\kappa_n^m\left(\frac{ir}{e}\right) = \kappa_n^m[r]$

From this we easily obtain

$$\frac{d(r^n \mathfrak{p}_m^n[r])}{dr} = n r^{n-1} \pi_m^n[r] \qquad \frac{d(r^{-n-1} \mathfrak{q}_m^n[r])}{dr} = -(n+1) r^{-n-2} \kappa_m^n[r]$$

and one recognizes without any difficulty that for the values of r taken into consideration here the functions κ_m^n and π_m^n as well as the earlier defined $\mathfrak{p}_m^n$ and $\mathfrak{q}_m^n$ are real and deviate little from 1. /10

I now put together the values of the three components as they are obtained for the same in accordance with the existing remarks for the points of the surface of the earth. For this, I still introduce the simplification, to write in place of $\mathfrak{p}_m^n[b]$ etc. in short the expression $\mathfrak{p}_m^n$.

$$(21)\begin{cases} X = \dfrac{b}{\sqrt{b^2+e^2\cos v^2}} \displaystyle\sum_{n=0}^{n=\infty} \sum_{m=0}^{m=n} \frac{dP_m^n(\cos v)}{dv} \left[\mathfrak{q}_m^n(c_m^n \cos m\lambda + s_m^n \sin m\lambda) + \mathfrak{p}_m^n(\gamma_m^n \cos m\lambda + \sigma_m^n \sin m\lambda)\right] \\ Y = -\dfrac{b}{\sin v\sqrt{b^2+e^2}} \displaystyle\sum_{n=0}^{n=\infty} \sum_{m=0}^{m=n} m P_m^n(\cos v) \left[\mathfrak{q}_m^n(s_m^n \cos m\lambda - c_m^n \sin m\lambda) + \mathfrak{p}_m^n(\sigma_m^n \cos m\lambda - \gamma_m^n \sin m\lambda)\right] \\ Z = \dfrac{\sqrt{b^2+e^2}}{\sqrt{b^2+e^2\cos v^2}} \displaystyle\sum_{n=0}^{n=\infty} \sum_{m=0}^{m=n} P_m^n(\cos v) \left[-(n+1)\kappa_m^n(c_m^n \cos m\lambda + s_m^n \sin m\lambda) + n\pi_m^n(\gamma_m^n \cos m\lambda + \sigma_m^n \sin m\lambda)\right] \end{cases}$$

All three expressions are quite similar to the corresponding ones which are valid for the sphere, and into which they can be transformed by the substitution e = 0. They differ from them in the first place by certain factors approaching unity which appear when ε is substituted for the quotient e:b

$$\frac{1}{\sqrt{1+\varepsilon^2\cos v^2}} \qquad \frac{1}{\sqrt{1+\varepsilon^2}} \qquad \frac{\sqrt{1+\varepsilon^2}}{\sqrt{1+\varepsilon^2\cos v^2}}$$

The quantity ϵ^2 is obviously an unnamed number which depends solely on the flattening of the earth. If we call the reciprocal value of the latter α, then $b = (\alpha - 1)\, a : \alpha$ and $\epsilon^2 = (2\alpha - 1) : (\alpha - 1)^2$.

An additional deviation of the formulas (21) referenced to the ellipsoid from the values valid for the sphere consists of the appearance in the so-called reduced instead of geographical polar distance. Strictly speaking, this cannot even be called a deviation since both angle quantities become identical for the sphere. (Therefore, there existed in the treatment used till now of the potential calculation under the assumption of a spherical earth shape a certain randomness, although justified by convenience considerations, in the fact that one introduced particularly the geographic latitude as argument for the development. As it turns out, i.e., without setting up a special investigation concerning the most effective establishment of the latitude argument, one might just as well have introduced the reduced or the geocentric latitude or any other angle quantity at all which differs from these only by the amount of the order of magnitude of the flattening.) Since for the observation points the geographical latitude and thus u is the originally given quantity, one must derive v therefrom. This can be done with the aid of equation (11) or more conveniently by means of the approximation formula $v = u + \frac{1}{4}\epsilon^2 \sin 2u$, resulting from the latter whose error nowhere exceeds $\frac{1}{2}''$. Furthermore it would not be difficult to transform equation (21) in such a way that u appears, not v; however, they would thereby lose considerably simplicity and clarity without this disadvantage being compensated for by a significant shortening of the numerical calculation.

One can readily see that through the circumstances touched upon till now the potential calculation experiences only a quite insignificant increase in mechanical computer work if one takes into account the deviation of the earth from a spherical shape. On the other hand, the determination of the coefficients p, q, π, x by which formulas valid for the ellipsoid finally differ from those referenced to the sphere, requires a rather drawn out numerical computation. However, this can once and for all obviously be carried out in advance since those coefficients depend neither on the results of the observation nor on the position of the observation points, but rather solely on the flattening of the earth. In fact, these constants denote the values of certain functions for the arguments $(ib:e)$, i.e.,

$(i:e)$ or $i(e-1):\sqrt{2e-1}$.

I have now carried out the calculation of the designated coefficients /11 and namely for the functions of the first 6 orders. The results are listed in the following tables. Since until now one did not go in the potential determination beyond terms of fourth order as a rule, and since on the other hand the only attempt to still take into account the spherical functions of this order showed that no significant improvement can be obtained in this way, the values listed here should be sufficient for a long time.

Regarding the question on what magnitude of the flattening the calculation should be based, I have decided to use the Bessel value which is still often being used for various reasons, i.e., α = 299.1528, but to make easier at the same time the transition to any other value of flattening by presenting suitable auxiliary values.

A few brief remarks are sufficient to explain the tables. Since these refer without any deviation to all four numerical tables, I will utilize a summarizing designation; I shall, instead of naming the quantities p, π, q, x always individually, use the letter z for each one of them without distinction.

The numbers 0.1...6 forming the headings of each column of each table denote the upper index n, the same ones shown at the beginning of the individual rows denote the lower index m. At the position characterized by the meeting of the column n and the horizontal row m we always find three numbers. These represent the values calculated with the use of the Bessel flattening number

$$s_m^n \qquad \frac{ds_m^n}{d\alpha} \qquad \frac{1}{2}\frac{d^2 s_m^n}{d\alpha^2}$$

- and namely the latter two in units of the sixth decimal place of the first. Thus if one wishes to find s_m^n for any other value of flattening which is represented by $1:(299.1528 + \Delta\alpha)$, then one must calculate with the aid of the numbers taken from the appropriate table the sum

$$s_m^n + \frac{ds_m^n}{d\alpha}\cdot\Delta\alpha + \frac{1}{2}\frac{d^2 s_m^n}{d\alpha^2}(\Delta\alpha)^2$$

this sum represents the desired value, and to be sure, as long as $\Delta\alpha$ taken as absolute value is not greater than 10, up to an error of maximum of one unit of the sixth decimal place. (In addition it should be stated that such an auxiliary calculation makes possible to derive the values of the coefficients p, π, q, x also for points which do not lie on the surface of the earth, but only close to it. This becomes directly evident if one considers that those coefficients do not depend on the absolute magnitude, but only on the flattening of the ellipsoid and that the latter possesses a different value for each ellipsoid confocal with the surface of the earth. The numerical tables listed here thus suffice completely for the calculation of the potential and the force components at all points at which magnetic measurements can be made.)

I shall still add several remarks concerning the calculation of these tables. The basis for them is naturally formed by the definition equations (5), (14), (16) and (20). The argument generally designated in it by x here becomes $ib:e$ or $i:\epsilon$, thus purely imaginary. On the other hand, in the infinite series which represent p, π, q, x, only whole negative numbers appear as exponents of x so that x^{-2} is best introduced as an argument. Since $x^{-2} = -\epsilon^2 = -(2\alpha-1):(\alpha-1)^2$, we obtain by using the Bessel flattening number

$$x^{-2} = 0.0067192187$$

1) Table of the coefficients p_m^n.

m; n	0			1			2			3			4			5			6		
0	1.000 000	0.0	0.00	1.000 000	0.0	0.00	1.002 240	− 7.5	+0.03	1.004 032	−13.5	+0.05	1.005 763	−19.4	+0.07	1.007 477	−25.2	+0.08	1.009 183	−30.9	+0.10
1				1.003 354	−11.2	+0.04	1.003 354	−11.2	+0.04	1.004 702	−15.8	+0.05	1.006 243	−21.0	+0.07	1.007 851	−26.4	+0.09	1.009 490	−32.0	+0.11
2							1.006 719	−22.6	+0.08	1.006 719	−22.6	+0.08	1.007 686	−25.8	+0.09	1.008 974	−30.2	+0.10	1.010 410	−35.1	+0.12
3										1.010 096	−34.0	+0.11	1.010 096	−34.0	+0.11	1.010 850	−36.5	+0.12	1.011 947	−40.3	+0.14
4													1.013 484	−45.5	+0.15	1.013 484	−45.5	+0.15	1.014 103	−47.6	+0.16
5																1.016 883	−57.0	+0.19	1.016 883	−57.0	+0.19
6																			1.020 293	−68.6	+0.23

2) Table of the coefficients π_m^n.

m; n	0			1			2			3			4			5			6		
0	1.000 000	0.0	0.00	1.000 000	0.0	0.00	1.000 000	0.0	0.00	1.001 344	− 4.5	+0.02	1.002 880	− 9.7	+0.03	1.004 482	−15.1	+0.05	1.006 115	−20.6	+0.07
1				0.996 657	+11.2	−0.04	1.000 006	0.0	0.00	1.001 568	− 5.3	+0.02	1.003 120	−10.5	+0.04	1.004 706	−15.8	+0.05	1.006 319	−21.3	+0.07
2							1.000 000	0.0	0.00	1.002 340	− 7.5	+0.03	1.003 840	−12.9	+0.04	1.005 378	−18.1	+0.06	1.006 931	−23.3	+0.08
3										1.003 354	−11.2	+0.04	1.005 039	−16.9	+0.06	1.006 500	−21.9	+0.07	1.007 953	−26.8	+0.09
4													1.006 719	−22.6	+0.08	1.008 072	−27.1	+0.09	1.009 384	−31.6	+0.11
5																1.010 096	−34.0	+0.11	1.011 227	−37.8	+0.13
6																			1.013 484	−45.5	+0.15

3) Table of the coefficients q_m^n.

m; n	0			1			2			3			4			5			6		
0	0.997 769	+7.5	−0.02	0.995 988	+13.4	−0.04	0.994 273	+19.1	−0.06	0.992 582	+24.8	−0.08	0.990 903	+30.3	−0.10	0.989 233	+35.9	−0.12	0.987 568	+41.4	−0.14
1				0.995 322	+15.6	−0.05	0.993 798	+20.7	−0.07	0.992 213	+26.0	−0.09	0.990 602	+31.3	−0.10	0.988 978	+36.7	−0.12	0.987 348	+42.1	−0.14
2							0.992 374	+25.4	−0.08	0.991 108	+29.7	−0.10	0.989 699	+34.3	−0.11	0.988 216	+39.2	−0.13	0.986 688	+44.3	−0.15
3										0.989 267	+35.7	−0.12	0.988 196	+39.3	−0.13	0.986 945	+43.4	−0.14	0.985 589	+47.9	−0.16
4													0.986 094	+46.2	−0.15	0.985 109	+49.3	−0.16	0.984 052	+53.0	−0.18
5																0.982 890	+56.8	−0.19	0.982 079	+59.5	−0.20
6																			0.979 672	+67.4	−0.22

4) Table of the coefficients x_m^n.

m; n	0			1			2			3			4			5			6		
0	0.993 326	+22.3	−0.07	0.991 995	+26.7	−0.09	0.990 476	+31.7	−0.11	0.988 896	+37.0	−0.12	0.987 291	+42.3	−0.14	0.985 672	+47.7	−0.16	0.984 048	+53.0	−0.18
1				0.990 669	+31.1	−0.10	0.989 688	+34.4	−0.11	0.988 346	+38.8	−0.13	0.986 871	+43.7	−0.15	0.985 335	+48.8	−0.16	0.983 766	+53.9	−0.18
2							0.987 326	+42.2	−0.14	0.986 694	+44.3	−0.15	0.985 612	+47.8	−0.16	0.984 322	+52.1	−0.17	0.982 920	+56.7	−0.19
3										0.983 947	+53.3	−0.18	0.983 516	+54.7	−0.18	0.982 635	+57.6	−0.19	0.981 513	+61.3	−0.20
4													0.980 588	+64.4	−0.21	0.980 278	+65.4	−0.22	0.979 545	+67.8	−0.22
5																0.977 254	+75.3	−0.25	0.977 021	+76.1	−0.25
6																			0.973 942	+86.1	−0.28

The calculation of the differential quotients taken after α of p, π, q, x was made with the aid of the rows serving for the definition of these quantities themselves by the introduction of the easily to be derived values

$$\frac{d x^{-6}}{d\alpha} = +22.574 \,.\, 10^{-6} \qquad \frac{1}{2}\frac{d^2 x^{-6}}{d\alpha^2} = -0.07584 \,.\, 10^{-6}$$

All calculations were made in such a way that the final results /13
were obtained accurately up to 7 decimal places. Before they were put into the tables printed here, they were shortened by one decimal place. If in this way we obtained as final number by rounding off in an upward direction the numeral 5, then the latter which for further rounding off simply had to be dropped, was designated by a horizontal line placed above it. In order to attain the specified sharpness of the results, it was generally sufficient to take into account the first four terms of the infinite series. Only in a few cases was it necessary to still take into account the term containing the factor x^{-8}. It goes without saying that all results were confirmed by means of sufficient control calculations; beyond that most of the numbers were calculated twice and namely always by two different methods.

It can easily appear that the maintaining of the accuracy of the results in this way represented great exaggeration, and indeed, one could easily carry out an additional rounding off when using these numbers. However, it seemed useful on the one hand to list the numbers so accurately that they would undoubtedly be sufficient in the future for the application of a perfected observation material - on the other hand it is always important in the mathematical treatment of experimentally determined quantities (here thus the earth-magnetic measurement results) to carry out all calculations independent of these empirical data so accurately that they could be considered as being absolutely accurate as compared with them.

The fact that the flattening of the earth is still uncertain by such an amount that the coefficients p, π, q, x depending on it actually do not possess the accuracy striven for here, is of no great significance here; it is primarily necessary that these coefficients correspond to one and the same exactly defined assumption concerning the shape of the earth.

I shall finally still remark that, as will be shown later (Ger. p. 23), the knowledge of the quotients $\pi_n^m : p_n^m$ and $\varkappa_n^m : q_n^m$ is sufficient for the purpose of the potential calculation. If despite this I still preferred to list the coefficients p, π, q, x individually, this was done primarily because of their simpler theoretical significance which lends to them a greater interest and a more extensive application capability than possessed by their quotients.

The Analytical Representation of the Magnetic Condition of the Earth Based on Observations

In the introduction I suggested that through the results of the preceding section the potential calculation for the ellipsoid is brought back to that for the spherical surface. Therefore, the numerical calculation to be made based on the observation experiences no significant changes neither in its arrangement nor in its scope if the flattening of the earth is taken into account. The changes occurring in this case are, once again listed briefly, the following. Instead of the geographical latitude the reduced latitude must be introduced, in place of the force components X, Y, Z derived directly from the observations of the elements one must obviously use the quantities deviating only insignificantly from them

$$(1) \ldots\ldots\ldots\ldots \quad \mathfrak{X} = X\sqrt{1+e^2\cos v^2} \qquad \mathfrak{Y} = Y\sqrt{1+e^2} \qquad \mathfrak{Z} = Z\sqrt{\frac{1+e^2\cos v^2}{1+e^2}}$$

as basis for the calculation, and finally the factors calculated in the preceding section must be added to the coefficients of the theories proceding in accordance with spherical functions. The latter are assimilated by the factors depending on the geographical coordinates and thus change the shape of the equations just as little as the two previously named deviations which represent a preliminary calculation. Thus, the following statements are generally valid whether one wishes to take into account the flattening of the earth or not.

In addition to the more exact taking into account of the shape of the earth, as I also already mentioned in the introduction, the independent treatments of the three components is a basic prerequisite for a continued progress of the theory of the earth's magnetism. Only by means of such a treatment is it possible to circumvent the introduction of hypotheses whose application, despite the great probability for their approximated validity, is unjustified as long as one can, supported solely by the observed facts, reach the goal /14
and can thus at the same time check their correctness. Of such hypotheses two are being considered: the assumption that an earth-magnetic potential exists and the second that the cause for the same has its seat exclusively in the interior of the earth.

Does the earth's magnetism actually have a potential? This question cannot necessarily be answered with yes. So far as the magnetic forces are generated by electric currents, they have indeed no potential within the volume through which these currents flow. Such a one exists only on the outside of the flow region and only to the extent that magnetic masses are acting; at least this is true according to our previous physical experiences. However, the possibility that in these cases minor deviations from the potential law, unnoticed until now only because of its smallness, could exist, can thus not be excluded. The difference in the movement of the individual parts caused by the rotation of the earth, a not quite complete effect or one first acting in measurable time of the magnetic force effective through the body of the earth - all this could possibly exert influence of the type indicated. In any case it seems to be wise not to assume the existence of the potential a priori. Gauss, who did not fail to notice this, shows how one can check the correctness of the assumption of a potential based on observances, and carries out the checking by means of an approximation example. The latter is based on the statement that the integral

$$\int T \cos \theta \, ds$$

extending over the closed curve in which T is the intensity of the magnetic force and θ the angle between its direction and that of the curved element ds, takes on the value of zero then and only then whenever a single-value potential exists at all places on the curve. If one obtains a value different from zero for this integral, then one can draw a conclusion concerning the causes of the deviation as long as it cannot be attributed to calculations of the observation errors. If one assumes in accordance with experiences gained till now that such causes can be found exclusively in electrical currents, then one can determine the total intensity of the current flow which penetrates any area bounded by such a curve. If the latter is measured in electromagnetic units, designated by J, then we get

$$\text{(2)} \quad \int T \cos \theta \, ds = 4 \pi J.$$

The positive direction of J is oriented toward that side of the area from which, viewed from it, the curve will penetrate it during the integration in a counterclockwise direction. Except for the total intensity one cannot learn much of importance concerning the makeup of the flow from the distribution of the magnetic forces on the curve. A complete determination of the latter within a volume region is possible only when in all its points the magnetic force is known with respect to direction and magnitude.

With respect to the second, previously mentioned hypothesis the following should be stated. That part of the earth's magnetic force which possesses a potential, thus possibly the entire force, can be calculated from the potential in a known manner, and conversely this can be determined from the force components. As long as the latter, such as is presently the case, can be observed accurately only at the surface of the earth, potential values should be calculated only for points on this surface. This can now be done rigorously by using one of the horizontal components without any other assumption. (It is only necessary, if Y is used, to also know X for all points of a line connecting both poles.) Using the vertical components alone the same is only possible, however, if one either knows that the seat of the earth's magnetic force should be looked for only within or outside of the surface of the earth. On the other hand, by suitable utilization of all three components one is then in a position to differentiate not only between these two possibilities, but also to present these parts separately if the origin of a part of the earth's magnetic force is in an internal volume and the other in an external volume. For the investigation of the periodic fluctuations of the earth's magnetism this is an assumption-free method, the only one allowable. However, with respect to the only slowly changing main part of the earth's magnetic force the assumption of origins located in the external volume is not too probable. For that reason Gauss was able to ignore this possibility in his investigation and to satisfy himself to derive the justification of the opposite assumption belatedly from the satisfactory results which were obtained with their aid. The possibility that at least a small part of the earth's magnetic force can be attributed to causes external to the earth, however, still remains and for that reason the rigorous calculation must be carried out in accordance with the requirements postulated by Gauss himself as long as the empirical base is sufficient for that.

In the subject remarks we have already introduced the limitations of the magnetic phenomena at the surface of the earth resulting not so much from theoretical difficulties, but rather from the makeup of the observation material. Since the latter represent a sharply bounded and at the same time the most important expression of the earth's magnetic force, and since in addition they can serve as a most convenient starting point for every subsequent investigation of the phenomena occurring above or below that surface, the temporary retainment of that limitation is completely justified even from a methodical standpoint. To this we must still add that the key point of all later investigations which present no difficulties as long as sufficient experimental data have been gained, lies in a different direction. The changes which the earth's magnetic force exhibits for a distance from the surface of the earth are of interest less on their own account, but rather because of the conclusions which can be drawn regarding the origin of this force. If in particular the seat of the same were to be a thin layer in direct proximity of the surface of the earth, then one would obtain an exact knowledge of the distribution of the acting causes in this layer and thus to a sure determination of its physical foundation. If this contains, on the one hand, the request to set up extensive investigation in this direction, an independent treatment of the force distribution at the earth's surface seems to be that much more justified, on the other hand. (The only, but really quite important difficulty in such investigations is found in the obtaining of reliable observation material. Naturally one cannot determine at present whether it will ever be possible to obtain sufficiently sharp values of the magnetic elements at great altitudes above the surface of the earth or at great depths in the ocean. Measurements made in mines, which by themselves are not too difficult to carry out, are subjected more than all others to the difficultly to be determined effects of local disturbances. However, if one were able to determine at every single point of the earth's crust the magnitude and direction of the force, then the concept of the disturbance would either drop out completely or one would at least be able to define it sharply. This requirement, however, cannot be fulfilled and for that reason the latter will become possible only by an exact determination of the causes of the disturbances. This circumstance is connected with the fact that a small inaccuracy can at first not be avoided even for considerations limited to the surface of the earth. All measurements are carried out at points of the physical surface of the earth; the calculation, on the other hand, assumes knowledge of the magnetic elements at the points of the ellipsoid assumed for the latter. Therefore, one should really carry out a reduction of the measured values to the ellipsoid depending essentially on the height above sea level of the observation point. The latter can be easily determined theoretically, but only when one makes a certain assumption concerning the seat of the magnetic force. The corrections depend for the time being on not well-known circumstances; however, they also turn out to be in every case insignificant which is confirmed by individual observations - for that reason it would be best to neglect them for the time being.)

After these preliminary remarks I now turn to the task to specialise the general equation presented in (2) for the present case. As integration path I select the circumference of an infinitely small quadrangle whose corners are determined by the geographical coordinates

$$v, \lambda \qquad v, \lambda+d\lambda \qquad v+dv, \lambda+d\lambda \qquad v+dv, \lambda$$

The sequence listed till now should signify the direction of the increasing path length s. In accordance with the explanation given for equation (2) the electrical current which flows through the described quadrangle is considered to be positive if it is directed into the interior of the earth. The four sides of the figure then have the length

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$$\sqrt{b^2+c^2}\sin v\, d\lambda \qquad \sqrt{b^2+c^2\cos v^2}\, dv \qquad \sqrt{b^2+c^2}(\sin v+\cos v\, dv)\, d\lambda \qquad \sqrt{b^2+c^2\cos v^2}\, dv$$

If one now observes that T cos θ is reduced for each one of these pages to one of the two horizontal components or to their negative value, one can calculate the integral appearing in (2). After simple transformation one finds this to be

$$-\sqrt{b^2+c^2\cos v^2}\,\frac{dX}{d\lambda}\,dv\,d\lambda-\sqrt{b^2+c^2}\,\frac{d(Y\sin v)}{dv}\,dv\,d\lambda$$

By using the designation explained in (1) one thus obtains

$$(3)\ldots\ldots\ldots\left[\frac{dX}{d\lambda}+\frac{d(Y\sin v)}{dv}\right]b\,dv\,d\lambda=-4\pi J$$

The area content of the infinitely small quadrangle is

$$dw=b^2\sqrt{1+e^2}\,\sqrt{1+e^2\cos v^2}\,\sin v\,dv\,d\lambda$$

If we thus designate the specific current strength by i, which is expressed by the expression J = i dw, then we obtain

$$(4)\ldots\ldots\ldots i=-\frac{1}{4\pi b\sin v\sqrt{1+e^2}\,\sqrt{1+e^2\cos v^2}}\left[\frac{dX}{d\lambda}+\frac{d(Y\sin v)}{dv}\right]$$

If we specifically find at any one location

$$(5)\ldots\ldots\ldots\frac{dX}{d\lambda}+\frac{d(Y\sin v)}{dv}=0$$

then we have proven in this way that no electrical current normal to the surface of the earth exists. To maintain from the beginning that such currents are impossible would be unjustified even though all observations, except for the polar light phenomena at low altitudes above the earth's surface and no longer to be doubted, oppose the assumption of electrical conductivity of air of conventional density. However, one can only conclude with certainty from these observations that the conductivity lies below certain very low limits. Equation (4) now shows that only extraordinary weak currents come into play. Therefore, even for a very great conductive resistance of the air, a noticeable deviation from the state specified in equation (5) could occur.

For an exact investigation in this direction knowledge of the force distribution in a larger connected area is necessary which can only be obtained by magnetic land measurement. This is due to the fact that a sharp determination of the differential quotients under consideration here of the horizontal partial forces is impossible from isolated values of the latter. Here a favorable circumstance is the fact that for every larger area the determination of the possible currents flowing through the same is independent of the measurements at other locations. Therefore, an incomplete knowledge of the force distribution even on the largest part of the earth's surface does not inpede the sharp carrying out of the calculation for such areas whose magnetic state is known accurately. The results obtained here can then possibly be generalized to such an extent that they can also be carried over to other areas. Such a case could possibly exist if equation (5) would be satisfied at all locations of the earth's surface investigated more accurately and having potentially different positions and makeup.

A complete knowledge of the hypothetical current at every point /1
on the earth's surface could be obtained if the values of the horizontal components were known everywhere. In contrast, knowledge of the current does in no way provide information regarding to the forces exerted by them. The reason for this is the fact that we have no knowledge of the forces which arise from current elements - and only the latter ones are known to us here; we are now in a position to calculate the effect of closed currents. However, one can imagine that the quantities of electricity penetrating the earth's surface are continued in an infinitely variable manner in the internal and external volume in such a way that closed currents develop, and a different force system on the earth's surface corresponds to each one of these different possibilities. (The fact that closed currents are at all possible presupposes that the sum of the quantities of electricity flowing in each interval of time from the inside to the outside is equal to the sum of the currents flowing in the opposite direction. This is indeed the case now if one determines them not through general consideration, but also by integration of equation (3).) One could therefore also consider the actually observed system of the horizontal forces totally as an effect of currents of which we merely know the elements penetrating the earth's surface. Under this concept we would thus assume for a small part of the earth's magnetic

force, at least at the surface of the earth, a potential. Now the theory is simplified extraordinarily by the introduction of the potential, and therefore we will utilize the existing uncertainty exactly in reverse in order to bring back to a potential a part of the magnetic horizontal force as large as possible. In other words, we will thus consider the currents possibly flowing through the earth's surface as being closed to such an extent that their magnetic effects in this surface become as small as possible. Drawing conclusions from the previous observations and potential calculations these effects, insofar as they at all turn out to be disappearingly small, will be able to be reduced to a system of very small magnetic forces. By subtraction of the same from the much greater, actually observed forces, we would then obtain for the latter the main part that could be represented by a potential. The extent to which this originates from magnetic masses or closed electrical currents within and to what extent this originates from such outside of the earth, can finally be found from the investigation of the vertical components as has already been mentioned several times.

The points of view under which the processing of the observation material must occur are presented in the previous considerations. However, they can now still be carried out in different ways. Two methods are primarily conceivable which individually still require several modifications depending on whether the observation data refer to individual points or to several parallel circles.

The first method assumes the existence of a potential. The same is considered to be developed in a progressive series using spherical functions. Their coefficients form the desired unknowns. From the potential expression we obtained series for the individual components of the force. By introducing the observation results into these series one obtains numerous linear equations for the unknowns whose values can finally be found by a compensation calculation using the methods of least squares. This is the way the problem has been treated until now.

The characteristic of the second method rests in the fact that for every individual component an analytical representation of their values over the entire surface of the earth is derived after which the further theoretical development connects to the three thus obtained plots.

I now limit myself in the following discussions to the second method. The other method since all previous work is based on it, requires no explanation, and the changes which become necessary by taking into account the flattening of the earth are insignificant, and have been discussed sufficiently by the remarks at the beginning of this section. However, above all several reasons suggest that in the future the second method should be used. The latter is not only shorter and more convenient than the first because in general it avoids the use of the method of least squares; but it is also to be preferred from a theoretical standpoint. The assumption of the existence of a potential is avoided and the contradictions possibly existing between the individual components with the use of this assumption are not covered up by a compensation calculation, but are brought back to their physical causes. (Belatedly, however, this also could occur in the first method because of the differences between observation and calculation.)

In order not to interrupt the continuity I shall at first not /1
take up the question in which way the development of the different elements by means of spherical functions could proceed most effectively, but I temporarily assume that the necessary series are given.

As can be seen from the following mathematical discussions, the quantities $\bar{X} \sin v$, $\bar{Y} \sin v$ and $\bar{Z}$ must be developed by means of spherical functions; on the other hand it is not necessary to represent $\bar{X}$ and $\bar{Y}$ themselves in this form. Since in the meantime these latter ones are of interest by themselves since they can be used in other ways and since they can finally be determined without excessively large additional expenditure of computer work, I shall

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also take them into account.

By the use of the abbreviations defined by

$$\sum f(n, m) = \sum_{n=0}^{n=\infty} \sum_{m=0}^{m=n} f(n, m) \qquad \text{and} \qquad P_n^m = P_n^m(\cos v)$$

we obtain

$$(6)\ldots\ldots \begin{cases} X = \sum P_n^m (b_n^m \cos m\lambda + c_n^m \sin m\lambda) & X \sin v = \sum P_n^m (B_n^m \cos m\lambda + C_n^m \sin m\lambda) \\ Y = \sum P_n^m (d_n^m \cos m\lambda + e_n^m \sin m\lambda) & Y \sin v = \sum P_n^m (D_n^m \cos m\lambda + E_n^m \sin m\lambda) \\ Z = \sum P_n^m (j_n^m \cos m\lambda + k_n^m \sin m\lambda) & \end{cases}$$

For the case of the existence of a potential the two integrals

$$U = \int_0^v X dv \qquad W = -\int_0^\lambda Y \sin v \, d\lambda$$

represent if we add to the second the part of the first independent of λ, $\psi(v)$, always one and the same function which is exactly the potential. If we should now choose not to introduce the assumption that such a one exists, then a separate calculation of U and W is necessary. Let me now precede the representation of the course which this calculation must take with a remark resulting from the nature upon which the physical problem is based. The horizontal component of the earth's magnetic force has everywhere a finite singular value and, wherever this is not equal to zero, a definite direction; furthermore, it can be constantly changed everywhere. (Strong local disturbances which are limited to small areas, naturally must be considered to be evened out for a consideration encompassing the entire surface of the earth; if one should take them into account, they would act approximately as discontinuities.) From this it follows that $\bar{X}$ and $\bar{Y}$ are everywhere finite and constant except at the poles. Near the North Pole these components are respectively of the form ($a \cos \lambda + b \sin \lambda$), ($-b \cos \lambda + a \sin \lambda$); at the South Pole they approximate the values ($a' \cos \lambda + b' \sin \lambda$), ($b' \cos \lambda - a' \sin \lambda$), whereby a, b, a', b' are certain constants.

The series given in (6) for the representation of $\bar{X}$ and $\bar{Y}$ can accurately express these quantities discontinuous at the pole only if we consider them to be continued to infinity. In order to obtain a representation with a finite number of terms whose approximation can be made arbitrarily far, one must separate from $\bar{X}$ the expression containing a discontinuity such as

$$\left(\frac{a+a'}{2}+\frac{a-a'}{2}\cos v\right)\cos\lambda+\left(\frac{b+b'}{2}+\frac{b-b'}{2}\cos v\right)\sin\lambda = (\alpha+\alpha'\cos v)\cos\lambda+(\beta+\beta'\cos v)\sin\lambda$$

as well as separate a corresponding expression, $(\alpha' + \alpha \cos v) \sin \lambda - (\beta' + \beta \cos v) \cos \lambda$ from $\bar{Y}$. This expression contains no spherical functions, but it can be developed by means of them into an infinite series whose coeffients do not approximate zero, but that of $\bar{X}$ respectively $\bar{Y}$.

The series given for $\bar{X} \sin v$ and $\bar{Y} \sin v$ must, since $\bar{X}$ and $\bar{Y}$ are /1
finite everywhere, satisfy the requirements that they represent the value zero at both poles. Small deviations which indeed can be caused by errors in the observations upon which they are based and by the incomplete reproduction of the same for the case of an infinite series, must be removed by compensation. Under the assumption that this has been done we obtained for each of the coefficient series B and D two for both equally formed equations of which I shall therefore present only the one valid for B:

$$(7) \ldots\ldots\ldots \begin{cases} B_0^0+\frac{2}{3}B_0^2+\frac{8}{35}B_0^4+\frac{16}{231}B_0^6+\frac{128}{6435}B_0^8+\ldots\ldots = 0 \\ B_0^1+\frac{2}{5}B_0^3+\frac{8}{63}B_0^5+\frac{16}{429}B_0^7+\frac{128}{12155}B_0^9+\ldots = 0 \end{cases}$$

These equations state that $\bar{X} \sin v$ have the value zero at every pole. The coefficients appearing in them are the values of the function P_0^m for v = 0 and v = π in the first equation of P_0^{m+1} for v = 0 and $-P_0^{m+1}$ for v = π in the second.

I now return to the problem to calculate the integrals U and W. With regard to the latter we obtained immediately

$$(8)\ldots W = -(D_0^0 + D_0^1 P_0^1 + D_0^2 P_0^2 + \ldots)\lambda + \sum P_m^n \left(\frac{1}{m} E_m^n \cos m\lambda - \frac{1}{m} D_m^n \sin m\lambda\right) = -\sin v . \varphi(v) . \lambda + W_0$$

The portion containing the factor λ which according to the remarks made for (7) disappears at the poles and therefore can be written as the product from sin v and an everywhere finite quantity $\varphi(v)\lambda$, is everywhere finite and, if λ is limited to the values from 0 to 2 π, a single valued function which becomes discontinuous at the initial meridian (but not along the same). It can therefore be developed by means of spherical functions and can be combined with W_0 whereby at the discontinuity the average of both limiting values is represented. However, such a development would be unusable because the coefficients do not converge towards zero so that accordingly the numerical calculation cannot be carried out. Therefore the breakup of W given in (8) is necessary. The here appearing function, denoted as $\varphi(v)$ can be gained, as one can readily see, from the development of $\bar{Y}$. It is

$$\varphi(v) = d_0^0 + d_0^1 P_0^1 + d_0^2 P_0^2 + \ldots\ldots$$

The determination of U is not quite so simple. It is first clear that no simple relation can exist between the coefficients of the two series for U and $\bar{X}$. On the other hand, such a relation appears to be possible if one introduces $\overline{X}$ sin v in the place of $\bar{X}$. This can be attributed to the fact that in each development proceding in accordance with spherical functions $\cos m\lambda$ or $\sin m\lambda$ always appears exactly multiplied by m-th power of $\sin v$, which factor is not retained during the integration or differentiation, but goes over into $\sin v^{m-1}$. (For the last statement the case that in the integral the arc v itself appears has not been taken into account.)

In order to now derive U from $Y \sin v$, I make use of an identity which I will provide in addition to two other ones used later without proof since this would take too long. Insofar as the latter can be used for the above purpose, that is for the first values of n, it can be easily verified by substitution of the known trigonometric expressions in place of the spherical functions. We have

$$(9)\ldots\left\{\begin{array}{l} \text{for } (n)_m = \dfrac{n^2-m^2}{4n^2-1} \\ \dfrac{d(\sin v P_m^n)}{dv} = -n(n)_m P_m^{n-1} + (n+1) P_m^{n+1} \qquad \sin v \dfrac{dP_m^n}{dv} = -(n+1)(n)_m P_m^{n-1} + n P_m^{n+1} \\ \sin v^2 P_m^n = -(n)_m (n-1)_m P_m^{n-2} + \left[1-(n)_m-(n+1)_m\right] P_m^n - P_m^{n+2} \end{array}\right.$$

With the aid of the second of these formulas we now find when a /20
function U_o is given by equation

$$(10)\ldots\ldots\ldots\ldots\ldots\ldots \quad U_0 = \sum P_m^n (G_m^n \cos m\lambda + H_m^n \sin m\lambda) = \sum P_m^n F_m^n$$

that

$$(11)\ldots\ldots\ldots\ldots \quad \sin v \frac{dU_0}{dv} = \sum P_m^n \Big((n-1) F_m^{n-1} - (n+2)(n+1)_m F_m^{n+1}\Big)$$

If the quantities F could be determined in such a way that the subject series would be identical with one representing $Y \sin v$, then we would have $U = U_o$. This is actually possible; but in general, as can be easily shown, the quantity F would always become larger with increasing n so that a completely useless series would be obtained. However, that part which proceeds toward infinity can be represented in closed form, similar to the development of W.

Since the following considerations allow the sums of the form $(B_m^n \cos m\lambda + C_m^n \sin m\lambda)$ to remain unchanged, I shall for the sake of brevity introduce in its place a more simple designation for $A_m^n \sin$. For the same reason I wrote shortly before F_m^n in place of $(G_m^n \cos m\lambda + H_m^n \sin m\lambda)$. If we now place everywhere - and in conducting

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the numerical calculation one will do this - in place of A_m^n and F_m^n either B_m^n and G_m^n or C_m^n and H_m^n, then one gain obtains correct equations. C_m^n and H_m^n naturally do not come into play at all; if one wishes to introduce them for the sake of formal uniformity, then one should simply give all of them the value zero.

For U = U_0, as remarked, the right side of equation (9) would coincide with $\Sigma \sin v$. By comparison of the coefficient of corresponding spherical functions one would then obtain in a number of equations, all of them of the form

$$(12)\ldots\ldots\ldots\ldots\ldots (n-1)F_m^{n-1}-(n+2)(n+1)_m F_m^{n+1} = A_m^n$$

If reversely these equations are satisfied, then we get U = U_0. Now it is first easy to prove that $F_m^{n-1}<F_m^{n+1}<F_m^{n+3}<\ldots$ applies, if all A_m^n vanish starting from a given value of n. This follows from the fact that generally, as long as only (n+m) at least has the value of 3, $(n-1)>(n+2)(n+1)_m$. A useful development of U is obtained only when $F_m^{n-1}=0$, in which case also $F_m^{n+1}=F_m^{n+3}=\ldots=0$ applies. In this way the following is obtained. If $\Sigma \sin v$ is given as a finite series, which breaks off with the spherical function of ν-th order, then U can be represented by a series which also ends already the function of $(\nu-1)$-th order, or by an infinite series whose coefficients do not approach zero, but on the contrary become greater and greater. In the second case we can now give, similar as in the development of W, a closed expression for that part of U extending to infinity. If we designate this expression which is a function of v and λ by $f(v,\lambda)$, then $U-f(v,\lambda)$, which difference we shall now call U_0, extends just as U in the first case to the spherical function of $(\nu-1)$-th order. With the proof for this statement U_0 and $f(v,\lambda)$ will be obtained simultaneously.

Among the equations (12) those that belong to a certain value of m can be separated into two groups:

$$
(13)\ldots\ldots
\begin{cases}
-(m+2)(m+1)_m F_m^{m+1} = A_m^m \\
(m+1) F_m^{m+1} - (m+4)(m+3)_m F_m^{m+3} = A_m^{m+2} \\
\quad\vdots \\
(m+2\mu-1) F_m^{m+2\mu-1} - (m+2\mu+2)(m+2\mu+1)_m F_m^{m+2\mu+1} = A_m^{m+2\mu} \\
\\
m F_m^m - (m+3)(m+2)_m F_m^{m+2} = A_m^{m+1} \\
(m+2) F_m^{m+2} - (m+5)(m+4)_m F_m^{m+4} = A_m^{m+3} \\
\quad\vdots \\
(m+2\mu-2) F_m^{m+2\mu-2} - (m+2\mu+1)(m+2\mu)_m F_m^{m+2\mu} = A_m^{m+2\mu-1} \\
(m+2\mu) F_m^{m+2\mu} - (m+2\mu+3)(m+2\mu+2)_m F_m^{m+2\mu+2} = A_m^{m+2\mu+1}
\end{cases}
$$

The fact that a term is missing in the first equation can be explained because F_m^{m-1} does not appear at all and can be set equal to zero. This is connected with the circumstance that the identical equations (9) remain valid even when they contain spherical functions with smaller upper than lower index on the righthand side, for which reason one assumes only that one be equal to zero. /21

The last equation of the second series disappears if the order of the spherical function with which $Y_{\sin\nu}$ breaks off, is equal to $\nu \equiv m \pmod{2}$; however, it must be taken into account for $\nu \equiv (m+1) \pmod{2}$. In the first case the last non-disappearing coefficients are $A_m^{m+2\mu}$ and $F_m^{m+2\mu-1}$, in the second case they are $A_m^{m+2\mu+1}$ and $F_m^{m+2\mu}$. The second equation of the second group is thus either

$$(m+2\mu-2) F_m^{m+2\mu-2} = A_m^{m+2\mu-1} \quad \text{or} \quad (m+2\mu) F_m^{m+2\mu} = A_m^{m+2\mu+1}$$

Thus, one can first calculate $F_m^{m+2\mu-2}$ or $F_m^{m+2\mu}$ and then with the aid of the preceding equation in succession $F_m^{m+2\mu-4} \ldots F_m^{m+2}, F_m^m$ in a very simple manner. An exception is formed by the further treated case m = 0 because in it the coefficient F_m^m multiplied by m drops out. The last equation of the first group becomes

$$(m+2\mu-1) F_m^{m+2\mu-1} = A_m^{m+2\mu}$$

and thus one can also find $F_m^{m+2\mu-1}$ and after that going backwards $F_m^{m+2\mu-3} \dots F_m^{m+3}, F_m^{m+1}$. Here, however, the first equation is not taken into account; therefore the latter will generally not be able to be satisfied by the value of F_m^{m+1} derived from the second. This value suffices for it, as one finds by elimination of F, only when the condition

$$A_m^m + \frac{m+2}{m+1}(m+1)_m A_m^{m+2} + \frac{(m+2)(m+4)}{(m+1)(m+3)}(m+1)_m (m+3)_m A_m^{m+4} + \dots\dots\dots = 0$$

is satisfied. The same one can be written in a somewhat different form. From the second of the often employed identities (9) it follows for v = 0

$$-(n+1)(n)_m P_m^{n-1}(\cos 0) + n P_m^{n+1}(\cos 0) = 0 \qquad \text{or} \qquad P_m^{n+1}(\cos 0) = \frac{n+1}{n}(n)_m P_m^{n-1}(\cos 0)$$

With the aid of this formula in which one places in succession m+1, m+3 etc. for n, one can transform the above conditional equation after multiplication of the same by $P_m^m(\cos 0)$ without any problem to the form

$$(14) \dots\dots\dots P_m^m(\cos 0) \,.\, A_m^m + P_m^{m+2}(\cos 0) \,.\, A_m^{m+2} + \dots\dots\dots + P_m^{m+2\mu}(\cos 0) \,.\, A_m^{m+2\mu} = 0$$

However, in general one will not use this equation at all in numerical calculations for checking to see if the values of the quantity A are compatible with one another. However, it makes possible the derivation of an important consequence to which still a second one is added if one considers the following expression, which can be proven similarly to (14) and which makes good sense directly as the result of the preceding statements

$$(15) \dots\dots P_m^{m+1}(\cos 0) \,.\, A_m^{m+1} + P_m^{m+3}(\cos 0) \,.\, A_m^{m+3} + \dots\dots\dots\dots = m\, P_m^{m+1}(\cos 0) \,.\, F_m^m$$

For, it results from the previously derived equation (7) whose numerical coefficients are the values of $P_\nu^n(\cos 0)$, that for m = 0 the two conditions (14) and (15) - equation (15) becomes a requirement in this case since the righthand side takes on the value of zero - are always satisfied.

If m is different from zero then one finds, as already remarked, even without the use of (14), by successive calculation of F whether the equations of the first group in (13) contradict one another or not. If a contradiction occurs, that is if $-(m+2)(m+1)_m F_m^{m+1}$ is found to deviate from A_m^m, then one must separate from $\bar{X}$ sin v a part formed from the difference of these quantities as coefficients and the corresponding spherical functions. The remaining part then obviously agrees with the conditional equation (14).

I picture that this is carried out for all values of m and place /22

$$(16)\ldots X \sin v = X_\nu \sin v + \sum_{m=1}^{m=\nu} P_m^m \left(A_m^m + (m+2)(m+1)_m F_m^{m+1}\right) = X_\nu \sin v + \sum_{m=1}^{m=\nu} \sin v^m \left(A_m^m + (m+2)(m+1)_m F_m^{m+1}\right)$$

whereby ν again denotes the order of the spherical function considered last. The integral

$$U_\nu = \int_0^v X_\nu \, dv$$

then takes on the simple form which has often been stated. Now for the sake of brevity I again put

$$\int_0^v \frac{P_m^m}{\sin v} v \, dv = \int_0^v \sin v^{m-1} \, dv = \Pi_m^m$$

thus

$$(17)\ldots\ldots \begin{cases} \Pi_1^1 = v & \Pi_2^2 = 1 - \cos v \\ \Pi_3^3 = \frac{1}{2} v - \frac{1}{4} \sin 2v & \Pi_4^4 = \frac{2}{3} - \frac{3}{4} \cos v + \frac{1}{12} \cos 3v \\ \Pi_5^5 = \frac{3}{8} v - \frac{1}{4} \sin 2v + \frac{1}{32} \sin 4v & \Pi_6^6 = \frac{8}{15} - \frac{5}{8} \cos v + \frac{5}{48} \cos 3v - \frac{1}{80} \cos 5v \end{cases}$$

etc.

By the use of this designation it finally follows that

$$(18) \ldots\ldots\ldots\ldots \quad U = \sum_{m=1}^{m=\infty} \Pi_m^n \left(A_m^n + (n+2)(n+1)_m F_m^{n+1}\right) + \sum P_m^n F_m^n = f(v,\lambda) + U_0$$

Here F_0^0 remains indeterminate, but also arbitrary, and thus can be set most simply equal to zero. It must still be remarked that $f(v,\lambda)$ can also be determined in a different way. One can separate from the coefficients $A_m^n \ldots\ldots A_m^{n+2\mu}$ any particular parts so that the remaining quantities satisfy the condition (14), and can then proceed in a similar way as stated previously. The different thus possible functions $f(v,\lambda)$, of which the one chosen here is the simplest and at the same time makes the quadratic average of all values of $X \sin v - X_0 \sin v$ the smallest, naturally differ only by expressions which can be represented by finite series proceeding by means of spherical functions.

I now return to the most important special case by assuming that the previous calculations whose results are contained in equations (8) and (18), give the existence of a potential for the entire horizontal force. This then is the case if at the same time

$$\varphi(v) = 0 \qquad f(v,\lambda) = 0 \qquad U_0 = W_0 + \psi(v)$$

is found. The potential V is the common value of bU_0 and $b(W_0 + \psi(v))$. I shall designate the coefficients in the expression for V, following the example of Gauss and the previous convention resulting therefrom, by g and h; thus I shall write

$$(19) \ldots\ldots\ldots\ldots \quad V = b \sum P_m^n (g_m^n \cos m\lambda + h_m^n \sin m\lambda)$$

By keeping together the series found for the vertical components

$$Z = \sum P_m^n (j_m^n \cos m\lambda + k_m^n \sin m\lambda)$$

I can finally decide whether a part of the earth's magnetic force is based on causes to be found outside of the earth's surface, and can if this is the case, separate the two parts.

By comparison of the two existing series with the developments given for V and Z through equations (18) and (21) of the first section, I find the following by dropping off the equal indices n and m associated with each letter /23

$$g = qc + p\gamma \qquad h = qs + p\sigma$$
$$j = -(n+1)\varkappa c + n\pi\gamma \qquad k = -(n+1)\varkappa s + n\pi\sigma$$

From these equations it follows

$$\text{for } n\pi q + (n+1)p\varkappa = \frac{1}{\Delta}$$

$$(20)\ldots\ldots\ldots\ldots\left\{\begin{aligned} c &= \Delta\big((n+1)\varkappa g + qj\big) \qquad & s &= \Delta\big((n+1)\varkappa h + qk\big) \\ \gamma &= \Delta(n\pi g - pj) \qquad & \sigma &= \Delta(n\pi h - pk) \end{aligned}\right.$$

As I already stressed earlier, it is somewhat more convenient for the numerical calculation to combine the factors p and q with the coefficients γ, σ and c, s; I will therefore also set up these formulas under the assumption that this has been done. So as not to increase still further the already plentiful amount of designations, I write instead of $p\gamma, p\sigma, qc, qs$ simply γ, σ, c, s. We then obtain

$$(21)\ldots\ldots\ldots\ldots\left\{\begin{aligned} &\text{for } n\frac{\pi}{p} + (n+1)\frac{\varkappa}{q} = \frac{1}{\delta} \\ c &= \delta\left((n+1)\frac{\varkappa}{q}g + j\right) \qquad & s &= \delta\left((n+1)\frac{\pi}{q}h + k\right) \\ \gamma &= \delta\left(n\frac{\pi}{p}g - j\right) \qquad & \sigma &= \delta\left(n\frac{\pi}{p}h - k\right) \end{aligned}\right.$$

In this way the solution of the problem for the case of the existence of a potential of the whole force is taken care of.

In the opposite case only a part of the force can be attributed to a potential whereby, however, as shown earlier, a certain amount of arbitrariness is possible. At first it is clear that those components of U and W which are not obtained in the form of spherical functions, - that is $f(v,\lambda)$ and $\sin v\,\varphi(v)\lambda$ - must not be considered because they produce discontinuities of the values $\bar{X}$ and $\bar{Y}$ at the pole respectively at the initial meridian. If now in addition $U_0 = W_0 + \psi(v)$, whereby again $\psi(v)$ denotes the part of U independent of λ, then one will set naturally $V = bU_0$. However, if this is not the case, one must assume for V an average value for whose specification a certain amount of tolerance remains depending on expediency reasons. In accordance with the principles of the compensation calculation it would be necessary to demand that the sum of the squares, extending over the entire surface of the earth, of the deviations between the observed (or, which is the same, resulting from U and W) and the values of $\bar{X}$ and $\bar{Y}$ calculated from V be a minimum. The problem to determine V in accordance with this requirement can, although in a rather awkward way, be solved by using the series developments for $\bar{X}$ and $\bar{Y}$; on the basis gained till now which leans on $\bar{X}$ sin v and $\bar{Y}$ sin v, the solution seems to be subjected to larger difficulties. On the other hand, another requirement can be fulfilled easily, namely the one that V itself shall deviate as little as possible from U_0 and $W_0+\psi(v)$, whereby the total deviation naturally is again determined by the sum of the squares of the individual differences. One recognizes immediately that this requirement leads to the arithmetic mean of U_0 and $W_0+\psi(v)$ and thus produces a very simple solution. One can well assume that at least for the average values of the earth's magnetic force both compensation methods lead to practically coinciding results. If thus also the first, on the one hand, brings about the best compensation of the observation errors and, on the other hand, the bringing back of a part as large as possible of the earth magnetic phenomena to a potential, the second one will still not remain far behind, and since according to former statements (German p. 17) it is as justified as the former, it should obtain preference because of its much greater simplicity.

The continued treatment of V, namely the breaking-up to be performed /2
by taking into account $\overline{Z}$ into a part arising within and a part rising external to the earth, is done in the same manner as in the previously treated special case by means of equations (20) or (21). However, to describe the magnetic state of the surface of the earth the specification of the two parts of V is no longer sufficient; rather in addition differences (U - V) and (W - V) or, as is more effective, the magnetic forces resulting from them, directed either toward the north or the east, must be given. These represent that part of the total horizontal force which has no potential.

The analytical representation of the magnetic phenomena has been taken care of by the preceding; thus there still remains the task to provide a development of the electrical currents penetrating the surface of the earth in a normal direction. (The easily to be obtained presentation of the currents parallel to the surface of the earth which are capable of producing the potential on it, lies outside of the scope of this report which merely proposes to present the mathematical description of the observed phenomena. I intend to publish at a later date a calculation and a cartographic presentation of these currents.)

With the aid of equation (4) one can easily calculate $i\sqrt{1+e^2\cos v^2}$ as well as $i\sin v\sqrt{1+e^2\cos v^2}$. Let $i\sqrt{1+e^2}\sqrt{1+e^2\cos v^2} = \imath$, in accordance with the designations $\overline{X}$, $\overline{Y}$, $\overline{Z}$. I shall indicate as briefly as possible the developments for both cases. We have

$$\imath \sin v = -\frac{1}{4\pi b}\left[\frac{d\overline{X}}{d\lambda} + \frac{d(\overline{Y}\sin v)}{dv}\right]$$

By the introduction of the values of $\overline{X}$ and $\overline{Y}$ given in (6) I obtain with the use of the first formula in (9) after simple transformations and with separation of the discontinuous part at the poles

$$(22)\ldots\quad \imath \sin v = -\frac{\sin v^2}{2\pi b}(\beta\cos\lambda - \alpha\sin\lambda) - \frac{1}{4\pi b}\sum P_m^n\Big[\left(n\,d_m^{n-1} - (n+1)(n+1)_m\,d_m^{n+1} + m\,c_m^n\right)\cos m\lambda$$
$$+\left(n\,e_m^{n-1} - (n+1)(n+1)_m\,e_m^{n+1} - m\,b_m^n\right)\sin m\lambda\Big]$$

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In order to describe $\mathfrak{t}$ itself I first calculate, starting from $\overline{X}$ sin v and $\overline{Y}$ sin v, with the aid of the second identity (9) the product $\mathfrak{t} \sin v^2$. We obtain

$$(28)\ldots\ldots \quad \mathfrak{t} \sin v^2 = -\frac{1}{4\pi\delta}\sum P_m^n\Big[\big((n-1)\,D_m^{n-1}-(n+2)\,(n+1)_m\,D_m^{n+1}+m\,C_m^n\big)\cos m\lambda$$
$$+\big((n-1)\,E_m^{n-1}-(n+2)\,(n+1)_m\,E_m^{n+1}-m\,B_m^n\big)\sin m\lambda\Big]$$

From this $\mathfrak{t}$ can now be derived in the following way. It is represented by a series proceeding according to spherical functions with at first indeterminate coefficients and subsequently from it with the aid of the third identity (9) a new one for $\mathfrak{t} \sin v^2$ is derived. By comparison of the latter with the former one obtains since both must agree in the corresponding coefficients, a number of equations from which one can calculate the quantities which were left unknown. If $\mathfrak{t}$ equal to $\overline{X}$ sin v and $\overline{Y}$ sin v, is to be represented by a finite series, similar difficulties arise as in the calculation of U which can be solved in a similar manner.

The fact that one reaches exactly the same value for $\mathfrak{t}$, respectively $\mathfrak{t} \sin v$ and $\mathfrak{t} \sin v^2$, if one introduces into the calculation not $\overline{X}$ and $\overline{Y}$, but only those parts therefrom which possess no potential, is self evident. If $U_0 = W_0 + \psi(v)$, equation (23) can be simplified considerably because of this remark which need not be gone into in any greater detail.

In the preceding pages it has been shown in what way, starting from the series specified in (6), one can obtain knowledge of the potential, the part of the horizontal force not included in this, and of the electrical currents considered to be the cause for the same. I only still have to explain in what way those series can be derived from the observed numerical values of the components $\overline{X}$, $\overline{Y}$, $\overline{Z}$.

The ideal of the development of these quantities by means of spherical functions would be attained if one were to possess the values for the same for every point on the surface of the earth and were to derive from them the known definite integrals extending over the entire surface which, except for constant numerical factors, are equal to the desired series coefficients. Of course this method, since it is based on the most complete material, is the most accurate one; it is at the same time the simplest one and possesses the additional advantage that a subsequent continuation of the series interrupted at any one place does not affect the already calculated coefficients. Naturally it can never be employed in a completely rigorous manner since it prescribes an infinite process; however, one will be able to make use of it with ever increasing approximation as the observations become more complete and more accurate. /2

The actual conduct of the method described naturally presupposes a cartographic representation, at a scale as large as possible, of the element to be developed in series form. On this basis, the individual integrals can be evaluated in different ways through graphic procedural methods. One can use for this purpose, for example, planimetric measurements if one establishes a special chart for the integral whose area element is proportional to the corresponding spherical function (or, for technical reasons we won't develop in greater detail in this place, to a sum several of them). Graphical representations and operations, however, are not as accurate as calculations unless the latter are carried out with considerable rounding off. Nevertheless, mechanical integration carried out in the above mentioned or in similar manner could produce quite satisfactory values because of the thus resulting compensation of the individual inaccuracies, naturally always with the requirement that the quantity to be represented is known at every point on the surface of the earth. At the present time, however, this prerequisite is satisfied in no way for the earth-magnetic element; just the same, in my opinion, a quite satisfactory result could now already be obtained by the use of this method, i.e., one at least as good as those obtained in another way. In regions from which few or no observations are available, such as in the Antarctic Zone, one should be able to use the results of the previous potential determinations for interpolations without worry. Naturally this auxiliary means can lead to no improvement of the result, neither does it introduce any noticeable errors into it, but rather makes the application of the previous method possible at all. If, on the other hand, an exact magnetic survey of a larger region is available, then the results from this method can be utilized more readily than with any other method. One can either determine, be it through calculation, be it through special measurements, the part of this region from the individual integral values with particular accuracy or can at least enter the curves which intersect it without any compensation whatsoever and generalization into the working charts. However, the thus obtained gain in accuracy will not come into play as long as the accurately surveyed regions are considerably smaller in size than those nearly totally unknown.

In direct contrast with the preceding methods we find that one which presupposes knowledge of the quantity to be represented only in a finite number of individual points. Here naturally only calculation can lead to the goal. By introducing the geographical coordinates of every point and the associated function value into the corresponding series development, one obtains a number of linear equations from which with the aid of the method of least squares the coefficient for the series can be derived. The observation material upon which this is based here is infinitely smaller in scope than the one used for the previous method; however, it can considerably exceed the former with respect to acuity and certainty, and for that reason this method also, as long as the observation points are distributed effectively and are sufficiently numerous, can also lead to an exact determination principally of the first coefficient. A special

advantage for this method develops if one imagines that the repeated potential determinations are carried out with constant utilization of the same observation points, e.g. of the available magnetic values. If in this case the formation and the solution of the system of equations which serves to calculate the coefficients is carried out generally, then one obtains simple end formulas with whose aid each individual potential determination can be carried out in a few hours by introducing the observation values referenced to the particular point in time. The development of this idea and the presentation of the advantages which the realization of the same promises to provide for the earth-magnetic research, constituted, as I should like to point out as an aside, the content of the memorial paper mentioned in the introduction which I prepared for the Inter- /26
national Polar Commission.

Until now we used exclusively a third path which started with the distribution of the magnetic elements on a number of parallel circles. This distribution was represented by trigonometric series as a function of the geographic longitude; by comparison of those series with the ones applying for the same parallel circles, and resulting from the potential expression, equations were obtained which were used to calculate the potential coefficients by means of a least squares compensation calculation. In a very similar manner one could now determine also the coefficients of those series which represent the quantities $\overline{X}$, $\overline{X}$ sin v, etc. themselves. However, these can be solved in a more convenient way which circumvents the application of the method of least squares. The latter corresponds completely to the known method to determine the coefficients of a trigonometric series development from the function values in a number of points distributed over the period. The coefficients are obtained here as sums of those values containing certain factors whereby the factors alone depend on the position of the points within the period. The corresponding method for the development of a changeable quantity given for the spherical surface by means of spherical functions, a method which was pointed out to me three years ago under the circumstances of the subject investigation, has been known for a very long time, as I subsequently found out during a literature search, but seems to have received little attention. It was published by F. Neumann in the year 1838 with particular reference to its applicability for presenting the earth-magnetic phenomena.* The intention stated at the end, to produce such a representation himself at a later time, seems to have not been fulfilled by the author - probably because of the soon thereafter appearing of "Allgemeinen Theorie.." (General Theory).

Since Neumann's method, as noted, does not seem to be generally known, I shall insert here a brief presentation of the same without any proof - naturally with the transformation into the designation method used exclusively in the subject paper.

Let the function to be represented, say $\overline{X}$, be expressed temporarily as a number of parallel circles, assumed to be an odd number at (2p+1), with the polar distances $v_1, v_2 \cdots v_{2p+1}$ by trigonometric series, which proceed to a p-fold multiple of the geographic longitude. On the parallel circle v let

$$(24) \ldots\ldots\ldots\ldots X = \sum_{m=0}^{m=p} (C_{m,\nu} \cos m\lambda + S_{m,\nu} \sin m\lambda) \qquad \nu = 1, 2 \ldots (2p+1)$$

If one now determines (2p+1) quantities $a_1, a_2 \cdots a_{2p+1}$ in such a way that the equations

* Concerning a new property of the Laplace function $Y^{(n)}$ and its application for the analytical representation of those phenomena which are functions of the geographical longitude and latitude. By F. Neumann in Koenigsberg. Schumacher's Astronomische Nachrichten, Vol. 15, page 313. Reprinted in "Mathematischen Annalen", Vol. 14, p. 567 (1879).

$$
(25)\ldots\left\{\begin{array}{llll}
\sum_{\nu=1}^{\nu=2p+1} a_\nu = 1 & \sum_{\nu=1}^{\nu=2p+1} a_\nu \cos v_\nu^2 = \frac{1}{3} & \ldots\ldots\ldots & \sum_{\nu=1}^{\nu=2p+1} a_\nu \cos v_\nu^{2p} = \frac{1}{2p+1} \\
\sum_{\nu=1}^{\nu=2p+1} a_\nu \cos v_\nu = 0 & \sum_{\nu=1}^{\nu=2p+1} a_\nu \cos v_\nu^3 = 0 & \ldots\ldots\ldots & \sum_{\nu=1}^{\nu=2p+1} a_\nu \cos v_\nu^{2p+1} = 0
\end{array}\right.
$$

which, however, are compatible only under a condition connecting the angles $v_1, v_2 \ldots v_{2p+1}$, then we obtain the coefficients b_m^n and c_m^n of the development of $\bar{X}$ through the following formulas:

$$
(26)\ldots b_m^n = \frac{2n+1}{2} a_m^n \sum_{\nu=1}^{\nu=2p+1} a_\nu P_m^n(\cos v_\nu) . C_{m,\nu} \quad c_m^n = \frac{2n+1}{2} a_m^n \sum_{\nu=1}^{\nu=2p+1} a_\nu P_m^n(\cos v) . S_{m,\nu} \quad n = 1, 2 .. p
$$

If the number of parallel circles is even, such as 2 p, the last two equations of system (25) drop out and the angles $v_1, v_2 .. v_\nu$ thus need satisfy no conditions in this case. However, the development of $\bar{X}$ can then be expanded only up to spherical functions of order (p-1).

If the polar distances $v_1, v_2 \ldots v_{2p+1}$ thus are arbitrary (up to the /27
requirement that in the first case they must cause the disappearance of the determinants of the equations (25)), it is still in the interest of a representation as accurate as possible of the function on the entire surface of the earth, to not choose it to be too irregular between 0 and π. It is most advantageous, as Neumann has shown, if one assumes for it the q roots of the equation

$$
(27)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots P_q^q(\cos v) = 0
$$

whereby q may be an even or an uneven number. In this way the development of X can be continued in that in the same way all coefficients can be calculated in accordance with the formulas (26), in which the sum of the two indices is smaller than q. Thus one obtains here a development which breaks off with a certain value not of n, but of (n+m).

The essential advantage of Neumann's method consists of the fact that, similar to the first method based on the evaluation of the integrals, with which it is by the way closely connected, provides the individual coefficients independent of one another. Thus one can, if the number of the parallel circles used is very great, break off the series at an earlier place than required by the number without bringing about any changes in the first coefficients. And one can also, on the other hand, if the initially selected expansion of the series turns out to be insufficient, continue the latter with little effort.

The presentation shown can be recognized as an additional advantage of this method in that it requires a relatively low work expenditure. However, it makes necessary the solution of a system of equations with a large number of unknowns; the legitimate formation of the coefficients of these equations, however, simplifies the solution considerably. If one introduces in place of the powers of cos v the cosine of the multiples of v, one can make use of with a few changes of the general formulas recently presented by Weihrauch* for the calculation of the unknown. Furthermore the fact that the same values of a can be used for all elements is of advantage.

In the application to the investigation of the earth-magnetic phenomena it is useful to take into account the function values at the two poles in the calculation, thus to introduce the poles into the number of those parallel circles in which one considers the function to be given. With respect to $\overline{X}$ sin v and $\overline{Y}$ sin v, which quantities disappear for $v = 0$ and $v = \pi$, this is possible without any problem; on the other hand, it seems to be impossible to carry it out for $\overline{X}$, $\overline{Y}$, $\overline{Z}$ since the observation of these elements

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* New investigations concerning the Bessel formula and its application in meteorology. By K. Weihrauch (paper published by the Naturf.-Ges. at the University of Dorpat, 1888.)

will probably be impossible for a long time yet at the North- and South Pole. However, as will be shown later, this gap can be filled, and this is useful for two reasons. In the first place, one is thus not forced to solve the system of equation (25) twice - once with the values of $v = 0$ and $v = \pi$, and the second time without them. Secondly, the series thus obtained will express with greater approximation the continuity of the direction and magnitude of the earth-magnetic force at the pole than would otherwise be impossible.

A remark considering the calculation of $\bar{X} \sin v$ and $\bar{Y} \sin v$ must still be made at this point. Although we put into the equation the equation (24) the requirement that both values should be zero for $v = 0$ and $v = \pi$, that requirement need not be strictly satisfied in the result - except if for q parallel circles (with inclusion of the poles) the development is carried out to the term P_0^{q-1}. For if this does not happen, the development of the representation expressed in the prescribed equations (24) can differ by spherical functions of higher order than the last one still taken into account. Now for a somewhat considerable number q the series will always probably be broken off sooner, and therefore it could happen indeed that $\bar{X} \sin v$ to which I shall limit myself would be represented by the same for $v = 0$ and $v = \pi$ as not disappearing. Let the values which the series will produce instead be designated by (0) and (π). The two sums appearing in equation (7) would then be equal respectively to $\frac{1}{2}\left[(0)+(\pi)\right]$ and $\frac{1}{2}\left[(0)-(\pi)\right]$, which I will write for short as s and d. Now, however, as previously mentioned, it is absolutely necessary to shape the series development of $\bar{X} \sin v$ /28 in such a way that it produces at the poles the value of zero in all acuity. Thus the coefficients B_0^0, B_0^1 must be brought into harmony by changes as small as possible with the equations (7). For this purpose I specify the quantity β^0, β^1 such that $B_0^0-\beta^0, B_0^1-\beta^1$, satisfy these equations, and that the sum of the squares of the thus introduced changes over the whole surface of the earth becomes a minimum. The same can be represented under consideration of the identical relation $a_0^n (P_0^n(\cos 0))^2 = 1$ by the following simple system of formulas:

$$(28) \dots\dots\dots \begin{cases} \beta^0 (1+5+\dots+(4\nu+1)\dots) = s & \dfrac{\beta^1}{3}(3+7+\dots(4\nu+3)\dots) = d \\ \beta^{2\nu} = \dfrac{4\nu+1}{P_0^{2\nu}(\cos 0)}\beta^0 & \beta^{2\nu+1} = \dfrac{4\nu+3}{3P_0^{2\nu+1}(\cos 0)}\cdot\beta^1 \end{cases}$$

A similar compensation must necessarily be made for $\overline{Y}$ sin v. One should still point out here that it is more effective if one expands the development of $\overline{X}$ sin v up to the terms of the $(\nu+1)$-th order, if one breaks off for $\overline{Y}$ sin v with those of the ν-th order because one then obtains according to the earlier discussions U and W equally far, namely both up to the spherical functions of ν-th order.

For $\overline{Z}$, it is simplest to introduce two indeterminate quantities z and z' to designate its value at the two poles and one then requires that from the final result the same values for $\overline{Z}$ are produced for v = 0 and v = π. This request leads to two equations of the form

$$(29) \dots z = \frac{(\nu+1)^2}{2} a_1 z + (-1)^\nu \frac{\nu+1}{2} a_0 z' + Const. \qquad z' = (-1)^\nu \frac{\nu+1}{2} a_1 z + \frac{(\nu+1)^2}{2} a_0 z' + Const.'$$

which, as can be seen, generally produced a definite solution.

The treatment of $\overline{X}$ and $\overline{Y}$ is not quite as simple. Let the sums of the terms multiplied in it by $\cos m\lambda$ and that by $\sin m\lambda$ be designated by X_m resp. Y_m; we then have

$$X = X_0 + X_1 + X_2 + \dots + X_\nu \qquad Y = Y_0 + Y_1 + Y_2 + \dots + Y_\nu$$

Each single one of the parts shown here separately, with the exception of $\overline{X}_1$ and $\overline{Y}_1$, becomes equal to zero at the poles; in its development by means of spherical functions its value can be considered to be known at these two points. However, one must generally carry out here with reference to $\overline{X}_0$ and $\overline{Y}_0$ a similar compensation as was described shortly before in the treatment of $\bar{X}$ sin v. It is considerably different for $\bar{X}_1$ and $\bar{Y}_1$. For these quantities which become discontinuous at the poles, it is necessary to calculate the previously (German page 18) stated expressions through which their values are represented in the vicinity of the pole. The part remaining after the subtraction of these expressions from $\bar{X}_1$ and $\bar{Y}_1$ can then be developed by means of spherical functions. In order to now solve this problem with the auxiliary means which were offered by the previous discussion, one develops two quantities composed of $\bar{X}_1$ and $\bar{Y}_1$ which are free of discontinuities, by means of spherical functions, then one derives based on this development their values at the poles, and finally concludes in reverse order as to the values of $\bar{X}$ and $\bar{Y}$. Two such quantities are for example

$$\bar{X}_1 \cos\lambda + \bar{Y}_1 \cos v \sin\lambda \quad \text{and} \quad \bar{X}_1 \sin\lambda - \bar{Y}_1 \cos v \cos\lambda$$

The first one becomes equal to a at the North Pole, at the South Pole equal to a'; the second one assumes at these two points the values b and b'. By developing these quantities by means of the method given for the representation of $\bar{Z}$ by means of spherical functions, one obtains directly a, b, a', b' and thus the coefficients α, α', β, β' of the desired expressions to be separated from $\bar{X}$ and $\bar{Y}$.

The two above listed functions of $\bar{X}_1$ and $\bar{Y}_1$ are not the only ones which satisfy the specified conditions; there are numerous other ones which could be put in their place. Although one could obtain for different selection of these quantities approximate but not exactly the same values of α, α', β, β', this is based on the fact that the calculation of the latter is essentially an interpolation. There thus exists a certain randomness which can be removed in a /2
rigorous fashion only through compensation calculations and whose influence on the determination of α, α', β, β' is presumably of no practical significance. I therefore feel that I can limit myself to this brief suggestion especially since the representation of $\bar{X}$ and $\bar{Y}$, as shown in previous discussions, is of smaller importance for the further calculation than the development of $\bar{X}$ sin v, $\bar{Y}$ sin v and $\bar{Z}$.

I summarize the results of the preceding discussion in a short overview which reproduces the course of the calculation to be carried out in this main point and the significance of the results to be gained thereby.

Given are the quantities $\bar{X}$, $\bar{Y}$, $\bar{Z}$, deviating from the rectangular components of the earth-magnetic force only insignificantly, whether this is at individual points or at a number of parallel circles or in an approximate manner over the whole surface of the earth. In accordance with one of the methods described in the last pages we then developed $\bar{X} \sin v$, $\bar{Y} \sin v$ and $\bar{Z}$ and if desired also $\bar{X}$ and $\bar{Y}$ into finite series progressing by means of spherical functions of which the first ones exactly have the value of zero at every pole. (The series for $\bar{X} \sin v$ is best continued by one more order than the other two).

The three series which represent the values of

$$\bar{X} \sin v \qquad \bar{Y} \sin v \qquad \bar{Z}$$

form for all continuing calculations the basis given by experience. A difference which consists between the observed magnetic elements and those to be calculated from these series is based on the fact that the latter were stopped at a definite place; this difference can be made arbitrarily small. It also includes the errors of the observations as well as the effects of local disturbances to a very large extent. From this we obtain the requirement that those series must be continued sufficiently far that the differences between observation and calculation are reduced to the amount of the potential errors and disturbances. This is no doubt possible; one need only, as long as the desired goal has not yet been obtained, continue with the development. Probably one can get to a satisfactory representation of the observations already through an expansion of the series which goes no farther than the ones customarily employed.

The further calculation introduces no new differences. In a completely rigorous manner the functions U and W are derived from $\bar{X}$ sin v and $\bar{Y}$ sin v; from them we obtain the potential V which is dispersed by the addition of $\bar{Z}$ into its two parts which have often been mentioned. In addition by an independent calculation we can calculate [illegible] and thus [illegible]. For all these quantities we obtain similarly finite series of spherical functions with whom, however, an expression which can be represented not in this form, but also in a closed form, is connected.

The result of the total calculation thus is finally represented by three functions: the potential of the magnetic masses or closed galvanic currents within the surface of the earth, the potential of just such agents outside of the same and the intensity of those currents which lead to one of these volume regions and enter the other. These three functions contain everything which can be stated concerning the causes of the magnetic forces observed at the surface of the earth as long as one utilizes no other experiences than the magnetic measurements in this surface. The force distribution defined by them is identical with that which is expressed by the series for $\bar{X}$ sin v, $\bar{Y}$ sin v and $\bar{Z}$; it thus differs from those determined by observations by the differences already occurring in these series. However, to the extent that the observations themselves are insufficient or subjected to errors, the functions gained as final results must naturally deviate from the actually existing magnetic state, also disregarding those differences. Just how high these deviations can get at a maximum can be determined by estimation. If the latter is greater than the effect of one of the three causes to which the earth-magnetic total force is attributed, one can naturally not conclude that this cause is actually in effect. Thus one will be justified to disregard the latter and to attribute the part of the force seemingly assigned to it to our insufficient knowledge of the actual state.