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PSEUDOSPECTRAL SOLUTION OF TWO-DIMENSIONAL GAS-DYNAMIC PROBLEMS

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PSEUDOSPECTRAL SOLUTION OF TWO-DIMENSIONAL GAS-DYNAMIC PROBLEMS

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ABSTRACT

Chebyshev pseudospectral methods are used to compute two-dimensional smooth compressible flows. Grid refinement tests show that spectral accuracy can be obtained. Filtering is not needed if resolution is sufficiently high and if boundary conditions are carefully prescribed.

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1. Introduction

Pseudospectral approximations to the compressible Euler equations have recently been studied as an alternative to second or fourth order finite differences [1], [2]. The motivation is to obtain the superior accuracy characteristic of spectral solutions to smooth incompressible flows. For simple linear hyperbolic models it is easy to demonstrate that spectral approximations are indeed far superior to finite difference approximations. (See, for example, Turkel [1], Hussaini, Salas and Zang [3].)

Hussaini, et. al. [4] have shown that spectral approximations do work in a variety of compressible flows. However, it has not been demonstrated that spectral accuracy is obtained in actual problems. Typically, more complicated flows have shocks and shock capturing spectral approximations introduce global oscillations which must be smoothed. In such cases a definite advantage in accuracy or convergence rate of spectral over finite difference approximations has not yet been established [4], [5]. For this reason we examine the use of spectral methods in conjunction with shock fitting algorithms. In the region where the solution is smooth there is hope of obtaining spectral accuracy.

In this paper we examine the accuracy of spectral methods when applied to smooth compressible flows. Even in such cases, spectral solutions sometimes exhibit "wiggles". We look at the need for smoothing the solutions to such problems. Two benchmark problems which are non-trivial and two-dimensional are used. The first is the interaction of a plane wave with a shock. The second is the classical Ringleb flow.

2. Pseudospectral Method

The novelty of the pseudospectral methods is that the solution defined at each grid point is represented by a global high order interpolating polynomial. Derivatives, when computed from the interpolant, couple all the points. For periodic problems Fourier interpolation is appropriate. Boundary value problems can use Chebyshev polynominals. Computation of the coefficients is done efficiently through the use of the Fast Fourier Transform. For continuous solutions the interpolation error decays faster than any polynominal of the number of mesh points.

The problems in the next two sections use the Euler equations in non-conservation form

$$Q_t + BQ_X + CQ_Y = 0 \quad X,Y\varepsilon[0,1],\ t \geq 0 \tag{1}$$

where Q is the column vector of the unknowns and $B(Q)$, $C(Q)$ are square matrices. For the shock/plane wave interaction problem of Section 3, the Y direction is periodic. Q is approximated by a Chebyshev-Fourier expansion

$$Q(X,Y,t) = \sum_{p=0}^{M} \sum_{q=-N/2}^{N/2-1} Q_{pq}(t)T_p(\xi)e^{2\pi iqY}, \tag{2}$$

where $\xi = 2X-1$. The coefficients Q_{pq} are products of the Chebyshev and Fourier coefficients computed from the values of Q at the mesh points. The derivatives of the interpolant are

$$Q_X = 2\sum_{p=0}^{M} \sum_{q=-N/2}^{N/2-1} Q_{pq}^{(1,0)}(t)T_p(\xi)e^{2\pi iqY} \tag{3}$$

$$Q_Y = 2\pi\sum_{p=0}^{M} \sum_{q=-N/2}^{N/2-1} Q_{pq}^{(0,1)}(t)T_p(\xi)e^{2\pi iqY} \tag{4}$$

The coefficients $Q_{pq}^{(1,0)}$ are computed with the standard recursion formula [3] and

$$Q_{pq}^{(0,1)} = iqQ_{pq} \tag{5}$$

For the Ringleb problem a double Chebyshev approximation is used and the solution is approximated by

$$Q(X,Y,t) = \sum_{p=0}^{M} \sum_{q=0}^{N} Q_{pq}(t)T_p(\xi)T_q(\eta), \tag{6}$$

Where $\eta = 2Y-1$. The derivatives in both directions are evaluated in a manner analogous to eq. (3).

While the approximation of derivatives at boundaries often requires points outside the mesh, this is not the case for the spectral approximations. The derivatives use only points within the mesh and hence do not require special treatment.

The time discretization used is the second order modified Euler. Let $L(Q)$ denote the spatial discretization of $B\partial_X + C\partial_Y$ and let $t = n\Delta t$. Then

$$\begin{aligned} \tilde{Q} &= [1-\Delta t L^n]Q^n \\ Q^{n+1} &= \tfrac{1}{2}[Q^n+(1-\Delta t\tilde{L})\tilde{Q}] \end{aligned} \tag{7}$$

where $\tilde{L} = L(\tilde{Q})$.

3. Shock/Plane Wave Interaction

The first benchmark problem is the time-dependent interaction of a plane wave with an infinite normal shock. We use this to demonstrate the appearance

of wiggles in a case where the relevant features are not resolved. A detailed discussion of the problem and a comparision of finite difference computations with linear theory predictions can be found in Zang, et. al. [6]. We comment here only that for low amplitude waves whose wave fronts are nearly parallel to the moving shock the linear theoretical solutions are quite accurate.

Let $x_s(y,t)$ denote the position of an infinite shock moving from left to right into a gas which is quiescent except for a specified pressure wave of amplitude, $A(x)$. We allow the amplitude to vary smoothly from zero to a constant value so that the shock interacts with a smooth perturbation. In the absence of the pressure wave the shock would remain plane and move with a shock Mach number M_s.

The computational domain lies between some arbitrarily chosen left boundary x_L and the shock on the right. The y direction is periodic, $-\infty < y < \infty$. This domain is mapped to the unit square by

$$X = (x-x_L)/(x_s-x_L), \quad Y = y/y_\ell \tag{8}$$

where y_ℓ is the period in y. The dependent variables are $Q = (P\ u\ v\ S)^T$ where P is the logarithm of the pressure, u and v are the velocities in x and y, and S is the entropy divided by the specific heat at constant volume.

The boundary conditions at $Y = 0$ and 1 are periodic. The right side is bounded by the moving shock and a shock fitting algorithm is used to determine the flow variables and move the shock. The left boundary is supersonic inflow so all variables are specified.

Table I shows the RMS error for the acoustic transmission coefficient of an incident 10^o pressure wave for $A = 0.001$ with an $M_s = 3$ shock on three

different Chebyshev grids. The error is defined by $e^2 = \sum_{p=1}^{M} (A'_p - A'_e)^2/M$. The transmission coefficient A'_p is taken as the fundamental Fourier amplitude computed by a Fourier transform in the Y direction at each grid point in X. The linearized solution is A'_e.

TABLE I

Rms Error in Acoustic Transmission Coefficient

Number of Chebyshev Modes	RMS Error
8	13.0
16	2.4
32	0.062

Figure 1 shows A' as a function of x behind the shock for the N = 16 and 32 Chebyshev grids. The solid line shows the linear theory results for a constant amplitude wave. The numerical wave is started up smoothly. Because the 16 point mesh cannot resolve the startup rise, large oscillations are present. (Note that at the time chosen the beginning of the wave occurs near the coarsest grid spacing.) If the solution is adequately resolved as in the 32 point calculation, the oscillations are almost eliminated.

For this problem it has not been necessary to smooth or filter for the purpose of stability. Aesthetic reasons may lead one to cosmetically filter the solution. However, it must be remembered that these oscillations indicate that the solution is not adequately resolved. See also Gottlieb et. al. [7] who show that stability can be obtained without filtering if resolution is adequate.

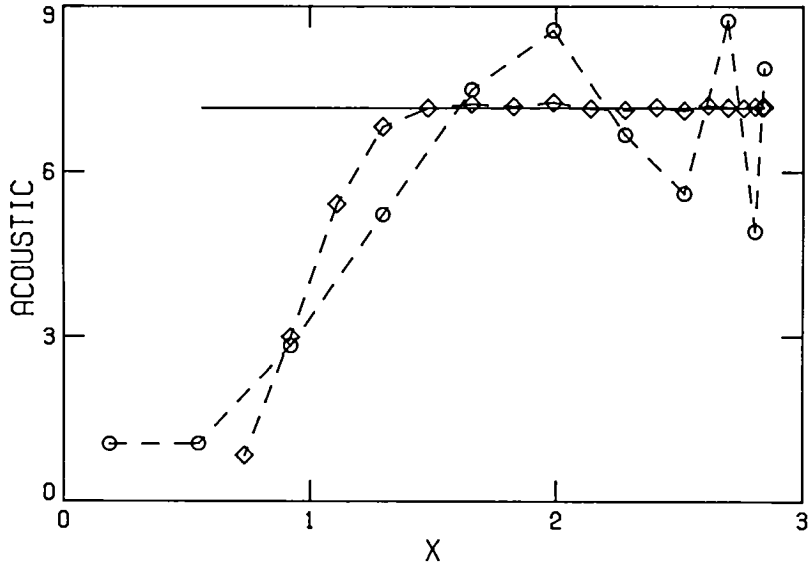

Figure 1: Acoustic transmission coefficient computed with a 16 (circles) and 32 (diamonds) point Chebyshev grid. The solid line is the linear theory prediction.

4. Ringleb Flow

The classical Ringleb flow is used for the second benchmark problem. We use this flow to test the algorithm on a smooth, steady, two-dimensional supersonic and transonic problem for which an exact solution is available. We also use it to bring out aspects associated with the specification of boundary conditions for Chebyshev methods.

The dependent variables $Q = (P\ u\ v)^T$ were chosen where (u,v) are the Cartesian velocity components. Since the flow is homentropic we simply set the entropy constant. The flow geometry is computed from the exact solution by specifying the Mach numbers at the inflow and outflow along the lower streamline and the outflow Mach number along the upper streamline. Figure 2 shows the Mach contours of the transonic problem used. The geometry is mapped

onto a square by a transformation to (ϕ,ψ), the potential-streamfunction coordinates, which are computed from the exact solution. A double Chebyshev grid is then used in these coordinates. Figure 3 shows a 17 × 9 point grid for the flow of fig. 2. The supersonic flow uses only the exit portion of the channel.

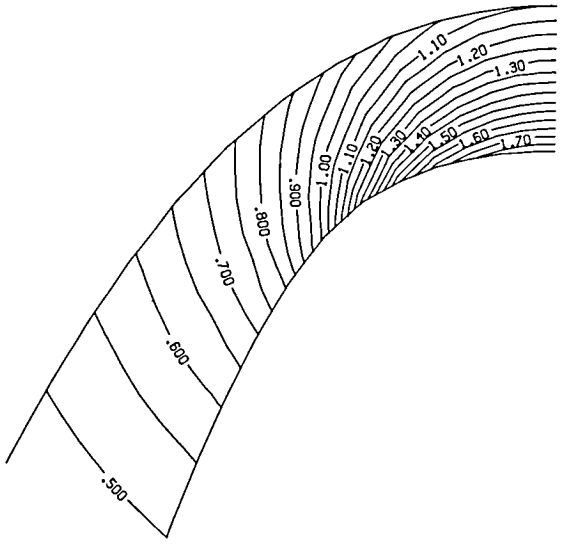

Figure 2: Mach number contours for the exact solution to the Ringleb flow.

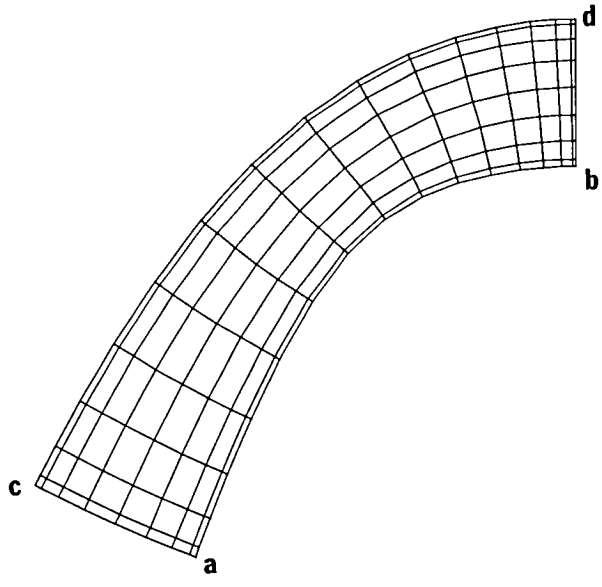

Figure 3: The 17 x 9 Chebyshe-Chebyshev grid for the flow in fig. 2.

In the mapped coordinate system (ϕ,ψ) correspond to (X,Y) in eq. (1). The coefficient matrices are

$$B = \begin{bmatrix} U & \gamma\phi_x & \gamma\phi_y \\ \frac{a^2}{\gamma}\phi_x & U & 0 \\ \frac{a^2}{\gamma}\phi_y & 0 & U \end{bmatrix} \quad C = \begin{bmatrix} V & \gamma\psi_x & \gamma\psi_y \\ \frac{a^2}{\gamma}\psi_x & V & 0 \\ \frac{a^2}{\gamma}\psi_y & 0 & V \end{bmatrix} \qquad (11)$$

where a is the sound speed and γ is the ratio of specific heats. The contravariant velocity components are

$$U = u\phi_x + v\phi_y$$
$$V = u\psi_x + v\psi_y \tag{12}$$

The specification of the boundary conditions has turned out to be a most important aspect of computing the Ringleb problem. Reference [7] details several studies applied to finite difference methods. We now describe the approach which works best with the Chebyshev methods.

The Ringleb problem requires several types of boundary conditions. First, the upper and lower streamlines (**ab** and **cd** in fig. 3) are treated as impermeable boundaries, hereafter referred to as walls. The outflow boundary **bd** is chosen to be supersonic. Finally, the inflow boundary **ac** can be either a subsonic or supersonic boundary, depending on where it is placed along the channel.

For the wall boundaries the tangential momentum equation can be written as

$$U_t + U\left(u_\phi \phi_x + v_\phi \phi_y\right) + \frac{a^2}{\gamma}|\nabla\phi|^2 P_\phi = 0 \tag{13}$$

The equation is left in this form without explicitly writing the ϕ derivative of the contravariant velocity, U, because the derivatives of u and v are available from the Chebyshev interpolant. The spatial derivatives directly at the wall are computed as described in Section 2. The time discretization is performed as in eq. (7). From the fact that the contravariant velocity, $V = 0$ along the wall, u and v can be determined.

Particular care must be used in specifying the wall pressure when using spectral methods. An example of the disastrous results which can occur when boundary conditions are overspecified can be seen in reference [4]. Computing the pressure from the enthalpy or directly from the pressure equation are also unsatisfactory. Such boundary conditions produce wiggles even for finite difference computations. The only approach which works effectively is to use the compatibility relation for the characteristics intersecting the wall from the interior of the flow.

By combining the pressure and normal momentum equations, an equation for the pressure is

$$P_t = \mp a|\nabla\psi|P_\psi - \left[UP_\phi + \gamma\left(u_\psi\psi_x + u_\phi\phi_x + v_\psi\psi_y + v_\phi\phi_y\right)\right.$$
$$\left. \pm \frac{\gamma U}{a|\nabla\psi|}\left(u_\phi\psi_x + v_\phi\psi_y\right)\right] \tag{14}$$

where the upper sign applies to the lower wall and the lower sign to the upper wall boundary. The spatial derivatives are again computed from the Chebyshev interpolant and no special treatment is needed. The equation is updated according to eq. (7).

The supersonic outflow and inflow boundaries pose no difficulties. At the inflow all the quantities are specified. The outflow requires no boundary condition, either physical or numerical. Unlike typical finite difference methods, particularly high order ones, the Chebyshev discretization does not require any so-called "numerical" boundary conditions.

Finally, for the subsonic inflow we specify the total enthalpy and the angle of the flow. Typically this leads to a faster approach to the steady state. A compatibility condition combining the normal momentum equation and the pressure equation is

$$P_t + (U-a|\nabla\phi|)P_\phi - \frac{\gamma}{|\nabla\phi|a}\left[U_t+U\left(u_\phi\phi_x+v_\phi\phi_y\right)\right]$$

$$= -\gamma\left(u_\psi\psi_x+u_\phi\phi_x+v_\psi\psi_y+v_\phi\phi_y\right) \tag{15}$$

Since the total enthalpy is taken to be a constant along the inflow boundary, another relation between P and U can be obtained by differentiating the total enthalpy equation in time

$$P_t = -\frac{U}{|\nabla\phi|^2}\, e^{-\frac{\gamma-1}{\gamma}P}\, U_t \tag{16}$$

Solving eq. (14) and (15) allows both P_t and U_t to be computed. They too are updated according to eq. (7). From the computed U and the fact that $V = 0$, the Cartesian velocities are calculated.

TABLE II

Maximum Error in p for MacCormack and Spectral Computation of Supersonic Ringleb Flow

Grid	MacCormack	Spectral
5 x 5	2.2×10^{-2}	7.5×10^{-4}
9 x 9	4.1×10^{-3}	1.1×10^{-6}
17 x 17	1.0×10^{-3}	6.6×10^{-11}

The fully supersonic flow is a relatively easy problem to compute. A 9 x 9 grid used is shown in fig. 4. The exact solution was chosen as the initial condition and the computations were run long enough for errors to propagate out of the mesh. The time steps were kept small so that the errors would be dominated by the spatial discretization. For the 9 x 9 computation 2000 time steps were used. The Mach contours for that solution are shown in

fig. 5. A grid refinement study is presented in Table II where the maximum error in the pressure from the spectral calculations are compared to second order MacCormack finite difference results. The superior error convergence for the spectral computations is clear.

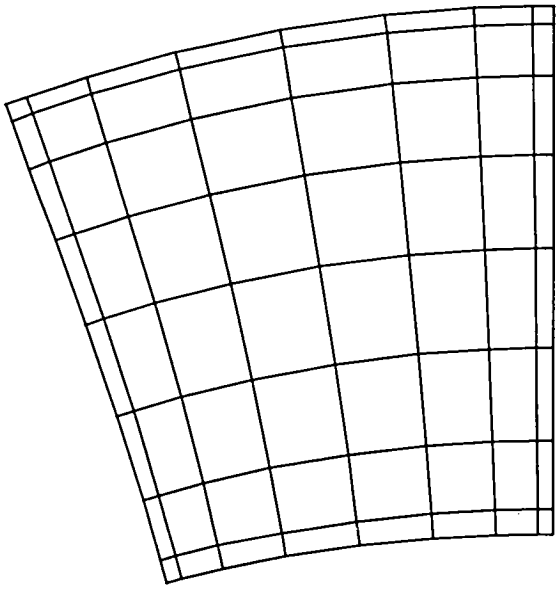

Figure 4: The 9 x 9 Chebyshev-Chebyshev grid used for the supersonic computation.

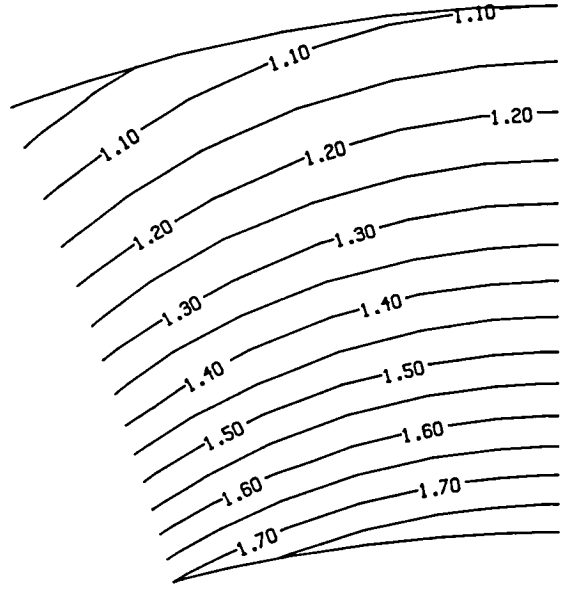

Figure 5: Computed Mach number contours of the supersonic flow for the 9 x 9 grid.

The computation of the transonic flow depicted in fig. 2 is more difficult than the supersonic flow of fig. 5. The reason is the presence of the sonic line and the rapid expansion to sonic conditions along the inner wall. The computations were started with the exact solution and run for approximately the same length of physical time. The slow, explicit time integration method used does not allow relaxation to convergence. The Mach contours of a 17 x 9 point calculation are shown in fig. 6 and can be compared directly with fig. 2. The largest errors occur near the high curvature

section of the lower wall near the sonic line, and at the lower inflow corner. A grid refinement study is shown in Table III. Though the results are not as spectacular as the supersonic case, the spectral still outperforms the finite difference computations.

Finally, no filtering was needed for the Ringleb problem either for the supersonic or the transonic cases. Solutions with wiggles result from boundary conditions other than the ones which we described. Application of the compatibility relations at the boundaries appears to be the best approach.

TABLE III

Maximum Error in p for MacCormack and Spectral Computation of Transonic Ringleb Flow

Grid	MacCormack	Spectral
9 x 5	2.6×10^{-2}	2.2×10^{-2}
17 x 9	1.1×10^{-2}	1.9×10^{-3}
33 x 17	3.2×10^{-3}	5.0×10^{-5}

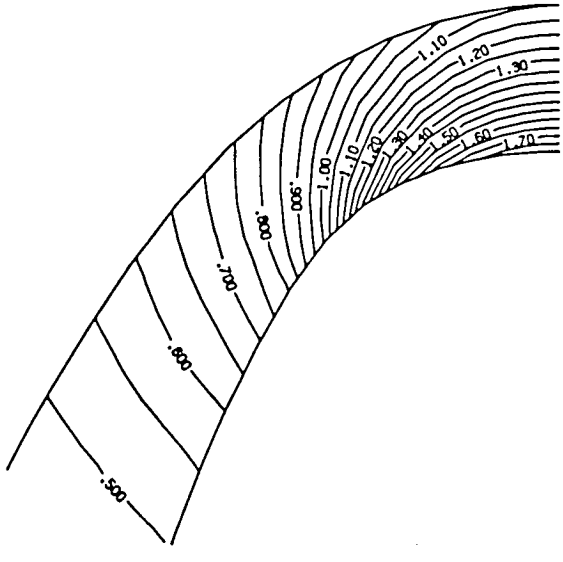

Figure 6: Computed Mach number contours of the transonic flow for the 17 x 9 grid.

5. CONCLUSIONS

We have shown that for two-dimensional smooth flows it is possible to obtain spectral accuracy characteristic of more simple problems. The first problem considered, that of the shock/plane wave interaction, needed no smoothing for stability. Oscillations were significant only if the flow was not well resolved. The Ringleb problem provided a more general boundary value test. Careful specification of the boundary conditions allowed the computation to be performed without smoothing.

Though the spectral method is superior to the finite difference method in terms of accuracy, the computation times for the spectral are far longer. One major difficulty comes from the use of an explicit time differencing procedure. Even for finite difference computations convergence to a steady state is very slow without some acceleration procedure. The spectral computations have the added disadvantage that the time step, which depends on the grid spacing, varies with N^2 rather than N. The widespread use of the spectral methods will even more strongly depend on the development of fast relaxation methods for the Euler equations.

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