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UNIAXIAL AERODYNAMIC ATTITUDE CONTROL OF ARTIFICIAL SATELLITES

V. V. Sazonov

Translation of "Odnoosnaya Aerodinamicheskaya Oriyentatsiya Iskusstvennykh Sputnikov", Academy of Sciences USSR, Institute of Applied Mathematics imeni M.V. Keldysh, Moscow, Preprint No. 151, 1982, pp. 1-27.

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16. Abstract

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Annotation

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1. Introduction

The aerodynamic moment can be used on circular and slightly elliptical orbits in the altitude range of as much as 500 km for attitude control of the longitudinal axis of an artificial earth satellite in regard to the oncoming air stream, the velocity of which is practically directed along the orbital tangent. For this purpose, the shape of the exterior shell of the satellite is chosen so that, when the normal attitude control is disturbed, pitch and yaw restorative moments are created that strive to line up the longitudinal satellite axis with the velocity vector of the oncoming current. Since the projection of the aerodynamic moment on the longitudinal satellite axis is equal to zero,[1] in itself the aerodynamic moment can only provide a uniaxial attitude control of the satellite. /3*

This attitude control principle was used aboard the Soviet satellites Kosmos-149 and Kosmos-320 [2-4]. However the attitude control system of these satellites was not a purely aero-

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[1] For certain configurations of the outer satellite shell the projection of the aerodynamic moment on the longitudinal axis is different from zero. This projection may result in a twisting of the satellite (propeller effect [1]) and somersaulting. Therefore the shape of the outer shell of a satellite that functions under conditions of aerodynamic attitude control should be chosen so that this projection is identically equal to zero.

*Numbers in the margin indicate pagination in the foreign text.

dynamic one. In order to create a restorative moment about the longitudinal axis, two-degree gyroscopes were installed aboard these satellites, also serving as absorbers of the perturbing motion. By using the gyroscopes and the gravitational moment it was possible to obtain a triaxial attitude control of these satellites. But if it is enough for the satellite to have a uniaxial attitude control in terms of the orbital tangent to accomplish its flight mission, the principle of aerodynamic attitude control in the pure form can be used. The present work investigates the possibilities of such a method of attitude control.

The satellite discussed below is regarded as a solid with outer shell in the form of a sphere. The center of the sphere does not coincide with the center of masses of the satellite and lies on one of its main central axes of inertia. Such a satellite may serve as a model of that shown in Fig. 1. Here the instrument package has small geometrical dimensions, and we can ignore the influence of the atmosphere on it. The sphere acts as an aerodynamic stabilizer and has a mass much less than that of the overall satellite. /4

The equations of motion of the satellite take account of the gravitational and restorative aerodynamic moments. It is presumed that the aerodynamic moment is much larger than the gravitational, and the motion equations contain a large parameter. A two-parameter integrated surface of these equations is constructed in the form of formal series in terms of negative powers of the large parameter, describing the oscillations and rotations of the satellite about its lengthwise axis, approximately oriented along the orbital tangent. It is proposed to treat such movements as nominal undisturbed motions of the satellite under conditions of aerodynamic attitude control. A numberical investigation is made for the above integrated surface.

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2. The Motion Equations

Let us consider a satellite (solid), the center of masses of which is moving in a circular orbit about the Earth. To write the equations of motion of the satellite relative to the center of masses we shall introduce two clockwise Cartesian systems of coordinates.

Ox_1, x_2, x_3 - a coordinate system rigidly bound to the satellite. The point O is the center of masses of the satellite. The axes Ox_1, Ox_2, Ox_3 are the major central axes of inertia of the satellite.

$OX_1X_2X_3$ - the orbital system of coordinates. The axis OX_3 is directed along the radius vector of the point O with respect to the center of Earth, the axis OX_1 is directed along the orbital tangent in the direction of movement of the satellite.

We shall assign the position of the coordinate system $Ox_1x_2x_3$ relative to the system $OX_1X_2X_3$ by using the angles α, β, γ (Fig. 2). The matrix for conversion from one of these systems to the other has the appearance:

	x_1	x_2	x_3
X_1	a_{11}	a_{12}	a_{13}
X_2	a_{21}	a_{22}	a_{23}
X_3	a_{31}	a_{32}	a_{33}

$$a_{11} = \cos\alpha\cos\beta,$$
$$a_{21} = \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma,$$
$$a_{31} = \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma,$$

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$$a_{12} \quad \sin\beta, \qquad a_{13} = \sin\alpha\cos\beta,$$
$$a_{22} = \cos\beta\cos\gamma, \qquad a_{23} \quad \sin\alpha\sin\beta\cos\gamma \quad \cos\alpha\sin\gamma,$$
$$a_{32} = \cos\beta\sin\gamma, \qquad a_{33} = \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma.$$

We introduce the definitions: $(\omega_1, \omega_2, \omega_3)$ - the absolute angular velocity of the satellite (hereafter the components of

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the vectors are given in the coordinate system $Ox_1x_2x_3$); /5
A,B,C - the moments of inertia of the satellite with respect to the axes Ox_1, Ox_2, Ox_3; (M_1, M_2, M_3) - the major moment of the external forces acting on the satellite; ω_0 = const > 0 - is the angular velocity of orbital motion; t is the time. The movement of the satellite relative to the center of masses is described by equations:

$$\begin{aligned}
&A\frac{d\omega_1}{dt} + (C-B)\omega_2\omega_3 = M_1,\\
&B\frac{d\omega_2}{dt} + (A-C)\omega_1\omega_3 = M_2,\\
&C\frac{d\omega_3}{dt} + (B-A)\omega_1\omega_2 = M_3,\\
&\frac{d\gamma}{dt} = \frac{1}{\cos\beta}(\omega_1\cos\alpha + \omega_3\sin\alpha) - \omega_0\, tg\beta\cos\gamma,\\
&\frac{d\alpha}{dt} = \omega_2 + tg\beta(\omega_1\cos\alpha + \omega_3\sin\alpha) - \omega_0\frac{\cos\gamma}{\cos\beta},\\
&\frac{d\beta}{dt} = -\omega_1\sin\alpha + \omega_3\cos\alpha + \omega_0\sin\gamma.
\end{aligned} \tag{1}$$

Of the external moments acting on the satellite we shall only consider the gravitational and restorative aerodynamic moments. The components of the gravitational moment have the form:

$$M_{g1} = 3\omega_0^2(C-B)a_{32}a_{33},\ M_{g2} = 3\omega_0^2(A-C)a_{31}a_{33},\ M_{g3} = 3\omega_0^2(B-A)a_{31}a_{32}.$$

In calculating the restorative aerodynamic moment we shall consider that the outer shell of the satellite has the shape of a sphere of radius R with center at point:

$$(d, 0, 0).$$

Here d < 0. With respect to the interaction between the satellite and the atmosphere we assume the following: 1) the atmosphere is immobile in absolute space; 2) the action of the atmosphere on the satellite reduces to a resistance force applied at the center of pressure and directed opposite the velocity of the center of masses of the satellite; 3) the resistance of the atmosphere does not affect the evolution of the orbit. Under these assumptions the components of the aerodynamic moment are: /6

$$M_{a1} = 0,\quad M_{a2} = -\frac{1}{2} c_x \rho V^2 \pi R^2 d\, a_{13},\quad M_{a3} = -\frac{1}{2} c_x \rho V^2 \pi R^2 d\, a_{12}.$$

Here c_x is the coefficient of aerodynamic resistance, ρ is the density of the oncoming air stream, V is the velocity of the center of masses of the satellite.

Until now the orbit of the center of masses of the satellite was taken as circular. Such an assumption is perfectly justified when the eccentricity is ≤ 0.01, although the calculation of the density of the oncoming air stream should allow for even a small orbital ellipticity. We shall assume that this density depends on the height of the satellite h above the surface of the Earth as:

$$\rho = \rho_\pi \exp\left(\frac{h_\pi - h}{H}\right). \tag{2}$$

where ρ_π is the density of the atmosphere at perigee, h_π is the height of perigee, H is the height of the homogeneous atmosphere. For small orbital eccentricity formula (2) can be represented as:

$$\rho = \rho_\pi \exp[\eta(1 - \cos\omega_0(t - t_0))].$$

Here:

$$\eta = \frac{1}{2} \ln(\rho_\alpha / \rho_\pi).$$

ρ_α is the density of the atmosphere at apogee, t_0 is the time it takes for the satellite to pass through perigee.

Let us consider equations (1) when:

$$M_i = M_{gi} + M_{ai} \quad (i = 1,2,3).$$

Using the nondimensional quantities: **ORIGINAL PAGE IS OF POOR QUALITY**

$$\lambda = \frac{A}{C}, \quad \mu = \frac{B-C}{A}, \quad æ = -\frac{c_x \rho_\pi V^2 \pi R^2 d}{2C\omega_0^2},$$
$$\Omega_i = \frac{\omega_i}{\omega_0} \ (i = 1,2,3), \quad \tau = \omega_0 (t - t_0)$$

these equations can be written as:

$$\begin{aligned}
\dot{\Omega}_1 &= \mu(\Omega_2\Omega_3 - 3a_{32}a_{33}), \\
\dot{\Omega}_2 &= \frac{1-\lambda}{1+\lambda\mu}(\Omega_1\Omega_3 - 3a_{31}a_{33}) - \frac{æ\varphi(\tau,\eta)a_{13}}{1+\lambda\mu}, \\
\dot{\Omega}_3 &= -(1-\lambda+\lambda\mu)(\Omega_1\Omega_2 - 3a_{31}a_{32}) + æ\varphi(\tau,\eta)a_{12}, \\
\dot{\gamma} &= \frac{1}{\cos\beta}(\Omega_1\cos\alpha + \Omega_3\sin\alpha) - tg\beta\cos\gamma, \\
\dot{\alpha} &= \Omega_2 + tg\beta(\Omega_1\cos\alpha + \Omega_3\sin\alpha) - \frac{\cos\gamma}{\cos\beta}, \\
\dot{\beta} &= -\Omega_1\sin\alpha + \Omega_3\cos\alpha + \sin\gamma.
\end{aligned} \tag{3}$$

Here the dot indicates a differentiation with respect to τ,

$$\varphi(\tau,\eta) = exp[\eta(1-\cos\tau)].$$

Equations (3) are simplifications. For example, they do not allow for so important a fact as the difference in atmospheric density values at the sunlit orbital segment and that in the Earth's shadow. Nevertheless these equations can reveal the main features of the dynamics of uniaxial aerodynamic attitude control. /7

We shall point out several properties of equations (3). To shorten the notation we shall use the vector conventions. We introduce the vector:

$$z = (\Omega_1, \Omega_2, \Omega_3, \gamma, \alpha, \beta)^T$$

and define the function:

$$F(z,\tau,\eta) \in R^6$$

such that set (3) can be written as:

$$\dot{z} = F(z, \tau, \eta). \tag{3'}$$

Set (3') has the property (E) [5] with respect to the matrices:

$$S = diag(1,1,-1,-1,-1,1), \quad S' = diag(-1,1,1,1,-1,-1),$$

i.e. the function $F(z,\tau,\eta)$ satisfies relations:

$$SF(Sz,-\tau,\eta) = -F(z,\tau,\eta), \quad S'F(S'z,-\tau,\eta) = -F(z,\tau,\eta). \tag{4}$$

Moreover:

$$F(z,\tau+2\pi,\eta) = F(z,\tau,\eta), \quad F(z+\pi e_4,\tau,\eta) = S''F(S''z,\tau,\eta), \tag{5}$$

where:

$$e_4 = (0,0,0,1,0,0)^T, \quad S'' = diag(1,-1,-1,1,-1,-1).$$

When $\eta = 0$, set (3') is autonomous: $\partial F(z,\tau,0)/\partial\tau \equiv 0$.

Set (3) admits of stationary solutions:

$$\Omega_1 = 0; \ \Omega_2 = \cos\gamma_0; \ \Omega_3 = -\sin\gamma_0; \ \alpha = \beta = 0; \ \gamma_0 = 0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2}, \tag{6}$$

that describe the position of equilibrium of the satellite in the orbital coordinate system. In all these positions of equilibrium the axis Ox_1 coincides with OX_1, the axis Ox_2 (Ox_3) coincides with the axis OX_2 or OX_3.

3. The Integral Surface

We shall call the movement of the satellite in which the angle θ between axes Ox_1 and OX_1 ($\theta = \arccos(\cos\alpha\cos\beta)$) does not exceed a certain value Δ the regime of uniaxial aerodynamic attitude control. For example, we may take Δ = 15°. Depending on the values of the satellite parameters, different approaches are possible for realization of the regime of aerodynamic attitude control and its investigation. In particular, we may consider one of the equilibrium positions (6) to be the nominal undisturbed motion of the satellite in this regime. In this case analysis of the regime of aerodynamic attitude control will primarily consist in investigating the stability of the chosen position of equilibrium. Such an approach is natural when the aerodynamic and gravitational moments acting on the satellite are comparable in magnitude $(\varkappa \sim 1)$. But if the aerodynamic moment preponderates $(\varkappa \gg 1)$, it is advisable to conduct the investigation of the regime of aerodynamic attitude control in different fashion. We discuss below one of the possible methods of such an investigation. The analysis will be based on the infinitesimal $\varepsilon = 1/\varkappa$. /8

We shall write set (3) as:

$$
\begin{aligned}
&\dot{\Omega}_1 = \mu(\Omega_2\Omega_3 - 3a_{12}a_{13}),\\
&\varepsilon(1+\lambda\mu)\dot{\Omega}_2 = \varepsilon(1-\lambda)(\Omega_1\Omega_3 - 3a_{11}a_{13}) - \varphi(\tau,\eta)a_{13},\\
&\varepsilon\dot{\Omega}_3 = \varepsilon(1-\lambda+\lambda\mu)(\Omega_1\Omega_2 - 3a_{11}a_{12}) + \varphi(\tau,\eta)a_{12},\\
&\dot{\gamma} = \frac{1}{\cos\beta}(\Omega_2\cos\alpha + \Omega_3\sin\alpha) - tg\beta\cos\gamma,\\
&\dot{\alpha} = \Omega_2 + tg\beta(\Omega_1\cos\alpha + \Omega_3\sin\alpha) - \frac{\cos\gamma}{\cos\beta},\\
&\dot{\beta} = -\Omega_1\sin\alpha + \Omega_3\cos\alpha + \sin\gamma.
\end{aligned}
\tag{7}
$$

When ε = 0 this set has solutions for which:

$$
\Omega_2 = \cos\gamma,\ \Omega_3 = -\sin\gamma,\ \alpha = \beta = 0,
\tag{8}
$$

while the variables γ and Ω_1 are determined by equations:

$$
\dot{\gamma} = \Omega_1,\quad \dot{\Omega}_1 = -4\mu\sin\gamma\cos\gamma.
\tag{9}
$$

Fig. 3 shows the phase picture of the set (9) in the case of $\mu = 0.2$. The stationary solutions (6) of set (7) at $\varepsilon = 0$ correspond to the stationary solutions of this set:

$$\gamma_0 = 0, \pi/2, \pi, 3\pi/2; \Omega_1 = 0$$

Motions of type (8) and (9) can be used as nominal undisturbed motions of the satellite in the regime of uniaxial aerodynamic attitude control when the aerodynamic moment acting on the satellite is infinitely large. In these motions:

$$\alpha = \beta = 0 ,$$

i.e. the axis Ox_1 is directed exactly along the orbital tangent. When $\varepsilon \neq 0$, set (7) does not have solutions (8), (9), but when $\varepsilon \ll 1$ it is possible to construct its integral surface, similar to the family of such solutions, in the form of formal power series. The motions that belong to this integral surface shall be regarded as nominal undisturbed movements of the satellite in the regime of uniaxial aerodynamic attitude control. /9

We shall seek the integral surface of the set (7) that merges into the family of solutions (8), (9) when $\varepsilon = 0$ in the form:

$$\begin{aligned}
\Omega_2 &= \Omega_2^*(\gamma,\Omega_1,\tau,\eta,\varepsilon) = \cos\gamma + \varepsilon\Omega_{21}(\gamma,\Omega_1,\tau,\eta) + \\
&\qquad + \varepsilon^2\Omega_{22}(\gamma,\Omega_1,\tau,\eta) + \cdots , \\
\Omega_3 &= \Omega_3^*(\gamma,\Omega_1,\tau,\eta,\varepsilon) = \sin\gamma + \varepsilon\Omega_{31}(\gamma,\Omega_1,\tau,\eta) + \\
&\qquad + \varepsilon^2\Omega_{32}(\gamma,\Omega_1,\tau,\eta) + \cdots , \\
\alpha &= \alpha^*(\gamma,\Omega_1,\tau,\eta,\varepsilon) = \varepsilon\alpha_1(\gamma,\Omega_1,\tau,\eta) + \varepsilon^2\alpha_2(\gamma,\Omega_1,\tau,\eta) + \cdots , \\
\beta &= \beta^*(\gamma,\Omega_1,\tau,\eta,\varepsilon) = \varepsilon\beta_1(\gamma,\Omega_1,\tau,\eta) + \varepsilon^2\beta_2(\gamma,\Omega_1,\tau,\eta) + \cdots ,
\end{aligned} \tag{10}$$

where:

$$\begin{aligned}
\dot{\gamma} &= P(\gamma,\Omega_1,\tau,\eta,\varepsilon) = \Omega_1 + \varepsilon P_1(\gamma,\Omega_1,\tau,\eta) + \varepsilon^2 P_2(\gamma,\Omega_1,\tau,\eta) + \cdots , \\
\dot{\Omega}_1 &= Q(\gamma,\Omega_1,\tau,\eta,\varepsilon) = 4\mu\sin\gamma\cos\gamma + \varepsilon Q_1(\gamma,\Omega_1,\tau,\eta) + \cdots ,
\end{aligned} \tag{11}$$

We shall regard the above series as formal, i.e. we are not concerned about their convergence.

Inserting series (10), (11) into set (7) and equating the expressions for identical powers of ε on the left and right, we obtain a sequence of recurrent relations:

$$\alpha_k = g_{1k},$$
$$\beta_k = g_{2k},$$
$$\Omega_{2k} = -\beta_k \Omega_1 + \frac{\partial \alpha_k}{\partial \gamma}\Omega_1 - \frac{\partial \alpha_k}{\partial \Omega_1} 4\mu \sin\gamma\cos\gamma + \frac{\partial \alpha_k}{\partial \tau} + g_{3k},$$
$$\Omega_{1k} = \alpha_k \Omega_1 + \frac{\partial \beta_k}{\partial \gamma}\Omega_1 - \frac{\partial \beta_k}{\partial \Omega_1} 4\mu \sin\gamma\cos\gamma + \frac{\partial \beta_k}{\partial \tau} + g_{4k},$$
$$P_k = -(\alpha_k \sin\gamma + \beta_k \cos\gamma) + g_{5k},$$
$$Q_k = \mu(\Omega_{1k}\cos\gamma - \Omega_{2k}\sin\gamma) + g_{6k}$$
$$(k = 1, 2, \ldots).$$

Here g_{jk} ($j = 1,\ldots,6$) are known functions that depend on:

$$\tau,\ \gamma,\ \Omega_1,\ \alpha_\ell,\ \partial\alpha_\ell/\partial\gamma,\ \partial\alpha_\ell/\partial\Omega_1,\ \partial\alpha_\ell/\partial\tau,\ \ldots,\ \Omega_{i\ell},$$
$$\partial\Omega_{i\ell}/\partial\gamma,\ \partial\Omega_{i\ell}/\partial\Omega_1,\ \partial\Omega_{i\ell}/\partial\tau,\ P_\ell,\ Q_\ell$$

when $\ell = 1, 2, \ldots, k-1$.

It is easy to see that this sequence uniquely defines the unknown coefficients of series (10), (11). The coefficients of ε in these series have the form:

$$\alpha_1 = \frac{\lambda(1+\mu)\Omega_1 \sin\gamma}{\psi},\quad \beta_1 = \frac{\lambda(1-\mu)\Omega_1 \cos\gamma}{\psi},$$
$$\Omega_{21} = \frac{2\lambda\mu\Omega_1\cos\gamma}{\psi} - \frac{\lambda(1+\mu)\sin\gamma}{\psi^2}(\dot\psi\Omega_1 + 4\mu\psi\sin\gamma\cos\gamma),$$
$$\Omega_{11} = \frac{2\lambda\mu\Omega_1\sin\gamma}{\psi} - \frac{\lambda(1-\mu)\cos\gamma}{\psi^2}(\dot\psi\Omega_1 + 4\mu\psi\sin\gamma\cos\gamma),$$
$$P_1 = -\alpha_1\sin\gamma - \beta_1\cos\gamma,$$
$$Q_1 = \mu(\Omega_{11}\cos\gamma - \Omega_{21}\sin\gamma)$$

Let us point out several properties of the integral surface (10), (11). To shorten the notation we introduce the vector function:

$$\bar{\varphi}(\gamma,\Omega_1,\tau,\eta,\varepsilon) = (\Omega_1, \Omega_2^*(\gamma,\Omega_1,\tau,\eta,\varepsilon), \Omega_3^*(\gamma,\Omega_1,\tau,\eta,\varepsilon), \gamma, \alpha^*(\gamma,\Omega_1,\tau,\eta,\varepsilon), \beta^*(\gamma,\Omega_1,\tau,\eta,\varepsilon))^T.$$

Using relations (4) we can prove that:

$$\begin{aligned} S\bar{\varphi}(-\gamma,\Omega_1,-\tau,\eta,\varepsilon) &= \bar{\varphi}(\gamma,\Omega_1,\tau,\eta,\varepsilon), \\ P(-\gamma,\Omega_1,-\tau,\eta,\varepsilon) &= P(\gamma,\Omega_1,\tau,\eta,\varepsilon), \\ Q(-\gamma,\Omega_1,-\tau,\eta,\varepsilon) &= -Q(\gamma,\Omega_1,\tau,\eta,\varepsilon); \end{aligned} \tag{12}$$

$$\begin{aligned} S'\bar{\varphi}(\gamma,-\Omega_1,-\tau,\eta,\varepsilon) &= \bar{\varphi}(\gamma,\Omega_1,\tau,\eta,\varepsilon), \\ P(\gamma,-\Omega_1,-\tau,\eta,\varepsilon) &= -P(\gamma,\Omega_1,\tau,\eta,\varepsilon), \\ Q(\gamma,-\Omega_1,-\tau,\eta,\varepsilon) &= Q(\gamma,\Omega_1,\tau,\eta,\varepsilon). \end{aligned} \tag{13}$$

By virtue of relations (5) we have:

$$\begin{aligned} \bar{\varphi}(\gamma,\Omega_1,\tau+2\pi,\eta,\varepsilon) &= \bar{\varphi}(\gamma,\Omega_1,\tau,\eta,\varepsilon), \\ P(\gamma,\Omega_1,\tau+2\pi,\eta,\varepsilon) &= P(\gamma,\Omega_1,\tau,\eta,\varepsilon), \\ Q(\gamma,\Omega_1,\tau+2\pi,\eta,\varepsilon) &= Q(\gamma,\Omega_1,\tau,\eta,\varepsilon); \end{aligned} \tag{14}$$

$$\begin{aligned} \bar{\varphi}(\gamma+\pi,\Omega_1,\tau,\eta,\varepsilon) &= S''\bar{\varphi}(\gamma,\Omega_1,\tau,\eta,\varepsilon) + \pi e_4, \\ P(\gamma+\pi,\Omega_1,\tau,\eta,\varepsilon) &= P(\gamma,\Omega_1,\tau,\eta,\varepsilon), \\ Q(\gamma+\pi,\Omega_1,\tau,\eta,\varepsilon) &= Q(\gamma,\Omega_1,\tau,\eta,\varepsilon). \end{aligned} \tag{15}$$

We can show that when $\gamma_0 = 0, \pi/2, \pi, 3\pi/2$:

$$\begin{aligned} \bar{\varphi}(\gamma_0,0,\tau,\eta,\varepsilon) &= (0, \cos\gamma_0, -\sin\gamma_0, \gamma_0, 0, 0)^T, \\ P(\gamma_0,0,\tau,\eta,\varepsilon) &= Q(\gamma_0,0,\tau,\eta,\varepsilon) \equiv 0, \end{aligned}$$

i.e. the stationary solutions of (6) lie on the constructed integral surface.

When $\eta = 0$, series (10), (11) do not contain τ, and the integral surface specified by these series can be numerically investigated rather easily. Let us consider the case $\eta = 0$ in more detail. With precision down to terms of order $O(\varepsilon)$ we can write set (11) in form (9). This set is equivalent to the equation of mathematical pendulum:

$$2\ddot{\gamma} + 4\mu \sin 2\gamma = 0$$

and is integrated in elliptical functions. Let us consider the periodic oscillatory and rotational[1] solutions of set (11) ($\eta = 0$), similar to the periodic oscillatory and rotation solutions of set (9). Without restricting the generality we shall consider that $\mu > 0$ and $\gamma(0) = 0$.

Let us first deal with the oscillatory periodic solutions. It is easy to prove that the T-periodic solution of set (9) $\gamma(\tau)$, $\Omega_1(\tau)$, for which $\gamma(0) = 0$, satisfies relations:

$$\begin{aligned} &\gamma(-\tau) = -\gamma(\tau), \quad \gamma\left(\tau + \tfrac{T}{2}\right) = -\gamma(\tau), \\ &\Omega_1(-\tau) = \Omega_1(\tau), \quad \Omega_1\left(\tau + \tfrac{T}{2}\right) = -\Omega_1(\tau). \end{aligned} \tag{16}$$

by virtue of the last two equations, $\Omega_1(T/4) = 0$. For the set (11) ($\eta = 0$) let us examine the boundary-value problem:

$$\gamma(0) = \Omega_1\left(\tfrac{T}{4}\right) = 0 \qquad (T > 0). \tag{17}$$

Using the properties (12), (13) of functions P and Q, we can prove [5,6] that any given solution $\gamma(\tau)$, $\Omega_1(\tau)$ of this problem satisfies relations (16) and, consequently, is T-periodic. For the corresponding solution of set (3'): /12

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[1] In the case of $\eta = 0$ we shall call the solution $\gamma(\tau), \Omega_1(\tau)\,(z(\tau))$ of the set (11) ((3')) the rotational periodic solution if there exists a number $T \gtrless 0$ (period) such that: $\gamma(\tau + T) = \gamma(\tau) + 2\pi$,

$$\Omega_1(\tau + T) = \Omega_1(\tau)\,(z(\tau + T) = z(\tau) + 2\pi e_1).$$

The T values of different sign correspond to rotations in different directions.

$$(\eta=0)\quad z(\tau)=\bar{\varphi}[\gamma(\tau),\Omega_1(\tau),\tau,0,\varepsilon]$$

the equations are valid (cf. (12), (13)):

$$Sz(-\tau)=z(\tau),\ S'z(-\tau+\tfrac{T}{4})=z(\tau+\tfrac{T}{4}), \tag{18}$$

by virtue of which:

$$z(\tau+\tfrac{T}{2})=SS'z(\tau).$$

It follows from this that the second and fifth components of the vector $z(\tau)$ (corresponding to variables Ω_2 and α) are T/2-periodic functions of τ, while the remaining components are T/2-antiperiodic functions. Setting $\tau = 0$ in (18), we find the boundary conditions:

$$Sz(0)=z(0),\ S'z(\tfrac{T}{4})=z(\tfrac{T}{4}), \tag{19}$$

which are satisfied by the solution of $z(\tau)$. The scalar form of these conditions is:

$$\Omega_3(0)=\gamma(0)=\alpha(0)=\Omega_1(\tfrac{T}{4})=\alpha(\tfrac{T}{4})=\beta(\tfrac{T}{4})=0.$$

It can be shown [5,6] that in the case $\eta = 0$ for any given solution of $z(\tau)$ of the boundary-value problem (3'), (19) relations (18) are valid and, consequently, this solution is T-periodic.

Let us turn to the rotational periodic solutions. Any given rotational T-periodic solution of set (9) $\gamma(\tau)$, $\Omega_1(\tau)$, for which $\gamma(0) = 0$, satisfies the relations:

$$\begin{gathered}\gamma(-\tau)=-\gamma(\tau),\ \gamma(\tau+\tfrac{T}{2})=\gamma(\tau)+\pi,\\ \Omega_1(-\tau)=\Omega_1(\tau),\ \Omega_1(\tau+\tfrac{T}{2})=\Omega_1(\tau).\end{gathered} \tag{20}$$

By virtue of the first two of these relations we have $\gamma(T/4) = \pi/2$. For the set (11) ($\eta = 0$) let us consider the boundary-value problem:

$$\gamma(0) = \gamma\left(\frac{T}{4}\right) - \frac{\pi}{2} = 0 \quad (T \geq 0). \tag{21}$$

It can be proved that any given solution $\gamma(\tau)$, $\Omega_1(\tau)$ of this problem satisfies conditions (20) and thus is T-periodic. For the corresponding solution of set (3') ($\eta = 0$):

$$z(\tau) = \tilde{\varphi}[\gamma(\tau), \Omega_1(\tau), \tau, 0, \varepsilon]$$

the equations are valid (cf. (12), (15)):

$$Sz(-\tau) = z(\tau), \quad z(\tau + T/2) = S''z(\tau) + \pi e_4, \tag{22}$$

by virtue of which:

$$z\left(-\tau + \frac{T}{4}\right) = SS''z\left(\tau + \frac{T}{4}\right) + \pi e_4.$$

Setting $\tau = 0$ we find the boundary conditions: /13

$$Sz(0) = z(0), \quad SS''z\left(\frac{T}{4}\right) = z\left(\frac{T}{4}\right) - \pi e_4, \tag{23}$$

which are satisfied by the solution $z(\tau)$. The scalar forms of these conditions is:

$$\Omega_3(0) = \gamma(0) = \alpha(0) = \Omega_2\left(\frac{T}{4}\right) = \gamma\left(\frac{T}{4}\right) - \frac{\pi}{2} = \beta\left(\frac{T}{4}\right) = 0.$$

It can be proved that, when $\eta = 0$, any given solution $z(\tau)$ of the boundary problem (3'), (23) satisfies relations (22) and, consequently, is a rotational T-periodic solution.

Numerically solving the boundary problems (19), (23) for the set (3') in the case $\eta = 0$, we can construct periodic solutions in explicit form that lie on the integral surface (10), (11). If there exists a family of such solutions for T values that belong to a certain interval $T_1 < T < T_2$, this family will form a subset of the investigated integral surface that wholly consists of periodic solutions. The subset has dimensions of

2, and its parameters may be the period and phase (in the solutions of boundary problems (19) and (23) the phase is fixed, but due to the autonomy of set (3') when $\eta = 0$ this parameter can be chosen at will).

4. Numerical Investigation of the Integral Surface (10), (11) When $\eta = 0$

A numerical construction of the solutions of the boundary problem (3'), (19) in the case $\eta = 0$ was done as follows. A certain value of the period T was assigned and for this value the problem (19) was solved by the zeroing method. As the first approximation of the unknown initial conditions $\Omega_1(0)$, $\Omega_2(0)$, $\beta(0)$ we used the values:

$$\Omega_1(0) = 2k\sqrt{\mu},\ \Omega_2(0) = 1,\ \beta(0) = 0,$$

where k is the root of the equation:

$$K(k) = \frac{T}{2}\sqrt{\mu}. \tag{24}$$

K(k) is a complete elliptical integral of the first kind. Such values of $\Omega_1(0)$, $\Omega_2(0)$, $\beta(0)$ are obtained by converting from the initial conditions of the solution of boundary problem (9), (17) to variables Ω_2 and β as per formulas (8). We observe that equation (24) has real roots only when $T > \pi/\sqrt{\mu}$. /14

For $\lambda = 0.25$, $\mu = 0.2$, $\kappa = 10.30$, Fig. 4 and 5 show the dependence on T of the initial conditions $\Omega_1(0)$, $\Omega_2(0)$, $\beta(0)$ of the solutions of boundary-value problem (19) (for $\kappa = 30$ Fig. 4 shows only $\Omega_1(0)$ as a function of T). By virtue of the second relation (4), the curves obtained from the curves in Fig. 4 and 5 by the transformation:

$$\Omega_1(0) \to -\Omega_1(0),\ \Omega_2(0) \to \Omega_2(0),\ \beta(0) \to -\beta(0),\ T \to T\ ,$$

will also specify the initial conditions for the solutions of boundary problem (19). The curves shown in Fig. 4 and 5 consist of separate pieces. In the scale of the figures the breaks between certain pieces are unremarkable and for clarity we show them by circles. The Jacobian:

$$J = \frac{\partial[\Omega_1(\frac{T}{4}), \alpha(\frac{T}{4}), \beta(\frac{T}{4})]}{\partial[\Omega_1(0), \Omega_2(0), \beta(0)]}, \tag{25}$$

calculated for solutions belonging to an identical piece, has an identical sign. The sign of J changes in moving to an adjacent piece.

Fig. 7 and 8 show graphs of the functions:

$$\gamma(\tau), \alpha(\tau), \beta(\tau)\ (0 \leq \tau \leq T)$$

for several of the calculated T-periodic solutions of set (3). An analysis of these figures reveals that the breaks of the curves in Fig. 4 and 5 are due to resonances between the slow (with frequency $\sim h$) and fast (with frequency $\sim\sqrt{\kappa}$) oscillations of the satellite. The presence of such resonances suggests a divergence of the series (10) and (11). This situation is typical for periodic solutions of sets of differential equations with a large parameter [7,8] [1].

In order to construct the rotational periodic solutions that lie on the integral surface (10), (11) when $\eta = 0$, the boundary-value problem (3'), (23) was solved. The solution was done by the method of zeroing. As a first approximation of the

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[1] In [7] an independent set of differential equations is examined with small positive parameter μ that regularly figures in it, and periodic solutions of this set with a period of $\sim 1/\sqrt{\mu}$ were sought. If, in the set of [7], we replace the independent variable $\tau \rightarrow \sqrt{\mu}$ and make another series of manipulations, we can obtain a set with the large parameter $1/\sqrt{\mu}$, similar to the set (3).

unknown starting conditions $\Omega_1(0)$, $\Omega_2(0)$, $\beta(0)$ the value was used:

$$\Omega_1(0) = \frac{\operatorname{sign} T}{k} 2\sqrt{\mu}, \quad \Omega_2(0) = 1, \ \beta(0) = 0,$$

where k is the root of the equation: /15

$$K(k) = \frac{|T|}{2k}\sqrt{\mu}.$$

Such values of $\Omega_1(0)$, $\Omega_2(0)$, $\beta(0)$ are obtained by changing from the initial conditions of the solution of the boundary-value problem (9), (21) to variables Ω_2 and β by formulas (8). For $\lambda = 0.25$, μ 0.2, $\kappa = 10.30$, Fig. 4 and 9 show the dependence on T of the initial conditions $\Omega_1(0)$, $\Omega_2(0)$, $\beta(0)$ of the solutions of the boundary problem (23) (for $\kappa = 30$ Fig. 4 shows only $\Omega_1(0)$ as a function of T). By virtue of the second relation of (4), the curves obtained from those in Fig. 4 and 9 by the transformation:

$$\Omega_1(0) \to -\Omega_1(0), \ \Omega_2(0) \to \Omega_2(0), \ \beta(0) \to -\beta(0), \ T \to -T.$$

will also specify the initial conditions of the solutions of the boundary problem (23). The curves shown in Fig. 4 and 9 consist of separate pieces. In the scale of the figures the breaks between certain of the pieces are not notable and for clarity we indicate them by circles. The reason for the occurrence of these breaks is the same as in the case of the oscillations: resonances between the slow and fast motions of the satellite. The behavior of the sign of the Jacobian:

$$\frac{\partial[\Omega_2(\frac{T}{4}), \dot{\Omega}_2(\frac{T}{4}), \beta(\frac{T}{4})]}{\partial[\Omega_1(0), \Omega_2(0), \beta(0)]}$$

for the calculated solutions of problem (23) is similar to that described above for the sign of the Jacobian (25). Fig. 11 and 12 show examples of the rotational periodic solutions of the set (3).

In order to investigate the stability of the discovered periodic solutions of (3), the characteristic equation of the corresponding set of equations in variations was examined. Using the property of symmetry of the set (3) and of the investigated periodic solutions, it can be proven that this characteristic equation is reciprocal. Furthermore, by virtue of the independence of set (3) in the case $\eta = 0$, this equation has a root equal to 1 with multiplicity of at least 2. In view of these remarks, this particular characteristic equation can be represented as:

$$(p-1)^2(p^2-a_1p+1)(p^2-a_2p+1)=0, \tag{26}$$

where a_1 and a_2 are certain coefficients. If a_1 and a_2 are real and:

$$|a_1| \leq 2, \ |a_2| \leq 2,$$

all the roots of equation (26) will lie on the circle $|p| = 1$ and the necessary conditions of orbital stability of the investigated periodic solution will be fulfilled. Otherwise, this solution is unstable. Figs. 6 and 10 show graphs for the /16 dependence on T of the coefficients a_1 and a_2 for several of the calculated periodic solutions of (3). As was found for the solutions shown in Fig. 4, 5 and 9, the necessary conditions of orbital stability are fulfilled for all values of T, except for the small neighborhoods of the points of discontinuity of the graphs of the initial conditions and narrow zones (with a width $\Delta T \lesssim 0.05$) of parametric resonance. The latter are produced in the neighborhood of the points specified by the relations $a_1 \approx -2$, $a_2 \approx -2$, $a_1 \approx a_2$ (cf. Fig. 6,10).[1] Since the

[1]By an appropriate replacement of the variables, equations (3) can be reduced to the Hamiltonian form: the external moment acting on the satellite is susceptible of a force function:

$$U = -\frac{3}{2}\omega_0^2(A\alpha_{11}^2 + B\alpha_{12}^2 + C\alpha_{13}^2) - \frac{1}{2}c_x\rho V^2\pi R^2 d\alpha_{11}$$

Therefore, in keeping with the theorem of Krein-Helfand-Lidskiy [9], zones of parametric resonance will occur in the neighborhood of only some of the points $a_1 \approx a_2$.

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zones of parametric resonance on the T axis are very narrow, it is difficult to determine them with reliability. Two such zones have been found with certainty. In Fig. 6 and 10 they are shown by the letter Π. For the oscillations (Fig. 6) when $T = 16.16$ we have $a_2 = -2.023$; for the rotations (Fig. 10) when $T = 13.86$ we have:

$$Im\, a_1 = -Im\, a_2 = 0.103.$$

Using the first terms of the series (10) for α and β we can establish that, when $\eta = 0$, the maximum angle θ_m between the axes Ox_1 and OX_1 in the case of solutions that belong to the integral surface (10), (11) will not exceed:

$$\lambda\, æ^{-1}(1+|\mu|)|\Omega_1|_{max}.$$

As shown by computations, this estimate is rather good for the nonresonant periodic solution. According to such estimate, when $\lambda = 0.25$, $\mu = 0.2$, $\kappa \geqslant 10$ and $|\Omega_1|_{max} \leqslant 2$*, we have $\theta_m < 4°$. The obtained result suggests the possibility of using nonresonant periodic solutions as the nominal undisturbed movements of the satellite in the regime of uniaxial aerodynamic attitude control.

5. Numerical Investigation of the Integral Surface (10), (11) When $\eta \neq 0$

When $\eta \neq 0$ the solutions that belong to the integral surface (10), (11), in general, are not periodic, and the numerical investigation of this surface is complicated. To check for the existence of such a surface the set (3) was numerically integrated on large intervals of time. On the segment $0 \leqslant \tau \leqslant 200\pi$** solutions of this set were calculated with initial conditions (cf. (8)):

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*For nonresonant solutions of the boundary problems (19) and (23), $|\Omega_1|_{max} = |\Omega_1(0)|$.

**For orbits with an altitude above 250 km, this interval comprises more than 6 days.

$$\gamma(0)=\gamma_0,\ \Omega_1(0)=\Omega_{10},\ \Omega_2(0)=\cos\gamma_0,\ \Omega_3(0)=-\sin\gamma_0,\ \alpha(0)=\beta(0)=0.$$

The computations were done for $\lambda = 0.25$, $\mu = 0.2$, $\kappa = 10$, $\eta = -0.1$ and $\eta = -0.3$. The findings are shown in Fig. 13-24. On each figure in the plane:

$$(\gamma(\mathrm{mod}\,2\pi),\Omega_1)$$

crosses indicate the points:

$$(\gamma(2\pi n)(\mathrm{mod}\,2\pi),\Omega_1(2\pi n))\ (n=0,1,\ldots,100).$$

Such figures are usually known as stroboscopic pictures. These figures also show for each solution the maximum angle θ_m between the axes Ox_1 and OX_1. This angle is found from formula:

$$\theta_m = \max_{0\le\tau\le 200\pi} \arccos\left(\cos\alpha(\tau)\cos\beta(\tau)\right).$$

A comparison of Figs. 13-24 with Fig. 3 reveals a rather good agreement between the obtained stroboscopic pictures of the solutions of (3) and the phase pattern of (9). A comparison by pairs of Fig. 3 with Fig. 13 and 14, 15 and 16, 17 and 18, 19 and 20 suggests that, for identical parameters γ_0 and Ω_{10}, when $\eta = -0.1$ the agreement between the stroboscopic pictures and the phase curves in Fig. 3 is more accurate than when $\eta = -0.3$. In all the versions of the analysis it was found that $\theta_m < 15°$. This result, as well as the results of §4, testify to the possibility of using the satellite motions that belong to the integral surface (10), (11) as nominal undisturbed motions in the regime of uniaxial aerodynamic attitude control.

The above-considered mathematical model of a satellite is quite idealized. This is in keeping with the fact that, at any rate, it is described by differential equations that can be converted to the Hamiltonian form. Nevertheless an analysis of this model can produce substantive results in the dynamics of aerodynamic satellite attitude control systems. The methods used in the work can be employed to investigate a broad class of such systems.

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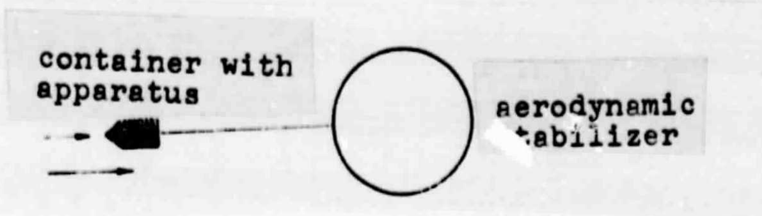

Fig. 1.

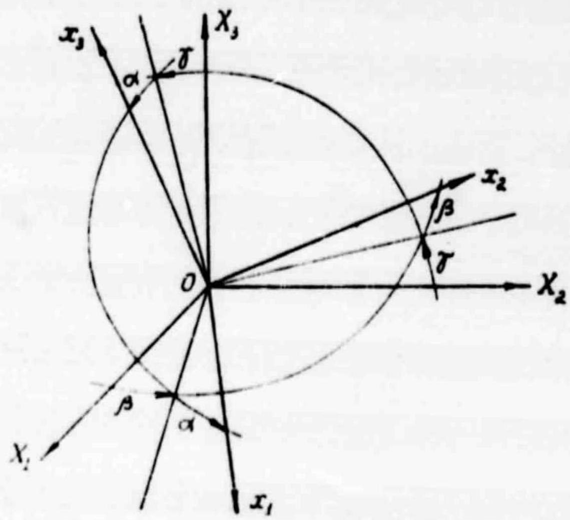

Fig. 2.

Fig. 3.

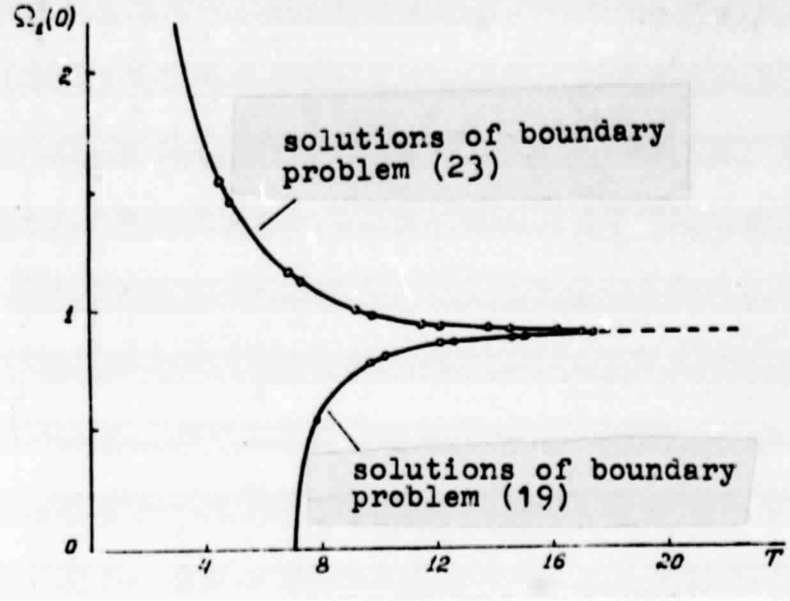

Fig. 4. κ = 30.

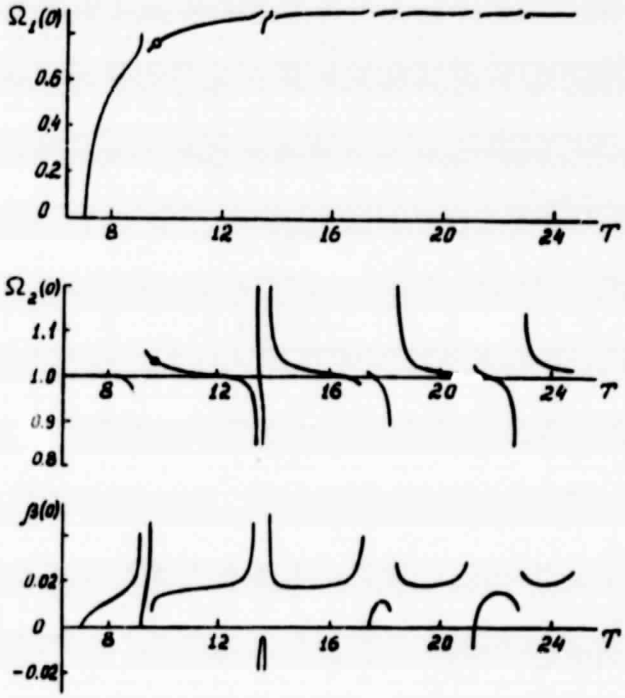

Fig. 5. Solutions of the boundary-value problem (19), κ = 10.

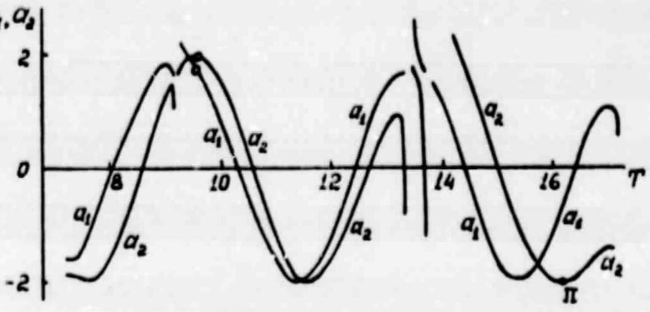

Fig. 6. Stability of solutions of the boundary problem (19), κ = 10.

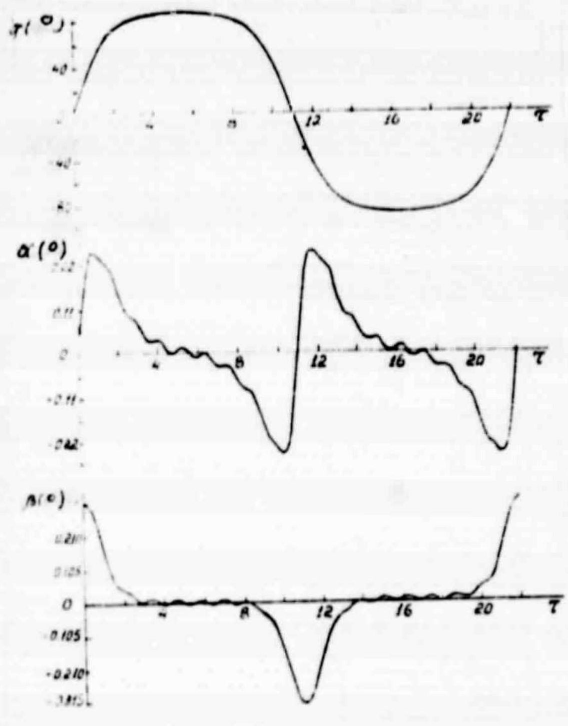

Fig. 7. Nonresonant solution of the boundary problem (19),

$\kappa = 30$, $\Omega_1(0) = 0.697411$, $\Omega_2(0) = 1.004878$, $\beta(0) = 0.005606$, $T = 22$.

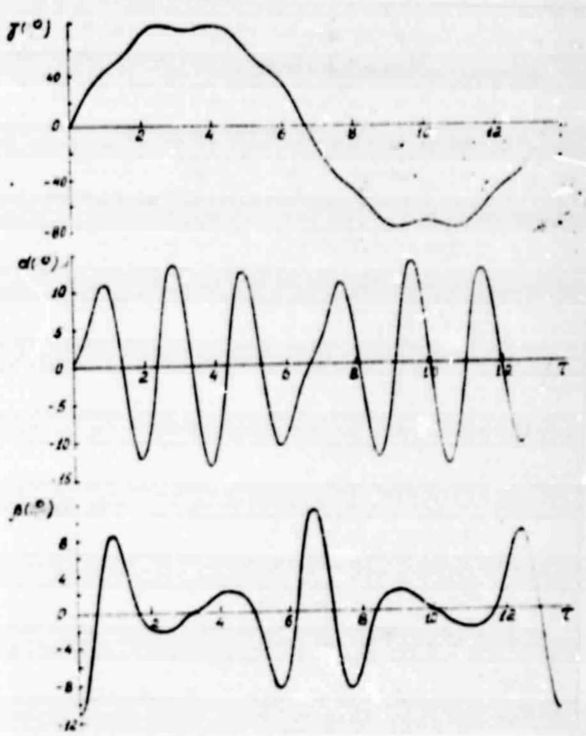

Fig. 8. Resonant solution of boundary problem (19).

$\kappa = 10$, $\Omega_1(0) = 0.787336$, $\Omega_2(0) = 1.327745$, $\beta(0) = -0.197119$, $T = 13.42$.

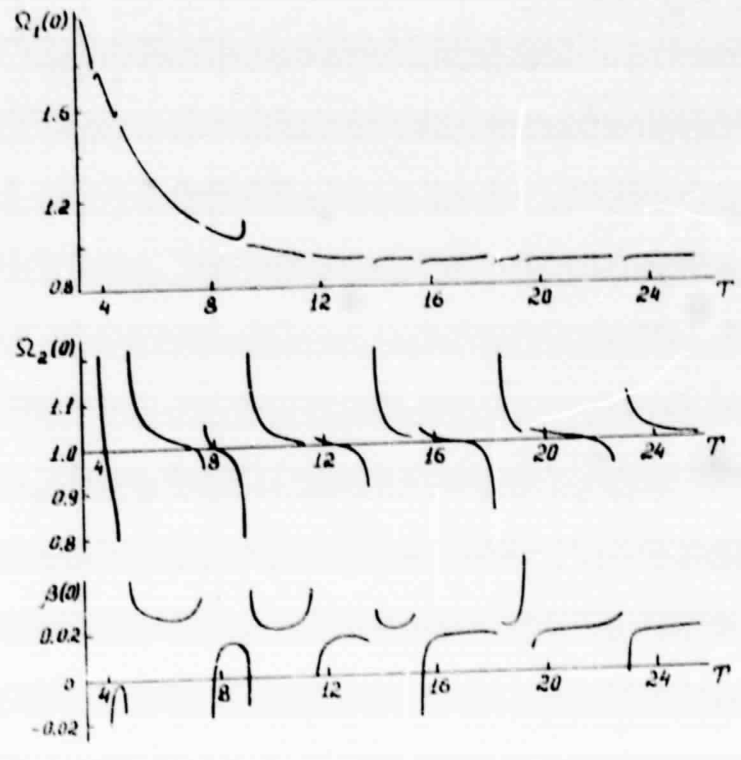

Fig. 9. Solutions of the boundary-value problem (23), κ = 10.

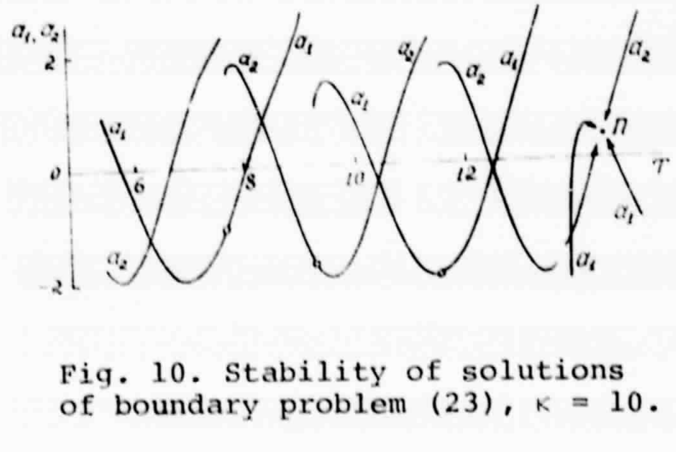

Fig. 10. Stability of solutions of boundary problem (23), κ = 10.

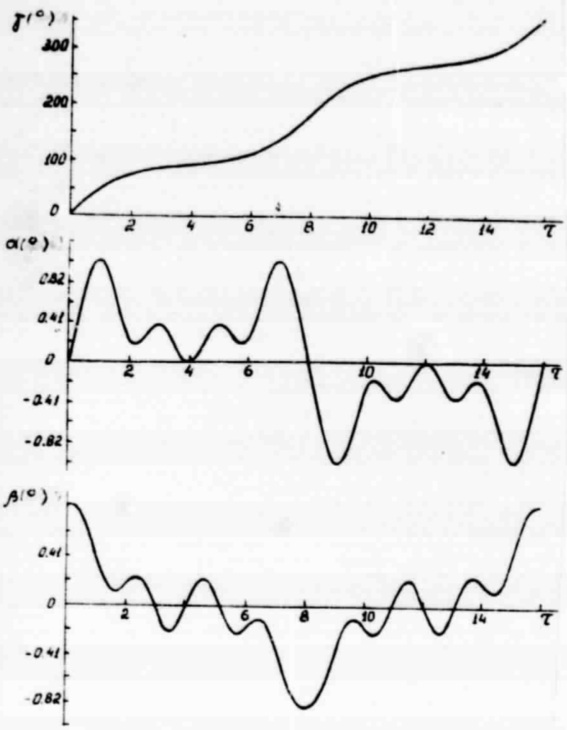

Fig. 11. Nonresonant solution of boundary problem (23),

$\varkappa = 10$, $\Omega_1(0) = 0.910185$, $\Omega_2(0) = 1.011661$, $\beta(0) = 0.014954$, $T = 16$.

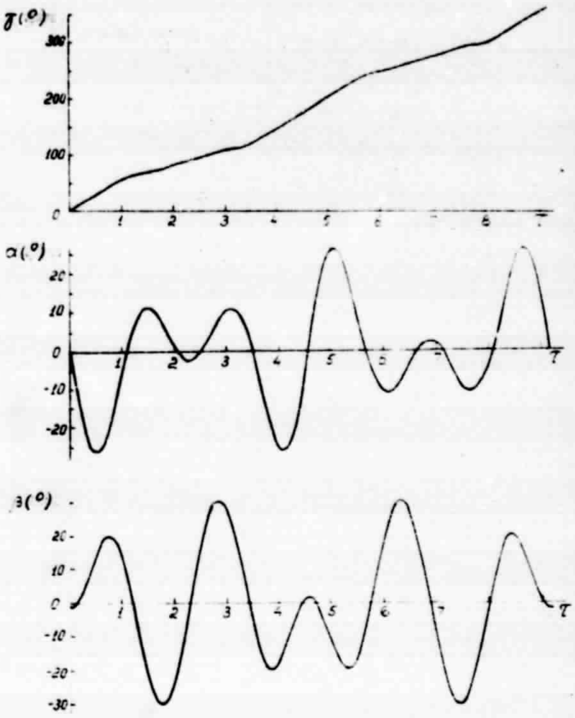

Fig. 12. Resonant solution of boundary problem (23),

$\varkappa = 10$, $\Omega_1(0) = 1.122122$, $\Omega_2(0) = -0.451623$, $\beta(0) = -0.033861$, $T = 9.186$.

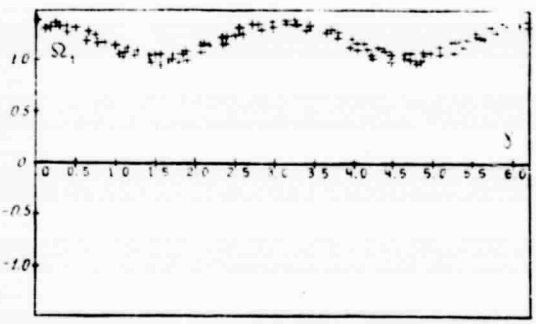

Fig. 13. $\eta = -0.1$, $\gamma_0 = 0$, $\Omega_{10} = 1.4$, $\theta_m = 6.17°$

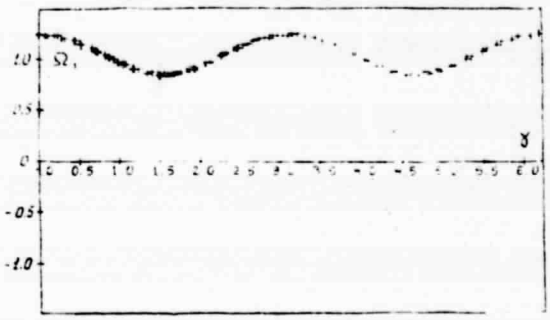

Fig. 15. $\eta = -0.1$, $\gamma_0 = 0$, $\Omega_{10} = 1.25$, $\theta_m = 3.69°$

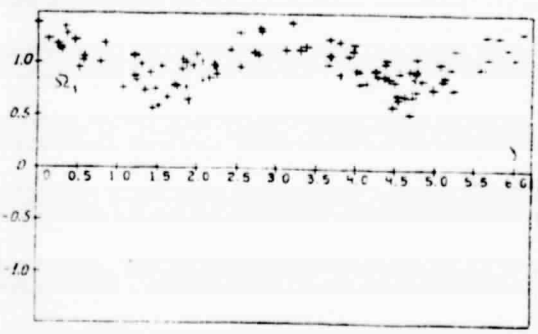

Fig. 14. $\eta = -0.3$, $\gamma_0 = 0$, $\Omega_{10} = 1.4$, $\theta_m = 12.01°$

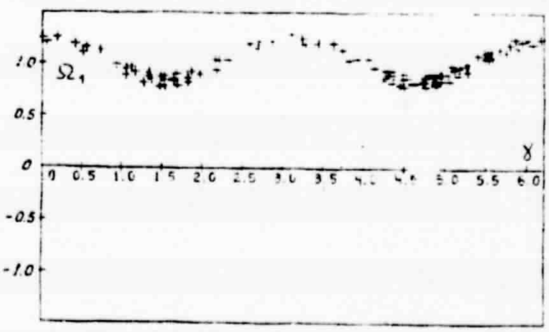

Fig. 16. $\eta = -0.3$, $\gamma_0 = 0$, $\Omega_{10} = 1.25$, $\theta_m = 6.59°$

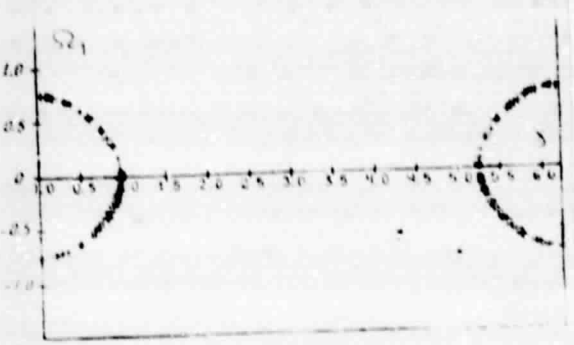

Fig. 17. $\eta = -0.1$, $\gamma_0 = 1$, $\Omega_{10} = 0$, $\theta_m = 1.81$

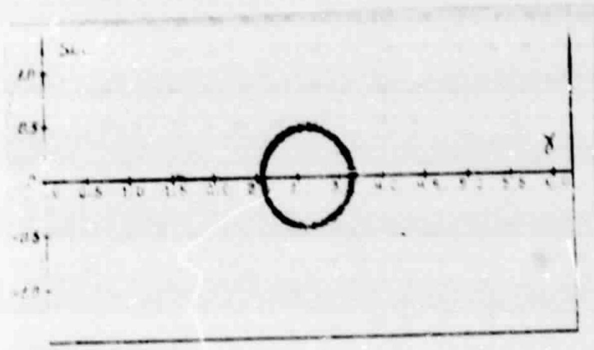

Fig. 19. $\gamma_0 = 2.6$, $\Omega_{10} = 0$, $\theta_m = 0.85°$

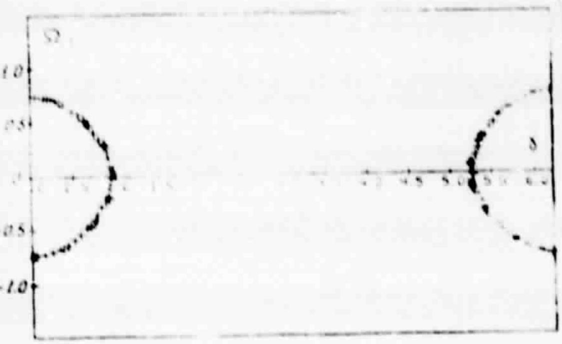

Fig. 18. $\eta = -0.3$, $\gamma_0 = 1$, $\Omega_{10} = 0$, $\theta_m = 2.94°$

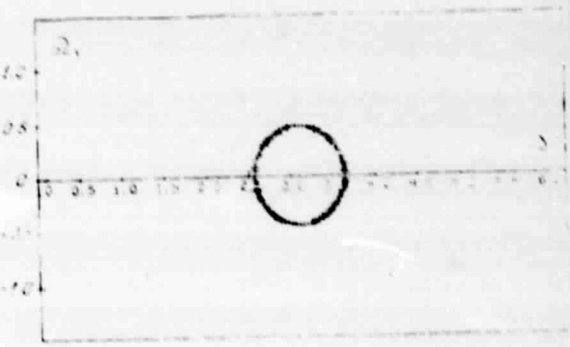

Fig. 20. $\eta = -0.3$, $\gamma_0 = 2.5$, $\Omega_{10} = 0$, $\theta_m = 1.62°$.

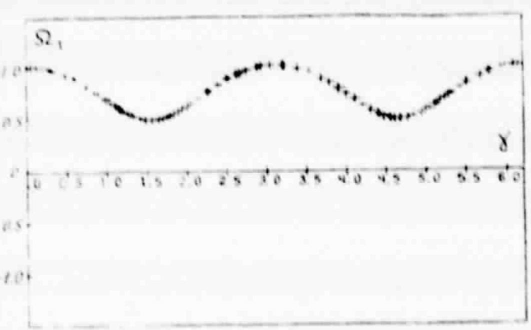

Fig. 21. $\eta = -0.1$, $\gamma_0 = 1.57$, $\Omega_{10} = 0.5$, $\theta_m = 2.34°$.

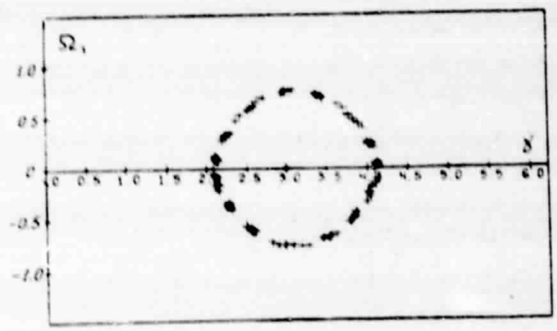

Fig. 23. $\eta = -0.1$, $\gamma_0 = 2.1$, $\Omega_{10} = 0$, $\theta_m = 2.64°$.

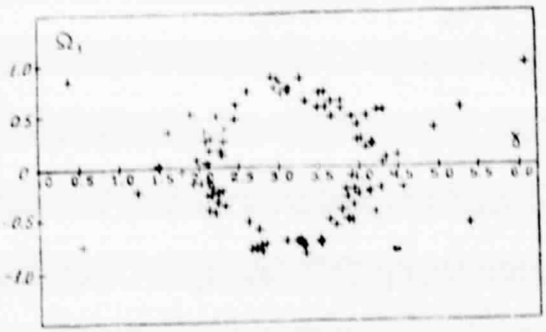

Fig. 22. $\eta = -0.3$, $\gamma_0 = 0$, $\Omega_{10} = 1$, $\theta_m = 10.77°$.

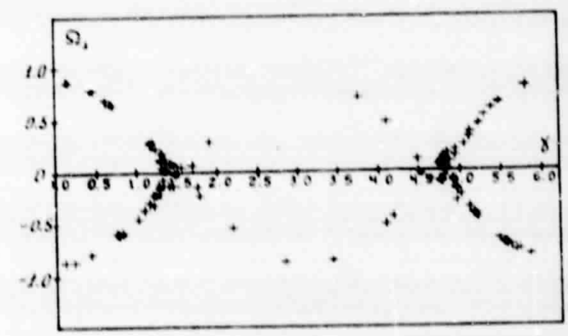

Fig. 24. $\eta = -0.1$, $\gamma_0 = 1.57$, $\Omega_{10} = 0.005$, $\theta_m = 3.23°$.