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Classical Linear-Control Analysis Applied to Business-Cycle Dynamics and Stability

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LIST OF SYMBOLS

F	linear model of forward path
G	linear model of feedback path
K	constant gain
m	rate of growth in money supply, %/yr
p	rate of growth in prices, %/yr
r	remnant
s	frequency domain operator, $\omega(-1)^{1/2}$
t	time, yr
u	unemployment rate, %
x	rate of growth in real GNP, %/yr
ω	oscillatory frequency, rad/yr
ζ	damping ratio
τ	time constant, yr
Δ	variation about long-term trend
$(\hat{\ })$	estimate
$(\dot{\ })$	rate of change with respect to time
$(\ )_{ic}$	initial condition
$(\ )_n$	natural
$(\ )_o$	long-term trend
$\|(\ )\|$	absolute value

SUMMARY

Classical linear-control analysis provides a framework for studying dynamic systems involving random disturbances. This framework is used to develop a set of equations that, in historical perspective, combine traditional concepts about the dynamics of economic systems and about the effects of random economic disturbances. This set of equations provides relationships among well-known ideas in general macroeconomics and provides a means to interrelate and examine ideas about stabilization policies. In this study, linear-control analysis is applied as an aid in understanding the fluctuations of business cycles in the past, and to examine monetary policies that might improve stabilization. The analysis shows how different policies change the frequency and damping of the economic system dynamics, and how they modify the amplitude of the fluctuations that are caused by random disturbances. Examples are used to show how policy feedbacks and policy lags can be incorporated, and how different monetary strategies for stabilization can be analytically compared. Representative numerical results are used to illustrate the main points.

INTRODUCTION

Since the 1800s economists have conducted a broad range of studies on the nature of business cycles and on the possible means of stabilizing these cycles. In the 1940s the maturing control systems theory was applied in studies on economic stabilization. As discussed in references 1-5, the recent trend in these studies is toward the use of large economic models with an increasing number of economic variables, along with the use of fairly sophisticated analysis methods.

Rather than going to more and more complex analysis to study the problems of business-cycle stabilization, an alternate approach might be to explore some basic ideas using simple linear methods. For example, Wicksell, a leading economist in the 1890s, suggested that the characteristics of the business cycle resembled a lightly damped system subjected to random disturbances (ref. 6); by following this idea, business-cycle stabilization might be analyzed using linear methods. The approach could be to start with a simple second-order dynamic model of the economy and then use linear-control analysis to examine how different policies affect the frequency and damping of the system and modify the amplitude of the fluctuations caused by random inputs. Although this alternate approach might start with simple linear models, this type of study can move in many directions. Linear-control analysis provides a direct method of adding different model elements and provides a way to determine closed-form relationships between different variables. These analytical relationships could provide links among the many variables that are important to the economy. Using simple linear methods, rather than continuing to use more complex analysis, the investigator or student may be able to better understand those economic policies, along with those linkages among the economic variables, that will have a major role in stabilizing the business cycle in the future.

The purpose of this working paper is to explore the use of linear-control analysis as a means of studying business cycles and stabilization. The economic system is formulated to include internal elements which respond to random external effects. Using this formulation, business-cycle dynamics and stability are studied through an examination of movements in the money supply. The movements of money supply during business cycles have been the topic of inquiry in studies such as those of references 7-13. The different ideas from these studies will be interrelated and combined with well-known concepts such as those from references 14 and 15. New insight is provided when these different concepts and ideas are linked within a framework of linear analysis, and leads to a series of results that have not appeared in the literature.

The analysis methods to be employed here are usually termed "classical linear-control methods" and are outlined in the many textbooks for introductory college courses on systems dynamics. These classical methods provide a framework for adding different dynamic elements (e.g., feedback) and superimposing different disturbances (e.g., random effects). As the paper proceeds, this framework is used in developing a set of linear equations that combine concepts and ideas in economics. Using these linear equations, standard feedback analysis is applied to aid in understanding the fluctuations of business cycles in the past, and to examine monetary policies that may stabilize the business cycle in the future. The results from this study are summarized at the conclusion.

The author thanks William Hindson, Thomas Mayer, Dallas Denery, Ralph Bach, and many colleagues for their helpful comments on early drafts of this paper.

MODELING

Textbooks (e.g., refs. 16 and 17) view the economy as operating about long-term growth trends where a stimulus such as an increase in the growth of money supply produces an interval of increased real output. During the interval of increased real output there is a decrease in the unemployment rate. This section formulates these economic relationships in the context of linear-analysis models. These models are illustrated using economic data from the United States.

Growth of Real GNP

The dynamic response in the growth of real GNP (x) from a stimulus in the growth of money supply (m) can be written using the standard convolution integral as

$$x(t) = \int_0^\infty f(\tau)m(t - \tau)d\tau + r_x(t) + x_o \qquad (1)$$

where $f(\tau)$ represents the impulse response function, $r_x(t)$ represents the remnant, and x_o represents the long-term growth trend. Using the Laplace transform

$$F(s) = \int_0^\infty f(\tau)e^{-\tau s}\, d\tau \qquad (2)$$

where $s = \omega(-1)^{1/2}$, equation (1) can be written as a function of frequency as

$$\Delta x(s) = F(s)m(s) + r_x(s) \quad (3)$$

where

$$\Delta x = x - x_o \quad (4)$$

Because the actual economic system is not an exact linear system, the f and F terms are forms of what are called "input-output describing functions." Input-output describing functions, as defined in reference 18, represent the linear correlation between sets of time-history data. All other (noncorrelated) effects are contained in the remnant, r. The describing function formulation provides a means of studying systems for which there is no exact mathematical model. For example, human-operator systems (e.g., refs. 19 and 20) representing the response of one individual or, in this paper, economic systems representing the response of a nation of individuals.

One way to represent the describing function in equation (3) is to draw upon background theory such as developed by Friedman (ref. 21). Friedman's theory was used in reference 22 to derive a linear second-order differential equation that can be represented as a function of frequency, by a model of the form

$$F(s) = (K_x s)/[(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1] \quad (5)$$

where K_x is the gain, ω_n is the natural frequency, and ζ is the damping ratio. Representative values for these parameters have been estimated from past economic data (1950-1981) as discussed in appendix A. The representative values, to be used for illustration, are

$$K_x = 2 \text{ (yr)}$$

$$\omega_n = 1.5 \text{ (rad/yr)}$$

$$\zeta = 0.8 \text{ (nondimensional)}$$

This model is illustrated in figure 1 using time-series data from 1950 through 1981. The upper chart in figure 1 presents the measured growth of money supply (m) over this 32-yr time period. (The money supply is based on the M1 measurement that includes currency and all checking accounts.) The lower chart presents the measured growth of real GNP (x) along with the output from the model ($\hat{x}$). Also in the lower chart, the long-term trend (x_o) is indicated by the representative value of 3 (%/yr).

As shown in the lower chart of figure 1, the growth of real GNP (x) tends to fluctuate around the long-term trend (x_o). These fluctuations (Δx) can be defined in terms of the following components

$$x = \underbrace{\overbrace{(\hat{x} - x_o)}^{\text{Model}} + \overbrace{(x - \hat{x})}^{\text{Remnant}}}_{\Delta x} + \overbrace{x_o}^{\text{Trend}}$$

The contributions to the fluctuations (Δx) include the model output ($\hat{x} - x_o$) and the remnant ($r_x = x - \hat{x}$). The remnant as discussed previously represents that portion of the actual output that remains after the model output is calculated. The remnant

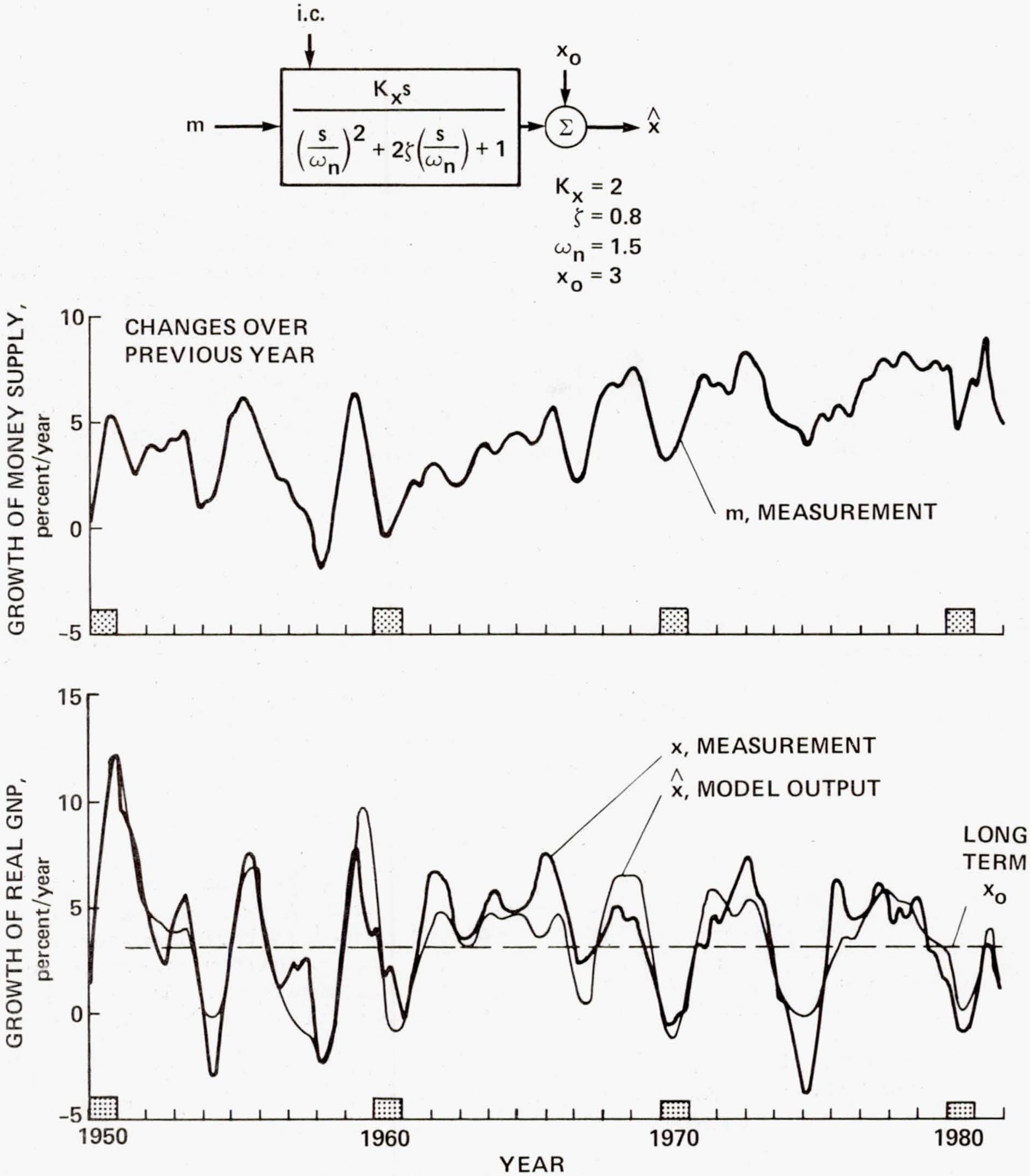

Figure 1.- Time history in the growth of real GNP x compared with the model output $\hat{x}$ and the long-term trend x_o.

contains nonmonetary effects outlined in the further discussion of business-cycle dynamics.

Unemployment Rate

The unemployment rate variable can be modeled through Okun's law. This relationship is based on the findings by Okun (ref. 15) that the growth in unemployment rate ($\dot{u}$) is associated with the growth of real GNP (x). Okun's law equation written as a function of time is

$$\dot{u}(t) = -K_u[x(t) - x_u] \tag{6}$$

or

$$\dot{u}(t) = -K_u[\Delta x(t) + x_o - x_u] \tag{7}$$

where K_u is a gain and x_u is a growth rate constant. Representative values for these parameters, based on the period 1950-1981, are

$$K_u = 0.4 \text{ (nondimensional)}$$

$$x_u = 3.6 \text{ (\%/yr)}$$

Okun's law equation is illustrated in figure 2 using the time-series data from 1950 through 1981. The upper chart in figure 2 presents the measured growth of real GNP (x) over this time period. The lower chart presents the measured growth in unemployment rate ($\dot{u}$) along with the computed value from Okun's law equation ($\hat{\dot{u}}$). As shown, there is close correspondence between Okun's law equation and the actual measurements.

BUSINESS-CYCLE DYNAMICS

This section first reviews some ideas about business cycles from the economic literature. This background material leads to a formulation of the overall economy as a feedback system subjected to random disturbances. Using this formulation, a set of linear-analysis equations are developed that will be used in the further examination of stabilization policies.

Historical Perspective

The movements of money supply during business cycles have been discussed in a number of studies (refs. 7-12). These studies indicate that, historically, there has been feedback from business activity into money supply. During those portions of the cycle when business activity has been increasing, the tendency has been for the growth in money supply to increase; conversely, when the business activity has been decreasing, the tendency has been for the growth in money supply to decrease. The studies in references 8, 9, and 12 outline a variety of factors that probably have combined to produce this feedback from business activity into money supply during past business cycles.

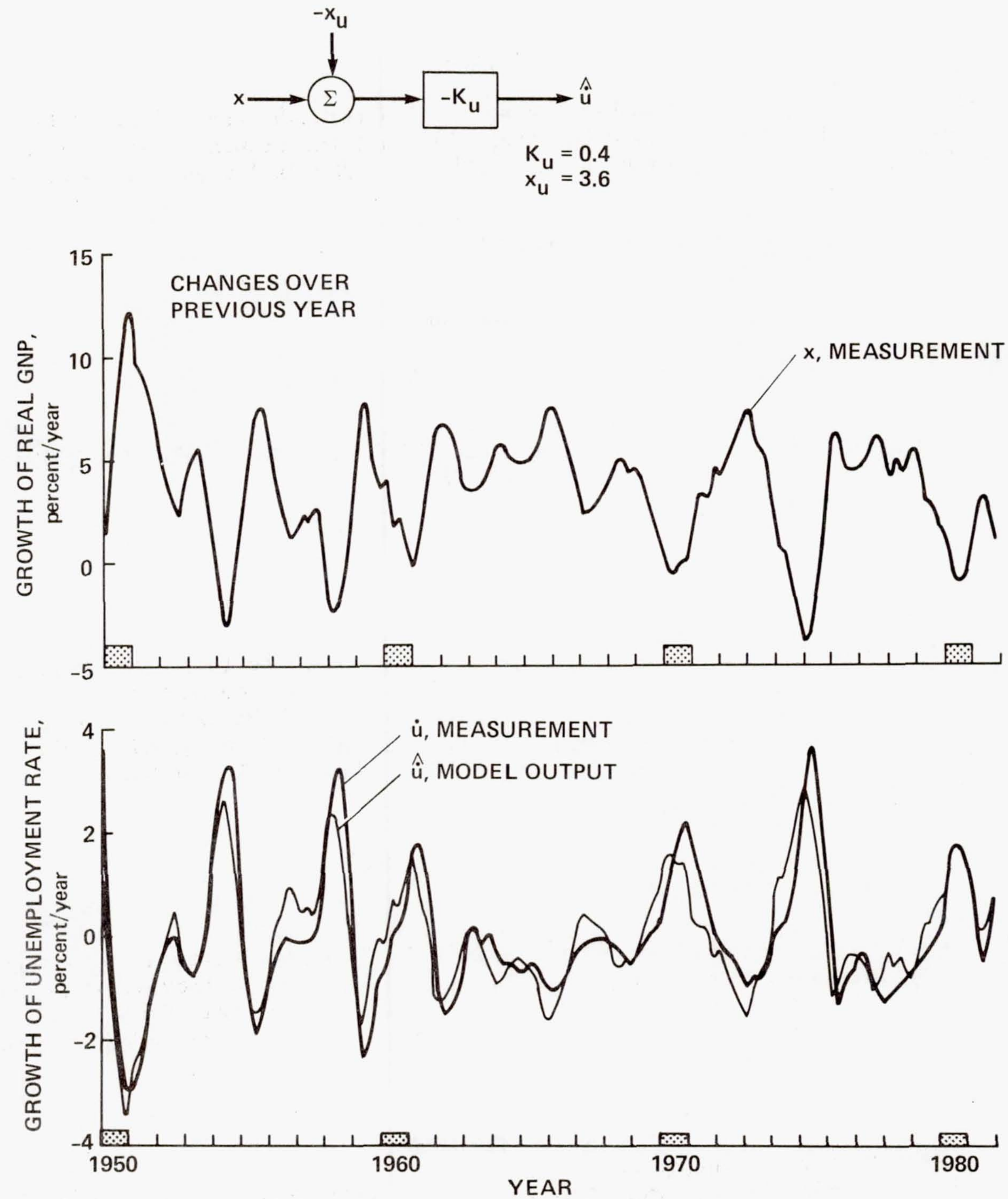

Figure 2.- Comparison of Okun's law equation with the measured growth in unemployment rate.

Friedman and Schwartz suggest that this feedback from business activity into money supply might reinforce cyclical fluctuations (ref. 7). The idea is that a change in business activity produces a change in money supply that produces a change in business activity that in turn continues the cycle. According to this idea, feedback from business activity into money supply can produce lightly damped motions in economic output.

Wicksell suggested that lightly damped dynamics when subjected to random disturbances may serve as a mechanism to produce the observed business cycles (ref. 6). This idea represents a combination of both internal and external factors. Samuelson summarizes the consensus about business cycles through a consideration of both internal and external factors (ref. 23). As a textbook example, he looks upon business cycles as not unlike a toy rocking horse subjected to occasional pushes. The pushes need not be regular; economic shocks seldom are. But just as the toy horse rocks with frequency and amplitude that depend partly on its internal nature, so too will the economic system respond to external pushes according to its internal nature. After many random pushes the observed set of fluctuations will be gathered about the natural frequency. For illustration, figure 3 presents a histogram of the 28 business cycles from 1857 to 1980 (see appendix B). Also shown for comparison is the natural frequency, ω_n = 1.5 rad/yr. As we can observe, the business cycles tend to gather about the natural frequency. (Most business cycles are in the frequency range from about 0.5 to 3 rad/yr. This will be termed the business-cycle frequency range in later discussions.)

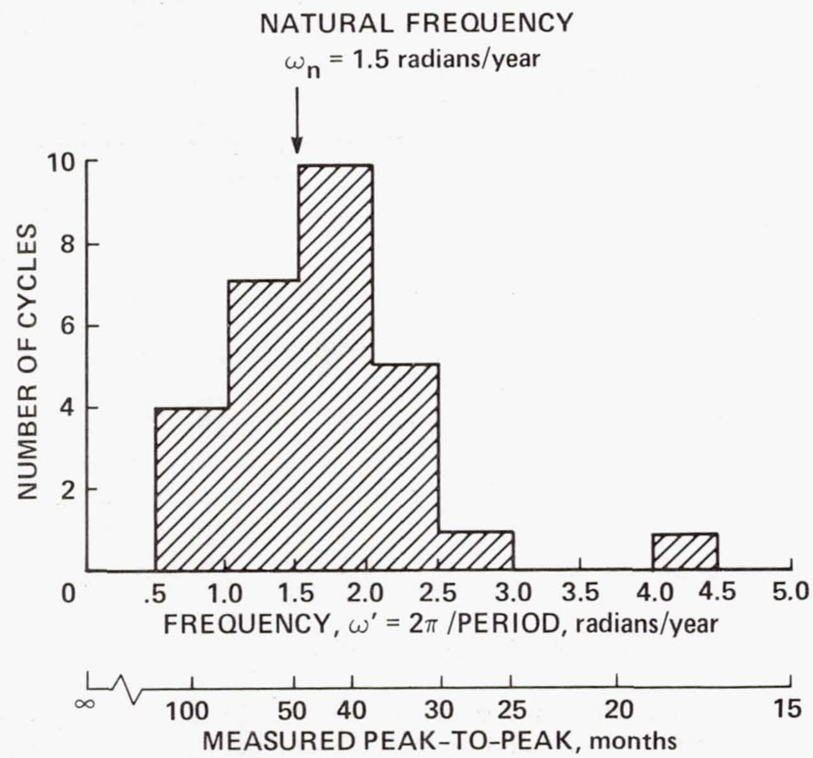

Figure 3.- Histogram of 28 business cycles, peak to peak, from 1857 to 1980 (appendix B).

This combination of ideas suggests that business cycles involve internal elements, containing both natural and feedback effects, along with external random effects. To study business-cycle dynamics, the economy will be formulated as a feedback system subjected to random disturbances. A flow diagram illustrating the closed-loop system to be analyzed is presented in figure 4. The system contains both a forward path (F) and a feedback path (G). The measured variables are growth of money supply (m) and growth of real GNP (x). Random disturbances follow the outline of "impulse and propagation" in the survey by Hansen (ref. 6), and involve nonmonetary effects (r_x introduced previously) and monetary effects (r_m). Random nonmonetary effects (r_x) include real factors such as harvest variations, inventions, industry/labor disputes, changes in foreign trade, and changes in fiscal policies, along with psychological factors such as errors in judgment and structural changes in expectations (e.g., the term r_x includes so-called "supply shocks" and "panics"). Random monetary effects (r_m) include, for example, the creation of money for wars, gold discoveries, international inflows/outflows, and changes in monetary policies.

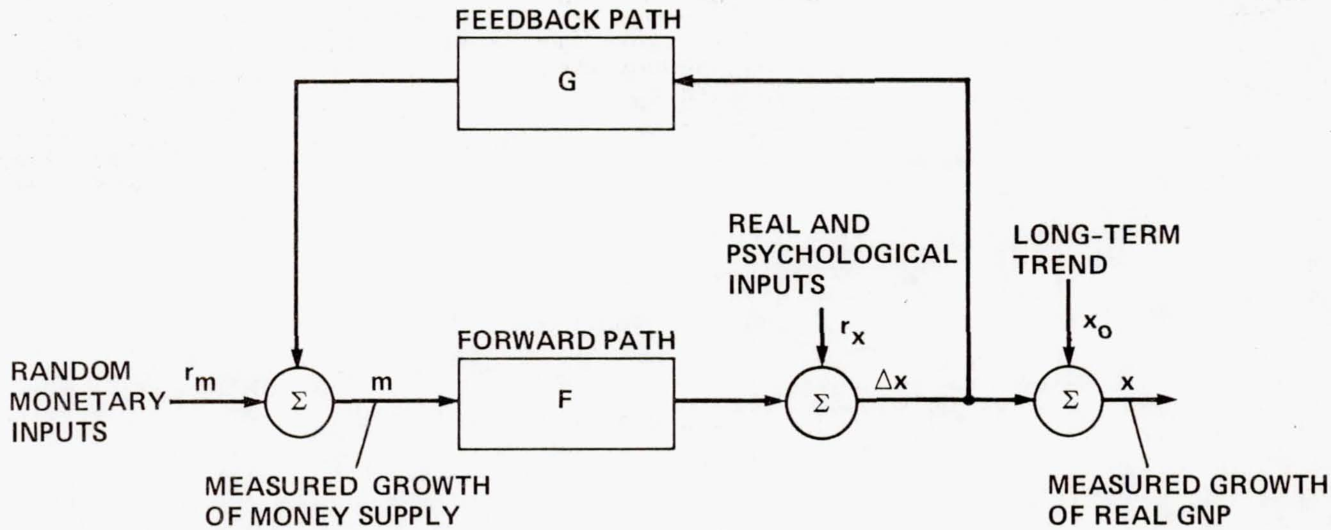

Figure 4.- Formulation of the closed-loop economic system.

The analysis first considers the influence of feedback on the stability of the closed-loop system. Feedback effects are then combined with random effects to: (1) provide a means to interrelate and illustrate the ideas about business cycles discussed previously; and (2) provide a background for the further examination of different stabilization policies.

Feedback and Stability

A standard method of studying the effects of feedback on stability is to examine the roots of the characteristic equation representing the closed-loop system. From linear-control analysis, the characteristic equation for the closed-loop economic system, figure 4, is

$$1 - F(s)G(s) = 0 \tag{8}$$

where $F(s)$ represents the forward path and $G(s)$ represents the feedback path. Substituting equation (5) for the forward path, we have

$$(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1 - K_x sG(s) = 0 \qquad (9)$$

There is a special solution to equation (9) for the case in which m moves in a direct response to any movements of the feedback state Δx. In this special case the feedback, G(s), is simply a constant, G_x, and the characteristic equation is

$$(s/\omega_n)^2 + (2\zeta - K_x G_x \omega_n)(s/\omega_n) + 1 = 0 \qquad (10)$$

We observe from equation (10) that any constant feedback G_x affects the damping, but does not affect the natural frequency. If G_x has a positive value (procyclical movement called positive feedback) then it will tend to undamp, or destabilize, the closed-loop system. On the other hand, if G_x has a negative value (countercyclical movement called negative feedback) then it will contribute to the damping of the closed-loop system.

The upper chart in figure 5 illustrates the root locations, in the complex plane, for both positive and negative values for G_x. As shown, positive values of G_x tend to move the operating point of the closed-loop system toward the right-half plane which is the region of dynamic instability. And negative values of G_x tend to move the operating point of the closed-loop system toward the negative axis which is the region of well-damped stability.

The lower chart in figure 5 illustrates the impulse response of the closed-loop system as a function of feedback G_x. As shown, a positive feedback ($G_x = +0.4$) results in a lightly damped response and a negative feedback ($G_x = -0.4$) results in a well-damped response. The period of the lightly damped oscillation shown in figure 5 is quite close to the 4.4-yr average of the 28 business cycles from 1857 to 1980 (appendix B).

Response to Random Inputs

The random inputs (r_m and r_x) are incorporated by considering the overall closed-loop response of the economic system in figure 4. From linear-control analysis, the output (Δx) can be written as

$$\Delta x(s) = [F(s)r_m(s) + r_x(s)]/[1 - F(s)G(s)] \qquad (11)$$

Equation (11) is solved to determine the amplitude of fluctuations in Δx due to random inputs (r_m and r_x).

The effect of feedback gain G_x on output response is illustrated in figure 6. Response to the monetary input r_m is presented in the upper chart, and response to the nonmonetary input r_x is presented in the lower chart. In these charts, the amplitude of the response in Δx to the inputs r_m and r_x is shown by the multipliers $|\Delta x/r_m|$ and $|\Delta x/r_x|$, respectively. The multiplier values were calculated from equation (11) as a function of the input sinusoidal frequency ω.

The closed-loop response characteristics in figure 6 follow the trends noted previously. Positive (procyclical) values of G_x tend to destabilize economic output; conversely, negative (countercyclical) values of G_x tend to stabilize economic output. These stability effects are particularly apparent at values of ω near the natural frequency, $\omega_n = 1.5$ rad/yr, where it is seen that positive values

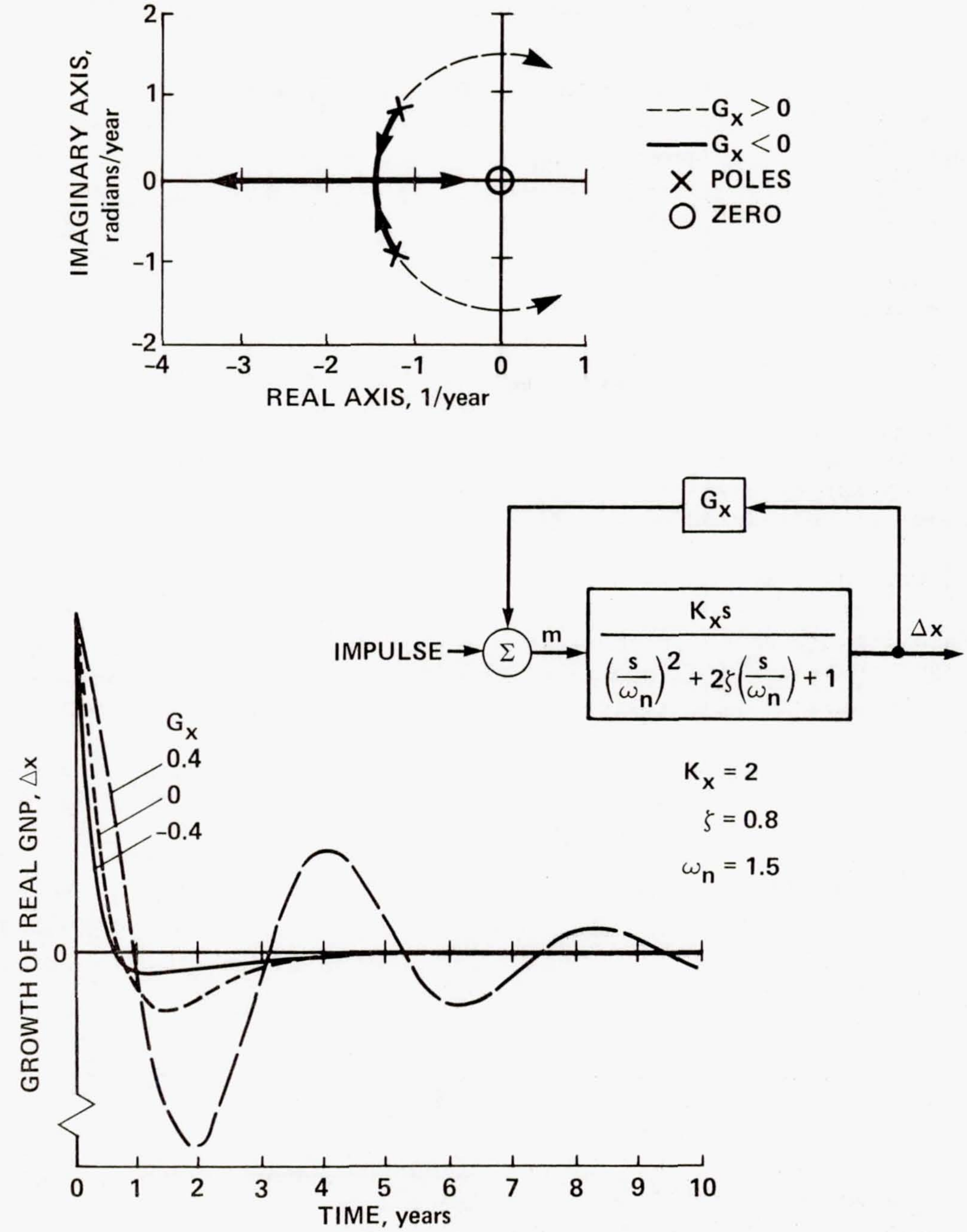

Figure 5.- Closed-loop dynamics as a function of feedback, G_x.

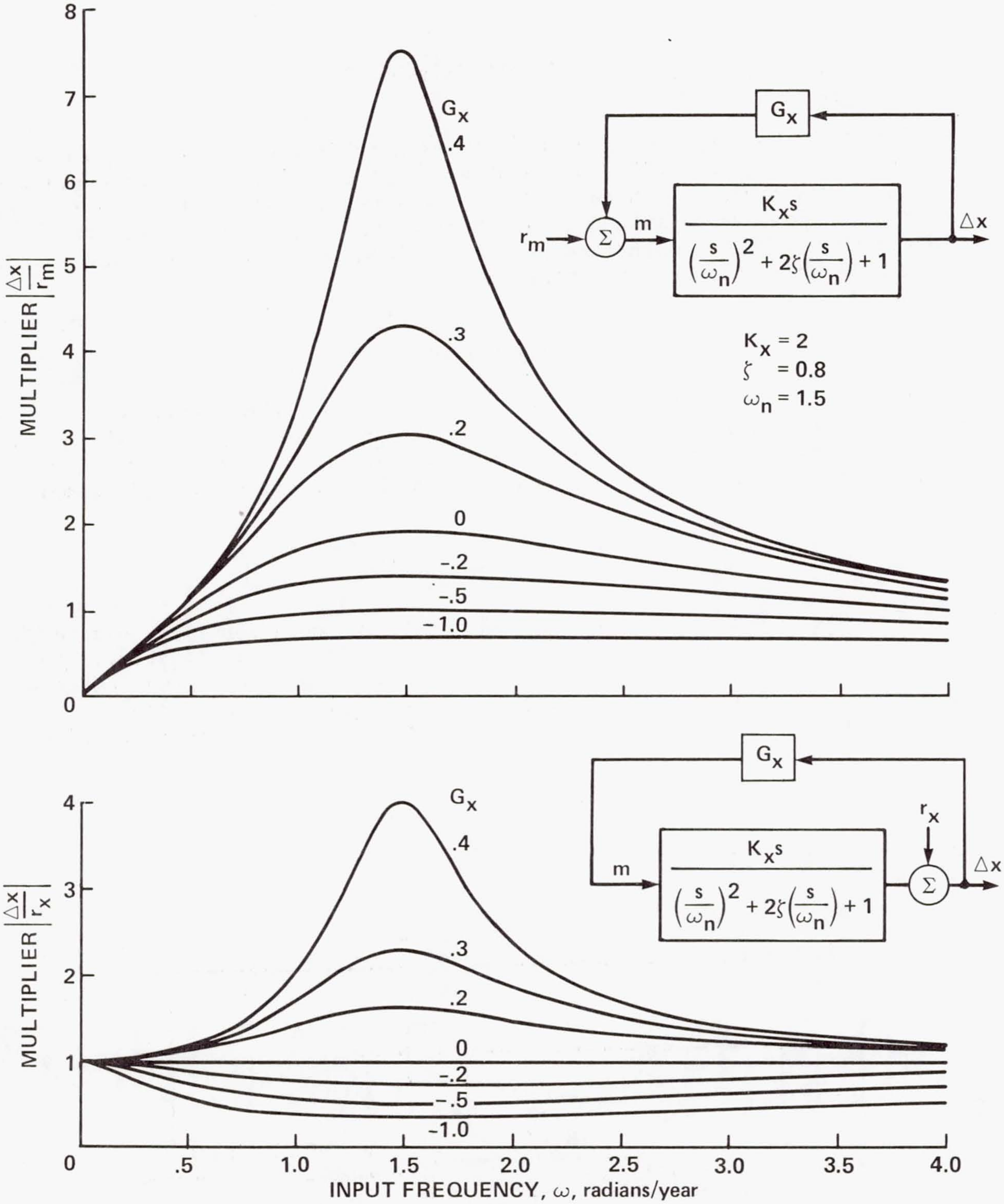

Figure 6.- Closed-loop response as a function of the feedback value G_x.

for G_x tend to compound the amplitude of the response in Δx to external inputs; conversely, negative values for G_x tend to reduce the amplitude of the response in Δx to external inputs.

STABILIZATION

The previous sections discussed how linear methods can be used in developing equations that combine concepts and ideas about the dynamics of economic systems and the effects of random disturbances. These equations will now be used to interrelate and then examine some concepts and ideas about stabilization policies. This section illustrates how policy feedbacks and policy lags can be incorporated, and how different policies can be analytically compared.

Policy Feedbacks

One means of stabilization discussed in recent textbooks (e.g., refs. 16 and 17) is for the Federal Reserve Board to use feedback from fluctuations in the unemployment rate. The fluctuations in unemployment rate are defined as $\Delta u = u - u_n$, where u is the measured unemployment rate and u_n is the long-term natural unemployment rate. Letting

$$\dot{u}_n = -K_u(x_o - x_u) \tag{12}$$

then from Friedman's model equation (5) and Okun's law equation (7), this feedback system can be formulated with the loop structure illustrated in figure 7. As illustrated, the feedback (G_u) from fluctuations in unemployment rate (Δu) represents an added (outer) loop to the single (inner) loop system discussed previously. Using standard block-diagram reduction, this combined feedback can be represented in the closed-loop system of figure 4 as

$$G(s) = G_x - G_u K_u / s \tag{13}$$

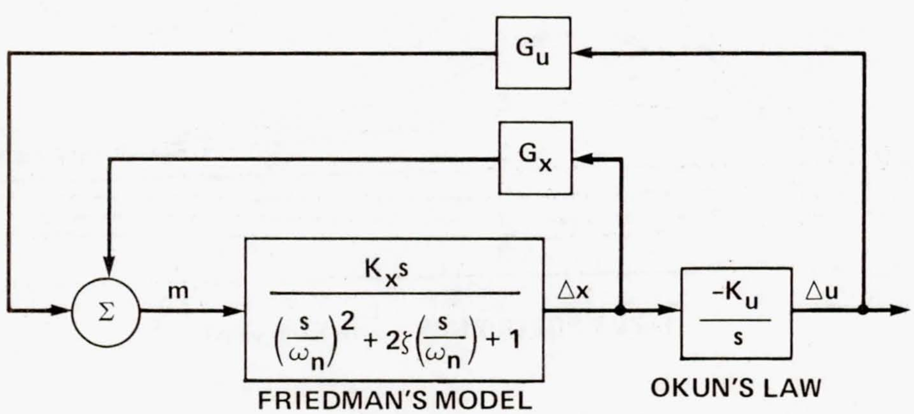

Figure 7.- Loop structure with feedback from unemployment rate and growth of real GNP.

And by substituting equation (13) into equation (9), the characteristic equation of the closed-loop system then becomes

$$(s/\omega_n')^2 + (2\zeta - G_x K_x \omega_n')(s/\omega_n') + 1 = 0 \qquad (14)$$

where

$$\omega_n' = [\omega_n^2(1 + G_u K_u K_x)]^{1/2}$$

We observe from equation (14) that feedback from unemployment rate (G_u) contributes to the frequency term (ω_n'). There is a phase shift between the feedback variables Δx and Δu because of the integration (1/s) in Okun's law equation relating these two variables. Also note that the feedback loop from unemployment rate contains the factor $K_u K_x$. As discussed in appendix C, the factor $K_u K_x$ is a linear representation of the Phillips curve slope, relating unemployment rate and growth of prices, based on the well-known study by Phillips (ref. 14).

The sign of the feedback gain G_u would be positive for a policy of increasing the growth of money supply when unemployment rate goes above the natural trend and decreasing the growth of money supply when unemployment rate goes below the natural trend. With a positive value for G_u, equation (14) indicates that the oscillatory frequency (ω_n') would increase. These results indicate that feedback from unemployment rate would not provide any damping to the closed-loop system, but, rather, would simply increase the frequency of the business-cycle fluctuations. As discussed previously, damping can be provided through inner-loop feedback from growth of real GNP. The following discussion will only consider the inner-loop feedback path, G_x.

Time Lags and Strategies

In the practical implementation of stabilization policies one of the important and continuing concerns has been in the effect of time lags, both in economic response and in policy response. As discussed in the textbooks (e.g., refs. 16 and 17), these are usually termed, respectively, outside and inside lags. The outside lag is inherent within the forward path, F. The inside lag is in the feedback path, G, and can come from many sources. The following examples include: (1) the time required to ascertain economic trends; (2) the time required for decisions; and (3) the time required for implementation. The sum of these time lags in the feedback path represents an overall policy lag (t').

When the policy lag, t', is combined with the feedback gain, G_x, the feedback path in the closed-loop system of figure 4 becomes

$$G(s) = G_x\, e^{-t's} \qquad (15)$$

Substituting equation (15) into equation (11), the fluctuations in the growth of real GNP (Δx) caused by random nonmonetary inputs (r_x) can be examined through the equation

$$\Delta x(s) = [r_x(s)]/[1 - F(s) G_x\, e^{-t's}] \qquad (16)$$

Equation (16) allows the strategy of countercyclical control to be analytically compared with the strategy of controlling to a constant growth in money supply.

In the constant growth concept, the money supply is to be controlled to a constant rate of growth independent of business-cycle fluctuations (ref. 24). This policy inherently attempts to insure that there is no feedback from random nonmonetary inputs (r_x) and the solution to equation (16) is simply $|\Delta x/r_x| = 1$ which means that any nonmonetary input r_x goes directly into Δx. (It is noted in references 11 and 13 that even with a constant growth in money supply there will be economic fluctuations caused by nonmonetary effects.)

For the countercyclical feedback concept the solution to equation (16) is a function of the values for both the time lag (t') and the feedback gain (G_x). Representative effects of the time lag and feedback gain are illustrated in figure 8.

The upper chart in figure 8 shows the multiplier $|\Delta x/r_x|$ for several values of time lag (t') with the feedback gain fixed at $G_x = -0.4$. This chart illustrates that if the time lag is less than about 0.2 yr, then countercyclical feedback can be effective in reducing the response from r_x. However, for larger values of time lag, on the order of 0.4 to 0.6 yr, countercyclical feedback may compound the response from r_x. These problems in countercyclical control, with the larger time lags, occur for input disturbance frequencies in the upper portion of the business-cycle frequency range.

This problem of time lags is a function of the value of feedback gain as illustrated in the lower chart of figure 8. This chart shows the multiplier $|\Delta x/r_x|$ for several values of feedback gain (G_x) with the time lag fixed at $t' = 0.5$ yr. This chart illustrates that with smaller values of feedback gain, there is less compounding of the response at the higher frequencies. Of course, with smaller values of feedback gain there is also less reduction in the response caused by random inputs in the lower portion of the business-cycle frequency range.

Essentially, the results in figure 8 illustrate a trade-off between the concept of countercyclical control and the concept of controlling to a constant growth in money supply. The use of countercyclical control has the potential to reduce the amplitude of the fluctuations in real GNP caused by random nonmonetary inputs. However, with significant time lags in the loop, countercyclical control tends to compound the response to random inputs in the upper portion of the business-cycle frequency range. (Reference 25 notes that if the countercyclical policy is consistent and anticipated, then firms might increase (or decrease) their inventories in a rational expectation of an increase (or decrease) in policy stimulus. Such anticipation might provide some "lead" that would tend to cancel out some of the policy "lag." For more general discussions about rational expectations and stabilization policies, see refs. 26-28.)

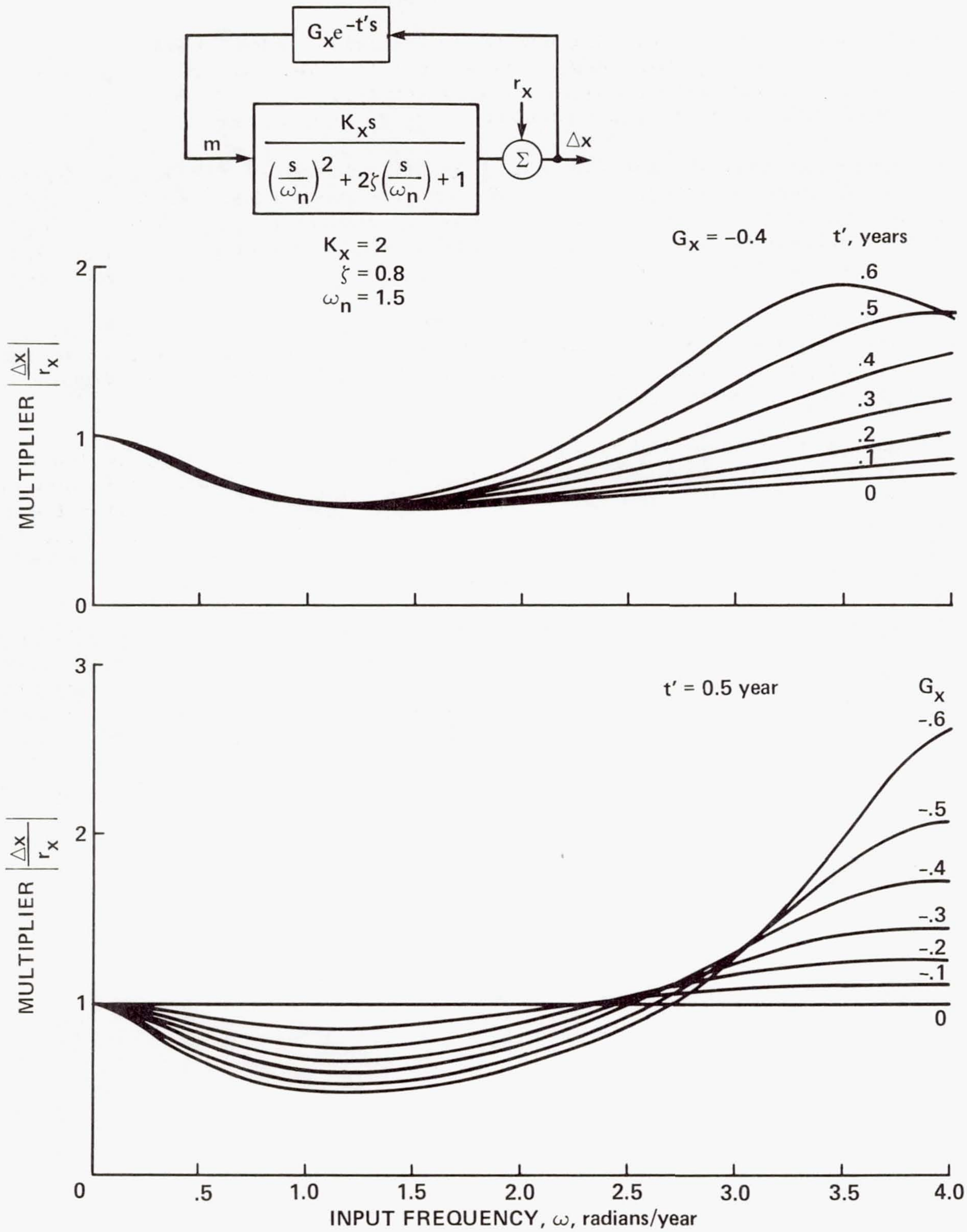

Figure 8.- Effects of time lag and feedback gain on closed-loop response.

CONCLUDING REMARKS

Some results based on the application of classical linear methods to the analysis of economic system dynamics have been presented. The closed-loop economic system was formulated to include internal elements which respond to random external disturbances. The disturbances include both monetary and nonmonetary effects.

A closed-loop analysis indicates that procyclical movements between the growth of money supply and the growth of real GNP tend to compound the amplitude of the fluctuations in the business cycle. The results suggest that to improve stabilization of the business cycle, a general rule is that any movements in the growth of money supply should be countercyclical with respect to the growth of real GNP.

The policy of controlling the growth of money supply in response to fluctuations in the unemployment rate was considered. The results indicate that this feedback policy will not provide damping, but, rather, will simply change the frequency of the business-cycle fluctuations.

The policy of controlling the growth of money supply countercyclical with respect to the growth of real GNP is compared to the policy of controlling to a constant growth of money supply. The countercyclical policy has the potential to provide the minimum reduction in the business-cycle fluctuations; however, with significant time lags in the loop, the countercyclical policy tends to compound the response to random disturbances in the upper portion of the business-cycle frequency range.

The applications of linear methods in this working paper appear promising. The series of examples illustrates how linear methods can be used in the development of equations that serve to combine well-known ideas in economics such as those from Wicksell, Friedman, Okun, and Phillips. This same set of equations can be used to interrelate and then examine ideas about stabilization policies. The possible extensions from this working paper are numerous, considering the many ideas and linear relationships that have been developed in the field of economics, and the many tools available in the field of linear analysis.

APPENDIX A

ESTIMATION AND FEEDBACK

This appendix discusses the estimation of economic models from empirical time-series data. The discussion first considers the closed-loop estimation problem and then outlines the estimation technique used to obtain the representative second-order model used in the main text.

For this discussion, let us consider the closed-loop system illustrated in figure A1. The forward path, F(s), is to be estimated from the measured time histories of m and x. By removing any long-term trends and using standard power-spectrum analysis we can write the estimate as

$$\hat{F}(s) = \Phi_{mx}(s)/\Phi_{mm}(s) \tag{A1}$$

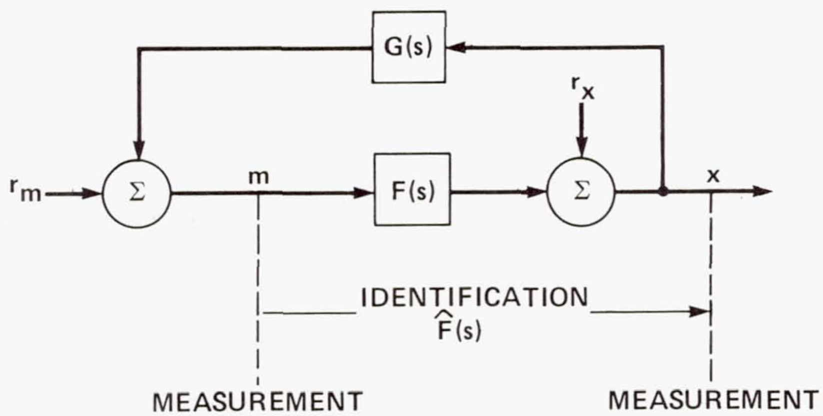

Figure A1.- Estimation within the closed-loop system.

where $\Phi_{mx}(s)$ is the cross-power spectrum between m and x, and $\Phi_{mm}(s)$ is the power-density spectrum of m. In this type of estimation, it is well known that the estimate, $\hat{F}(s)$, may not be the same as the actual F(s). This difference, an identification error, can be shown by delineating the components of the cross-power spectrum: $\Phi_{mx}(s) = F(s)\Phi_{mm}(s) + \Phi_{mr_x}(s)$. Substituting these components into equation (A1) yields

$$\hat{F}(s) = F(s) + \overbrace{\Phi_{mr_x}(s)/\Phi_{mm}(s)}^{\text{error}} \tag{A2}$$

Equation (A2) shows that any cross-correlation $\Phi_{mr_x}(s)$ may contribute to a bias error in estimation. Such a correlation will exist if there is feedback, because r_x

transfers through G and thus appears as a component of m. If r_x is much larger than r_m, then the ratio $\Phi_{mr_x}(s)/\Phi_{mm}(s)$ will be significant and the estimate $\hat{F}(s)$ will be different from F(s). Conversely, if r_m is much larger than r_x, then the estimate, $\hat{F}(s)$, will be near F(s).

The empirical evidence indicates that there have been large random monetary inputs, r_m, which aid in reducing the errors in identifying F(s). For instance, several types of random inputs (e.g., gold discoveries, international inflows/ outflows, creating money for wars, etc.) have been documented (ref. 7). Also, there have been continual changes in the policies of the monetary authorities which have contributed to r_m. These changing monetary policies, in effect, have provided input test signals. Recent test signals include, for example, the deliberate reductions in the growth of money supply in 1966, in 1969, and in 1980-1981. The random effects caused by changes in monetary policies appear in the flow diagram of the closed-loop system as illustrated in figure A2.

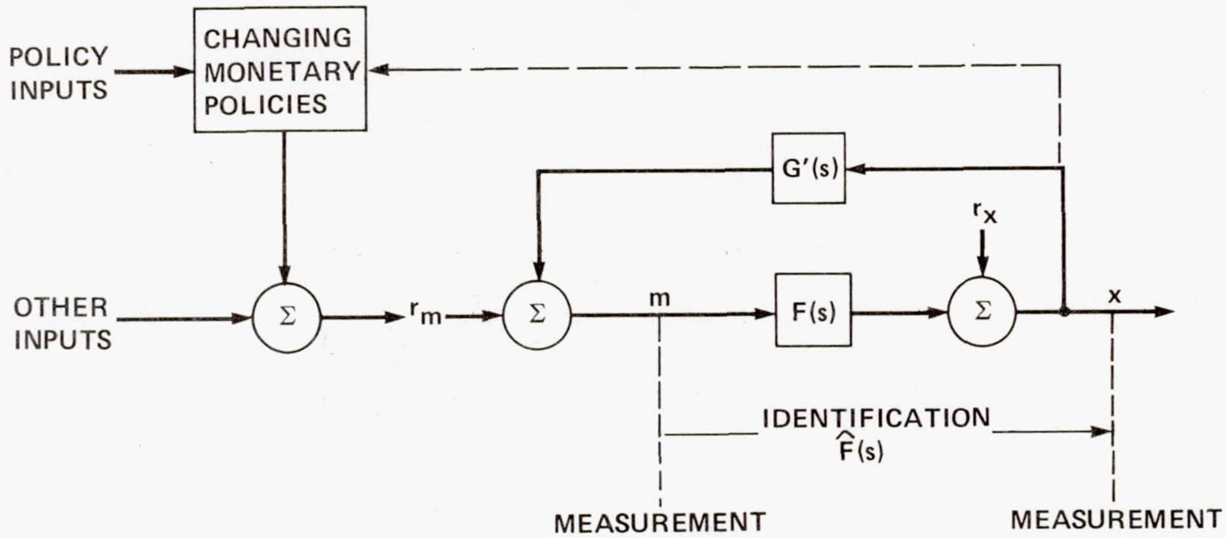

Figure A2.- Estimation using actual time-history records.

The changing policies of the monetary authorities, along with the other contributions to r_m, aid in reducing the size of the possible identification error; however, there still remains some residual feedback, G'(s), that could lead to some residual identification error.

The magnitude of the identification bias error due to residual feedback is a function of the constraints used in the empirical estimation process. For instance, the simple cross-spectrum method, equation (A1), contains no constraints and thus allows any residual-feedback identification error to appear in the estimate $\hat{F}(s)$. Other types of estimation methods, in which the allowable structure of $\hat{F}(s)$ is more constrained, will result in less feedback identification error. Reference 20 illustrates the amount that any residual-feedback identification error can be reduced to as a function of the constraints used in the empirical estimation process. Essentially, the study in reference 20 indicates that the feedback identification error is reduced when the structure of the estimation model, $\hat{F}(s)$, approaches the structure of the actual model, F(s).

To obtain representative results for this paper, the estimation model was constrained to the structure of Friedman's theoretical model (equation (5) in the main text). Representative values for the parameters in this model have been estimated using the standard quasilinearization method (e.g., Refs. 20 or 29). The state-vector equations used in the parameter identification were

$$\begin{bmatrix} \ddot{z}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -\hat{a} & -\hat{b} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} m(t) \; ; \qquad \begin{bmatrix} \hat{\dot{z}}_{ic} \\ \hat{z}_{ic} \end{bmatrix}$$

$$\hat{x}(t) = \begin{bmatrix} \hat{c} & 0 \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ z(t) \end{bmatrix} + \hat{x}_o$$

The unknown parameters include the model terms, $\hat{a}$, $\hat{b}$, and $\hat{c}$, along with the long-term constant $\hat{x}_o$ and the two initial conditions $\hat{\dot{z}}_{ic}$ and $\hat{z}_{ic}$. The input to the model is m(t), the dynamic variable is z(t), the estimated output is $\hat{x}(t)$, and the measured output is x(t). Using the quasilinearization method, values for the six unknown parameters were determined such as to minimize $[x(t) - \hat{x}(t)]^2$ over the time period from 1950 through 1981.

The economy is a continuous system, but the measurements (e.g., monthly or quarterly) represent averages over a given time span. To look at this problem (and the problem of measurement noise) a variety of data handling procedures were applied with the economic data. The results from all of these different types of data runs were near the following round-off values

$$c/b = K_x = 2 \;, \qquad (b)^{1/2} = \omega_n = 1.5 \;, \qquad a/[2(b)^{1/2}] = \zeta = 0.8$$

APPENDIX B

BUSINESS CYCLES IN THE UNITED STATES

The business cycles defined by the National Bureau of Economic Research are presented in table B1. To correlate this set of cycles with frequency domain methods, we can compute a term, $\omega' = 2\pi/\text{PERIOD}$, for each of the cycles, where PERIOD is defined as the peak-to-peak duration of each cycle (in years). A histogram of the 28 values of ω', within increments $\Delta\omega' = 0.5$ rad/yr, is presented in figure 3 in the main text.

TABLE B1.- MEASURED BUSINESS CYCLES IN THE UNITED STATES

Business cycle reference dates		Duration in months	
Trough	Peak	Trough from previous trough	Peak from previous peak
December 1854	June 1857	--	--
December 1858	October 1860	48	40
June 1861	April 1865	30	54
December 1867	June 1869	78	50
December 1870	October 1873	36	52
March 1879	March 1882	99	101
May 1885	March 1887	74	60
April 1888	July 1890	35	40
May 1891	January 1893	37	30
June 1894	December 1895	37	35
June 1897	June 1899	36	42
December 1900	September 1902	42	39
August 1904	May 1907	44	56
June 1908	January 1910	46	32
January 1912	January 1913	43	36
December 1914	August 1918	35	67
March 1919	January 1920	51	17
July 1921	May 1923	28	40
July 1924	October 1926	36	41
November 1927	August 1929	40	34
March 1933	May 1937	64	93
June 1938	February 1945	63	93
October 1945	November 1948	88	45
October 1949	July 1953	48	56
May 1954	August 1957	55	49
April 1958	April 1960	47	32
February 1961	December 1969	34	116
November 1970	November 1973	117	47
March 1975	January 1980	52	74
July 1980		64	--
Averages		52	53

Source: National Bureau of Economic Research, Inc.

APPENDIX C

CLOSED-FORM SOLUTIONS

This appendix considers some closed-form solutions that serve to interrelate model parameters, in the main text, with economic trend relationships. The gain in Friedman's model (K_x) is first linked with acceleration theory and then combined with the gain in Okun's law (K_u) to derive the slope of the Phillips curve (K_xK_u).

A steady-state solution to Friedman's model equation (5) can be written as

$$\Delta x = K_x sm = K_x \dot{m} \tag{C1}$$

which is consistent with the concept of acceleration theory (ref. 30). According to this theory, an increasing (or decreasing) trend in the growth of money supply produces real output effects, and along with this trend in money supply there is an equivalent trend in the growth of prices:

$$\dot{p} = \dot{m} \tag{C2}$$

Combining equation (C1) with Okun's law equation (7) we have

$$\dot{u} = -K_u(K_x\dot{m} + x_o - x_u) \tag{C3}$$

Equations (C2) and (C3) allow us to examine the trend in unemployment rate compared with the growth of prices. Such trend lines, or slopes, can be computed as a function of acceleration in money supply ($\dot{m}$).

There is a special-case solution from equations (C2) and (C3) when either

$$(x_o - x_u) << K_x\dot{m} \quad \text{or} \quad (x_o - x_u) \approx 0$$

In either of these special cases we obtain from equations (C2) and (C3) the following

$$\dot{u}/\dot{p} = -K_xK_u \tag{C4}$$

This is a linear representation of the Phillips curve slope, relating unemployment rate and the growth of prices, based on the well-known study by Phillips (ref. 14).

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16. Abstract

Classical linear-control analysis provides a framework for studying dynamic systems involving random disturbances. This framework is used to develop a set of equations that, in historical perspective, combine traditional concepts about the dynamics of economic systems and about the effects of random economic disturbances. This set of equations provides relationships among well-known ideas in general macroeconomics and provides a means to interrelate and examine ideas about stabilization policies. In this study, linear-control analysis is applied as an aid in understanding the fluctuations of business cycles in the past, and to examine monetary policies that might improve stabilization. The analysis shows how different policies change the frequency and damping of the economic system dynamics, and how they modify the amplitude of the fluctuations that are caused by random disturbances. Examples are used to show how policy feedbacks and policy lags can be incorporated, and how different monetary strategies for stabilization can be analytically compared. Representative numerical results are used to illustrate the main points.

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