

NASA-CR-172,196

NASA Contractor Report 172196

ICASE

NASA-CR-172196
19830028500

**NEW DISCRETIZATION AND SOLUTION TECHNIQUES FOR
INCOMPRESSIBLE VISCOUS FLOW PROBLEMS**

For Reference

NOT TO BE REMOVED FROM THIS ROOM

M. D. Gunzburger
R. A. Nicolaides
C. H. Liu

Contract No. NAS1-17130
August 1983



**INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665**

Operated by the Universities Space Research Association



**National Aeronautics and
Space Administration**

**Langley Research Center
Hampton, Virginia 23665**

LIBRARY COPY

OCT 18 1983

**LANGLEY RESEARCH CENTER
LIBRARY - NASA
HAMPTON, VIRGINIA**

NEW DISCRETIZATION AND SOLUTION TECHNIQUES FOR
INCOMPRESSIBLE VISCOUS FLOW PROBLEMS

M. D. Gunzburger* and R. A. Nicolaides*

Carnegie-Mellon University
Pittsburgh, PA 15213

C. H. Liu

NASA Langley Research Center
Hampton, VA 23665

Abstract

This paper considers several topics arising in the finite element solution of the incompressible Navier-Stokes equations. Specifically, the question of choosing finite element velocity/pressure spaces is addressed, particularly from the viewpoint of achieving stable discretizations leading to convergent pressure approximations. Following this, the role of artificial viscosity in viscous flow calculations is studied, emphasising recent work by several researchers for the anisotropic case. The last section treats the problem of solving the nonlinear systems of equations which arise from the discretization. Time marching methods and classical iterative techniques, as well as some recent modifications are mentioned.

*Research reported here was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17130 while the authors were in residence at The Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, Va 23665.

Discretization

Continuous Problem

Let Ω be a bounded region of \mathbb{R}^2 or \mathbb{R}^3 , the flow region, and let \underline{u} and p denote the velocity and pressure fields, respectively and ν the kinematic viscosity. Normalizing the pressure by the constant density, the stationary Navier-Stokes equations take the form

$$\underline{u} \cdot \nabla \underline{u} + \nabla p = \nu \Delta \underline{u} + \underline{f} \quad \text{in } \Omega \quad (1)$$

$$\operatorname{div} \underline{u} = 0 \quad (2)$$

$$\underline{u} = \underline{g} \quad \text{on } \partial\Omega \quad (3)$$

where \underline{f} and \underline{g} are given functions and $\partial\Omega$ denotes the boundary of Ω .

In conservation form (1) may be written

$$\operatorname{Div}(\underline{u} \underline{u}^T) + \nabla p = \nu \Delta \underline{u} + \underline{f} \quad (4)$$

where Div denotes the tensor divergence operator and the equivalence of (1) and (4) is shown using (2). Equation (2)-(4) are the equations to be solved.

The standard weak form of (2)-(4) is: find $\underline{u} \in \bar{H}^1(\Omega)$ satisfying (3) and $p \in L_0^2(\Omega)$ such that

$$\nu \int_{\Omega} \nabla \underline{u} : \nabla \underline{v} - \int_{\Omega} (\underline{u} \underline{u}^T) : \nabla \underline{v} \quad (5)$$

$$- \int_{\Omega} p \operatorname{div} \underline{v} = \int_{\Omega} \underline{f} \cdot \underline{v} \quad \forall \underline{v} \in \bar{H}_0^1(\Omega)$$

$$\int_{\Omega} q \operatorname{div} \underline{u} = 0 \quad \forall q \in L_0^2(\Omega). \quad (6)$$

In the above formulation, $\bar{H}^1(\Omega)$ is the usual Sobolev space of functions with one square integrable derivative, $\bar{H}_0^1(\Omega)$ is that subspace of $\bar{H}^1(\Omega)$ whose elements are zero on $\partial\Omega$ and $L_0^2(\Omega)$ consists of square integrable functions with zero mean over Ω .

A slightly different treatment of the convection term is given in the following weak form of the momentum equation¹

$$\begin{aligned} \nu \int_{\Omega} \underline{u} : \nabla \underline{v} - 1/2 \int_{\Omega} (\underline{u} \cdot \nabla \underline{u} \cdot \underline{v} - \underline{u} \cdot \nabla \underline{v} \cdot \underline{u}) \\ - \int_{\Omega} p \operatorname{div} \underline{v} = \int_{\Omega} \underline{f} \cdot \underline{v} \quad \forall \underline{v} \in \bar{H}_0^1(\Omega). \end{aligned} \quad (7)$$

Again, in view of (2), this formulation is equivalent to (5); however for computations, the form (7) possesses certain advantages,¹ and thus is recommended.

Discrete Problem

Proceeding in the usual way, one chooses finite dimensional subspaces $V^h \subset \bar{H}^1(\Omega)$, $S^h \subset L_0^2(\Omega)$, and seeks $\underline{u}^h \in V^h$ satisfying

$$\underline{u}^h = \underline{g}^h \quad \text{on } \partial\Omega \quad (8)$$

and $p^h \in S^h$ such that

$$\begin{aligned} \nu \int_{\Omega} \underline{u}^h : \nabla \underline{v}^h - 1/2 \int_{\Omega} (\underline{u}^h \cdot \nabla \underline{u}^h \cdot \underline{v}^h - \underline{u}^h \cdot \nabla \underline{v}^h \cdot \underline{u}^h) \\ - \int_{\Omega} p^h \operatorname{div} \underline{v}^h = \int_{\Omega} \underline{f} \cdot \underline{v}^h \quad \forall \underline{v}^h \in V_0^h \end{aligned} \quad (9)$$

$$\int_{\Omega} \underline{q}^h \operatorname{div} \underline{u}^h = 0 \quad \forall \underline{q}^h \in S^h. \quad (10)$$

In (8) \underline{g}^h denotes an approximation, in V^h restricted to $\partial\Omega$, to \underline{g} (e.g., an interpolant). V_0^h is the subspace of V^h of trial functions which are zero on $\partial\Omega$. Strictly speaking, this formulation is valid for polygonal domains but may be easily extended, e.g., by isoparametric techniques, to more general domains.

Unlike the standard positive definite elliptic case^{2,3} mere inclusion of $V^h \subset \bar{H}^1(\Omega)$ and $S^h \subset L_0^2(\Omega)$ is not sufficient to ensure convergence (or even existence) of the discrete solutions. In fact, the spaces V^h and S^h cannot be chosen independently of each other. Mathematically, the following condition is required⁴

$$\sup_{\substack{\underline{v}^h \in V_0^h \\ |\underline{v}^h| = 1}} \int_{\Omega} \underline{q}^h \operatorname{div} \underline{v}^h > \gamma \|\underline{q}^h\| \quad \forall \underline{q}^h \in S^h \quad (11)$$

where $\gamma > 0$ is independent of the discretization parameter h . Here, the norms used in (11) are defined by

$$|\underline{v}|^2 = \int_{\Omega} \nabla \underline{v} : \nabla \underline{v} \quad \forall \underline{v} \in H_0^1(\Omega)$$

$$\|\underline{q}\|^2 = \int_{\Omega} \underline{q}^2 \quad \forall \underline{q} \in L_0^2(\Omega).$$

There are other mathematical conditions on the discrete spaces which must be imposed in order to guarantee convergence,⁴ however it is (11) which can fail to hold in general, and anyway offers substantial difficulties in its

verification. Equation (11) is referred to as the condition of "div-stability".^{5,6}

Note that the analogous condition

$$\sup_{\substack{\underline{v} \in \bar{H}_0^1(\Omega) \\ |\underline{v}| = 1}} \int_{\Omega} q \operatorname{div} \underline{v} > \gamma \|q\| \quad \forall q \in L_0^2(\Omega)$$

is necessary to guarantee the existence and stability of the solution of the continuous problem, i.e., the Navier-Stokes equations. This condition is easily verified since it is well known that the problem

$$\operatorname{div} \hat{\underline{v}} = q \quad \text{in } \Omega$$

$$\hat{\underline{v}} = 0 \quad \text{on } \partial\Omega$$

has a solution $\hat{\underline{v}} \in \bar{H}_0^1(\Omega)$ for any $q \in L_0^2(\Omega)$ which satisfies

$$\gamma |\hat{\underline{v}}| < \|q\|$$

for some constant $\gamma > 0$. Then

$$\sup_{\substack{\underline{v} \in \bar{H}_0^1(\Omega) \\ |\underline{v}| = 1}} \int_{\Omega} q \operatorname{div} \underline{v} > \int_{\Omega} q \operatorname{div} \hat{\underline{v}} / |\hat{\underline{v}}|$$

$$= \int_{\Omega} q^2 / |\hat{\underline{v}}| > \gamma \|q\|.$$

div-Stable Elements

In practice, one would like to use low degree piecewise polynomial trial spaces V^h, S^h . Unfortunately these are the spaces which encounter the most difficulty in satisfying (11). For the higher order cases, a reasonably efficient test for showing that an element pair is div-stable is known.⁵ No such simple test is known for the low order cases. This is discussed further below.

Perhaps the best known low order pair is the piecewise bilinear velocity/piecewise constant pressure combination defined on a quadrilateral subdivision of Ω . This pair has been extensively used in engineering computations and been the object of much theoretical work. In particular, the checkerboard pressure mode is well documented.⁷ The effect of this mode is to make γ in (11) zero, so (11) is caused to fail in this case. Less well known⁸ is the existence of a mode such that $\gamma = O(h)$. This causes the pressure approximation to fail to converge in general, even when the standard checkerboard mode is filtered out. This is proved in Boland and Nicolaides⁶, where also a filtering technique applicable to many cases is given. Use of this filter enables the pressures to converge optimally.

Unfortunately, for arbitrary elemental subdivisions of Ω , the appropriate filters are not known. Therefore, it seems advisable to use element pairs known a priori to be div-stable. Here are presented two element pairs which have been used in practical calculations and have been shown rigorously⁶ to be stable.

For the first of these, let $\Omega = \{(x,y), 0 < x,y < 1\}$ and subdivide Ω by lines $x = ih, y = jh, i,j=0,1,\dots,2n$, so that $h = 1/2n$. For V^h one may use continuous piecewise bilinear vector fields, bilinear in each subsquare. The pressure field is a subspace of all piecewise constants with zero mean on

Ω , defined as follows: on the coarser grid $h' = 1/n$ whose subsquares each contain four of the smaller subsquares, one merely imposes one constraint on the four scalars. For example, referring to Fig. 1, one imposes the constraint $s_1 + s_2 = s_3 + s_4$. For the general case, isoparametric mapping onto this macroelement is used. The resulting pair is div-stable, requires no filtering and gives convergent approximations.⁶

Another stable pair, not requiring the use of isoparametric methods is the following. Let Ω be triangulated regularly and let τ_i be an arbitrary element of the triangulation. In τ_i , p^h is taken as a piecewise constant. To define \underline{v}^h , τ_i is further subdivided into four similar triangles and \underline{v}^h is then defined to be all continuous piecewise linear fields on the finer triangulation. This element pair is also div-stable and convergent without filtering. Fig. 2a shows a typical τ_i in the pressure triangulation and Fig. 2b depicts the resulting velocity elements derived from τ_i .

Since it is known that these elements are div-stable, i.e., (11) holds, then one may use finite element theory^{1,9} to obtain the following estimates:

$$\|\underline{u} - \underline{u}^h\|_{H^1(\Omega)} \leq c_1 h \quad (12)$$

$$\|p - p^h\|_{L^2(\Omega)} \leq c_2 h$$

where c_1 and c_2 do not depend on h . The duality method then shows that

$$\|\underline{u} - \underline{u}^h\|_{L^2(\Omega)} \leq c_3 h^2. \quad (13)$$

Thus, the methods are second order, in the root mean square sense, for the velocities and first order for the pressure. These are the best possible rates obtainable with these elements.

Upwinding

Since v^h and S^h are used as both test and trial spaces in (8)-(10), the discrete equations will be of centered type. Thus, for fixed h as $\nu \rightarrow 0$ one has to expect the numerical solution to develop increasingly large oscillations¹⁰ ("wiggles"). These can of course be eliminated by use of any of the numerous upwinding/artificial diffusion methods.¹¹ However, the very idea of letting $\nu \rightarrow 0$ while keeping h fixed precludes the possibility of accurate computation of viscous effects. For this one must, at the very least, permit h to go to zero with ν . The precise dependence of h on ν depends on what viscous phenomena are to be accurately simulated.

For example, simulation of shear or tangential boundary layers certainly will require $h = O(\nu^{1/2})$ in the neighborhood of these layers. Conceivably, especially at outflows, there may be other kinds of layers, maybe induced by numerical boundary conditions, requiring the more onerous (and less practical) restriction $h = O(\nu)$. Usually the latter layers appear not to be of physical interest and may therefore be smoothed by use of anisotropic artificial diffusion, chosen so that tangential layers are not smoothed. Such methods have recently come into prominence.^{12,13} Specifically these methods attempt to add diffusion only in the streamwise direction. One must emphasize that $h = O(\nu^{1/2})$ is still an essential requirement for resolution of physical layers and hence, convergence.

One streamwise artificial diffusion method is now presented. First, observe that (4) may be written in the form

$$\text{Div}[\underline{u} \underline{u}^T + pI - \nu \nabla \underline{u}] = \underline{f} \quad (14)$$

where I is the identity matrix. The quantity in the brackets is the momentum flux density. To this one adds the term

$$- \lambda \nabla \underline{u} (\underline{u} \underline{u}^T) \quad (15)$$

where λ is a mesh dependent parameter tending to zero with h . Then (14) becomes

$$\text{Div}[\underline{u} \underline{u}^T + pI - \nabla \underline{u} (\nu I + \lambda \underline{u} \underline{u}^T)] = \underline{f} \quad (16)$$

In the form (16) one may interpret (15) as an anisotropic perturbation of the viscosity tensor (νI) . Clearly the effect of this perturbation vanishes in directions normal to the velocity vector, and hence to streamlines. Alternately, by associating the perturbation with the convection term a connection may be made, in transient cases, with the method of characteristics.¹⁴ This is not pursued further here. Equation (16), along with (2)-(3), may now be discretized in the usual manner using the same test and trial spaces. Generally, in order to achieve the desired stabilizing effect, λ should be $O(h)$. The effect of such a choice of λ on the estimates (12)-(13) is not fully understood. When using higher order elements it is reasonable to expect that the accuracy would be degraded by this crude approach. Therefore it is of interest to note that a similar effect may be achieved by appropriate choices of distinct test and trial spaces.¹³

Normally, for internal flows, e.g., cavity flows, and for good choices for numerical outflow conditions,¹⁵ $O(v)$ layers are not present in the flow field. In such cases, the use of artificial streamwise diffusion methods is not necessary.

One is still left with the task of resolving tangential layers. If a uniform mesh is used throughout the flow field, the $h = O(v^{1/2})$ restriction results in an unacceptable number of degrees of freedom. However such a small h is needed only in the neighborhood of the layers themselves. Thus it is advantageous, in the sense of reducing the number of degrees of freedom, to use nonuniform grids. As an example consider the driven cavity problem. Here Ω is a unit square, the upper end of the cavity moves with velocity $\underline{u} = (1,0)$, and $v = 1/3200$. The results of three calculations are reported. The first¹⁶ uses an upwind finite difference technique, on a uniform mesh, for a streamfunction - vorticity formulation of (1) - (3). The second¹⁷ again solves the streamfunction - vorticity formulation by a finite element technique using a nonuniform grid. The grid spacing is determined by the functions

$$x = \sin^2(\pi\bar{x}/2)$$
(17)

$$y = \sin^2(\pi\bar{y}/2)$$

with a uniform grid spacing in the \bar{x} and \bar{y} coordinates. The third calculation uses the element pair of Fig. 2 in conjunction with the primitive variable formulation and a nonuniform grid again determined by (17). In neither finite element calculation were artificial diffusion/upwinding techniques used. In Fig.3 is given the y -component of velocity v at $y = 1/2$ as a function of x . The uniform grid calculation used a

(129 x 129) grid so that $h = 0.0078$. The nonuniform grid calculations used a (19 x 19) grid. With the spacing determined by (17), the minimum $h = 0.0076$ which is comparable to the uniform h of the calculation of Ghia, et.al..¹⁶ Clearly, one can achieve the same accuracy at a greatly reduced cost by using nonuniform grids.

Solution Technique

Whatever method is employed for discretization, the outcome is a large system of nonlinear equations which must be solved for the approximate solution. Concerning the solution of this system by some classical technique, such as Newton's method, one must first observe that the Jacobian of the system requires a remarkably large amount of storage, if the usual band storage scheme is adopted. For example, it is easy to verify that a driven cavity calculation on an $n \times n$ grid requires $27n^3$ words of storage (for the bilinear/constant element). Thus, even with $n = 20$, a rather large storage requirement is apparent. In 3D, the situation is catastrophic. Hence, the classical techniques may be of limited value (although some fairly fine mesh calculations have been reported¹¹). Anyway, it is evident that alternative techniques are of interest. Of course many have been proposed. They fall into the two natural classes of "false transient" or time marching algorithms on the one hand and general nonlinear equation solvers on the other.

Here is an example of a time marching technique used successfully by the authors. Let U, P denote the unknowns and U^k, P^k the k -th approximations to U and P , and consider

$$\frac{1}{\Delta t}(U^{k+1} - U^k) + N(U^k) + BP^{k+1} = F \quad (18)$$

$$k = 0, 1, \dots$$

$$B^T U^{k+1} = G \quad (19)$$

$$k = 0, 1, \dots$$

where $N(U)$ represents an approximate Laplacian plus convection terms and B an approximate gradient operator, generated by the particular finite elements used. The first term in (19) is a discretization of $\frac{\partial u}{\partial t}$, forward in time from the k^{th} time level. U^0 is arbitrary, and is not required to satisfy (18). In the steady state limit, $U^{k+1} = U^k$ and one recovers the solution of the original problem. To get U^{k+1} , given U^k , multiply (18) by B^T and use (19) to get

$$B^T BP^{k+1} = B^T F - B^T N(U^k) \quad k \geq 1 \quad (20)$$

which must be solved for BP^{k+1} , so that then

$$U^{k+1} = U^k + \Delta t [F - BP^{k+1} - N(U^k)]. \quad (21)$$

If $k = 0$, a term

$$- \frac{1}{\Delta t} (G - B^T U^0)$$

is added to the right hand side of (20). Notice that (20) is the usual "Poisson" equation for the pressure, but with the crucial feature that boundary conditions do not need to be supplied externally. They are

automatically present in $B^T B$. Notice also that in (21) not p^{k+1} but Bp^{k+1} is required. This is the least squares solution of the equation

$$B^T V^{k+1} = B^T F - B^T N(U^k) \quad (22)$$

where

$$V^{k+1} = Bp^{k+1}$$

which may be found by any of numerous techniques, direct or iterative. The following iterative method, a generalization of a method of Kaczmarz which may be called the row projections method, was used by the authors. For solving

$$B^T V = L$$

this method is the following:

$$V^{\ell+1} = V^{\ell} + \omega \alpha B_{\ell}$$

$$V^0 = B\xi, \quad \xi \text{ arbitrary, } \ell=1,2,\dots$$

where B_{ℓ} denotes the ℓ^{th} column of B (taken cyclically), ω is a relaxation parameter and α a number chosen so that when $\omega = 1$, the ℓ^{th} component of the residual $r^{\ell+1} = L - B^T V^{\ell+1}$ is zero, thus,

$$\alpha = \frac{r_{\ell}^{\ell}}{\rho_{\ell}^2}$$

where ρ_{ℓ}^2 is the sum of the squares of the elements in row ℓ of B^T . Geometrically, the error $(V - V^{\ell+1})$ is projected orthogonally in succession onto the planes whose normals are the rows of B^T . ω effects a kind of

overprojection. Generalization to projection onto the planes normal to several rows (corresponding to line, etc., relaxation) is straightforward.

Returning to (20), note that only few iterations of (22) are required for each time step, due to the availability of the solution from the earlier time level.

The problem with time marching based on first order time derivatives is the large number of time steps (regardless of the size of ν) needed to reach the steady state. Generally, several thousand steps are needed for each digit of accuracy required, for the kind of grid sizes encountered in practice, and the number of steps rapidly increases as the latter approach zero. Using higher order time derivatives would give considerable improvement, but does not appear to be a common idea. Among the other methods, great promise is shown by the reduced basis - continuation techniques.^{17,18} A nonlinear conjugate gradient method is used by Glowinski, et.al.¹⁹, although it does not appear to function well in all cases. Various iterative techniques have also been applied with mixed success by the authors. These will be reported elsewhere.

Conclusions

In connection with the incompressible Navier-Stokes equations discretization techniques via finite elements, accuracy questions, and methods for solving the algebraic systems of nonlinear equations, all for the \underline{u}, p (primitive) variable case, have been discussed. It is fair to say that the first topic is by now reasonably well understood. The main difficulty concerned the discretization of the incompressibility condition, and this topic now has a variety of theoretical analyses and tests whereby the

stability of the discretization can be verified before calculations are performed.

Less satisfactory is the state of affairs relating to solving the systems of nonlinear equations. Here, there are many problems still to be overcome, mostly concerning the efficiency of the solution procedures. The time marching method discussed, along with similar methods based on first order time derivatives really is too inefficient, requiring of the order of $10^3 - 10^4$ time steps to obtain each digit of the solution. On the other hand, the direct solution methods encounter problems caused by the large storage resources necessary for carrying out the matrix manipulations as well as domain of convergence problems as $\nu \rightarrow 0$. In the latter case, the starting approximation has to be closer and closer to the exact solution being computed, as ν becomes smaller. Continuation techniques are naturally suggested for dealing with this latter issue, as stated in the text.

References

1. V. Girault and P.-A. Raviart, Finite element approximation of the Navier-Stokes equations, Lecture Notes in Mathematics 749, Springer, New York, (1979).
2. G. Strang and G. Fix, An Analysis of the Finite Element Method, Prentice-Hall, Englewood Cliffs, (1972).
3. J. Oden and J. Reddy, An Introduction to the Mathematical Theory of Finite Element, Wiley, New York, (1976).
4. F. Brezzi, On the existence, uniqueness and approximation of saddle point problems arising from Lagrange multipliers, RAIRO, R2, pp. 129, (1974).
5. J. Boland and R. Nicolaides, Stability of finite element discretizations of incompressibility conditions, SIAM J. Numer. Anal., to appear, (1983).
6. J. Boland and R. Nicolaides, Stable and semi-stable low order finite element schemes for solving Navier-Stokes equations, SIAM J. Numer. Anal., to appear.
7. R. Sani, P. Gresho, R. Lee and D. Griffiths, The cause and cure (?) of the spurious pressures generated by certain FEM solutions of the incompressible Navier-Stokes equations: Part 1, Int. J. Numer. Meth. Fluids, 1, p. 17, (1981).

8. J. Boland and R. Nicolaides, Counterexample to the uniform div-stability of bilinear-constant velocity-pressure finite elements for viscous flows, Numer. Math., to appear.
9. M. Gunzburger and J. Peterson, On conforming finite element methods for the inhomogeneous stationary Navier-Stokes equations, Numer. Math., to appear.
10. P. Gresho and R. Lee, Don't suppress the wiggles - they're telling you something, Comput. & Fluids, 9, p. 223, (1981).
11. F. Thomasset, Implementation of Finite Element Methods for Navier-Stokes Equations, Springer, New York, (1981).
12. J. Dukowicz and J. Ramshaw, Tensor viscosity method for convection in numerical fluid dynamics, J. Comput. Phys., 32, p. 71, (1979).
13. A. Brooks and T. Hughes, Streamwise upwind/Petrov Galerkin formulation for convection-dominated flows with particular emphasis on the incompressible Navier-Stokes equations, Comput. Methods Appl. Mech. Engrg., 32, p. 199, (1982).
14. J. Douglas and T. Russell, Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures, SIAM J. Numer. Anal., 19, p. 871, (1982).

15. G. Fix and M. Gunzburger, Downstream boundary conditions for viscous flow problems, Comput. Math. Appl., 3, p. 53, (1977).
16. U. Ghia, K. Ghia and C. Shin, High-Re solutions for incompressible flow using the Navier-Stokes equations and a multigrid method, J. Comput. Phys., 48, p. 387, (1982).
17. J. Peterson, High Reynolds number solutions for incompressible viscous flow using the reduced basis technique, J. Comput. Phys., to appear.
18. H. Keller, Continuation methods in computational fluid dynamics, in Conf. Proc. on Numerical and Physical Aspects of Aerodynamic Flows, California State University, Long Beach, CA, T. Cebeci, ed., p. 3-34, Springer-Verlag, (1982).
19. R. Glowinski, B. Mantel, J. Periaux, O. Pironneau, and G. Poinien, An efficient preconditioned conjugate gradient method applied to nonlinear problems in fluid dynamics, Computing Methods in Applied Science and Engineering. North Holland, Amsterdam, (1980).

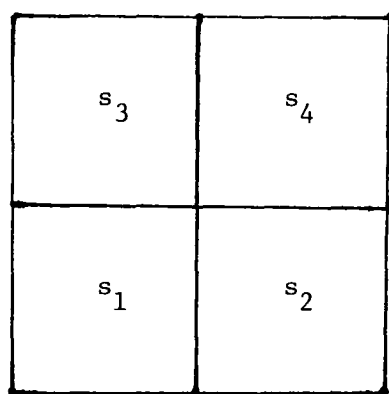


Fig. 1

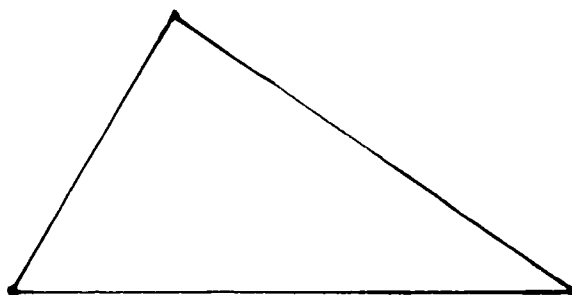


Fig. 2a

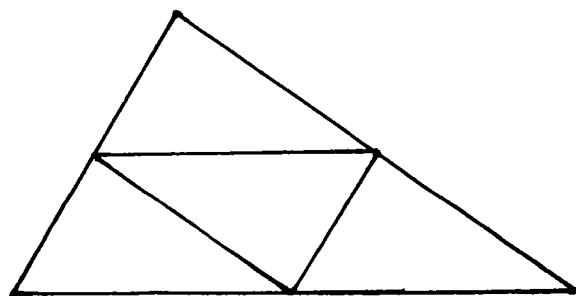


Fig. 2b

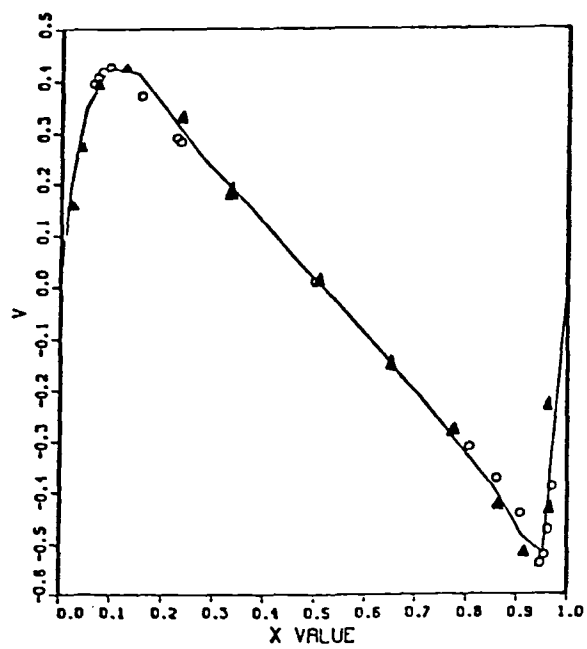


Fig. 3

—	nonuniform grid (19 x 19) ¹⁷
○ ○ ○ ○	uniform grid (129 x 129) ¹⁶
△ △ △ △	nonuniform grid (19 x 19)

1. Report No. NASA CR-172196		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle New Discretization and Solution Techniques for Incompressible Viscous Flow Problems				5. Report Date August 1983	
				6. Performing Organization Code	
7. Author(s) M. D. Gunzburger, R. A. Nicolaides, C. H. Liu				8. Performing Organization Report No. 83-44	
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665				10. Work Unit No.	
				11. Contract or Grant No. NAS1-17130	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546				13. Type of Report and Period Covered Contractor report	
				14. Sponsoring Agency Code	
15. Supplementary Notes Langley Technical Monitor: Robert H. Tolson Final Report					
16. Abstract This paper considers several topics arising in the the finite element solution of the incompressible Navier-Stokes equations. Specifically, the question of choosing finite element velocity/pressure spaces is addressed, particularly from the viewpoint of achieving stable discretizations leading to convergent pressure approximations. Following this, the role of artificial viscosity in viscous flow calculations is studied, emphasising recent work by several researchers for the anisotropic case. The last section treat the problem of solving the nonlinear systems of equations which arise from the discretization. Time marching methods and classical iterative techniques, as well as some recent modifications are mentioned.					
17. Key Words (Suggested by Author(s)) Navier-Stokes finite element			18. Distribution Statement 64 Numerical Analysis Unclassified-Unlimited		
19. Security Classif. (of this report) Unclassified		20. Security Classif. (of this page) Unclassified		21. No. of Pages 22	
				22. Price A02	

End of Document