

THEORY OF HYDROMAGNETIC TURBULENCE

David Montgomery
The College of William and Mary
Williamsburg, Virginia 23185 U. S. A.

Abstract

The present state of MHD turbulence theory as a possible solar wind research tool is surveyed. The theory is statistical, and does not make statements about individual events. It is unreasonable to expect ever to be able to "explain" individual events with turbulence theory. The ensembles considered typically have individual realizations which differ qualitatively, unlike equilibrium statistical mechanics. Most of the theory deals with highly symmetric situations; most of these symmetries have yet to be tested in the solar wind. The applicability of MHD itself to solar wind parameters is highly questionable; yet it has no competitors, as a potentially comprehensive dynamical description. The purposes of solar wind research require sharper articulation. If they are to understand radial turbulent plasma flows from spheres, laboratory experiments and numerical solution of equations of motion may be a cheap alternative to spacecraft. If "real life" information is demanded, multiple spacecraft with variable separation may be necessary to go further. The principal emphasis in the theory so far has been on spectral behavior for spatial covariances in wave number space. There is no respectable theory of these for highly anisotropic situations. A rather slow development of theory acts as a brake on justifiable measurement, at this point.

I. INTRODUCTION

One question solar wind research may ultimately have to answer for itself is whether it will resemble geography or dynamic meteorology more closely. Geography must concern itself with events which are specific and in a very real sense accidental. Its use of analytical mathematics is limited and may sometimes be simply decorative. Dynamic meteorology proceeds from the assumption that a largely complete mathematical description can be found and should be pursued, even if the complexity of the differential equations and the incompleteness of the boundary data guarantee that the program will be a long time coming to completion (see, e.g., Pedlosky, 1979).

Much of what happens in any individual rainstorm is extraordinarily striking, but it cannot usefully be considered in the light of a detailed mathematical theory. It will never happen in quite the same way again. Austere discipline is required to focus on those aspects of the weather which are at least statistically reproducible, and therefore susceptible to a mathematical theory.

Space physics has frequently taken the "event" as its unit of concern. Such-and-such a set of fluctuating field signals were seen on such-and-such a detector on such-and-such a day. Plausible hypotheses about what might have been responsible for the signals are produced, and are buttressed by such mathematics as lies ready to hand. The kind of boundary and initial data that would be necessary to extract sharp conclusions from the mathematics are invariably lacking, and the machinery for extracting the conclusions often also does not exist. A rather subjective opinion is usually necessary at the end as to whether or not the "event" has been satisfactorily "explained". This paradigm is by now deeply ingrained, and is an unconscious ingredient in the evaluation of many of the papers, say, which one finds in Journal of Geophysical Research.

The subject is at a natural stage to begin to ask what the possibilities are for making it into a mathematically tighter and more intellectually crisp area of endeavor. It is equally natural to inquire into the range of available models which have been pursued in comparable and more highly developed continuous-media situations such as meteorology, say, or oceanography.

The purpose of the following material is a consideration of the adequacy of the available solar-wind mathematical description to the task of

providing a comprehensive dynamical description. If precedents from nearby subjects are any guide, there would seem to be only one serious contender as a model for what such a mathematical description might look like. That is classical Navier-Stokes hydrodynamics, which is the basis for such theories as there are for the dynamics of the earth's atmospheres and oceans. Some exposure to hydrodynamic theory will be assumed--sufficient, at least, to take for granted the unquestioned role hydrodynamics plays as theoretical research tool in those subjects. The following pages are a survey of the present status of magnetohydrodynamic (hereafter: MHD) turbulence theory and its adequacy as an off-the-shelf research tool for describing solar wind measurements. The conclusions are not all rosy, and the analysis of the available mathematical descriptions and techniques leads to the belief that they should only be applied to solar wind data with extreme caution, and perhaps with a sense of humor.

In order of ascending complexity, the possible dynamical descriptions for the solar-wind plasma are: (1) one-fluid magnetohydrodynamics (MHD); (2) multi-species, charged-fluid hydrodynamics with assumed closures for the pressure tensors (equations of state); (3) the Vlasov description in terms of particle distribution functions; (4) Vlasov equations modified by adding Fokker-Planck collision terms on the right hand sides. Specialized models, such as the Chew-Goldberger-Low approximation, which rather arbitrarily drops heat flow along magnetic field lines, can be accommodated in various niches in the above list.

If the expected dynamics of the system were linear and non-turbulent, at least the first three models could be taken seriously as contenders. The controlled fusion (CTR) community has done so, gambling on the hope that laboratory experimentalists will be able to produce confined plasmas whose dynamics remain linear and at most weakly turbulent. But by anyone's definitions, the solar wind's behavior is unmistakably turbulent and nonlinear. The fluctuating magnetic fields, flow velocities, and electric fields are as large as anything that can be defined as averages in the local zero-momentum frame. The time history of any component of the fields behaves for all practical purposes like a random variable. This is the definition of "strong turbulence", if one is needed.

Because it is the only one of the four descriptions that is close to being manageable, even numerically, MHD assumes the role of the only serious

contender for a "strong turbulence" mathematical description. It is far simpler, mathematically, than any of the others, and yet the number of strong turbulence problems that we can handle with it will be seen to be extremely limited, even assuming its correctness.

The following material is intended as a brief look at nonlinear MHD turbulence theory, as it may be considered as a potential solar-wind research tool. Section II deals with the applicability of MHD itself to a plasma with solar-wind densities and temperatures, stressing the roles of incompressibility, collisionality, and the proper analytical form for the essential dissipative terms. Section III summarizes the status of incompressible MHD turbulence theory as it has been developed so far, emphasizing the high degrees of symmetry required if even the crudest theories are to have extractible consequences. Section IV suggests some tentative implications of Sections II and III for solar-wind research.

Anticipating the conclusions, one of them is that there is presently available at best only an outline of a theoretical framework in which kinds of solar wind data that have been collected could be sensibly interpreted. If we are serious about wanting to go beyond a largely descriptive understanding of the solar wind, a far higher fraction of our effort will have to go into understanding the basic plasma physics of the medium. The analytical and numerical tools now in hand are not adequate to the demands being placed on them by the sophisticated collection of vast quantities of data, whose quality is far higher than any framework available for making use of it.

II. THE APPLICABILITY OF MHD

The one-fluid MHD equations, in the simplest form in which they might be considered realistic, is

$$\nabla \cdot \underline{y} = 0 \quad (1)$$

$$\rho \left(\frac{\partial \underline{y}}{\partial t} + \underline{y} \cdot \nabla \underline{y} \right) = -\nabla p + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi} + \rho \nu \nabla^2 \underline{y} \quad (2)$$

$$\frac{\partial \underline{B}}{\partial t} + \underline{y} \cdot \nabla \underline{B} = \underline{B} \cdot \nabla \underline{y} + \eta \nabla^2 \underline{B}, \quad (3)$$

with a fluid velocity \underline{y} , and a magnetic field \underline{B} , a mass density ρ , a kinematic viscosity ν , and a magnetic diffusivity η . The solenoidal condition $\nabla \cdot \underline{B} = 0$, imposed initially, is preserved by Eq. (3). p is the total pressure, magnetic plus mechanical, and is obtained from the Poisson equation which results when the divergence of Eq. (2) is taken and use is made of Eq. (1).

Eqs. (1)-(3) are the simplest closed-form mathematical description known for the mechanical motions of a fluid which is both energetic enough and electrically-conducting enough to modify the magnetic field imbedded in it. Yet the simplicity of Eqs. (1)-(3) is misleading. We are far from being able to give analytical solutions except in highly simplified special cases, or in the linear limit. For reasons which are by now well known (although we will review them later), they make demands on computing capability which we cannot always expect to meet, even numerically.

Some of the assumptions which go into the derivation of Eq. (1)-(3) are widely known, such as the neglect of the displacement current relative to the conduction current, or the assumption that electrostatic forces are capable of keeping the electron and ion charge densities approximately equal. Three assumptions need to be singled out for mention in connection with the solar wind. They are not obviously fulfilled by solar wind parameters, and the serious failure of any one of them can leave us with a mathematical description which is even far less tractable than Eqs. (1)-(3). They are: (1) incompressibility ($\nabla \cdot \underline{y} = 0$); (2) scalar dissipation coefficients ν and η ; and (3) collision-dominated inequalities required in the derivation of Eqs. (1)-(3).

(1) Incompressibility ($\nabla \cdot \mathbf{y} = 0$)

Incompressibility is an undisputed feature of normal fluid mechanics that is difficult to justify rigorously. It is usually done (Landau and Lifshitz, 1959; Batchelor, 1967) by using estimates for the dominant force terms in the equation of motion and their effect, through the compressibility, on the density ρ of a moving fluid element. The change in density $\Delta\rho$ for a fluid element which experiences a change in pressure Δp may be taken to be:

$$\Delta\rho = \frac{\Delta p}{dP/d\rho} = \frac{\Delta p}{c_s^2} \quad (4)$$

If the medium obeys an equation of state $p = p(\rho)$. The sound speed is $c_s^2 \equiv dP/d\rho$. Δp may be estimated by using either the $\rho \mathbf{y} \cdot \nabla \mathbf{y}$ term in Eq. (2) or the $\mathbf{B} \cdot \nabla \mathbf{B}/4\pi$ term. (These are expected to dominate the $\rho \partial \mathbf{y}/\partial t$ term and the viscous term $\rho \nu \nabla^2 \mathbf{y}$ in cases which have significant amounts of turbulence.) For ∇ , we will use L^{-1} , where L is a characteristic length over which the fields vary. Δp may be estimated from the convective term, first, as of order $\sim \rho v^2$. In this case, the fractional variation in density is small for a typical fluid element if

$$\frac{\Delta\rho}{\rho} \sim \frac{\rho v^2}{\rho c_s^2} = v^2/c_s^2 \ll 1 \quad (5)$$

as in ordinary hydrodynamics. Then we may estimate Δp from the magnetic force term as $\Delta p \sim B^2/4\pi$, and instead of (5), we get

$$\frac{\Delta\rho}{\rho} \sim \frac{B^2}{4\pi\rho c_s^2} \ll 1 \quad (6)$$

or that the magnetic pressure shall be small compared to the mechanical pressure ($\beta \gg 1$, in conventional plasma physics jargon). If there is a strong mean field B_0 present which is large compared to the fluctuating B , $\Delta p \sim B_0 B/4\pi$, and (6) is replaced by

$$\frac{\Delta\rho}{\rho} \sim \frac{B^2}{4\pi\rho c_s^2} \frac{B}{B_0} \ll 1 \quad (7)$$

or that (again in conventional plasma terms) $B/B_0 \ll \beta$.

Though the fulfilment of conditions (5) and (7) might give some confidence in (say) the applicability of incompressible MHD for a tokamak plasma, no one of the three inequalities (5)-(7) can be said to characterize the solar wind. Yet the solar wind, except for occasional shock transitions, often shows surprisingly little density variation. From the point of view of considerations presently known, this tendency toward incompressibility is still slightly mysterious.

(2) Dissipation Coefficients

Derivations from first principles lead to far more elaborate dissipative terms than those which appear in Eqs. (1)-(3). Only those who have actually dragged themselves through a Chapman-Enskog calculation of magnetized-plasma transport coefficients can probably appreciate the fragility of the enterprise, but a widely-accepted derivation due to Braginskii (1965) [see also: Book, 1980] yields a considerably more involved term for the viscous dissipation than that given in Eq. (2). Reverting to component notation, $\rho v^2 v_i$ should be replaced by the ion viscosity term $-\sum_{j=1}^3 \partial P_{ji} / \partial x_j$, where, in a coordinate system with the z-axis along the magnetic field B ,

$$P_{xx} = -\frac{\eta_0}{2} (W_{xx} + W_{yy}) - \frac{\eta_1}{2} (W_{xx} - W_{yy}) - \eta_3 W_{xy}$$

$$P_{yy} = -\frac{\eta_0}{2} (W_{xx} + W_{yy}) + \frac{\eta_1}{2} (W_{xx} - W_{yy}) + \eta_3 W_{xy}$$

$$P_{xy} = P_{yx} = -\eta_1 W_{xy} + \frac{\eta_3}{2} (W_{xx} - W_{yy})$$

$$P_{xz} = P_{zx} = -\eta_2 W_{xz} - \eta_4 W_{yz}$$

$$P_{yz} = P_{zy} = -\eta_2 W_{yz} + \eta_4 W_{xz}$$

$$P_{zz} = -\eta_0 W_{zz}.$$

(8)

The coefficients in Eq. (8) are

$$\begin{aligned}\eta_0 &= 0.96 nk_B T_i \tau_i \\ \eta_1 &= \frac{3}{10} \frac{nk_B T_i}{\omega_{ci}^2 \tau_i} = \frac{1}{4} \eta_2 \\ \eta_3 &= \frac{1}{2} \frac{nk_B T_i}{\omega_{ci}} = \frac{1}{2} \eta_4\end{aligned}\quad (9)$$

$3k_B T_i/2$ is an ion thermal energy, and $\omega_{ci} = eB/m_i c$ is the proton gyrofrequency. τ_i is an ion collision time and is given by

$$\tau_i = \frac{3\sqrt{m_i} (k_B T_i)^{3/2}}{4\sqrt{\pi} n \lambda e^4} \quad (10)$$

where m_i is the ion (proton) mass, n is the proton number density, e is the proton charge, and λ is the Coulomb logarithm, typically 10 to 20. The rate of strain tensor W_{jk} is

$$W_{jk} = \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3} \delta_{jk} \nabla \cdot \underline{v} \quad (11)$$

The viscosity coefficients $\eta_0, \eta_1, \eta_2, \eta_3, \eta_4$ differ mainly by the numbers of powers of $\omega_{ci} \tau_i$ they contain in the denominators. The largest term for solar-wind plasmas is η_0 . A formal estimate of the η_0 -containing terms at 1AU, using measured values for the length scales and typical fluctuating velocities, leads to the conclusion that the η_0 terms are by orders of magnitude the largest terms in the equation of motion! There is nothing else in the equations of motion that they could be equated to unless the coefficients multiplying the η_0 were themselves small. These coefficients turn out to be linear combinations of $\nabla \cdot \underline{v}$ and $\nabla_{\perp} \cdot \underline{v}_{\perp}$ (here, the subscript "1" applied to a vector means the components perpendicular to \underline{B}). Only if $\nabla \cdot \underline{v} \approx 0$ and $\nabla_{\perp} \cdot \underline{v}_{\perp} \approx 0$ does it appear that the viscous terms can do anything but completely overpower every other term in the equation. This may be a more convincing argument for incompressibility than any that can be given in the conventional way, as in the previous subsection. It does, however, leave an additional constraint, incompressibility in the plane perpendicular to \underline{B} , which is not built into

Eqs. (1)-(3). The constraint $\nabla_{\perp} \cdot \underline{y}_{\perp} = 0$ does come up in the Strauss (1976; Montgomery, 1982) equations of "reduced" MHD, which are appropriate to the case of strong externally-imposed dc magnetic field ($\beta \ll 1$), but its content without the presence of such an externally-imposed \underline{B} -field is far from clear.

Finally, and perhaps most annoyingly, even if the divergences of \underline{y} and \underline{y}_{\perp} are small, that does not mean that the terms containing η_0 are negligible. The jungle of terms involved in Eqs. (8) and (9) does not lead, by any known asymptotic expansion, to a simple diffusion-like viscous term such as the last term of Eq. (2), at the time of this writing. It is possible that we will remain in the unpleasant position of settling for the relatively tractible $\rho v \nabla^2 \underline{y}$ term as a crude model of short-wavelength dissipation, knowing full well that it is not an accurate representation.

(3) Collisionality

Such expressions as Eqs. (8) and (9) are the output of lengthy, tedious Chapman-Enskog calculations which begin with a transport (e.g., Braginskii, 1965) equation with a Fokker-Planck collision term, and iterate about a local Maxwell distribution. The expansion parameter, assumed small, is the ratio of the mean collision time (τ_i for ions, τ_e for electrons) to the time scale T over which the macroscopic field variables vary, or equivalently, the ratio of mean free paths to macroscopic length scales. In the solar wind, these ratios, rather than being $\ll 1$, are $\gg 1$ if standard estimates are used for mean free paths and collision times. From one perspective, it is astonishing that MHD has any relevance to solar wind phenomena. It has been suggested, not unconvincingly, that the Fokker-Planck collision terms which are used to compute expressions such as Eqs. (8) and (9), are improper because of the observed high level of turbulence in the solar-wind magnetic field. Free-flight straight-line trajectories are used in evaluating collision integrals and are cut off at a Debye length, and these may be less than appropriate for a particle following a tangled field line. But these are no more than suggestions at this point, and what their implied modification of Eqs. (8) and (9) might be has not been suggested.

In summary, there are three respects at least in which the validity of incompressible MHD with scalar dissipation coefficients might legitimately

be doubted for solar-wind parameters. Yet it is the only contender among mathematical descriptions which have so far proved tractable enough to lead to any comprehensive theory of turbulent situations. Even then, we shall see in the following section that further severe restrictions are necessary in order to have concrete results emerge.

III. MAGNETOHYDRODYNAMIC TURBULENCE

If the previously-enumerated reservations about the validity of the MHD description are passed over, it may be noticed that a certain amount of relatively clean theory of MHD turbulence has emerged in the last two decades. The theory relies on certain idealizations that render it less than wholly applicable to real-life solar wind conditions. Applicable or not, it constitutes the only presently-existing framework in which statements about the solar wind can be made which are more than impressionistic or anecdotal. Virtually all of it is for the uniform-density ($\rho = \text{const.}$) case, and the incompressibility restriction is important. No significant body of strong turbulence theory exists for compressible fluids, even for ordinary neutral gases, and it would be unreasonable to expect MHD to yield where the simpler compressible system has not.

Use of the term "strong turbulence" in the preceding paragraph is intended to differentiate it from "weak turbulence" theory, which is a perspective which has shaped most thinking about nonlinear disordered processes in plasmas since about 1962. In weak turbulence theory (e.g., Montgomery, 1977), the emphasis is on systems whose dynamics may be considered to be the interaction of oscillatory normal modes, whose oscillation period is short compared to the characteristic time of transfer of excitations from one normal mode to another. Our reasons for discounting the value of weak turbulence theory in discussing the solar wind will become apparent when we write Eqs. (1)-(3) in appropriate dimensionless units.

We first observe that there are at least three physically distinct time scales represented in the dynamics described by Eqs. (1)-(3). If we call a typical rms flow speed U_0 (in a coordinate system moving with the local mean velocity of the solar wind), a typical rms magnetic field strength B_0 , a typical suitably defined mean magnetic field $\langle B \rangle$, and a typical length scale over which the fields vary $1/k$, then these three time scales may be defined as follows. There are: (1) the "eddy turnover time" $(kU_0)^{-1}$ associated with the fluid motions [in the solar wind, often $B_0 \sim U_0$]; (2) the "Alfvén transit time" $(k|\langle B \rangle|/\sqrt{4\pi\rho})^{-1}$; and (3) two dissipative time scales $(k^2\nu)^{-1}$ and $(k^2\eta)^{-1}$ which may be the same or different, depending upon the magnetic Prandtl number

v/η . The situation becomes more complex when we realize that there is not one length scale $\sim 1/k$, but a whole spectrum of scales, present at any instant, and the U_0 and B_0 may be defined locally in the wavenumber k as well. In the short wavelength range (large k), the dissipative effects may be dominant, while at large scales (small k) they may be negligible. There is no sharp dividing line where one passes from one regime to another.

Weak turbulence theory assigns orders of magnitude to its time scales of its excitations once and for all, and makes no provision for these to change. Its limitations are apparent in any situation in which there are fluxes of excitations in k space which move from one regime to another.

The point is that it is unacceptable to neglect any of the terms in Eqs. (1)-(3). It is important to resist the temptation to try to treat a limited range of k in dynamical isolation from the rest, making approximations there that do not apply elsewhere in k , because of some inequalities which obtain locally. Eqs. (1)-(3) are a package, no part of which can be ignored without peril. It might be argued, as in Sec. II, that more terms are needed in Eqs. (1)-(3) to do justice to the dynamics of the solar wind; if so, then the effect is to complicate an already almost prohibitively difficult problem. It cannot be argued that terms can be dropped because they may be "small" in certain ranges of k .

For the solar wind, U_0 and the Alfvén speed $C_A = B_0/\sqrt{4\pi\rho}$ are comparable in the zero-momentum frame. The coefficients v and η are uncertain for reasons already given, and may not even be well-defined. If the Spitzer formula for the conductivity σ is adopted, $\eta = c^2/4\pi\sigma$. If we use the Braginskii η_1 to estimate the viscosity, then $v = \eta_1/\rho$. We get, in cgs units,

$$\begin{aligned}\sigma &\sim 2.5 \times 10^{15} \text{ sec}^{-1} \\ v &\sim \eta_1/\rho \sim 3 \times 10^4 \text{ cm}^2/\text{sec},\end{aligned}$$

at a number density of $n \sim 10 \text{ cm}^{-3}$ and a temperature of 10^5 K . Both U_0 and C_A are typically 2 or $2.5 \times 10^6 \text{ cm/sec}$, and the most typical length scales have been measured to be $L \sim 10^{11} \text{ cm}$ [e.g., Matthaeus and Goldstein, 1982].

We rewrite all velocities in units of $U_0 = B_0/\sqrt{4\pi\rho}$, all lengths in units of L , all times in units of L/U_0 , and all magnetic fields in units of B_0 . The dimensionless version of Eqs. (1)-(3) becomes

$$\nabla \cdot \underline{y} = 0, \quad (12)$$

$$\frac{\partial \underline{y}}{\partial t} + \underline{y} \cdot \nabla \underline{y} = -\nabla p + \underline{B} \cdot \nabla \underline{B} + \frac{1}{R} \nabla^2 \underline{y}, \quad (13)$$

and

$$\frac{\partial \underline{B}}{\partial t} + \underline{y} \cdot \nabla \underline{B} = \underline{B} \cdot \nabla \underline{y} + \frac{1}{S} \nabla^2 \underline{B}. \quad (14)$$

The dimensionless numbers R and S are the Reynolds number and magnetic Reynolds number, respectively: $R \equiv U_0 L / \nu$ and $S \equiv C_A L / \eta = U_0 L / \eta$ (since $U_0 \sim B_0 / \sqrt{4\pi\rho}$, here). For the numbers cited, $R \sim 10^{12}$ and $S \sim 10^{13}$. These large values put us far into the regime of high Reynolds number turbulence, which is the domain of applicability of such theory as we have (e.g., Batchelor, 1970; Panchev, 1971).

The picture of high-Reynolds number fluid turbulence which has served as a model for the recent development of magnetofluid turbulence theory is due to Richardson, G. I. Taylor, and Kolmogoroff, and is elegantly summarized and developed in the classic monograph by Batchelor (1970). It does not make reference to specific solutions of the dynamical equations, which are regarded as irreproducible random variables. Instead, statements are made about ensemble averages, indicated by angular brackets $\langle \rangle$, which are hoped to be relatively smooth and reproducible. Thus B_i , a measured component of the magnetic field, might be divided up into a "mean" plus a "fluctuation" δB_i :

or

$$B_i = \langle B_i \rangle + \delta B_i, \quad (15)$$

$$v_i = \langle v_i \rangle + \delta v_i$$

for the velocity field, and so on.

What the brackets $\langle \rangle$ mean experimentally is a tricky question. Ideally, they should represent ensemble averages over a very large number of experiments prepared in the same way, based on measurements made after a fixed, elapsed time. Even in the laboratory this is difficult, and in the solar wind it is out of the question. What must be done is to conjecture something like an ergodic hypothesis, which makes it possible to equate phase space averages (or ensemble averages) and time averages. Because there is in the solar wind an inevitable relative velocity between the solar wind plasma and the measuring instruments, these time averages are really averages over a space-time trajectory,

in the zero-momentum frame. By the time the various symmetries necessary to interpret the data have been invoked, one has assumed a certain fraction of the consequences that one would, ideally, have liked for the experiment to demonstrate. A shaky consistency is often the most conclusive imaginable outcome.

Very nearly all the results so far on MHD turbulence concern the case of homogeneous turbulence, for which the statistical properties of the fields $B_i(\underline{x}, t)$, $v_i(\underline{x}, t)$ are independent of \underline{x} . One conventionally works in the zero-momentum frame, $\langle v_i \rangle = 0$. If the direction of the magnetic field is not externally constrained in some way by boundary conditions, then $\langle B_i \rangle = 0$. The quantities of theoretical interest then are mostly derivable from the covariances

$$R_{ij}^V(\underline{r}, t) \equiv \langle v_i(\underline{x}, t) v_j(\underline{x} + \underline{r}, t) \rangle \quad (16)$$

$$R_{ij}^B(\underline{r}, t) \equiv \langle B_i(\underline{x}, t) B_j(\underline{x} + \underline{r}, t) \rangle \quad (17)$$

$$R_{ij}^{BV}(\underline{r}, t) \equiv \langle v_i(\underline{x}, t) B_j(\underline{x} + \underline{r}, t) \rangle \quad (18)$$

which, by the assumption of spatial homogeneity, are independent of \underline{x} .

Virtually all serious theoretical attempts in both fluid and magneto-fluid turbulence so far have centered around such quantities as these covariances. Attempts to calculate R_{ij}^V , R_{ij}^B , R_{ij}^{BV} from a closed, deterministic dynamical description have displayed great ingenuity and some results, but nothing that is of obvious use for explanation of solar wind phenomena, so far.

Analytical approaches to data have been concerned with the rotationally isotropic case. In this case, the tensor description of Eqs. (16)-(18) contracts drastically. The $R_{ij}^B(\underline{x}, t)$, for example, may be Fourier-decomposed as

$$R_{ij}^B(\underline{x}, t) = \int d\mathbf{k} S_{ij}^B(\underline{k}, t) e^{i\mathbf{k} \cdot \underline{x}}$$

where

$$S_{ij}^B(\underline{k}, t) = E_B(k, t) \left(\frac{k_i k_j - k^2 \delta_{ij}}{k^2} \right) \quad (19)$$

with a single scalar variable $E_B(k, t)$ determining the evolution of the covariance. $E_B(k, t)$ is the energy spectrum, and is related to the rms fluctuating

field variable $\delta \underline{B}$ by

$$\frac{\langle (\delta \underline{B})^2 \rangle}{8\pi} = \int_0^\infty E_B(k, t) dk. \quad (20)$$

Eq. (19) does imply rotational isotropy, and the presence of a finite mean $\langle B_i \rangle$ will not in general permit this. Analytical impediments to a deductive theory are best illustrated by illustrating the dynamics in a Fourier decomposition of \underline{y} and \underline{B} over a large cubical box, assuming periodic boundary conditions:

$$\begin{aligned} \underline{y}(\underline{x}, t) &= \sum_{\underline{k}} \underline{y}(\underline{k}, t) e^{i\underline{k} \cdot \underline{x}} \\ \underline{B}(\underline{x}, t) &= \sum_{\underline{k}} \underline{B}(\underline{k}, t) e^{i\underline{k} \cdot \underline{x}} \end{aligned} \quad (21)$$

If we make up a large column vector whose i^{th} element X_i can be the real or imaginary part of any component of any one of the expansion coefficients $\underline{B}(\underline{k}, t)$ or $\underline{y}(\underline{k}, t)$, the Fourier decomposed structure of Eqs. (12)-(14) can be written symbolically as [see, e.g., Orszag 1977, or Kraichnan and Montgomery, 1980]:

$$\frac{dX_i}{dt} = \sum_{jk} C_{ijk} X_j X_k - \nu_i X_i \quad (22)$$

where the coupling coefficients C_{ijk} are known, and of a kinematical nature. The dissipation coefficients ν_i come from the viscous and resistive (linear) terms, and generally increase quadratically with increasing wavenumber magnitude.

The essential problem with any analytical approach to Eq. (22) is that the nonlinear (quadratic) terms are much larger, throughout most of \underline{k} -space, than the linear dissipative ones. No linearization can be justified. The inequality is measured by the Reynolds numbers R and S , so that, particularly for the solar wind, the nonlinearity may be expected to be strong.

Attempts at ensemble averaging moments of the X_i lead to an acute closure problem exemplified by (e.g., Orszag, 1977):

$$\begin{aligned} \frac{d}{dt} \langle X_i^2 / 2 \rangle + \nu_i \langle X_i^2 \rangle \\ = \sum_{jk} C_{ijk} \langle X_j X_k X_i \rangle \end{aligned} \quad (23)$$

with a corresponding equation for the time derivative of each n th moment in terms of the $(n + 1)$ st.

The situation is reminiscent of the BBGKY hierarchy derived from the Liouville equation, with the difference that no small parameters suggest themselves as bases for perturbation expansions. Great ingenuity has been brought to bear, particularly by R. H. Kraichnan (1959, 1964, 1975), on the problem of closure approximations for the moment hierarchy derived from Eqs. (22). The calculations are lengthy, require (Kraichnan, 1964) extensive numerical analysis, and so far have been limited to the isotropic case. Their generalization to anisotropic cases poses formidable problems, and has not been done.

Eq. (23) expresses the growth or decay of the energy in a particular Fourier mode as a sum of a large number of contributions from interacting triads of modes whose wave numbers sum to zero. Physical intuition is of limited utility in assessing the cumulative effect of the large number of these terms which contribute to each mode: the expansion in Fourier series (or other orthogonal functions) leave behind any simple resolution into forces and responses, "frozen-in field lines", or any of the other readily visualizable but often non-quantitative conceptualizations in terms of which MHD has often been discussed. The C_{ijk} , or modal interaction coefficients, are smoothly-varying functions of wavenumber where they are non-zero.

The statistical mechanics of the system (22) with all the dissipation coefficients ν_i set $\equiv 0$ is tractable. In the cases investigated (Navier-Stokes and MHD in two and three dimensions), truncation at a large but finite number of expansion coefficients and equations has led to systems which seem to be ergodic. Time averages of phase functions are predictable as ensemble averages (canonical or microcanonical) based on the constancy of those invariants which are still invariant after the truncation. These conclusions have been repeatedly verified numerically [Seyler, et al, 1975; Fyfe et al, 1977a,b; Kells and Orszag, 1978], and they need only to be alluded to here.

The difficulty is that the dissipative terms, if non-zero ($\nu_i \neq 0$), modify the dynamics qualitatively. Even though they may be relatively small over a good part of the wavenumber space, they in effect "pull the plug" at the high end of wavenumber space. Because they originate from terms like $\nu \nabla^2 \mathbf{y}$ and $\eta \nabla^2 \mathbf{B}$, they become arbitrarily large, when Fourier-represented, at the large

values of k . The effect of the (conservative) nonlinear terms is basically to scramble, in virtually a stochastic way, excitations from one value of k to another. Those excitations that find themselves at large values of $|k|$ get gobbled up by dissipation. The flow in k space tends to be toward those regions which are deficient, relative to the predictions of the non-dissipative equilibrium ensembles. The nonlinear scrambling terms continually try to replenish the excitations which are being drained away at high k . Raising the Reynolds numbers R and S in Eqs. (13)-(14) only increases the "dissipation wave number", at which the dissipation sets in, but does not make it go away. The prevailing opinion is that the integrated dissipation rate for Eqs. (13), (14) remains finite even as $R \rightarrow \infty$ and $S \rightarrow \infty$. This gives transfer from one part of the wave-number spectrum to another a central role in the dynamics that it does not have in linear, or nearly linear, systems.

Very large numbers of Fourier modes are required to resolve all the dynamically important spatial scales, as R, S become large. This provides severe limits on numerical attempts to solve Eqs. (13) and (14). A pessimistic rule of thumb is that one grid point (or finite element, or Fourier coefficient) per dimension per unit Reynolds number is required. Thus, a three-dimensional $(64)^3$ simulation (which will not quite fit in core on a CRAY-1) would be required to resolve turbulence with a Reynolds number of 64. This requirement can be relaxed somewhat, but not by an order of magnitude---a Reynolds number 1000 run could probably not ever be resolved on a $(64)^3$ grid, if the Reynolds number were to be based on the mean length scale in the flow. When one begins to talk about Reynolds numbers many orders of magnitude larger, the real limitations of foreseeable computers, in dealing with turbulence, become apparent.

Dimensional analysis, applied to isotropic, homogeneous situations, have led to predictions of power laws in wave number space for the energy spectra $E_B(k), E_V(k)$ in different situations. The predictions differ from fluids to magnetofluids, and from two to three dimensions. They are virtually the only simple, testable analytic predictions that four decades of turbulence theory have been able to come up with. There are ingenious closures of the hierarchy of which Eq. (23) is the first member, but they are not simple, and so far they all assume higher degrees of symmetry than the solar wind has been shown to possess. These dimensional analysis arguments can be grouped under the rubric of "cascade theory".

High Reynolds number turbulence theories and computations have been formalized around two general classes of situations: "forced" and "decay" situations. These are at best loose approximations to actual physical cases, but they are as close to a universal or situation-independent problem as can be isolated. Because the magnetofluid equations are dissipative, a turbulent field will eventually decay away, and one can seek features of the decay which may be insensitive to initial spectra. Steady-state situations require a source of excitations, or "forcing", that is balanced against the dissipation rate, averaged over time. The nature of the forcing, often regarded as band-limited in wavenumber space, is usually not restricted very specifically, and is often modelled by a random function. The search in turbulence theory, as elsewhere in physics, is for soluble situations from which a universal, reproducible, and transferable core of general behavior can be extracted.

Cascades and Inverse Cascades

Power laws and cascade processes are expected for forced situations, not for decaying ones, unless there is reason to believe that the lifetime of the long-wavelength components is sufficiently great that the short wavelength components cannot distinguish them from a maintained "source". Under circumstances that have been discussed at great length in the published literature, the following table (Table 1) shows what has been done so far in the way of conjecturing and establishing inertial subrange exponents for fluids and magnetofluids.

There is insufficient scope within this article to review in detail the evidence and arguments for and against inertial-subrange power laws which have been accumulated. There is little doubt that the question of exponents has come to occupy more of the territory than it deserves, to some extent because there are concrete theoretical predictions. The exponents derive not from any dynamical arguments but from conjectured similarity variables. Deriving them from dynamics has been the most pursued of all subjects in turbulence theory, but no wholly satisfactory resolution has been achieved. Even if it were achieved, relatively little light would be shed on the dynamics of the solar wind.

Table 1

Situation	NAVIER-STOKES, 3D	NAVIER-STOKES, 2D	MHD, 3D	MHD, 2D
Cascaded Quantity	ENERGY	ENERGY & ENSTROPHY	ENERGY & MAGNETIC HELICITY	ENERGY & MAGNETIC POTENTIAL
Direction of Cascade in k space	ENERGY UP	ENSTROPHY UP, ENERGY DOWN	ENERGY UP, HELICITY DOWN	ENERGY UP, MAGN. POT. DOWN
Predicted Power Law, Energy Spectrum	$k^{-5/3}$ KOLMOGOROFF- OBUKHOV (1941)	k^{-3} , $k^{-5/3}$ KRAICHNAN (1967), BATCHELOR, LEITH	$k^{-5/3}$ or $k^{-3/2}$, k^{-1} FRISCH et al (1975)*	$k^{-5/3}$, $k^{-1/3}$ FYFE et al (1977b)
Experimental Verification Attempted	GRANT, STEWART, & MOLLIET (1962)	NO	MATTHAEUS & GOLDSTEIN (1982)	NO
Computational Verification Attempted	NO	FYFE et al, (1977) [insufficient spatial resolu- tion]	NO*	FYFE et al (1977) [insufficient spatial resolu- tion]

* See also Meneguzzi et al (1981).

Table 1. Cascades, Inverse Cascades, and Power Law Predictions. Original references are cited in bibliography.

The strongest single limitation which the present cascade and inverse cascade theory may have when applied to the solar wind concerns the assumption of isotropy, which underlies all of the predictions listed in Table 1, and all of the dimensional analysis arguments formulated in the Kolmogoroff style since 1941. We are virtually certain that the solar wind is not isotropic, and the weaker assumption of axisymmetry may be regarded as open to serious legitimate doubt. It is naive to regard the removal of the isotropic restrictions on cascade power-law predictions as only a technical point which is sure to be overcome soon; its status is at present very dark, and no resolution is in sight.

Selective Decays

An even more tentative class of generalizations, not without implications for the solar wind, are those processes called selective decays, in which all the fields decay as in the initial value problem, but some of the global, non-dissipative invariants may decay less rapidly than others. Qualitatively, there are two possible reasons for this. First, the dissipation is effective only at the shorter wavelengths, and quantities transferred to long wavelengths may simply stay out of reach of the dissipation. Second, dissipation integrands for some variables may be peaked at higher wavenumbers than for others and to be more effective at dissipation for this reason. Arguments and computations for these possible "selective decay" processes have been presented by Montgomery, Turner and Vahala (1978), by Matthaeus and Montgomery (1980), and by Riyopoulos, Bondeson, and Montgomery (1982).

Each such selective decay process, if valid, would imply a temporally decreasing magnitude of the ratio of two of the ideal invariants: energy to magnetic helicity for 3D MHD, for example (Taylor 1974 made use of such an assumption in predicting asymptotic states of decaying toroidal Z pinches). A variational problem arises by minimizing this ratio, which often has for its solution a relatively simple Euler equation which predicts a quiescent state. Needless to say, this is an attractive possibility. If the tendency of highly disordered turbulent motions is to decay to some universal non-trivial quiescent state, regardless of the path of the decay, then this is indeed a wonderfully simple ingredient to add to the few pieces of general information we have about turbulence.

For example, if for 3D MHD, the energy-to-helicity ratio were to decay toward its minimum value, this is simply a force-free state, a solution of $\nabla \times \underline{B} = \lambda \underline{B}$, where λ is a Lagrange multiplier, and $\underline{v} = 0$ everywhere. For 2D MHD, the decay of energy to mean square magnetic potential again leads to a quiescent state with a mean magnetic field derivable from a vector potential $\underline{B} = \nabla \times \underline{A} = \nabla \times A \hat{e}_z$, where

$$(\nabla_{\perp}^2 + \lambda^2)A = 0.$$

Some numerical evidence has been presented for both of these kinds of selective decays. The results are encouraging but should be held with extreme caution; the conclusions are difficult to document and expensive, and very few runs have been carried out [Matthaeus and Montgomery 1980; Riyopoulos et al 1982].

A second class of decay hypothesis, not entirely consistent with the first, concerns the ratio of the cross-helicity (another ideal invariant) to the energy. The ratio

$$R_{Hc} \equiv \left| \frac{2 \int \underline{v} \cdot \underline{B} \, d\underline{x}}{\int (\underline{v}^2 + \underline{B}^2) \, d\underline{x}} \right|,$$

a constant in the absence of dissipation, has been shown under some circumstances to increase monotonically (Grappin et al, 1982; Matthaeus, Goldstein, and Montgomery 1982) with time in the presence of dissipation. This increase points to an equipartitioned state, certainly not quiescent, with $\underline{v} = \pm \underline{B}$. From the point of view of solar wind observations, this is an attractive possibility, because many observations, from Belcher and Davis (1971) on, have shown solar wind velocity fields and magnetic fields to be closely aligned or anti-aligned. These are sometimes referred to as "Alfvénic fluctuations".

The paradox of MHD turbulence's tending apparently both to states in which R_{Hc} is maximal and helicity to energy is also maximal is an example of the wide-open character of research into MHD turbulence. There is compelling evidence for both conjectures, but both cannot be simultaneously true. If either is true, it may well determine the asymptotic state toward which solar wind turbulence is trying to decay.

IV. SUMMARY

MHD turbulence theory provides the most nearly adequate framework in which to discuss the physics of solar wind turbulence. The collected data, however, are far superior both to the available justification of the MHD description and to its systematic development for turbulent fields which lack high degrees of symmetry such as rotational isotropy. Understanding the physics of the solar wind at the present time is probably more limited by the unanswered questions in turbulence theory than by any scarcity of measurements. Expanded experimental programs to probe solar wind turbulence, such as that advocated by the 1980 Plasma Turbulence Explorer Panel (Montgomery et al, 1980) would require a considerably broader attack than has so far been mounted on the basic plasma physics of the turbulent medium.

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