NASA Contractor Report 172251

ICASE

COMPLEX EIGENVALUES FOR THE STABILITY OF COUETTE FLOW

R. C. DiPrima

and

P. Hall

Contract No. NAS1-17070
October 1983

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

COMPLEX EIGENVALUES FOR THE STABILITY OF COUETTE FLOW

R. C. DiPrima*
Rensselaer Polytechnic Institute

and

P. Hall**
Imperial College, London, England
and
Institute for Computer Applications in Science and Engineering

ABSTRACT

The eigenvalue problem for the linear stability of Couette flow between rotating concentric cylinders to axisymmetric disturbances is considered. It is shown by numerical caluculations and by formal perturbation methods that when the outer cylinder is at rest there exist complex eigenvalues corresponding to oscillatory damped disturbances. The structure of the first few eigenvalues in the spectrum is discussed. The results do not contradict the "principle of exchange of stabilities"; namely, for a fixed axial wavenumber the first mode to become unstable as the speed of the inner cylinder is increased is nonoscillatory as the stability boundary is crossed.

*Research partially supported by the U.S. Army Research Office, National Science Foundation Grant No. MCS81-20839, a United Kingdom S.E.R.C. Grant and a Guggenheim Fellowship and done while a visitor at Imperial College.
**Research supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17070 and completed while in residence at ICASE, NASA Langley Research Center, Hampton, Va 23665.

N84-13492#

1. INTRODUCTION

In this paper we reconsider the classical eigenvalue problem for the linear stability of Couette flow between rotating concentric cylinder to axisymmetric disturbances. First we state the mathematical problem.

Let r, θ, z denote the usual cylindrical coordinates, and let R_1, Ω_1 and R_2, Ω_2 be the radii and angular velocities of the inner and outer cylinders, respectively. In the formulation of the eigenvalue problem we will not take Ω_2 equal to zero; however, in all the calculations Ω_2 is equal to zero. With $d = R_2 - R_1$, we define the following dimensionless parameters:

$$\eta = R_1/R_2, \quad \delta = d/R_2, \quad \mu = \Omega_2/\Omega_1,$$
$$\gamma = 2\eta(1 - \mu/\eta^2)/(1 + \eta), \quad R = \Omega_1 R_1 d/\nu, \tag{1.1}$$

where ν is the kinematic viscosity. We scale all length variables with respect to d, time with respect to d^2/ν, and all velocities with respect to $R_1\Omega_1$. Then the linear stability problem for the stability of Couette flow to disturbances proportional to $\exp(\sigma t + iaz)$, where a is real and positive, can be written in the form

$$(DD^* - a^2)^2 u - 2a^2 R F_\ell(r) v - \sigma(DD^* - a^2)u = 0 \ , \tag{1.2}$$

$$\gamma R u + (DD^* - a^2)v - \sigma v = 0 \ , \tag{1.3}$$

for $r_1 < r < r_2$ with

$$u = Du = v = 0 \text{ at } r = r_1, r_2. \tag{1.4}$$

Here

$$D=d/dr,\ D^{*} = D + r^{-1},\ r_1 = \eta/(1-\eta),\ r_2 = r_1 + 1; \tag{1.5}$$

$u(r)$ and $v(r)$ are proportional to the radial and azimuthal components of the disturbance velocity; and $F_{\ell}(r)$, the dimensionless Couette angular velocity, is

$$F_{\ell}(r) = -\frac{\gamma}{2} + \frac{\eta(1-\mu)}{(1+\eta)(1-\eta)^2}\ \frac{1}{r^2}\ . \tag{1.6}$$

Equations (1.2) - (1.4) define a non-selfadjoint eigenvalue problem

$$H(\eta\ ,\ \mu,\ a,\ R,\ \sigma) = 0. \tag{1.7}$$

DiPrima and Habetler (1969) have shown that for fixed values of a, R, μ, and η this eigenvalue problem has a countable spectrum $\{\sigma_j\}$, which can be ordered with $Re(\sigma_1) \geq Re(\sigma_2) \geq \cdots$, with no cluster points in the complex plane and that the corresponding (generalized) eigenfunctions span a certain Hilbert space.

In this paper we will show by numerical calculations and by formal peturbation methods that for fixed η and $\mu = 0$ (outer cylinder at rest) there are values of $a > 0$ and $R > 0$ such that some of the σ_j are complex. To the best of our knowledge, this is the first demonstration of the existence of complex eigenvalues for the boundary value problem defined by Eqs. (1.2) - (1.4). This result contradicts a proof by Yih (Main Theorem p. 299, 1972b) that all the eigenvalues are real when the cylinders rotate in the same direction and the circulation of the basic flow decreases in the outward radial direction. However, all of the complex eigenvalues we have found correspond to damped disturbances, $\mathrm{Re}(\sigma) < 0$. Thus the results do not contradict the conjectured (and widely believed) principle of exchange of stabilities; namely, for fixed values of η, $a > 0$, and $\mu > 0$ the first eigenvalue (σ_1) to cross the imaginary axis as R is increased is real. However, the present results show that a proof of the principle of exchange of stabilities must be restricted to a study of the behaviour of the first eigenvalue. The condition that σ_1 is real and simple is required in a rigorous proof of the existence of Taylor vortex flow following the instability of Couette flow; see Velte (1966) and Kirchgässner and Sorger (1969).

In Section 2 we give an example of how complex eigenvalues can arise due to small perturbations of a selfadjoint eigenvalue problem. The example is related to the eigenvalue problem (1.2)-(1.4), but is simple enough that one can carry out the calculations readily and explicitly. In Section 3 we give some results for the numerical calculation of complex eigenvalues of Eqs.(1.2)-(1.4) and confirm these results by formal perturbation calculations. In Section 4 we study the eigenvalue problem in the limit $a \to 0$ in order to obtain a better understanding of how the complex eigenvalues occur. Finally, the results are discussed in Section 5 where we also make a few remarks about the limiting case $a \to \infty$.

For many purposes, and especially for numerical work, it is convenient to write Eqs. (1.2) and (1.3) as a system of six first order equations. If we introduce the axial component of the perturbation $w = ia^{-1}D^{*}u$ and the pressure perturbation p and let $q = [p, dv/dr, dw/dr, u, v, w]$, we can write Eqs. (1.2) and (1.3) in the form

$$dq/dr - Aq - \sigma Bq = 0, \qquad r_1 < r < r_2 \tag{1.8}$$

with the boundary conditions

$$u = v = w = 0 \text{ at } r = r_1,\ r_2. \tag{1.9}$$

The matrices A and B are

$$A = \begin{pmatrix} 0 & 0 & -ia & -a^2 & 2RF_{\ell}(r) & 0 \\ 0 & -1/r & 0 & -R\gamma & a^2+1/r^2 & 0 \\ ia & 0 & -1/r & 0 & 0 & a^2 \\ 0 & 0 & 0 & -1/r & 0 & -ia \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.10}$$

Finally, for use later, it is helpful to have the results

$$\gamma = (1 - \mu) - \delta \frac{1 + 3\mu}{2} - \delta^2 \frac{1 + 7\mu}{2} + O(\delta^3), \tag{1.11}$$

$$F_\ell(r) = \frac{\delta}{2}\left[(1 + \mu) - (1 - \mu)x\right] + \frac{\delta^2}{2}\left[\frac{1 + 3\mu}{2} - \frac{3(1 - \mu)x}{2}\right.$$

$$\left. + (1 - \mu)x^2\right] + O(\delta^3) , \tag{1.12}$$

and, for $\mu = 0$,

$$2\gamma F_\ell(r) = \delta(1 - x) - \delta^2(x - x^2) + O(\delta^3) , \tag{1.13}$$

where $x = r - r_1 = r + 1 - 1/\delta$.

2. AN EXAMPLE

If we solve Eq. (1.3) for u, substitute in Eq. (1.2), let $x = r - r_1$, $T = R^2\delta(1 - \mu^2)$, and take the small-gap limit $\delta \to 0$ with σ, T, and a fixed we obtain the classic small-gap equation

$$(d^2/dx^2 - a^2 - \sigma)^2 (d^2/dx^2 - a^2)v$$
$$+ a^2T(1 - \epsilon x)v = 0, \quad 0 < x < 1, \tag{2.1}$$

where $\epsilon = (1 - \mu)/(1 + \mu)$. We consider Eq. (2.1) with the boundary conditions

$$v = v'' = v'''' = 0 \text{ at } x = 0 \text{ and } x = 1. \tag{2.2}$$

The eigenvalue problem defined by Eqs. (2.1) and (2.2) is the small-gap limit of the auxiliary problem introduced by Yih [Eqs. (11), (12), and (14), 1972a] for $a > 0$, $T > 0$, $0 < \epsilon < 1$. Yih (p. 296, 1972b) asserts that for $R^2 > 0$ $(T > 0)$, $a > 0$, and $0 < \epsilon < 1$ the eigenvalues σ of his auxiliary system are real. We will show that this is not true for the corresponding small-gap equations (2.1) and (2.2).

For $\epsilon = 0$, Eqs. (2.1) and (2.2) are simply those of the classic Bénard problem with free-free boundaries - a selfadjoint boundary value problem. The eigenvalues σ can be split into two sets $\{\sigma_n^+\}$ and $\{\sigma_n^-\}$ where

$$\sigma_n^{\pm} = - n^2\pi^2 - a^2 \pm \sqrt{\frac{a^2T}{n^2\pi^2 + a^2}}, \quad n = 1, 2, \cdots. \tag{2.3}$$

The corresponding eigenfunction for σ_n^+ and σ_n^- is $v_n(x) = \sin n\pi x$. Notice that if $T = 0$, then the eigenvalues are negative and each eigenvalue has multiplicity two ($\sigma_n^+ = \sigma_n^-$) with only one eigenfunction; however, this degeneracy is not of interest at the moment. More important is to observe that for any n, σ_n^+ increases monotonically with increasing T and eventually becomes positive while σ_n^- decreases montonically with increasing T.

Thus we can choose positive integers N and M with $N > M$ such that

$$\sigma_N^+ = \sigma_M^- = -N^2\pi^2 - a^2 + \sqrt{\frac{a^2 T_0}{N^2\pi^2 + a^2}} \tag{2.4}$$

where

$$T_0 = \frac{(N^2 - M^2)\pi^2}{a} \left[\frac{1}{(N^2\pi^2 + a^2)^{1/2}} + \frac{1}{(M^2\pi^2 + a^2)^{1/2}} \right]^{-1} . \tag{2.5}$$

For a given value of a with $\varepsilon = 0$ and $T = T_0$ the eigenvalue problem (2.1) and (2.2) has an eigenvalue of multiplicity two, $\sigma = \sigma_N^+ = \sigma_M^-$, with two linearly independent eigenvectors $v_N(x) = \sin N\pi x$ and $v_M(x) = \sin M\pi x$, $N \neq M$. We will now show by standard perturbation methods that for values of T close to T_0 and $\varepsilon \neq 0$, Eqs. (2.1) and (2.2) have complex eigenvalues.

We write

$$\begin{aligned} \sigma &= \sigma_0 + \varepsilon\sigma_1 + \cdots, \quad \sigma_0 = -(N^2\pi^2 + a^2) + \sqrt{\frac{a^2 T_0}{N^2\pi^2 + a^2}} , \\ T &= T_0 + \varepsilon T_1 + \cdots, \\ v(x) &= v_0(x) + \varepsilon v_1(x) + \cdots, \end{aligned} \tag{2.6}$$

where $v_0(x) = \alpha \sin N\pi x + \beta \sin M\pi x = \alpha v_N(x) + \beta v_M(x)$. The constants α and β are to be determined as part of the calculation. Substituting the series (2.6) in Eqs. (2.1) and (2.2), we obtain

$$Lv_0 \equiv [(d^2/dx^2 - a^2 - \sigma_0)^2 (d^2/dx^2 - a^2) + a^2 T_0]\, v_0 = 0 , \tag{2.7}$$

and

$$Lv_1 = 2\sigma_1(d^2/dx^2 - a^2)^2 v_0 - 2\sigma_0\sigma_1(d^2/dx^2 - a^2)v_0 - a^2 T_1 v_0 + a^2 T_0 x v_0 , \tag{2.8}$$

and the boundary conditions (2.2) for v_0 and v_1. Equation (2.7) with the boundary conditions (2.2) is automatically satisfied. In order for the boundary value problem for v_1 to have a solution, it is necessary that the right side of Eq. (2.8) be orthogonal to $v_N(x)$ and $v_M(x)$; see Courant and Hilbert (pp. 346-350, 1953) or Case I of the Appendix of this paper. This leads to two linear homogenous equations for α and β, and the condition that these equations have a non-trivial solution is

$$\left(T_1 - \tfrac{1}{2}T_0 - \frac{2\sigma_1}{a}\sqrt{(N^2\pi^2 + a^2)T_0}\right)\left(T_1 - \tfrac{1}{2}T_0 + \frac{2\sigma_1}{a}\sqrt{(M^2\pi^2 + a^2)T_0}\right) - \frac{16N^2M^2T_0^2}{\pi^4(N^2 - M^2)^4}\left[(-1)^{N+M} - 1\right]^2 = 0. \tag{2.9}$$

It is clear from Eq. (2.9) that if $N + M$ is odd and $(T_1 - T_0/2)^2 < 64N^2M^2T_0{}^2/\pi^4(N^2 - M^2)$ then σ_1 is pure imaginary. Thus for some, but not all, values of T_1 the perturbation in ε of the double eigenvalue, gives complex eigenvalues. For example, if $T_1 = T_0/2$, then

$$\sigma_1 = \pm i\nu, \quad \nu = \frac{4NMaT_0{}^{1/2}}{\pi^2(N^2 - M^2)^2(N^2\pi^2 + a^2)^{1/4}\,(M^2\pi^2 + a^2)^{1/4}} \tag{2.10}$$

Hence for $\varepsilon \to 0$ with $T = T_0(1 + \varepsilon/2)$ there are complex eigenvalues of the boundary value problem (2.1) and (2.2) of the form $\sigma_0 \pm i\varepsilon\nu + O(\varepsilon^2)$. While the above arguments do not provide a rigorous proof that the boundary value problem (2.1) and (2.2) has complex eigenvalues for certain positive values of a, T, and ε it is strongly suggestive of such an assertion. Moreover, it suggests that the corresponding problem without the small-gap limit, the auxiliary problem discussed by Yih, also has complex eigenvalues which contradicts his assertion.

3. COMPLEX EIGENVALUES

We have solved the eigenvalue problem (1.8), (1.9) for several different sets of parameter values using a standard shooting procedure. The integration routine was a fourth order Runge-Kutta procedure with a step size of 0.025, and Mueller's method was used for the eigenvalue search routine. In Figure 1 we show our calculation of the first five eigenvalues for η = 0.877, $\mu = 0$, and R = 150. For this value of η the critical value of R, above which there exist values of a such that $\sigma > 0$, is R_c = 119.3. The corresponding critical value of a is a = 3.13.

It can be seen in Figure 1 that the first eigenvalue is real and becomes positive for a finite band of wavenumbers (approximately $1.7 < a < 5.3$) corresponding to the band of wavenumbers inside the neutral curve at R = 150. The second and third eigenvalues are real and negative for R = 150 and $1 < a < 8$. The fourth and fifth eigenvalues merge at a $\cong$ 2.56 and form a pair of complex conjugate eigenvalues for $2.56 < a < 6.6$. Indeed for $a < 1$ the second and third eigenvalues cross over and for a very small interval of wavenumbers form a pair of complex conjugate eigenvalues. In Figures 2 and 3 we show the first five eigenvalues for $\eta = 0.75$ and $\eta = 0.5$, respectively, for R = 150. The overall structure of the eigenvalues remains virtually unchanged as η is varied, except that the interval of wavenumbers for which the fourth and fifth eigenvalues form a complex conjugate pair decreases with decreasing values of η. We found no indication of a possible crossing of any eigenvalue with the first eigenvalue in any of our calculations; note that $R = 150$ corresponds to $1.26R_c$, $1.75R_c$, and $2.20R_c$ for η equal to 0.877, 0.75, and 0.5, respectively.

At a value of a, say a_0, in Figures 1, 2, and 3 where two real eigenvalues merge to give an eigenvalue of multiplicity two our calculations show that there is one only eigenvector. A generalized eigenvection ψ can be obtained by numerical integration of the equation

$$d\psi/dr - A\psi - \sigma B\psi = Bq \qquad (3.1)$$

with the boundary conditions (1.9) and where q is the eigenvector. As a check on the consistency of the numerical calculations we can follow the procedure described in the Appendix (Case II) to calculate the variation of the two eigenvalues with a for $|a - a_0|$ small. In addition to calculating q and ψ we must also calculate the eigenvector and the generalized eigenvector of the adjoint system

$$dq^*/dr + A^{tr}q^* + \sigma B^{tr}q^* = 0 \qquad (3.2)$$

with the boundary conditions that the first, second, and third components of q^* vanish at $r = r_1$ and $r = r_2$.

We find for $\eta = 0.877$, $\mu = 0$, $R = 150$, and $a_0 = 2.5576$ that

$$\sigma_{4,5} = -92.1818 \pm 3.229\ (a_0 - a)^{1/2} + O(a - a_0)\ . \qquad (3.3)$$

Formula (3.3) shows that for $|a_0 - a|$ small, the eigenvalues σ_4 and σ_5 are real for $a < a_0$ and are complex conjugates for $a > a_0$. The agreement of the perturbation formula (3.3) with the numerical calculations is satisfactory, especially for σ_i, as can be seen in Figure 4.

It is impossible to carry out a complete analysis of the spectrum of the eigenvalue problem defined by Eqs. (1.2) - (1.4) or Eqs. (1.8) - (1.9) for all allowable values of η, μ, a, and R. Our primary goal was to show the existence of complex eigenvalues for $\mu = 0$ with values of a and R typical of those for which there exist eigenvalues which are real and positive (unstable modes). Having found complex eigenvalues we shall now consider the limiting case $a \to 0$. We will find that the structure of the spectrum found by numerical calculations for the full equations is preserved in this limit. As a consequence, we can gain some understanding of the origin of the complex eigenvalues.

4. THE STRUCTURE OF THE SPECTRUM IN THE LIMIT $a \to 0$

In this section we consider the eigenvalue problem (1.2) - (1.4) in the limit $a \to 0$. It is known that on the neutral curve $R = O(1/a)$ as $a \to 0$. If we take $R = O(1/a)$ then it is also clear from Eqs. (1.2) and (1.3) that in order to obtain a meaningful problem we must take $u = O(a)$. It is also convenient to introduce the scaling that is used for the small-gap problem. Thus we let

$$R_a = aR, \quad u = aU, \quad v = \gamma R_a V. \tag{4.1}$$

If we substitute these expressions in Eqs. (1.2) - (1.4), and then let $a \to 0$ with η, σ, R_a, U, and V fixed, we obtain the eigenvalue problem

$$\begin{pmatrix} (DD^*)^2 & R_a^2\delta G_\ell(r) \\ -1 & DD^* \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} - \sigma \begin{pmatrix} DD^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = 0 \tag{4.2}$$

with

$$U = DU = V = 0, \tag{4.3}$$

and where $G_\ell(r) = 2\gamma\delta^{-1}F_\ell(r)$.

We will first study the eigenvalue problem (4.2) and (4.3) for small values of δ and $R_a^2\delta$ using perturbation methods. For this purpose we set $x = r - r_1$, observe from Eq. (1.13) that $G_\ell(r) = (1 - x) + \delta(x - x^2) + O(\delta^2)$, and then let $\delta \to 0$ and $R_a^2\delta \to 0$ in

Eq. (4.2). We obtain

$$\begin{pmatrix} d^4/dx^4 & 0 \\ -1 & d^2/dx^2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} - \sigma \begin{pmatrix} d^2/dx^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = 0 \qquad (4.4)$$

with the boundary conditions

$$U = dU/dx = V = 0 \quad \text{at } x = 0 \text{ and } x = 1. \qquad (4.5)$$

This is a non-selfadjoint eigenvalue problem with the following sets of real eigenvalues and eigenvectors:

$$\sigma_{An} = -4n^2\pi^2, \; U_{An} = 1 - \cos 2n\pi x, \; V_{An} = \frac{1}{4n^2\pi^2} (1-\cos 2n\pi x) - \frac{x}{4n\pi} \sin 2n\pi x \; ; \qquad (4.6)$$

$$\sigma_{Bn} = -n^2\pi^2, \; U_{Bn} = 0, \; V_{Bn} = \sin n\pi x \; ; \qquad (4.7)$$

$$\sigma_{Cn} = -\lambda_n^2, \;\; U_{Cn} = 1 - \cos \lambda_n x - \frac{\sin \lambda_n}{1-\cos \lambda_n} (\lambda_n x - \sin \lambda_n x), \; V_{Cn}, \qquad (4.8)$$

$n = 1, 2, \ldots$. The λ_n are the roots of $\lambda/2 = \tan \lambda/2$ with $\lambda_1 \cong 8.986$, $\lambda_2 \cong 15.45$, and $\lambda_n \sim (2n + 1)\pi$ for $n \to \infty$. The functions V_{Cn} are the solutions of

$$(d^2/dx^2 - \lambda_n^2)\, V_{Cn} = U_{Cn}, \; V_{Cn}(0) = V_{Cn}(1) = 0. \qquad (4.9)$$

The first nine eigenvalues are

$$\sigma_1 = \sigma_{B1} = -\pi^2 \cong -9.870$$

$$\sigma_2 = \sigma_{A1} = -4\pi^2 \cong -39.48$$

$$\sigma_3 = \sigma_{B2} = -4\pi^2 \cong -39.48$$

$$\sigma_4 = \sigma_{C1} = -\lambda_1^2 \cong -80.75$$

$$\sigma_5 = \sigma_{B3} = -9\pi^2 \cong -88.83$$

$$\sigma_6 = \sigma_{A2} = -16\pi^2 \cong -157.9$$

$$\sigma_7 = \sigma_{B4} = -16\pi^2 \cong -157.9$$

$$\sigma_8 = \sigma_{C2} = -\lambda_2^2 \cong -238.7$$

$$\sigma_9 = \sigma_{B5} = -25\pi^2 \cong -246.7$$

The reason we have labeled the eigenvalues in the above manner will become clear later in this section. For the moment, notice that with the exception of the first eigenvalue, the B eigenvalues for n even pair with the A eigenvalues and for n odd pair with the C eigenvalues.

It is possible to study the behavior of the multiple eigenvalues at $\sigma = -4m^2\pi^2$, $m = 1, 2, \cdots$, for small values of $Ra^2\delta$ and δ using formal perturbation methods. In order to study the effect on these eigenvalues of comparable perturbations in $R_a{}^2\delta$ and δ, we set

$$R_a{}^2\delta = \Lambda^2\delta^2, \qquad \Lambda = O(1) \tag{4.10}$$

in Eq. (4.2), and then expand in δ following the procedure described in Case I of the Appendix. In order to carry out this calculation we need the adjoint eigenvectors for the eigenvalue problem (4.2) and (4.3). If we let $q = [U, V]^{tr}$ and introduce the inner product

$$< q_1, q_2 > = \int_0^1 (U_1U_2 + V_1V_2)\, dx, \tag{4.11}$$

then we find that the adjoint eigenvalue problem is given by Eq. (4.4), with each matrix replaced by its transpose and Eq. (4.5). The adjoint eigenvectors corresponding to $q_{An} = [U_{An}, V_{Bn}]^{tr}$ and $q_{Bn} = [U_{Bn}, V_{Bn}]^{tr}$ are

$$U_{An}{}^* = 1 - \cos 2n\pi x, \; V_{An}{}^* = 0, \qquad n = 1, 2, \ldots \tag{4.12}$$

and

$$U_{Bn}{}^* = \frac{1 - \cos n\pi x}{4n^3\pi^3} - \frac{\sin n\pi x}{2n^4\pi^4} + \frac{x \cos n\pi x}{2n^3\pi^3}, \; V_{Bn}{}^* = \sin n\pi x,$$

$$n = 1, 3, \cdots \tag{4.13}$$

$$U_{Bn}{}^* = \frac{x^3}{2n^3\pi^3} (\cos 2n\pi x - 1), \; V_{Bn} = \sin n\pi x, \quad n = 2, 4, \cdots.$$

If we let

$$P = \begin{pmatrix} d^2/dx^2 & 0 \\ 0 & 1 \end{pmatrix} \quad , \tag{4.14}$$

then one can readily verify the orthogonality relations

$$\langle Pq_{An}, q_{Am}^{*} \rangle = 0, \qquad \langle Pq_{Bn}, q_{Bm}^{*} \rangle = 0, \qquad n \neq m ,$$

and (4.15)

$$\langle Pq_{An}, q_{Bm}^{*} \rangle = 0, \qquad \langle Pq_{Bn}, q_{Am}^{*} \rangle = 0.$$

Returning to the eigenvalue problem (4.2) and (4.3) we now consider the perturbation of the double eigenvalue $\sigma = -4m^2\pi$ for small δ with $\Lambda = O(1)$. The series expansions are

$$\sigma = 4m^2\pi^2 + \delta\mu + \delta^2\nu + \cdots ,$$

$$q = (\alpha\, q_{Am} + \beta\, q_{B,2m}) + \delta q_1 + \delta^2 q_2 + \cdots , \tag{4.16}$$

for $m = 1, 2, \cdots$, where the ratio α/β, which depends on m, is to be determined in the course of the analysis. (We use μ and ν as perturbation coefficients. There should be no confusion with their earlier usage for the ratio Ω_2/Ω_1, and the kinematic viscosity.) Following the procedure described in Case I of the Appendix, we find at $O(\delta)$ that $\mu = 0$ and

$$q_1 = \alpha \begin{pmatrix} \frac{1}{2}\, x\, (\cos 2\, m\pi x - 1) \\ -\dfrac{x}{8m^2\pi^2}\,(1 - \cos 2m\pi x) - \dfrac{x^2}{8m\pi} \sin 2m\pi x \end{pmatrix}$$

$$+\beta \begin{pmatrix} 0 \\ -\frac{1}{2}\, x \sin 2m\pi x \end{pmatrix} . \qquad (4.17)$$

The function q_1 is only determined up to additive multiples of the eigenfunctions q_{Am} and $q_{B,2m}$, but as explained in the Appendix there multiples can be chosen to be zero. If they are included in q_1, they will not affect the calculations of ν. At $O(\delta^2)$ we obtain the following two linear homogeneous equations for α and β:

$$\alpha\left[-\frac{3}{32m\pi}+\frac{\Lambda^2}{256m^5\pi^5}\left(1+\frac{105}{16m^2\pi^2}\right)\right]+\beta\left(\frac{3}{8}+\frac{\nu}{2}\right)=0,$$

$$\alpha\left(\frac{3m^2\pi^2}{2}-\frac{3\Lambda^2}{16m^2\pi^2}-2m^2\pi^2\nu\right)+\beta\left(-\frac{3\Lambda^2}{8m\pi}\right)=0. \qquad (4.18)$$

These equations determine ν and the ratio α/β. There are two values of ν given by

$$\nu = \frac{1}{2}\left\{\frac{-3\Lambda^2}{32m^4\pi^4} \pm \left[\frac{9}{4}-\frac{27\Lambda^2}{64m^4\pi^4}+\frac{6\Lambda^4}{1024m^8\pi^8}\left(\frac{5}{2}+\frac{105}{16m^2\pi^2}\right)\right]^{1/2}\right\} . \qquad (4.19)$$

For $\delta \to 0$ with $\delta \gg R_a{}^2$, which corresponds to letting $\Lambda \to 0$, we find that $\nu = \pm\, 3/4$. Thus a perturbation in δ splits the double eigenvalue at $\sigma = -4m^2\pi^2$ into two real and unequal eigenvalues,

$$\sigma = -4m^2\pi^2 + \delta^2\left[\ \pm\frac{3}{4} + O(\Lambda^2)\right]. \tag{4.20}$$

For $R_a \to 0$ with $R_a^2 >> \delta$, which corresponds to letting $\Lambda \to \infty$, we find that

$$\nu = \frac{\Lambda^2}{64m^4\pi^4}\left[-\ 3 \pm \sqrt{6}\left(\frac{5}{2} + \frac{105}{16m^2\pi^2}\right)^{1/2} + O(1/\Lambda^2)\right] \ .$$

Hence, again the double eigenvalue at $\sigma = -4m^2\pi^2$ is split into two real eigenvalues,

$$\sigma = -4m^2\pi^2 + (R_a^2\delta)\ \frac{1}{64m^2\pi^2}\left[\ -\ 3\ \ \pm\sqrt{6}\left(\frac{5}{2} + \frac{105}{16m^2\pi^2}\right)^{1/2} + O(1/\gamma^2)\right]. \tag{4.21}$$

However, for each value of m there is a finite interval of values of Λ^2 given by

$$\left(\frac{\Lambda}{m^2\pi^2}\right)^4\left(\frac{5}{2} + \frac{105}{16m^2\pi^2}\right) - 72\left(\frac{\Lambda}{m^2\pi^2}\right)^2 + 384 < 0 \tag{4.22}$$

for which Eq. (4.19) yields complex conjugate values for ν. For $m = 1$, the values of ν are complex conjugates if

$$8.54\pi^4 < \Lambda^2 < 14.21\pi^4. \tag{4.23}$$

We can interpret this result as follows. Suppose we fix δ, $0 < \delta << 1$, and take $R_a = 0$ so $R_a^2\delta = 0$ and hence $\Lambda = 0$.

Then the double eigenvalue $\sigma = -4m^2\pi^2$ of Eqs. (4.2) and (4.3) when $\delta = 0$ splits into two real eigenvalues, $\sigma = -4m^2\pi^2 \pm 3\delta^2/4$ at leading order. Now we increase R_a, but with $R_a{}^2\delta << \delta^2$ and hence $\Lambda << 1$; then the dominant correction term to $\sigma = -4m^2\pi^2$ remains $\pm 3\delta^2/4$ and the two eigenvalues are still real. However, as R_a is increased still further until $R_a{}^2\delta = O(\delta^2)$ the correction term ν becomes complex when Λ reaches the lower value determined by Eq. (4.23) and ν stays complex until Λ reaches the larger value determined by Eq. (4.23). For $\delta^2 << R_a{}^2\delta << 1$ so $\Lambda^2 >> 1$ the eigenvalues are real and distinct differing from $-4m^2\pi^2$ by terms $O(R_a{}^2\delta)$ as given by Eq. (4.21).

In Figure 5 we show the results of the perturbation analysis for $\delta = 0.123$ ($\eta = 0.877$) for the eigenvalues σ_2 and σ_3 corresponding to $m = 1$. For $R_a = 0$ the eigenvalues are $\sigma_2 = -39.4897$ (corresponding to σ_{A1}) and $\sigma_3 = -39.4671$ (corresponding to σ_{B2}); as R_a is increased they coalesce at $R_a = 10.10$ ($\Lambda^2 = 8.54\pi^4$) and become complex conjugates until $R_a = 13.05$ ($\Lambda^2 = 14.21\pi^4$) at which point they coalesce again; and for $R_a > 13.05$ they are again real. Also in Figure 5 we show the corresponding results obtained by solving the full equations (4.2) and (4.3) numerically. We have labeled the modes according to their ordering at $\delta = 0$ and $R_a = 0$. Note the crossing of the eigenvalues. The perturbation and numerical results are in reasonable agreement even though $R_a{}^2\delta$ is not numerically small for $\Lambda^2 = 8.54\pi^2$ and larger values of Λ^2. We note that it would not have been easy to determine these eigenvalue curves numerically without the information gained from the perturbation analysis. For the other double eigenvalues $\sigma = -4m^2\pi^2$, $m = 2, 3, \cdots$, at $\delta = 0$ and $R_a = 0$ the splitting structure for perturbations in δ and R_a is similar to the case $m = 1$ just discussed.

For a fixed value of δ and moderate values of R_a it is necessary to resort to standard numerical procedures (discussed in Section 3) to solve the eigenvalue problem (4.2) and (4.3) for the variation of the eigenvalues σ with R_a. In Figure 6 we show the development of the first five eigenvalues with R_a for $0 < R_a < 1200$ for $\eta = 0.877$ ($\delta = 0.123$). The modes are labeled accordingly to their ordering at $R_a = 0$. Along the branch denoted by * - ** the eigenvalues σ_4 and σ_5 are complex conjugates. The corresponding modes interchange their order at the point ** just as was the case for the σ_2 and σ_3 eigenvalues. Our numerical calculations (and perturbation calculations for σ_2 and σ_3) show the following continuous mode association: $\sigma_1 \sim q_{B1}$, $\sigma_2 \sim q_{A1}$, $\sigma_3 \sim q_{B2}$, $\sigma_4 \sim q_{C1}$, and $\sigma_5 \sim q_{B3}$.

For the modes we have investigated, we find that the B eigenvalues initially increase monotonically with R_a, while the A and C eigenvalues initially decrease with R_a. When a B and an A eigenvalue or a B and a C eigenvalue coalesce (recall the pairing mentioned earlier) the eigenvalues become complex conjugates for some finite interval of values of R_a, the eigenvalues then coalesce again and split as two real eigenvalues with the corresponding modes interchanging their order. The only eigenvalue which does not have such a crossing (and hence for some R_a an association with a complex eigenvalue) is the first, or most unstable mode, $\sigma_1 \sim q_{B1}$. None of our numerical calculations showed the existence of a complex eigenvalue $\sigma = \sigma_r + i\sigma_i$, with $\sigma_r > 0$; however we did not make an exhaustive search.

Finally, we want to mention a not so obvious way in which the B and A modes can cross. This is associated with a branching at points at which $d\sigma/dR_a = 0$. At such points the inverse eigenvalue relation $R = R(\sigma)$ has a double eigenvalue which can split apart as a pair of complex conjugate eigenvalues as σ (real) is varied. Thus in Figure 6 the points ○ and ◎ at which $d\sigma/dR = 0$ are connected by a branch on which σ is real and there are two complex conjugate values of R_a. In this way the A1 and the B3 modes interchange their order with σ remaining real but R_a being complex. Of course for the physical problem R, and hence R_a, is real.

Corresponding to the segment * - ** in Figures 5 and 6 the eigenvalues σ_2 and σ_3 and σ_4 and σ_5, respectively, are complex conjugates. From the numerical calculations these intervals correspond approximately to

$$\begin{aligned} &\sigma_2 \text{ and } \sigma_3\,, \quad 7.8 < R_a < 9.8 \quad \text{or} \quad 7.8/a < R < 9.8/a; \\ &\sigma_4 \text{ and } \sigma_5\,, \quad 320 < R_a < 720 \quad \text{or} \quad 320/a < R < 720/a. \end{aligned} \tag{4.24}$$

With this in mind we investigated the full problem for $\eta = 0.877$ to determine regions in the a-R plane where complex eigenvalues would be found. In Figure 7 we show this region (between the two curves) for the eigenvalues σ_4 and σ_5. The asymptotic results for $a \to 0$ are in reasonable agreement with the numerical calculations even for moderate values of a. There is a similar region for the eigenvalues σ_2 and σ_3, and, we believe, for the other B eigenvalues as they cross the A or C eigenvalues . Again we note that the numerical calculations were motivated by the results of the asymptotic analysis.

5. DISCUSSION

We have demonstrated by numerical computation and formal perturbation methods that the eigenvalue problem for the linear stability of Couette flow to axisymmetric disturbances (the Taylor problem) has complex eigenvalues when the outer cylinder is at rest. However, it is not inherent in the analysis that the outer cylinder is at rest (for example the asymptotic analysis for $a \to 0$ in Section 4) and we anticipate the occurence of complex eigenvalues when the outer cylinder also rotates. All of the complex eigenvalues which we have found correspond to damped oscillatory modes; and in particular there is no indication that the first eigenvalue is complex for any values of η, a, and R with the outer cylinder at rest. Thus it seems likely that the principle of exchange of stabilities holds. However, the existence of complex eigenvalues shows that, unlike the situation for the Benard problem, a proof of this principle cannot be constructed by showing that all the eigenvalues are real.

It is interesting to speculate on whether an experimental investigation of the Taylor problem could confirm the existence of decaying oscillatory axisymmetric disturbances. In any experimental situation there are small imperfections and random fluctuations present which, presumably, will excite all the modes of linear thoery. However, all of the oscillatory decaying modes have large decay rates. The least damped oscillatory modes have $\sigma_r \cong -4\pi^2$ (corresponding to σ_2 and σ_3 coalescing), but this occurs for only a small area in the a-R plane. The eigenvalues σ_4 and σ_5 can be complex conjugates for a significant set of values of a and R, as is shown in Figure 7. For $\eta = 0.877$, $a = 5$, and $R = 150$ we find the complex conjugate eigenvalues* $\sigma = -110.2 \pm i3.72$. The physical time T for such a mode to decay to e^{-1} times its initial amplitude is given by $T = 0.19/\Omega_1$, which may allow sufficient time for observation.

We have seen that the structure of the spectrum of the linear problem for $a = O(1)$ is preserved in the limiting problem discussed in Section 4 for $a \to 0$. Moreover, for the latter problem we found that the complex eigenvalues occur because there are two denumerable sets of negative eigenvalues at $R_a = 0$ which separate each other and move in opposite directions as R_a, and hence R, is increased. One set of eigenvalues have σ_r increasing, and ultimately correspond to unstable modes, while σ_r decreases for the other set. It follows that there will be an infinite number of intersections and at these points of intersection we can expect the occurence of complex conjugate eigenvalues. As we have noted these points of intersection correspond to damped oscillatory disturbances. It is of course possible that the eigenvalues corresponding to two unstable modes can grow at different rates with R, and hence eventually intersect so as to give rise to complex conjugate eigenvalues and growing oscillatory

---

* This calculation was confirmed by Dr. P.M. Eagles using an independent program.

modes; however, we have not explored such possibilities.

It is also possible to study the full eigenvalue problem, Eqs. (1.2) and (1.3), in the limit $a \to \infty$ with η and R fixed. In this limit we can expect that $\sigma = O(-a^2)$ for all the modes so an expansion will take the form

$$\sigma = a^2(-1 + a^{-1}\mu + \ldots).$$

It turns out that the eigenfunctions have a boundary layer behaviour at the cylinder walls and satisfy a fourth order differential equation in the interior. The parameter μ and the higher order coefficients are determined by solving a sequence of fourth order equations with boundary conditions at the cylinder walls determined by matching conditions. We did not study this problem in detail, but our few calculations for the small-gap case did show that μ can be complex and that the curves of Figure 7 will asymptote to horizontal straight lines as $a \to \infty$. It is interesting to note that if $a \to \infty$ with $R = O(a^2)$, corresponding to the behaviour of R on the neutral curve, the asymptotic analysis of Hall (1982) shows that all the eigenvalues σ are real. This result gives additional credence to the principle of exchange of stabilities, but a proof is yet to be constructed.

Acknowledgement

We wish to express our thanks to Professor George Habetler for several helpful discussions.

APPENDIX

Consider a boundary value problem consisting of a system of ordinary differential equation of the form

$$(L + \varepsilon M + \varepsilon^2 N + \cdots)\,\bar{q} - \bar{\sigma}\,(A + \varepsilon B + \varepsilon^2 C + \cdots)\,\bar{q} = 0, \qquad \text{(A1)}$$

where L, M, N, ⋯, A, B, C, ⋯ are matrix differential operators with non-singular coefficients; $\bar{q}$ is a vector function satisfying separated homogeneous boundary conditions at the end points of some finite interval; say $0 < x < 1$; ε is a small non-negative parameter; $\bar{\sigma}$ is the eigenvalue; and the order of the system (A1) is equal to the order of the operator L. The boundary value problem (A1), including the boundary conditions, need not be selfadjoint. Our discussion is formal; moreover, we do not consider all possible cases.

We assume that the boundary value problem

$$(L - \sigma A)\, q = 0 \qquad \text{(A2)}$$

has an infinite denumerable set of eigenvalue $\{\sigma_n\}$ with $\sigma_1 = \sigma_2$ while all the other eigenvalues are simple. We are interested in how the corresponding eigenvalues $\bar{\sigma}_1$ and $\bar{\sigma}_2$ of (A1), whose limits are σ_1 and σ_2 as $\varepsilon \to 0$, vary with ε. We consider two cases

I. Corresponding to $\sigma_1 = \sigma_2$ there are two linearly independent vectors q_1 and q_2 of (A2),

and

II. Corresponding to $\sigma_1 = \sigma_2$ there is one eigenvector q_1 of (A2) and a generalized eigenvector q_{11} satisfying

$$(L - \sigma_1 A)\ q_{11} = Aq_1. \tag{A3}$$

We assume that q_1, q_2, q_3, ... in Case I and q_1, q_{11}, q_3, ... in Case II are complete in an appropriate function space. Let $\langle \cdot\ ,\ \cdot \rangle$ denote the inner product, and let

$$(L^* - \sigma A^*)\ q^* = 0 \tag{A4}$$

with appropriate boundary conditions be the adjoint boundary value problem corresponding to (A2). We also assume that $\bar{\sigma}_1 \neq \bar{\sigma}_2$ for ε small and non-zero.

<u>Case I</u>. For this case the discussion is similar to that given in Courant and Hilbert (1953, pp. 346-350) for a multiple eigenvalue of a second order selfadjoint boundary value problem. We assume that the eigenvectors and adjoint eigenvectors have been chosen so that

$$\langle Aq_i,\ q_j^* \rangle = \langle q_i,\ A^* q_j^* \rangle = \delta_{ij} \quad . \tag{A5}$$

As part of the perturbation calculation it is necessary to determine the "directions" of the eigenvectors $\bar{q}_1$ and $\bar{q}_2$ of (A1) as $\varepsilon \to 0$. Thus we choose as our eigenvectors of (A2) the set

$$u_1 = \alpha_{11} q_1 + \alpha_{12} q_2\ ; \qquad u_2 = \alpha_{21} q_1 + \alpha_{22} q_2\ ;$$

$$u_j = q_j\ ,\ j = 3,\ 4,\ \ldots, \tag{A6}$$

where the α's are to be chosen so that $\bar{q}_1 \to u_1$, $\bar{q}_2 \to u_2$ as $\varepsilon \to 0$.

We expand $\overline{\sigma}_n$, $\overline{q}_n$ in the series

$$\overline{\sigma}_n = \sigma_n + \varepsilon\mu_n + \varepsilon^2\nu_n + \cdots, \qquad \overline{q}_n = u_n + \varepsilon v_n + \varepsilon^2 w_n + \cdots,$$

$$n = 1, 2, \cdots. \tag{A7}$$

If we substitute (A7) in (A1), we obtain

$$(L - \sigma_n A)v_n = (-M + \sigma_n B + \mu_n A)u_n, \tag{A8}$$

$$(L - \sigma_n A)w_n = (-N + \sigma_n C + \mu_n B + \nu_n A)u_n$$

$$+ (-M + \sigma_n B + \mu_n A)v_n. \tag{A9}$$

Next we assume that we can write

$$v_n = \sum_{j=1}^{\infty} a_{nj}\, q_j \quad , \qquad w_n = \sum_{j=1}^{\infty} b_{nj}\, q_j\ . \tag{A10}$$

If we substitute for v_n (A8) and take the inner product with q_m^*, we obtain

$$a_{nm}\ (\sigma_m - \sigma_n) = \langle(-M + \sigma_n B)u_n,\ q_m^*\rangle + \mu_n\ \delta_{nm}. \tag{A11}$$

For n = 1, 2 and m = 1, 2 the left side of (A11) vanishes since $\sigma_1 = \sigma_2$ and we obtain the equation

$$\begin{vmatrix} \langle(-M + \sigma_1 B)q_1,\ q_1^*\rangle + \mu & \langle(-M + \sigma_1 B)q_2,\ q_1^*\rangle \\ \langle(-M + \sigma_1 B)q_1,\ q_2^*\rangle & \langle(-M + \sigma_1 B)q_2,\ q_2^*\rangle + \mu \end{vmatrix} = 0 \tag{A12}$$

for μ_1 and μ_2. For $n = m = 3, 4, \cdots$, (A11) gives $\mu_n = \langle (M - \sigma_n B)q_n, q_n^* \rangle$. If we only need the first order corrections to the eigenvalues $\bar{\sigma}_n$ we can stop at this point. However, if we also require the first order corrections to the eigenvectors and the second order corrections to the eigenvalues, it is advantageous to reorganize our computations.

We also find from (A11) that

$$\alpha = \frac{\alpha_{12}}{\alpha_{11}} = -\frac{\langle (M - \sigma_1 B)q_1, q_1^* \rangle - \mu_1}{\langle (M - \sigma_1 B)q_2, q_1^* \rangle}$$

$$= -\frac{\langle (M - \sigma_1 B)q_1, q_2^* \rangle}{\langle (M - \sigma_1 B)q_2, q_2^* \rangle - \mu_1}, \tag{A13}$$

and

$$\beta = \frac{\alpha_{21}}{\alpha_{22}} = -\frac{\langle (M - \sigma_1 B)q_2, q_1^* \rangle}{\langle (M - \sigma_1 B)q_1, q_1^* \rangle - \mu_2}$$

$$= -\frac{\langle (M - \sigma_1 B)q_2, q_2^* \rangle - \mu_2}{\langle (M - \sigma_1 B)q_1, q_2^* \rangle}, \tag{A14}$$

where we have assumed $\langle (M - \sigma_1 B)q_2, q_1^* \rangle \neq 0$ and $\langle (M - \sigma_1 B)q_2, q_2^* \rangle \neq 0$. Now introduce new adjoint eigenvectors $u_1^* = q_1^* - \beta q_2^*$, $u_2^* = -\alpha q_1^* + q_2^*$, $u_j^* = q_j^*$ for $j = 3, 4, \cdots$, and choose $\alpha_{11} = \alpha_{22} = (1 - \alpha\beta)^{-1}$. As a result we have

$$\langle Au_i, u_j^* \rangle = \langle u_i, A^* u_j^* \rangle = \delta_{ij} . \tag{A15}$$

We can also write

$$v_n = \sum_{j=1}^{\infty} c_{nj} u_j \quad , \qquad w_n = \sum_{j=1}^{\infty} d_{nj} u_j . \tag{A16}$$

Substituting this expansion for v_n in (A8) and taking the inner product with u_m^*, we obtain

$$c_{nm}(\sigma_m - \sigma_n) = - F_{nm} + \mu_n \delta_{nm} \tag{A17}$$

where $F_{nm} = \langle (M - \sigma_n B)u_n, u_m^* \rangle$. We will restrict our attention to the eigenvalues $\bar{\sigma}_1$ and $\bar{\sigma}_2$ and the corresponding eigenfunctions; that is, n = 1 and n = 2. For n = 3, 4, ... the calculation is straightforward. Equation (A17) yields the following information for n = 1, 2 and m = 1, 2

$$\mu_1 = F_{11}, \quad \mu_2 = F_{22}, \quad F_{12} = 0, \quad F_{21} = 0, \tag{A18}$$

and

$$c_{nm} = - F_{nm}/(\sigma_m - \sigma_n), \quad n = 1, 2, \quad m = 3, 4, \cdots. \tag{A19}$$

Equations (A18) are consistent with our choice of μ_1, μ_2, u_1, u_2, u_1^*, and u_2^* as can be verified by doing the necessary algebra. Alternatively, if we had chosen the u_j according to (A6), the u_j^* so that the biorthogonality condition (A15) is satisfied, and then used the expansions (A16) we would have obtained (A18) for μ_1, μ_2, and the ratios α_{12}/α_{11} and α_{21}/α_{22}.

The coefficients c_{1m} and c_{2m} in the series expansions for v_1 and v_2 are given by (A19) for m = 3, 4, $\cdots$. We must determine c_{11}, c_{12}, c_{21}, and c_{22}. Imposition of the normalization condition

$$\langle A\bar{q}_n, u_n^* \rangle = 1 \quad , \quad n = 1, 2, \cdots \tag{A20}$$

yields $c_{11} = c_{22} = 0$. To determine c_{12} and c_{21} we must consider (A9) for w_n.

Substituting the expansion for w_n given in (A16) and (A9) and taking the inner product with $u_m{}^*$ gives

$$d_{nm}\,(\sigma_m - \sigma_n) = -\langle (N - \sigma_n C - \mu_n B)u_n,\ u_m{}^*\rangle + \nu_n \delta_{nm} + \sum_{j=1}^{\infty} c_{nj}(-F_{jm} + \mu_n \delta_{jm}). \tag{A21}$$

For $n = m = 1$ and $n = m = 2$ we obtain

$$\nu_n = \langle (N - \sigma_n C - \mu_n B)u_n,\ u_n{}^*\rangle - \sum_{j=3}^{\infty} \frac{F_{nj}\,F_{jn}}{\sigma_j - \sigma_n}\,; \tag{A22}$$

and for $n = 1$, $m = 2$ and $n = 2$, $m = 1$ we obtain

$$c_{nm} = \frac{1}{\mu_n - \mu_m}\left(\langle (N - \sigma_n C - \mu_n B)u_n,\ u_m{}^*\rangle - \sum_{j=3}^{\infty} \frac{F_{nj}\,F_{jm}}{\sigma_j - \sigma_n} \right) . \tag{A23}$$

This completes the calculation of $\bar{\sigma}_1$ and $\bar{\sigma}_2$ through terms $O(\varepsilon^2)$ and the corresponding eigenfunctions $\bar{q}_1$ and $\bar{q}_2$ through terms $O(\varepsilon)$. To compute corrections to the other eigenvalues and eigenvectors as well as higher order corrections for $\bar{\sigma}_1$, $\bar{\sigma}_2$, $\bar{q}_1$, and $\bar{q}_2$ is straight forward in principle. In practice the calculations would probably not be carried out using eigenfunction expansions, but rather the nonhomogeneous equations for v_1 and v_2 would be solved numerically after the solvability conditions (A18) had been used.

<u>Case II</u> For this case the discussions by Wilkinson (1965, pp. 62-70) for the matrix eigenvalue problem, Friedman (1956, pp. 110-113 and 131-133), and Kato (1980, Chapter 2) are useful. Let q_1^*, q_3^*, ... and q_{11}^* be the corresponding eigenvectors and generalized eigenvector of (A4). We assume that the vectors have been normalized (see Friedman for the case A is the identity operator) so that

$$\langle Aq_1, q_1^* \rangle = \langle Aq_{11}, q_{11}^* \rangle = 0, \ \langle Aq_1, q_{11}^* \rangle = \langle Aq_{11}, q_1^* \rangle = 1,$$

$$\langle Aq_1, q_j^* \rangle = \langle Aq_{11}, q_j^* \rangle = 0, \quad j = 3, 4, \cdots,$$

$$\langle Aq_i, q_j^* \rangle = \delta_{ij}, \quad i, j = 3, 4, \cdots. \tag{A24}$$

We will only consider the eigenvalues $\bar{\sigma}_1$, and $\bar{\sigma}_2$ of (A1) which coalesce at $\varepsilon = 0$. The form of the expansion is

$$\bar{\sigma} = \sigma_1 + \varepsilon^{1/2}\mu + \varepsilon\nu + \varepsilon^{3/2}\zeta + \cdots,$$

$$\bar{q} = q_1 + \varepsilon^{1/2}u + \varepsilon v + \varepsilon^{3/2}w + \cdots. \tag{A25}$$

If we substitute (A25) in (A1) we obtain

$$O(\varepsilon^0): \quad (L - \sigma_1 A)q_1 = 0 \ , \tag{A26}$$

$$O(\varepsilon^{1/2}): \quad (L - \sigma_1 A)u = \mu A q_1 \ , \tag{A27}$$

$$O(\varepsilon): \quad (L - \sigma_1 A)v = -Mq_1 + \mu A u + (\nu A + \sigma_1 B)q_1 \ , \tag{A28}$$

$$O(\varepsilon^{3/2}): \quad (L - \sigma_1 A)w = -Mu + \mu A v + (\nu A + \sigma_1 B)u$$

$$+ (\zeta A + \mu B)q_1 \ . \tag{A29}$$

Equation (A26) is the eigenvalue problem for σ_1, q_1. The solution of (A27) is $u = \mu q_{11}$ plus a multiple of the eigenvector q_1 which we can take to be zero. The parameter μ is still to be determined. Next we write

$$v = a_1 q_1 + a_{11} q_{11} + \sum_{j=3}^{\infty} a_j q_j \ , \tag{A30}$$

and substitute for v and u in (A28) to obtain

$$a_{11} A q_1 + \sum_{j=3}^{\infty} a_j (\sigma_j - \sigma_1) A q_j = - M q_1 + \mu^2 A q_{11}$$

$$+ (\nu A + \sigma_1 B) q_1 \ . \tag{A31}$$

If we take the inner product of (A31) with respect to q_1^*, q_{11}^*, and q_m^* for $m = 3, 4, \cdots$, we obtain, respectively,

$$\mu^2 = \langle M q_1, q_1^* \rangle - \sigma_1 \langle B q_1, q_1^* \rangle \ , \tag{A32}$$

$$a_{11} = - \langle M q_1, q_{11}^* \rangle + \nu + \sigma_1 \langle B q_1, q_{11}^* \rangle \ , \tag{A33}$$

$$a_m (\sigma_m - \sigma_1) = - \langle M q_1, q_m^* \rangle + \sigma_1 \langle B q_1, q_m^* \rangle \ ,$$

$$m = 3, 4, \cdots . \tag{A34}$$

Notice that in order to calculate μ^2 we need only know q_1, q_1^*, and q_{11}^* (which is needed in the normalizing condition $\langle Aq_1, q_{11}^*\rangle = 1$). If μ is real and μ^2 is positive, then the perturbed eigenvalues are real $\bar{\sigma}_1, \bar{\sigma}_2 = \sigma_1 \pm\sqrt{\varepsilon}\ |\mu| + O(\varepsilon)$, and if μ^2 is negative then the perturbed eigenvalues are complex conjugates $\bar{\sigma}_1, \bar{\sigma}_2 = \sigma_1 \pm i\sqrt{\varepsilon}\ |\mu| + O(\varepsilon^2)$. If $\mu = 0$ then also $u = 0$ and the splitting of the eigenvalues occurs at $O(\varepsilon)$ or a higher order. Assuming that μ is real and non-zero, then the corresponding eigenvectors of (A1) at this order are $\bar{q}_1, \bar{q}_2 = q_1 \pm\sqrt{\varepsilon}\ |\mu|\ q_{11}$ or $\bar{q}_1, \bar{q}_2 = q_1 \pm i\sqrt{\varepsilon}\ |\mu|\ q_{11}$. Equation (A34) provides a linear nonhomogeneous equation for a_{11} and ν, while the a_m for $m = 3, 4, \ldots$ are given by (A34). The coefficient a_1 ia arbitrary; however if we impose the normalization conditions $\langle A\bar{q}_1, q_{11}^*\rangle = \langle A\bar{q}_2, q_{11}^*\rangle = 1$ then $a_1 = 0$. In order to determine ν and a_{11} we turn to (A29) for w.

We expand w in the series

$$w = b_1 q_1 + b_{11} q_{11} + \sum_{j=3}^{\infty} b_j\, q_j\ , \tag{A35}$$

substitute in (A29) for w, v, and u, and take the inner product with respect to q_1^*. We obtain

$$0 = \mu \Big(-\langle Mq_{11}, q_1^*\rangle + a_{11} + \nu + \langle Bq_{11}, q_1^*\rangle + \langle Bq_1, q_1^*\rangle \Big). \tag{A36}$$

Again, assuming $\mu \neq 0$ and solving (A36) and (A33) for ν and a_{11} we obtain

$$\nu = \frac{1}{2}\left(\langle Mq_{11}, q_1^*\rangle + \langle Mq_1, q_{11}^*\rangle\right) - \frac{\sigma_1}{2}\left(\langle Bq_{11}, q_1^*\rangle + \langle Bq_1, q_{11}^*\rangle\right) - \frac{1}{2}\langle Bq_1, q_1^*\rangle , \tag{A37}$$

$$a_{11} = \frac{1}{2}\left(\langle Mq_{11}, q_1^*\rangle - \langle Mq_1, q_{11}^*\rangle\right) - \frac{\sigma_1}{2}\left(\langle Bq_{11}, q_1^*\rangle - \langle Bq_1, q_{11}^*\rangle\right) - \frac{1}{2}\langle Bq_1, q_1^*\rangle . \tag{A38}$$

This completes the calculation of the eigenvalues $\bar{\sigma}_1$ and $\bar{\sigma}_2$ and the corresponding eigenvectors through terms $O(\varepsilon)$. Note that both μ and ν can be calculated once q_1, q_{11}, q_1^*, and q_{11}^* are known. The calculation of higher order corrections for $\bar{\sigma}_1$ and $\bar{\sigma}_2$ is straightforward.

References

Courant, R and Hilbert, D., 1953 Methods of Mathematical Physics, vol. 1, Interscience, New York.

DiPrima, R.C. and Habetler, G.J. 1969 A completeness theorem for non-selfadjoint eigenvalue problems in hydrodynamic stability, Arch. Rat. Mech. Anal. 34, 218-227.

Friedman, B. 1956 Principles and Techniques of Applied Mathematics, J. Wiley, New York.

Hall, P. 1982 Taylor-Görtler vortices in fully developed or boundary-layer flows: linear theory, J. Fluid Mech. 124, 475-494.

Kato, T. 1980 Perturbation Theory for Linear Operators (corrected printing of 2nd ed.), Springer, Berlin.

Kirchgässner, K. and Sorger, P. 1969 Branching analysis for the Taylor problem, Quart. J. Mech. Appl. Math. 22, 183-190.

Velte, W. 1966 Stabilität und Verzweigung stationärer Lösungen der Navier-Stokesschen Gleichungen beim Taylor-Problem, Arch. Rat. Mech. Anal. 22, 1-14.

Wilkinson, J.W. 1965 The Algebraic Eigenvalue Problem, Clarendon Press, Oxford.

Yih, C.S. 1972a Spectral theory of Taylor vortices. Part I Structure of unstable modes, Arch. Rat. Mech. Anal. 47, 218-240.

Yih, C.S. 1972b Spectral theory of Taylor vortices. Part II Proof of nonoscillation, Arch. Rat. Mech. Anal. 47, 288-300.

Legends for Figures

Figure 1. The variation of the first five eigenvalues with a for $1 < a < 8$, $\eta = 0.877$ and $R = 150$. $R_c = 119.3$. Eigenvalues σ_4 and σ_5 are complex conjugates for $2.6 < a < 6.5$. The imaginary parts of σ_4 and σ_5 are shown by the dashed curve with the scale given by the axis at the right. The labeling of the eigenvalues corresponds to their ordering at $R = 0$.

Figure 2. The same as Figure 1 except for $\eta = 0.75$ and $R_c = 85.79$

Figure 3. The same as Figure 1 except for $\eta = 0.5$ and $R_c = 68.18$

Figure 4. The real and imaginary parts of the eigenvalues σ_4 and σ_5 for $\eta = 0.877$, $R = 150$, and a near $a_0 = 2.5576$.
○ Numerical calculation. ——— Perturbation formula (3.3)

Figure 5 Behaviour of the second and third eigenvalues of Eqs. (4.2) and (4.3) for $\eta = 0.877$ and $R_a\delta^{-1/2} = O(1)$. The eigenvalues correspond to a double eigenvalue $\sigma = -4\pi^2$ for $\delta = 0$ and $R_a = 0$. The perturbation analysis, Eqs. (4.16) and (4.19), is given by the dashed curve. The numerical calculation using Eqs. (4.2) and (4.3) is shown by the solid curve. On the branch * - ** the two eigenvalues are complex conjugates and only the real part is shown.

Figure 6. The behaviour of the first five eigenvalues of Eqs. (4.2) and (4.3) for $\eta = 0.877$ ($\delta = 0.123$) and $0 < R_a < 1200$. The eigenvalues σ_4 and σ_5 are complex conjugates on the branch * - ** and only the real part is shown. A branch connects the points ○ and ◎ along which σ is real corresponding to two complex conjugate values of R_a.

Figure 7. For points (a, R) between the two curves the eigenvalues σ_4 and σ_5 of the full problem, Eqs. (1.2)-(1.4), are complex conjugates. ——— Numerical calculation. -------- Asymptotic formula (4.24).

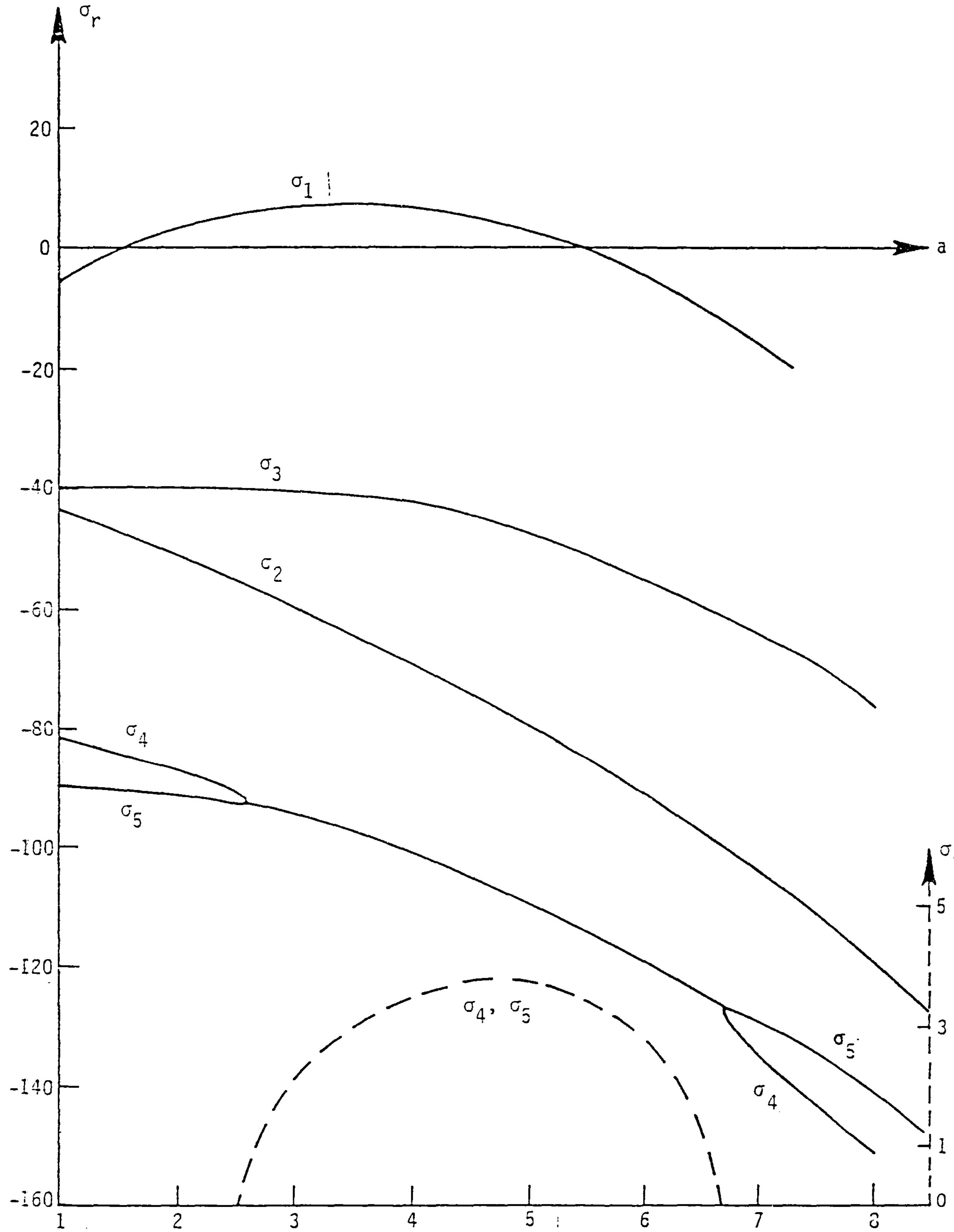
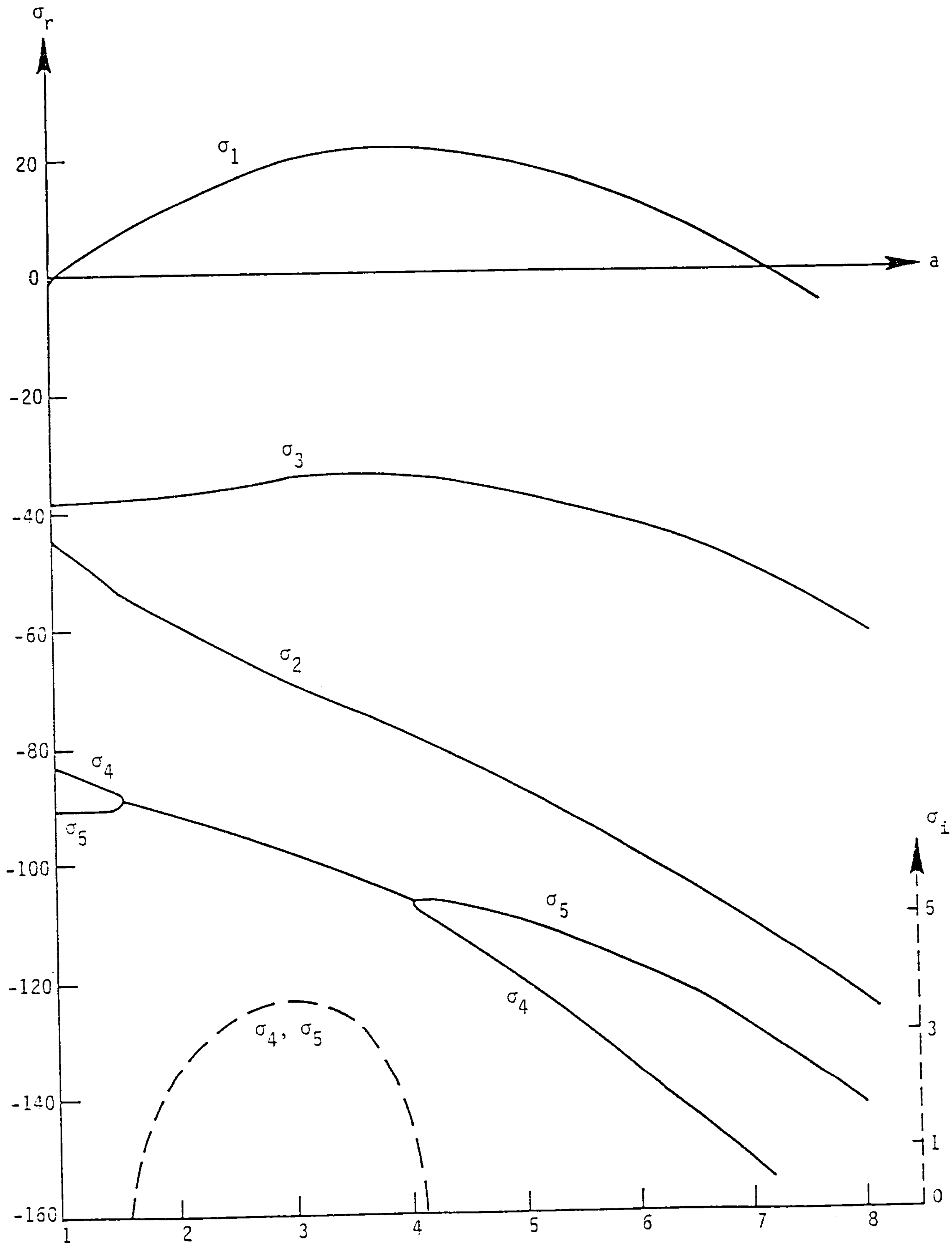
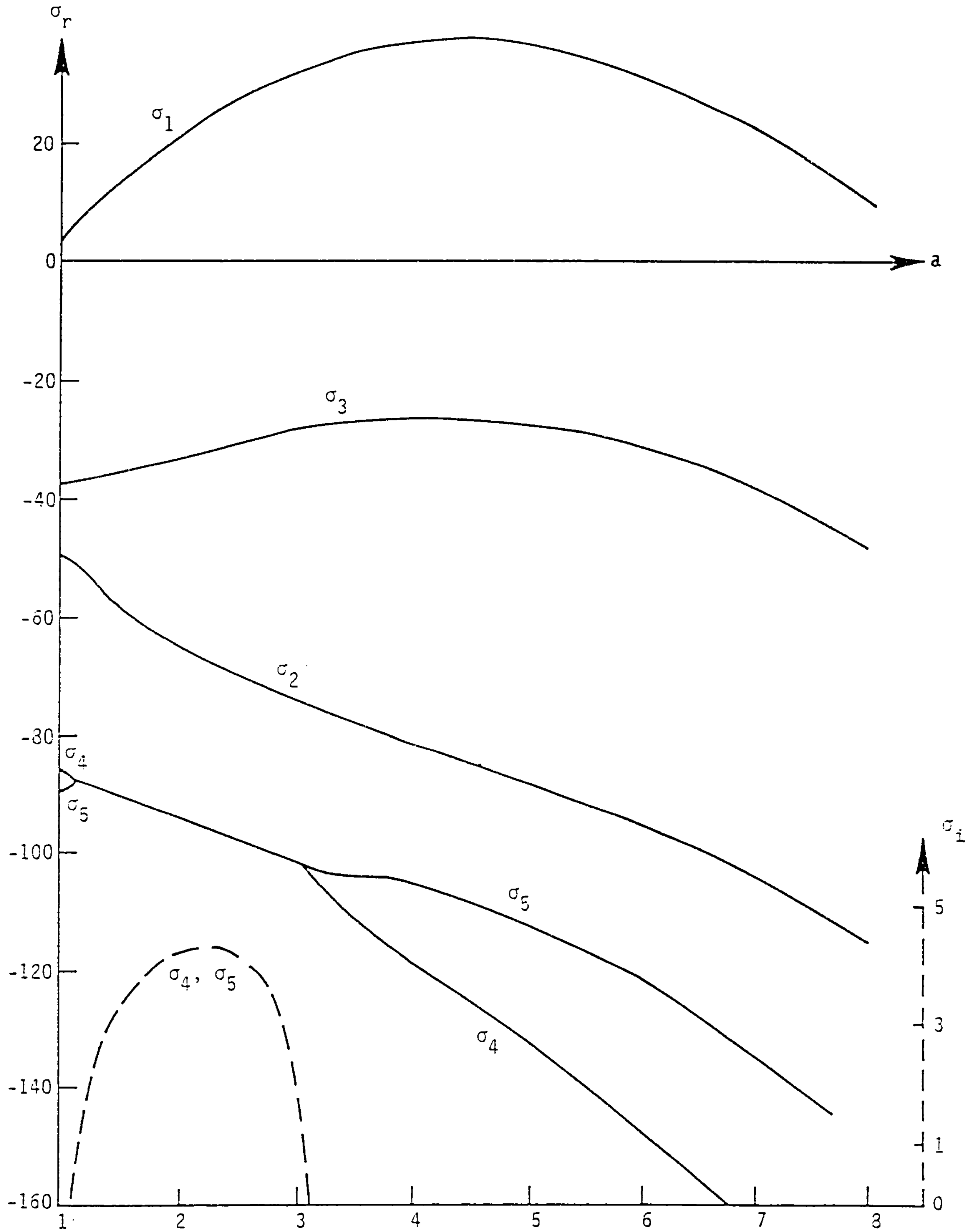

σ_r
20
0
-20
-40
-60
-80
-100
-120
-140
-160
σ_1
σ_3
σ_2
σ_4
σ_5
σ_4, σ_5
σ_5
σ_4
a
σ_i
5
3
1
0
1
2
3
4
5
6
7
8

Figure 1



σ_r
20
0
-20
-40
-60
-80
-100
-120
-140
-160
σ_1
σ_3
σ_2
σ_4
σ_5
σ_5
σ_4
σ_4, σ_5
a
σ_i
5
3
1
0
1
2
3
4
5
6
7
8

Figure 2



σ_r
20
0
-20
-40
-60
-80
-100
-120
-140
-160
σ_1
σ_3
σ_2
σ_4
σ_5
σ_5
σ_4, σ_5
σ_4
a
σ_i
5
3
1
0
1
2
3
4
5
6
7
8

Figure 3

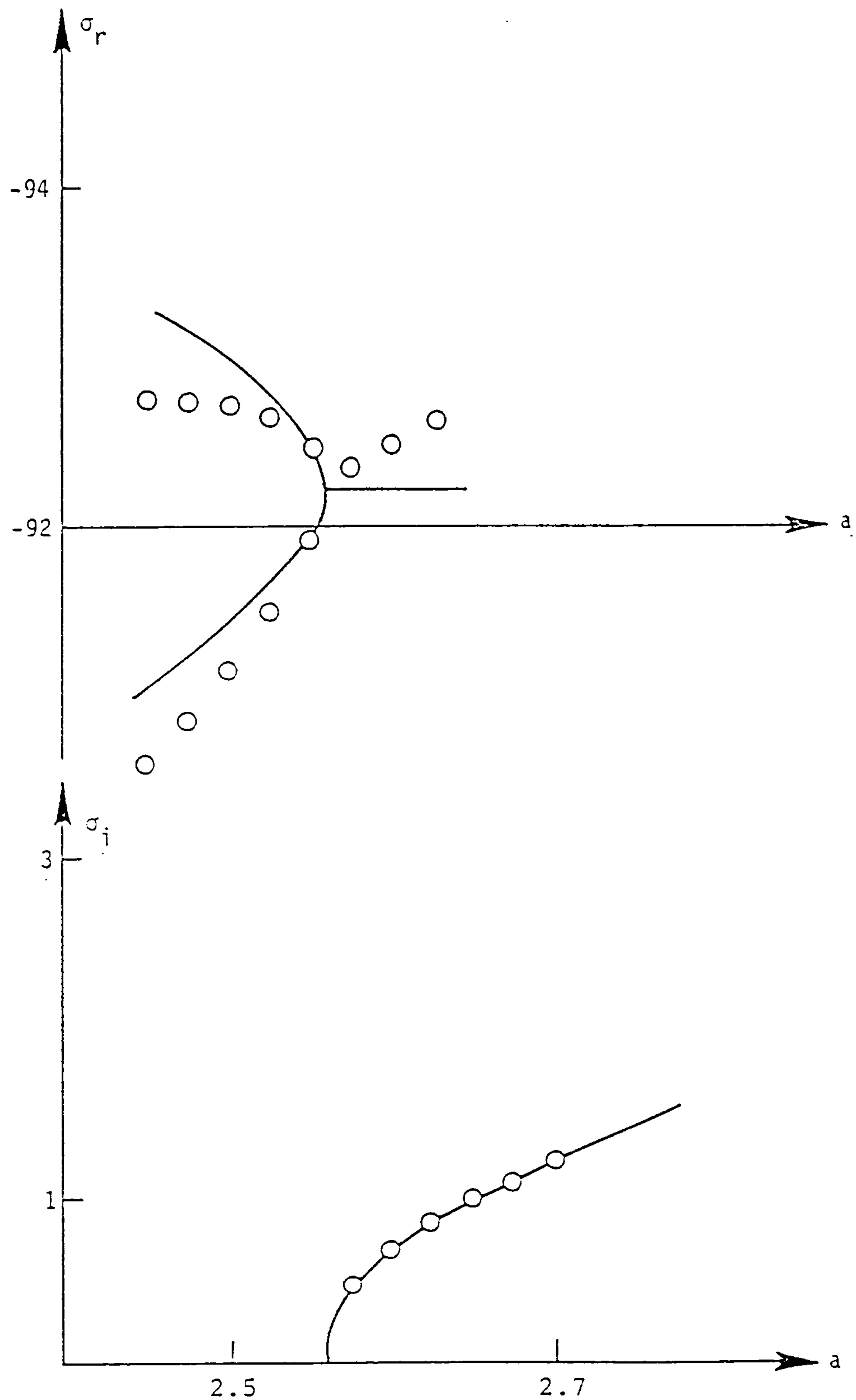


Figure 4

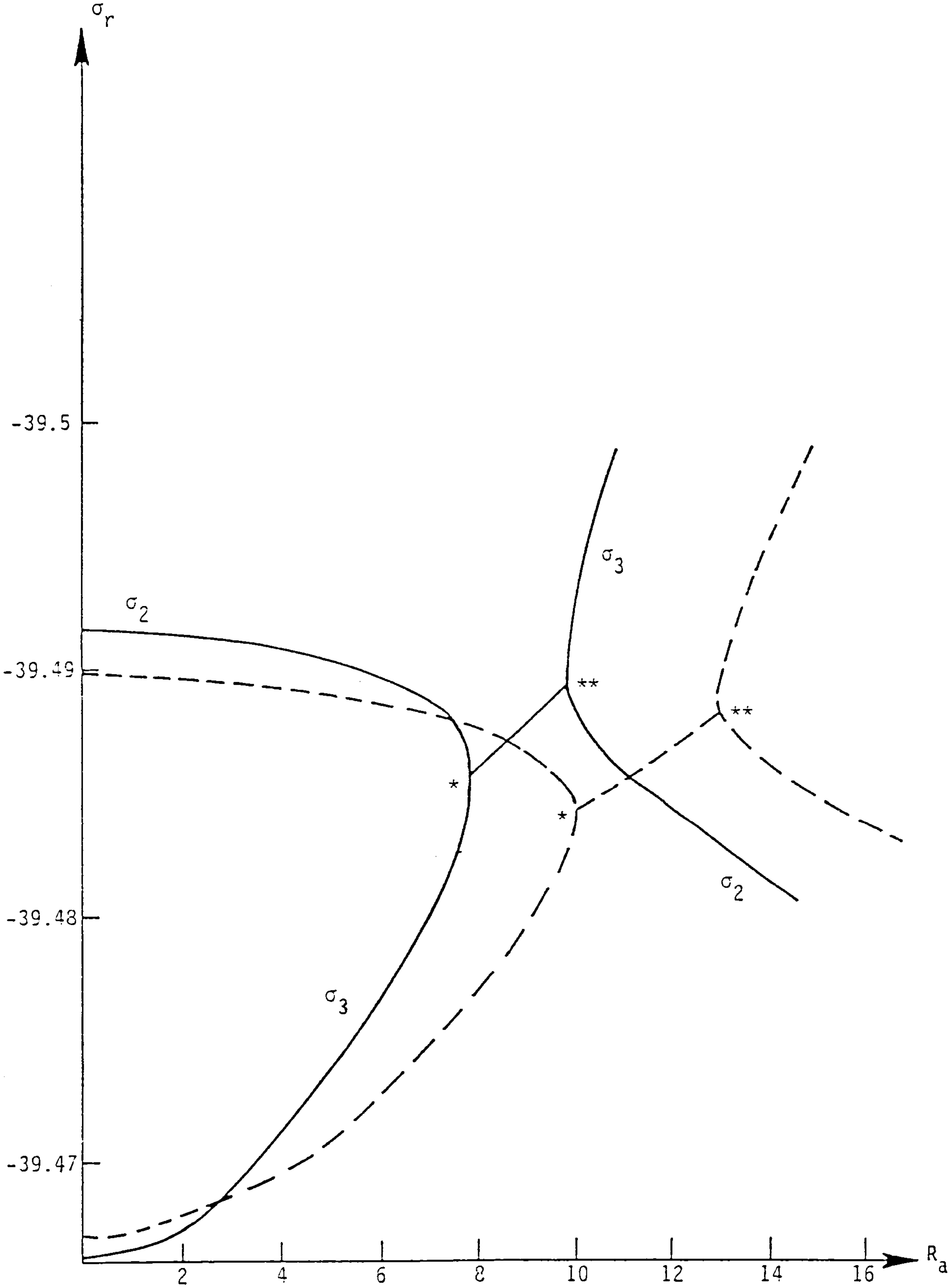
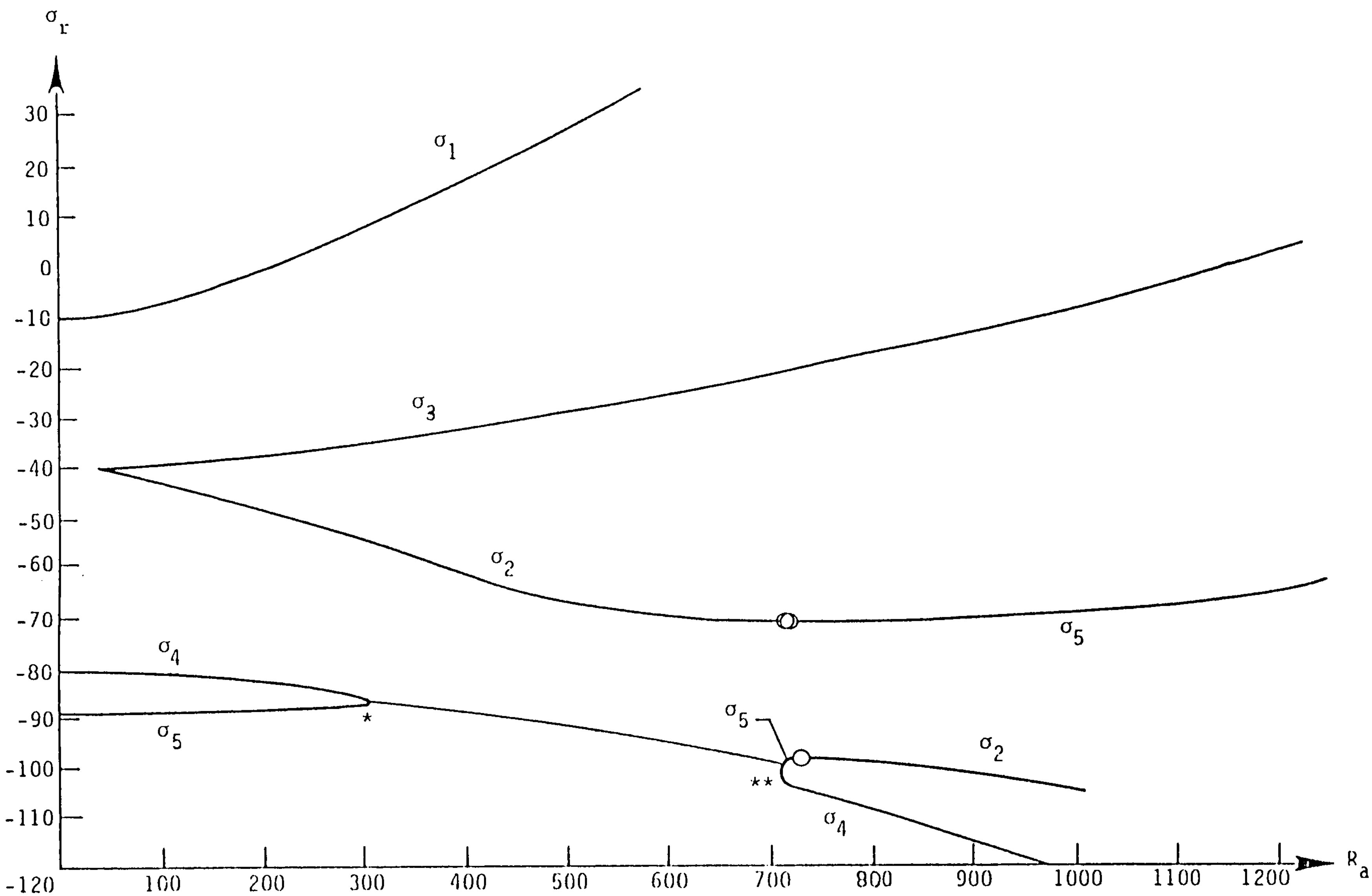

σ_r
-39.5
-39.49
-39.48
-39.47
σ_2
σ_3
**
*
σ_3
σ_2
2
4
6
8
10
12
14
16
R_a

Figure 5



σ_r
30
20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
-110
-120
100
200
300
400
500
600
700
800
900
1000
1100
1200
R_a
σ_1
σ_3
σ_2
σ_5
σ_4
σ_5
*
σ_5
σ_2
**
σ_4

Figure 6

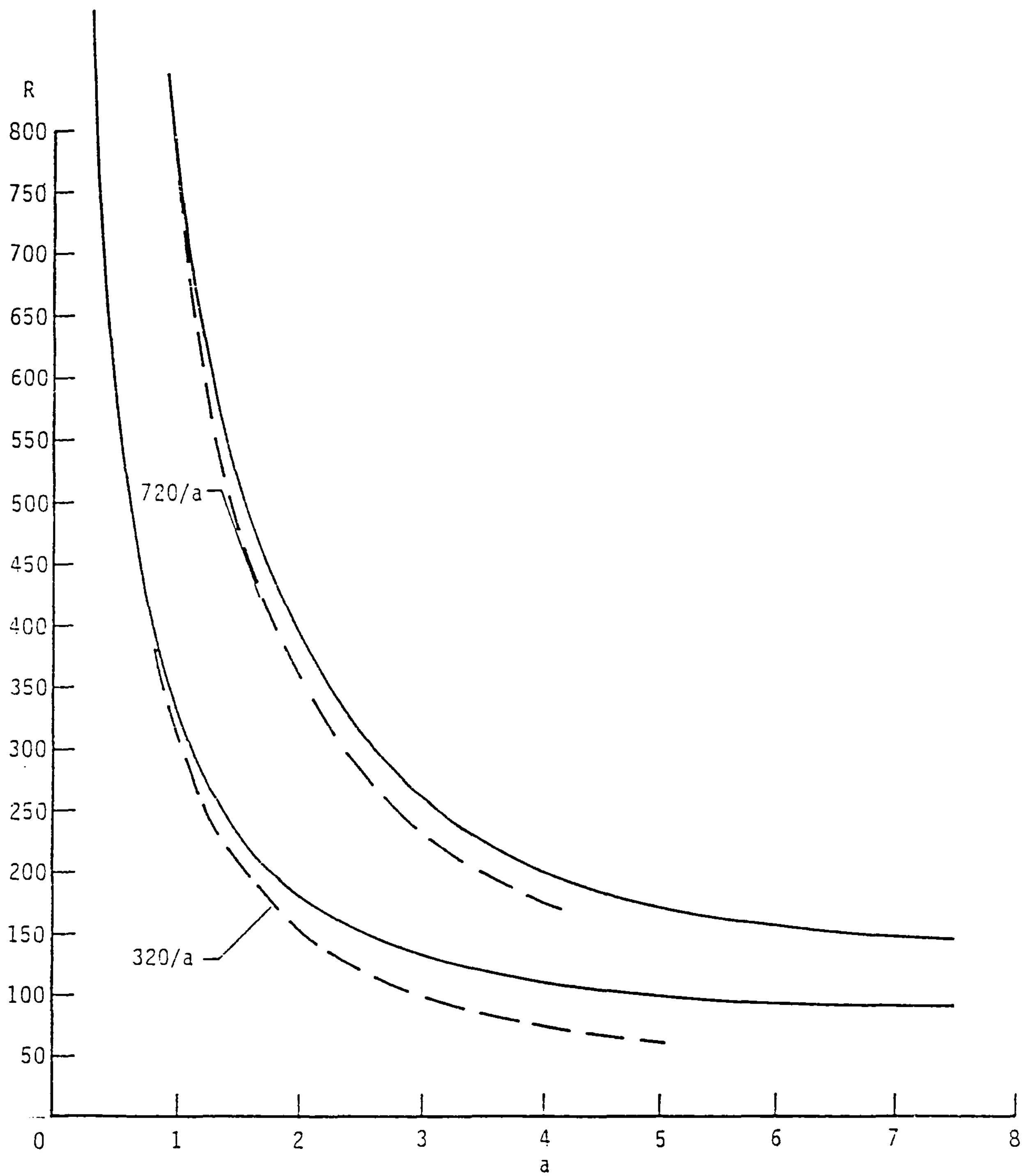

R
800
750
700
650
600
550
500
450
400
350
300
250
200
150
100
50
720/a
320/a
0
1
2
3
4
5
6
7
8
a

Figure 7

1. Report No. NASA CR-172251	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Complex Eigenvalues for the Stability of Couette Flow		5. Report Date October 1983
		6. Performing Organization Code
7. Author(s) R. C. DiPrima and P. Hall		8. Performing Organization Report No. 83-57
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665		10. Work Unit No.
		11. Contract or Grant No. NAS1-17070
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered contractor report
		14. Sponsoring Agency Code

15. Supplementary Notes Additional support: U.S. Army Research Office, National Science Foundation Grant No. MCS81-20839, United Kingdom S.E.R.C. Grant, Guggenheim Fellowship.
Langley Technical Monitor: Robert H. Tolson
Final Report

16. Abstract

The eigenvalue problem for the linear stability of Couette flow between rotating concentric cylinders to axisymmetric disturbances is considered. It is shown by numerical calculations and by formal perturbation methods that when the outer cylinder is at rest there exist complex eigenvalues corresponding to oscillatory damped disturbances. The structure of the first few eigenvalues in the spectrum is discussed. The results do not contradict the "principle of exchange of stabilities"; namely, for a fixed axial wavenumber the first mode to become unstable as the speed of the inner cylinder is increased is nonoscillatory as the stability boundary is crossed.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
hydrodynamic stability	34 Fluid Mechanics and Heat Transfer Unclassified-Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	48	A03

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1983