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LOW FREQUENCY ACOUSTIC AND ELECTROMAGNETIC SCATTERING

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Abstract

This paper deals with two classes of problems arising from acoustics and electromagnetics scattering in the low frequency situations. The first class of problem is solving Helmholtz equation with Dirichlet boundary conditions on an arbitrary two-dimensional body while the second one is an interior-exterior interface problem with Helmholtz equation in the exterior. Low frequency analysis show that there are two intermediate problems which solve the above problems accurate to $O(k^2 \log k)$ where k is the frequency. These solutions greatly differ from the zero frequency approximations. For the Dirichlet problem numerical examples are shown to verify our theoretical estimates.

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Introduction

The "soft-obstacle" problem in acoustics involves finding a solution of the Helmholtz equation, $\Delta u + k^2 u = 0$, in the exterior Ω^+ of a bounded region Ω with $u = f$ on $\partial\Omega$. In general, f itself will depend on k, $f = f^k$. A natural question to ask is whether it is true that if $f^k \to f^0$ as $k \to 0$ then the solution u^k will tend to u^0 the solution of $\Delta u^0 = 0$ in Ω^+, $u = f^0$ on $\partial\Omega$. The limit problem is much easier to solve numerically than that for u^k.

It is known that the answer to the above question is yes. For two dimensions, however, it is also known that the solution u^k has a logarithmic branch point at $k = 0$. This in turn leads to the fact that $u^k - u^0 = O\left((\log k)^{-1}\right)$ so that the convergence to u^0 is very slow. These ideas are discussed in [1], [2] and [7].

The purpose of this paper is to present a modified low frequency approximation. We illustrate with the important special case of the field produced by line sources in the exterior of a cylinder. We give an approximation which is accurate to $O(k^2 \log k)$ but which still involves solving only the limit case of Laplace's equation in Ω^+.

Our method can also be applied to the two-dimensional eddy current problem discussed in [3]. Here one is again solving $\Delta u + k^2 u = 0$ in Ω^+ but now one has $\Delta u + i\alpha^2 u = 0$ in Ω^+ and there are transition conditions across $\partial\Omega$. Solving the problem with $\Delta u = 0$ in Ω^+ again produces a solution accurate only to $O\left((\log k)^{-1}\right)$. Our revised method again gives accuracy of $O(k^2 \log k)$ while keeping $\Delta u = 0$ in Ω.

Let us give a precise statement of the problems under considerations. Ω is a bounded region in $\mathbb{R}^2$ with smooth boundary Γ and Ω^+ the exterior of $\overline{\Omega}$. In all our problems we will have given incident field U_k which is that due to line sources at m points $\underline{x}_1,\cdots,\underline{x}_m$;

$$U_k(\underline{x}) = -\frac{i}{4}\sum_{j=1}^{m} a_j\, H_0^{(1)}\left(k|\underline{x} - \underline{x}_j|\right) \tag{1.1}$$

where $H_0^{(1)}$ is the Hankel function. For $k > 0$ all our fields, including (1.1), are to satisfy the Sommerfeld radiation condition. For any f we will write f_+,f_- for the limits from Ω^+ and Ω. We write down the problem for the scattered fields u^k; that is the total fields in Ω^+ will be $u^k + U_k$:

Problem (P_k): Exterior Dirichlet Problem:

Find u^k such that:

$$\Delta u^k + k^2 u^k = 0 \qquad \text{in } \Omega^+ \tag{1.2}$$

$$u^k_+ = \Phi \;:\; \Phi = -(U_k)_+ \qquad \text{on } \Gamma. \tag{1.3}$$

Problem $(P_{k\alpha})$: Interface Problem:

Find u^k such that:

$$\Delta u^k + k^2 u^k = 0 \quad \text{in } \Omega^+; \quad \Delta u^k + i\alpha^2 u^k = 0 \quad \text{in } \Omega,\ \alpha > 0 \tag{1.4}$$

$$u^k_- = u^k_+ + \Phi; \quad (u^k_n)_- = (u^k_n)_+ + \Psi; \quad \Phi = (U_k)_+,\ \Psi = (U_{k,n})_+ \quad \text{on } \Gamma. \tag{1.5}$$

We obtain the limit problems by formally setting $k = 0$ in (P_k) and $(P_{k\alpha})$ but some care is necessary in the behavior at infinity. To motivate this we consider u_k more carefully. For the Hankel function $H_0^{(1)}$ we have,

$$-\frac{i}{4}H_0^{(1)}(kz) = \frac{1}{2\pi}\log z + \beta + \sum_{n=1}^{\infty} a_n(kz)^{2n}\log kz + b_n(kz)^{2n}, \qquad (1.6)$$

$$\beta = \frac{1}{2\pi}\left(\gamma - \frac{\pi}{2}i + \log\frac{k}{2}\right) \qquad (1.7)$$

γ is Euler's constant and we note that $\operatorname{Im}\beta \neq 0$. From (1.1) and (1.6),

$$U_k(\underline{x}) = U_0(\underline{x}) - A\beta + (k^2\log k)V_k(\underline{x}) \qquad (1.8)$$

$$U_0(\underline{x}) = \frac{1}{2\pi}\sum_{j=1}^{m} a_j \log|x - x_j|; \quad A = -\sum_{j=1}^{m} a_j. \qquad (1.9)$$

When $k = 0$ we no longer have the radiation condition but rather we require that the total fields remain bounded at infinity. We recall that u^0 is supposed to represent the scattered field. We see from (1.9) that U_0 becomes logarithmically infinite for large $|x|$ and to compensate we must allow u^k to do likewise. This suggests that in the limit problem for both (P_k) and $(P_{k\alpha})$ we should require that,

$$u^0(\underline{x}) - A\log|x| \quad \text{is bounded as} \quad |\underline{x}| \to \infty. \qquad (1.10)$$

Problem (P_0) (P_0): Find u^0 satisfying (1.10) and,

$$\Delta u^0 = 0 \text{ in } \Omega^+, \quad u^0 = \Phi \text{ on } \Gamma.$$

Problem $(P_{0\alpha})$: Find u^0 satisfying (1.10) and,

$$\Delta u^0 = 0 \text{ in } \Omega^+, \quad \Delta u^0 + i\alpha^2 u^0 = 0 \quad \text{in } \Omega$$
$$u^0_- = u^0_+ + \Phi; \quad (u^0_n)_- = (u^0_n)_+ + \Psi \quad \text{on } \Gamma.$$

The following two results are proved in [5] and [3] respectively.

Theorem 1.1. <u>There exists a unique solution</u> u^k <u>for problem</u> (P_k), $(k \geq 0)$.

Theorem 1.2. <u>There exists a unique solution</u> u^k <u>for problem</u> $(P_{k\alpha})$, $(k \geq 0)$.

Our aim is to give approximate procedures for the solution of (P_k), $(P_{k\alpha})$ in which one has to solve only (P_0), $(P_{0\alpha})$. The crucial question then is how to choose $\Phi(\Phi,\Psi)$ in (P_0), $(P_{0\alpha})$.

The straightforward low frequency approximation involves putting

$$\Phi = -(U_0)_+ \text{ in } (P_0) \text{: } \Phi = (U_0)_+ \text{; } \Psi = (U_{0,n})_+ \text{ in } (P_{0\alpha}). \tag{1.11}$$

We will see that this yields $u^k - u^0 = O\left((\log k)^{-1}\right)$. Our procedure involves a more sophisticated choice of Φ or (Φ,Ψ) which accounts for the term $A\beta$ in (1.8). This involves a very careful analysis of solution processes for (P_0) and $(P_{0\alpha})$ which we carry out in the next section. In Section 3 we give our modified process for (P_k) and in Section 4 for $(P_{k\alpha})$. Section 5 gives the results of some simple numerical experiments.

2. INTEGRAL OPERATORS

Our analysis involves use of singular integral equations theory. We represent solutions of (P_k), $(P_{k\alpha})$, $(k \geq 0)$ by means of simple layer potentials. The properties of these are well known in the literature.

Let

$$G_\delta(\underline{x},\underline{y}) = \begin{cases} -\frac{i}{4} H_0^{(1)}(\delta|\underline{x}-\underline{y}|) & \text{if } \delta \neq 0 \\ \\ \frac{1}{2\pi}\log|\underline{x}-\underline{y}| & \text{if } \delta = 0 . \end{cases} \quad (2.1)$$

$$\begin{cases} \mathscr{S}_\delta[\sigma](\underline{x}) = \int_\Gamma \sigma(\underline{y}) G_\delta(\underline{x},\underline{y}) ds_{\underline{y}} & \text{if } \underline{x} \in \Omega^+ \\ \\ S_\delta[\sigma](\underline{x}) = \mathscr{S}_\delta[\sigma](\underline{x}) & \text{if } \underline{x} \in \Gamma \\ \\ M_\delta[\sigma](\underline{x}) = \int_\Gamma \sigma(\underline{y}) \frac{\partial}{\partial n_{\underline{x}}} G_\delta(\underline{x},\underline{y}) ds_{\underline{y}} & \text{if } x \in \Gamma \end{cases} \quad (2.2)$$

We note that,

$$\mathscr{S}_0[\sigma](\underline{x}) = \frac{1}{2\pi}\Gamma[\sigma]\log|\underline{x}| + 0\left(\frac{1}{|\underline{x}|}\right) \quad \text{as} \quad |\underline{x}| \to \infty \quad (2.3)$$

where

$$\Gamma[\sigma] \equiv \int_\Gamma \sigma(\underline{y}) ds_y .$$

Formulas (1.6) and (1.7) give the following expansions to our integral operators

$$\mathscr{S}_k[\sigma](\underline{x}) = \mathscr{S}_0[\sigma](\underline{x}) + \beta\Gamma[\sigma] + k^2 \log k \mathscr{R}_k[\sigma](\underline{x}) \quad (2.4)$$

$$S_k[\sigma](\underline{x}) = S_0[\sigma](\underline{x}) + \beta\Gamma[\sigma] + k^2 \log k\, R_k[\sigma](\underline{x}) \quad (2.5)$$

$$M_k[\sigma](\underline{x}) = M_0[\sigma](\underline{x}) + k^2 \log k\, N_k[\sigma](\underline{x}), \quad (2.6)$$

where

$$\mathscr{R}_k[\sigma](\underline{x}) = \int_\Gamma \sigma(\underline{y})\mathscr{K}_k(\underline{x},\underline{y})ds_{\underline{y}} \qquad \text{if} \quad \underline{x} \in \Omega^+$$

$$R_k[\sigma](\underline{x}) = \mathscr{R}_k[\sigma](\underline{x}) \qquad \text{if} \quad \underline{x} \in \Gamma,$$

and $\mathscr{K}_k$ is a regular kernel whose leading term is of the order $O(|\underline{x} - \underline{y}|^2 \log |\underline{x} - \underline{y}|)$. Similarly, N_k is an integral operator whose kernel is regular.

We have the following properties concerning the integral operators S_k, R_k, M_k and N_k. Let $\Gamma \in C^s$, $s \geqslant |r| + 2$ and then for any $r \in \mathbb{R}$ we have the following:

<u>Properties</u> (Hsiao and MacCamy, Hsiao and Wendland, Hariharan and Stephan)

1. $S_k : H^r(\Gamma) \to H^{r+1}(\Gamma)$ $\qquad k \geqslant 0$
2. $R_k : H^r(\Gamma) \to H^{r+s}(\Gamma)$ $\qquad k > 0$
3. S_k is bijective for $k > 0$
4. $M_k : H^r(\Gamma) \to H^{r+(s-|r|-2)}$ $\qquad k \geqslant 0$
5. $N_k : H^r(\Gamma) \to H^{r+(s-|r|)}$ $\qquad k > 0.$
6. If $\Gamma \in C^\infty$, then for any $r \in \mathbb{R}$ and for any $(g,A) \in H^r(\Gamma) \times \mathbb{R}$, there exists unique $(\sigma,c) \in H^{r-1}(\Gamma) \times \mathbb{R}$ such that

$$\begin{cases} S_0[\sigma] + c = g \\ \Gamma[\sigma] = A. \end{cases} \tag{2.7}$$

The solution of (2.7) has the form

$$\sigma = L[g] + \alpha A\psi \qquad \text{for some } \alpha$$

$$c = \ell[g] + \lambda A \qquad \text{for some } \lambda$$

and where

$$L : H^r(\Gamma) \to H^{r-1}(\Gamma)$$

$$\ell : H^r(\Gamma) \to \mathbb{R}.$$

As a consequence of Properties 1 and 6 we have the following theorem.

THEOREM 2.1 Let Γ be C^∞ and $r \in \mathbb{R}$.

(i) There exists a unique $(f_0,c_0) \in H^r(\Gamma) \times \mathbb{R}$ such that

$$\begin{cases} S_0[f_0] + c_0 = 0 \\ \Gamma[f_0] = 1. \end{cases} \tag{2.8}$$

(ii) For any $g \in H^r(\Gamma)$, there exists unique $f \in H^{r-1}(\Gamma)$ such that

$$S_0[f] + \beta\Gamma[f] = g \qquad (\mathrm{Im}\ \beta \neq 0). \tag{2.9}$$

Proof:

(i) Follows directly from Property 6.

(ii) (Existence): Define $(f_1,c_1) \in H^{r-1}(\Gamma) \times \mathbb{R}$ by

$$S_0[f_1] + c_1 = g$$

$$\Gamma[f_1] = 1.$$

This has a unique solution by Propery 6. Let

$$f = f_1 + p\, f_0 \qquad \text{for some } p.$$

This will satisfy (2.9) provided

$$S_0[f_1] + pS_0[f_0] + \beta\Gamma[f_1] + p\beta\Gamma[f_0] = g,$$

or

$$g - c_1 - pc_0 + \beta + p\beta = g$$

or

$$p = \frac{c_1 - \beta}{\beta - c_0}, \tag{2.10}$$

and $\beta - c_0 \neq 0$ since $\operatorname{Im} \beta \neq 0$.

(iii) (Uniqueness): Suppose $S_0[f] + \beta\, \Gamma[f] = 0$.

<u>CASE (i)</u>: $\Gamma[f] = 0$, then $f = 0$ by Property 6.

<u>CASE (ii)</u>: $\Gamma[f] \neq 0$.

Let

$$\hat{f} = \frac{f}{\Gamma(f)},$$

then

$$S_0[\hat{f}] + \beta = 0$$

$$\Gamma[\hat{f}] = 1.$$

Thus by (i), $\hat{f} = f_0$, $\beta = c_0$. This is not possible since $\operatorname{Im} \beta \neq 0$.

NOTATION: For the solution of (2.9) we write

$$f = J_\beta[g].$$

Then we have the following:

Corollary 2.1.

(i) $J_\beta : H^r(\Gamma) \to H^{r-1}(\Gamma)$

(ii) J_β is bijective.

3. LOW FREQUENCY RESULTS FOR THE DIRICHLET PROBLEM

In this section we consider problems (P_k) and (P_0). We prove the following result. Let the total field (i.e. incident plus scattered part) of (P_k) and (P_0) be $\bar{u}^k$ and $\bar{u}^0$ then:

THEOREM 3.1: For problem (P_k), there exists an intermediate solution v such that, (i) $\bar{u}^k - v = 0(k^2 \log k)$; (ii) $v - \bar{u}^0 = 0(\frac{1}{\log k})$. Moreover v can be obtained by solving (P_0) for appropriate Φ.

According to the notations developed in the previous section we can represent the solutions of (P_k) and (P_0) as follows.

For (P_k):

$$u^k(\underline{x}) = \mathscr{S}_k[\sigma^k](\underline{x}) \qquad (\underline{x} \in \Omega^+). \tag{3.1}$$

For (P_0):

$$u^0(\underline{x}) = \mathscr{S}_0[\sigma^0](\underline{x}) + c \qquad (\underline{x} \in \Omega^+ \text{ and } c \in \mathbb{R}) \tag{3.2}$$

Formula (3.1) will be the solution of P_k provided

$$S_k[\sigma^k] + U_k = 0 \qquad \text{on } \Gamma \tag{I_k}$$

and (3.2) will be the solution of P_0 provided

$$\begin{aligned} S_0[\sigma^0] + U_0 + c &= 0 \qquad \text{on } \Gamma \\ \Gamma[\sigma^0] &= A. \end{aligned} \tag{I_0}$$

By Property 3, (I_k) has a unique solution and by Property 6, (I_0) has a unique solution. We expand the operator S_k and the incident field U_k in (I_k) according to (2.5) and (2.7) respectively. This yields

$$S_0[\sigma^k] + \beta\Gamma[\sigma^k] + U_0 - A\beta = k^2 \log k\, V_k - k^2 \log k\, R_k[\sigma^k]. \tag{3.3}$$

Define σ by

$$S_0[\sigma] + \beta\Gamma[\sigma] + U_0 - A\beta = 0. \tag{3.4}$$

By Theorem 2.1 this has a unique solution and is given by

$$\sigma = J_\beta\left[- U_0 + A\beta\right].$$

We set $f = \sigma^k - \sigma$ and subtract (3.4) from (3.3) to give

$$S_0[f] + \beta\Gamma[f] = k^2 \log k \left(V_k - R_k[f]\right).$$

By Theorem 2.2 this again has a unique solution by successive approximation with

$$f = O\left(k^2 \log k\right).$$

Hence,

$$\sigma^k = \sigma + O(k^2 \log k). \tag{3.5}$$

Define v by

$$v(\underline{x}) = \mathscr{S}_0[\sigma](\underline{x}) + \beta\Gamma[\sigma] + U_0(\underline{x}) - A\beta \qquad (\underline{x} \in \Omega^+). \tag{3.6}$$

Since σ is a uniquely defined function the function v is also uniquely determined for given U_0 and β. Since $\bar{u}^k = u^k + U_k$ we have,

$$\bar{u}^k - v = k^2 \log k(\mathscr{R}_k[f] + V_k).$$

Hence

$$\bar{u}^k - v = O(k^2 \log k). \tag{3.7}$$

This proves the first part of Theorem 1.3.

Now we consider $(I_0)_1$ and (3.4). Subtracting the first from the second equation, we have:

$$S_0[\sigma - \sigma^0] + \beta\Gamma[\sigma] - A\beta - c = 0,$$

since $A = \Gamma[\sigma^0]$, it follows that

$$S_0[\sigma - \sigma^0] + \beta\Gamma[\sigma - \sigma^0] - c = 0.$$

Set

$$\Gamma[\sigma - \sigma^0] = a$$

and

$$c^* = \beta\ a - c. \tag{3.8}$$

Then

$$S_0[\sigma - \sigma_0] + c^* = 0$$

$$\Gamma[\sigma - \sigma_0] = a.$$

By Theorem 2.1(i)

$$\begin{cases} \sigma - \sigma_0 = a\ f_0 \\ \\ \quad c^* = a\ c_0. \end{cases} \tag{3.9}$$

Now (3.8) and (3.9) yield

$$\begin{cases} a = \dfrac{c}{\beta - c_0} & (\beta - c_0 \neq 0, \text{ since } \operatorname{Im}\beta \neq 0) \\ \sigma = \dfrac{c}{\beta - c_0} f_0 + \sigma_0. \end{cases} \tag{3.10}$$

Note that $\sigma - \sigma_0 = O(\frac{1}{\log k})$. Now taking into account that $\bar{u}^0 = u^0 + U_0$ subtract (3.2) from (3.6).

$$v - \bar{u}^0 = \mathscr{S}_0[\sigma - \sigma_0] + \beta\Gamma[\sigma] - A\beta - c$$

$$= \frac{c}{\beta - c_0} \mathscr{S}_0[f_0] + \frac{\beta c}{\beta - c_0} + \beta A - \beta A - c$$

$$v - \bar{u}^0 = \frac{c}{\beta - c_0} [\mathscr{S}_0[f_0] + c_0]. \tag{3.11}$$

Thus

$$v - \bar{u}^0 = O(\frac{1}{\log k}).$$

Hence the proof of Theorem 1.3 is complete.

<u>REMARK 2.1</u>: The intermediate solution v differ from the zero frequency solution exactly by the function given in the right hand side of (3.11) while v itself differ from $\bar{u}^k$ only by $O(k^2 \log k)$. Thus v is the solution we are looking for and is determined by simply solving (3.4).

<u>REMARK 2.2</u>: From (3.6) the definition of v we see that it satisfies $\Delta v = 0$ and v bounded at infinity. Thus in principle v satisfies (P_0). However it has to be computed by solving the single integral equation (3.4).

4. LOW FREQUENCY RESULTS FOR THE INTERFACE PROBLEM

In this section we present similar results for problem $(P_{k\alpha})$ and $(P_{0\alpha})$. As before it will be seen later that there exists an intermediate problem which solves $(P_{k\alpha})$ to $0(k^2 \log k)$. The analysis is a little more complicated. Let $\bar{u}^k = u^k + U_k$ and $\bar{u}^0 = u^0 + U_0$. We shall prove:

THEOREM 4.1: <u>For problem</u> $(P_{k\alpha})$ <u>there exists an intermediate solution</u> v <u>such that</u>

$$\text{(i)} \quad \bar{u}^k - v = 0(k^2 \log k)$$

$$\text{(ii)} \quad v - \bar{u}^0 = 0\left(\frac{1}{\log k}\right).$$

<u>Moreover,</u> v <u>can be obtained by solving</u> $(P_{0\alpha})$ <u>for appropriate</u> Φ,Ψ.

We begin with simple layer representations for solution. For $(P_{k\alpha})$ we seek

$$u^k(\underline{x}) = \begin{cases} \mathscr{S}_k[\psi^k](\underline{x}) & \underline{x} \in \Omega^+ \\ \mathscr{S}_{\sqrt{i\alpha}}[\phi^k](\underline{x}) & \underline{x} \in \Omega \end{cases} . \tag{4.1}$$

and for $(P_{0\alpha})$ we seek

$$u^0(\underline{x}) = \begin{cases} \mathscr{S}_0[\psi^0](\underline{x}) + c & \underline{x} \in \Omega^+ \ (c \in \mathbb{C}) \\ \mathscr{S}_{\sqrt{i\alpha}}[\phi^0](\underline{x}) & \underline{x} \in \Omega \end{cases} . \tag{4.2}$$

The boundary conditions that u^k and u^0 satisfy yield

$$S_{\sqrt{i\alpha}}[\phi^k] = S_k[\psi^k] + U_k$$

$$-\frac{1}{2}\phi^k + M_{\sqrt{i\alpha}}[\phi^k] = \frac{1}{2}\psi^k + M_k[\psi^k] + \frac{\partial U_k}{\partial n}\,, \qquad (II_k)$$

$$S_{\sqrt{i\alpha}}[\phi^0] = S_0[\psi^0] + U_0 + c$$

$$-\frac{1}{2}\phi^0 + M_{\sqrt{i\alpha}}[\phi^0] = \frac{1}{2}\psi^0 + M_0[\psi^0] + \frac{\partial U_0}{\partial n} \qquad (II_0)$$

$$\Gamma[\psi^0] = A.$$

Existence of unique solutions of (II_k) and (II_0) is given by Hariharan and MacCamy [3] and is also found in [4] for (II_0). We need the following theorem relating to the general situation of (II_0):

THEOREM 4.2 (Hariharan and MacCamy). For $r(\geq 1) \in \mathbb{R}$ and for any $(g,h,a) \in H^r(\Gamma) \times H^{r-1}(\Gamma) \times \mathbb{R}$ there exists a unique solution $(\phi^0,\psi^0,c) \in H^{r-1}(\Gamma) \times H^{r-1}(\Gamma) \times ¢$ such that

$$S_{\sqrt{i\alpha}}[\phi^0] = S_0[\psi^0] + g + c$$

$$-\frac{1}{2}\phi^0 + M_{\sqrt{i\alpha}}[\phi^0] = \frac{1}{2}\psi^0 + M_0[\psi^0] + h$$

$$\Gamma[\psi_0] = a.$$

COROLLARY 4.1 In particular we have for $r(\geq 1) \in \mathbb{R}$ there exists unique $(\phi_0, \psi_0, c_0) \in H^{r-1}(\Gamma) \times H^{r-1}(\Gamma) \times \mathbb{C}$, such that

$$\begin{aligned} S_{\sqrt{i\alpha}}[\phi_0] &= S_0[\psi_0] + c_0 \\ -\frac{1}{2}\phi_0 + M_{\sqrt{i\alpha}}[\phi_0] &= \frac{1}{2}\psi_0 + M_0[\psi_0] \qquad (4.3) \\ \Gamma[\psi_0] &= 1. \end{aligned}$$

We also need the following results to establish our low frequency results.

THEOREM 4.3 For any (g,h) $H^r(\Gamma) \times H^{r-1}(\Gamma)$ there exists unique $(\phi,\psi) \in H^{r-1}(\Gamma) \times H^{r-1}(\Gamma)$, such that

$$\begin{aligned} S_{\sqrt{i\alpha}}[\phi] &= S_0[\psi] + \beta\Gamma[\psi] + g \\ -\frac{1}{2}\phi + M_{\sqrt{i\alpha}}[\phi] &= \frac{1}{2}\psi + M_0[\psi] + h \qquad (\mathrm{Im}\,\beta \neq 0). \end{aligned} \qquad (4.4)$$

Proof: Existence

Define (ϕ_1, ψ_1, c_1) by

$$\left\{\begin{aligned} S_{\sqrt{i\alpha}}[\phi_1] &= S_0[\psi_1] + g + c_1 \\ -\frac{1}{2}\phi_1 + M_{\sqrt{i\alpha}}[\phi_1] &= \frac{1}{2}\psi_1 + M_0[\psi_1] + h \\ \Gamma[\psi_1] &= 1. \end{aligned}\right. \qquad (4.5)$$

This has a unique solution $(\phi_1,\psi_1,c_1) \in H^{r-1}(\Gamma) \times H^{r-1}(\Gamma) \times \mathbb{C}$ by Theorem 4.2. Let

$$(\phi,\psi) = (\phi_1,\psi_1) + p(\phi_0,\psi_0).$$

When this substituted in (4.4) together with the use of (4.3) and (4.5) we find that p is determined by

$$p = \frac{c_1 - \beta}{\beta - c_0}. \tag{4.6}$$

Note that c_0 is a complex constant but independent of k (or β). Thus c_0 would possibly equal to one value of k which we rule out.

<u>Uniqueness</u>

Consider

$$\begin{aligned} S_{\sqrt{i\alpha}}[\psi] &= S_0[\psi] + \beta\Gamma[\psi] \\ -\frac{1}{2}\phi + M_{\sqrt{i\alpha}}[\phi] &= \frac{1}{2}\psi + M_0[\psi]. \end{aligned} \tag{4.7}$$

<u>CASE (i):</u> $\Gamma[\psi] = 0$.

This is the homogeneous case of Theorem 4.2 and we find $(\phi,\psi) \equiv (0,0)$.

<u>CASE (ii):</u> $\Gamma(\psi) \neq 0$.

Set

$$\hat{\psi} = \frac{\psi}{\Gamma[\psi]}, \qquad \hat{\phi} = \frac{\phi}{\Gamma[\psi]},$$

then

$$\Gamma[\hat{\psi}] = 1.$$

Therefore

$$\begin{aligned} S_{\sqrt{i\alpha}}[\hat{\phi}] &= S_0[\hat{\psi}] + \beta \\ -\frac{1}{2}\hat{\phi} + M_{\sqrt{i\alpha}}[\hat{\phi}] &= \frac{1}{2}\hat{\psi} + M_0[\hat{\psi}]. \end{aligned} \tag{4.8}$$

Thus Corollary 4.1

$$\beta = c_0, \quad \hat{\psi} = \psi_0 \quad \text{and} \quad \hat{\phi} = \phi_0.$$

$\beta = c_0$ could not happen since c_0 is independent of k. Hence the theorem.

Let us write this soluton as

$$(\phi,\psi) = \mathcal{J}_\beta(g,h). \tag{4.9}$$

COROLLARY 4.3

(i) $\mathcal{J}_\beta : H^r(\Gamma) \times H^{r-1}(\Gamma) \to H^{r-1}(\Gamma) \times H^{r-1}(\Gamma)$.

(ii) $\mathcal{J}_\beta$ is bijective.

With these results we can now show the low frequency estimates. We return to integral equations (II_k). Using the expansion given in Section 2 we have

$$\begin{aligned} S_{\sqrt{i\alpha}}[\phi^k] &= S_0[\psi^k] + \beta\Gamma[\psi^k] - A\beta + U_0 + k^2 \log k\big(V_k + R_k[\psi^k]\big) \\ -\frac{1}{2}\phi^k + M_{\sqrt{i\alpha}}[\phi^k] &= \frac{1}{2}\psi^k + M_0[\psi^k] + \frac{\partial U_0}{\partial n} + k^2 \log k\Big(\frac{\partial V_k}{\partial n} + N_k[\psi^k]\Big). \end{aligned} \tag{4.10}$$

Define (ϕ,ψ) by

$$S_{\sqrt{i\alpha}}[\phi] = S_0[\psi] + \beta\Gamma(\psi) - A\beta + U_0$$

$$-\frac{1}{2}\phi + M_{\sqrt{i\alpha}}[\phi] = \frac{1}{2}\psi + M_0[\psi] + \frac{\partial U_0}{\partial n}. \tag{4.11}$$

This has a unique solution by Theorem 4.3 and the solution is

$$(\phi,\psi) = \mathcal{J}_\beta\left(U_0 - A\beta, \frac{\partial U_0}{\partial n}\right). \tag{4.12}$$

Define $(f,g) = \left(\phi^k - \phi, \psi^k - \psi\right)$. Subtracting (4.11) from (4.10) we have,

$$S_{\sqrt{i\alpha}}[f] = S_0[g] + \beta\Gamma[g] + k^2 \log k\left(V_k - R_k[\psi^k]\right)$$

$$-\frac{1}{2} f + M_{\sqrt{i\alpha}}[f] = \frac{1}{2} g + M_0[g] + k^2 \log k\left(\frac{\partial V_k}{\partial n} - N_k[\psi^k]\right).$$

This system has a unique solution by Theorem 4.3 and given by

$$(f,g) = \mathcal{J}_\beta\left(k^2 \log k\{V_k + R_k[\psi^k]\}, k^2 \log k\{\frac{\partial V_k}{\partial n} + N_k[\psi^k]\}\right).$$

This has solutions with successive approximations

$$(f,g) = O(k^2 \log k).$$

Therefore

$$\psi^k = \psi + O(k^2 \log k).$$

Now define v by

$$v(\underline{x}) = \begin{cases} \mathcal{S}_0[\psi](\underline{x}) + \beta\Gamma[\psi] + u_0(\underline{x}) - A\beta, & \underline{x} \in \Omega^+ \\ \mathcal{S}_{\sqrt{i\alpha}}[\psi](\underline{x}) & \underline{x} \in \Omega \end{cases} . \tag{4.13}$$

Also we write the expansion of (4.1) plus the incident field U_k i.e.,

$$\bar{u}^k(x) = \begin{cases} \mathcal{S}_0[\psi^k](\underline{x}) + \beta\Gamma[\psi^k] + U_0(\underline{x}) - A\beta + k^2 \log k\big(V_k + \mathcal{R}_k[\psi^k]\big) & x \in \Omega^+ \\ \mathcal{S}_{\sqrt{i\alpha}}[\phi^k](\underline{x}) & \underline{x} \in \Omega \end{cases} \tag{4.14}$$

Thus

$$\bar{u}^k(\underline{x}) - v(\underline{x}) = O(k^2 \log k). \tag{4.15}$$

This proves the first part of Theorem 4.1.

Now we consider system (4.11) and (II_0). Subtracting the second from the first we have

$$S_{\sqrt{i\alpha}}[\phi - \phi^0] = S_0[\psi - \psi^0] + \beta\Gamma[\psi - \psi^0] - c$$

$$-\frac{1}{2}[\psi - \psi^0] + M_{\sqrt{i\alpha}}[\phi - \phi^0] = \frac{1}{2}\psi - \psi^0 + M_0[\psi - \psi^0].$$

Let $\Gamma[\psi - \psi^0] = a$. Thus by Corollary 4.2 we have

$$\phi - \phi^0 = a\,\phi_0$$

$$\psi - \psi^0 = a\,\psi_0$$

$$\beta - \frac{c}{a} = c_0$$

yielding

$$a = \frac{c}{\beta - c_0}\ .$$

Thus

$$\phi = \phi^0 + \frac{c}{\beta - c_0}\,\phi_0$$

$$\psi = \psi^0 + \frac{c}{\beta - c_0}\,\phi_0\ ,$$

yielding $(\phi,\psi) - (\phi^0,\psi^0) = 0(\frac{1}{\log k})$.

Now we subtract (4.2) plus the incident field U_0 from (4.13) to give

$$v(\underline{x}) - \overline{u}_0(\underline{x}) = \begin{cases} \frac{c}{\beta - c_0}\{\mathscr{S}_0[\psi_0](\underline{x}) + c_0\} & \underline{x} \in \Omega^+ \\ \frac{c}{\beta - c_0}\ \mathscr{S}_{\sqrt{1}\alpha}[\psi_0](\underline{x}) & \underline{x} \in \Omega \end{cases}, \qquad (4.16)$$

which establishes that

$$v(\underline{x}) - \overline{u}_0(\underline{x}) = 0(\frac{1}{\log k}).$$

Hence the proof of Theorem 4.1.

REMARK 4.13: Analogous to the exterior problem we see that v defined in (4.13) satisfies $(P_{0\alpha})$ in principle. However actual solution has to be computed by solving the system (4.11).

5. CONCLUSIONS

In both problems we have shown that there exists an approximate solution v which solves the original problem within $O(k^2 \log k)$. This solution as we have shown in (3.4) and (4.16) differ from the solution of the zero frequency cases by the functions given in the right hand sides of these equations. These functions can be large depending on the frequency. Thus the best approximations for these problems are the functions v.

In order to obtain v in the Dirichlet case one should solve the singular integral equation (3.4) to obtain σ and use (3.6) to obtain solution in the exterior. Similarly to obtain v in the interface problem we solve the system (4.11) to obtain the densities (Φ,Ψ) and use (4.13) to construct solution at a desired point. We emphasize that our numerical process need be applied only once for the problem (P_0) or $(P_{0\alpha})$. This process is much simpler than the corresponding one for (P_k) or $(P_{k\alpha})$ given in [3].

We performed some numerical experiments to verify this theory. To this end we performed computations for an exterior problem. The numerical method is the same as the one in [5] and we refer the reader to that paper. The method is general enough to compute all geometries which are polar representable. If $r = R(\theta)$ is the function which characterize the geometry then point on the boundary Γ is given by $(x,y) = \left(R(\theta)\cos\theta,\ R(\theta)\sin\theta\right)$. We performed calculations for a body $R(\theta) = 3 + 2\cos\theta$, (see Figure 1), with a single source on the x axis namely at (7,0).

The solutions for different frequencies at points equi-distant $\theta_i = (i-1)\frac{\pi}{4}$, $(i = 1,\cdots,5)$ are computed at a radius 6 which is between the scatterer and the source. The results are presented in Table 1. In the Table H•E denotes the solution of u^k (Helmholtz equation) and A•E denotes the solution of v (approximate equation) and L•E denotes the solution of Laplace equation. Re and Im denote the real and imaginary part. The solutions clearly indicate that the difference between u^k and v is $O(k^2 \log k)$ and they becomes almost identical for a frequency of .01. However, the difference between the solution of Laplace equation is substantial relative to the magnitudes of the solution. These results confirm our low frequency results and validate the fact that v provides a reasonably accurate result.

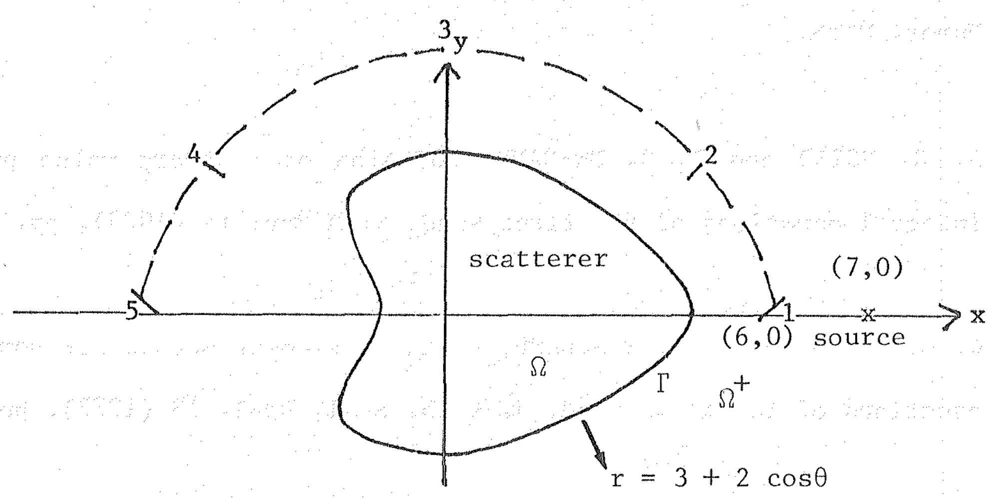

Figure 1

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Table I

$\theta_i = \frac{i\pi}{4}$	Re H•E	Re A•E	L•E	Im H•E	Im A•E	Im L•E
k = .50						
1	−.2017	−.2055	−.2026	−.1235	−.0114	0.0000
2	+.0327	−.0496	−.0454	−.0310	−.0169	0.0000
3	−.0024	−.0496	−.0315	−.0134	−.0274	0.0000
4	−.0026	−.0387	−.0299	−.0031	−.0356	0.0000
5	+.0028	−.0393	−.0298	−.0011	−.0385	0.0000
k = .10						
1	−.2011	−.1967	−.2026	−.0141	−.0076	.0000
2	−.0347	−.0367	−.0454	−.0162	−.0112	.0000
3	−.0071	−.0174	−.0315	−.0145	−.0182	.0000
4	+.0031	−.0116	−.0299	−.0091	−.0236	.0000
5	+.0056	−.0100	−.0298	−.0065	−.0255	.0000
k = .05						
1	−.1985	−.1967	−.2026	−.0066	−.0049	.0000
2	−.0367	−.0366	−.0454	−.0086	−.0072	.0000
3	−.0141	−.0172	−.0315	−.0107	−.0117	.0000
4	−.0054	−.0113	−.0299	−.0109	−.0152	.0000
5	−.0027	−.0097	−.0298	−.0106	−.0165	.0000
k = .01						
1	−.1983	−.1981	−.2026	−.0021	−.0020	.0000
2	−.0388	−.0387	−.0454	−.0030	−.0030	.0000
3	−.0205	−.0207	−.0315	−.0048	−.0040	.0000
4	−.0154	−.0158	−.0299	−.0061	−.0063	.0000
5	−.0140	−.0146	−.0298	−.0065	−.0068	.0000

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16. Abstract

This paper deals with two classes of problems arising from acoustics and electromagnetics scattering in the low frequency situations. The first class of problem is solving Helmholtz equation with Dirichlet boundary conditions on an arbitrary two-dimensional body while the second one is an interior-exterior interface problem with Helmholtz equation in the exterior. Low frequency analysis show that there are two intermediate problems which solve the above problems accurate to $O(k^2 \log k)$ where k is the frequency. These solutions greatly differ from the zero frequency approximations. For the Dirichlet problem numerical examples are shown to verify our theoretical estimates.

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