NASA-CR-172,255

**NASA Contractor Report** 172255



# ICASE

RESTRICTED MAXIMUM PRINCIPLES FOR ELASTIC BODIES

Milton E. Rose

Contract No. NAS1-17070
October 1983

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

**Langley Research Center**
Hampton, Virginia 23665

# RESTRICTED MAXIMUM PRINCIPLES
# FOR ELASTIC BODIES

Milton E. Rose
Institute for Computer Applications in Science and Engineering

## Abstract

This paper describes a maximum principle for the equilibrium of an elastic material body which is free of body forces. We show that not all of the components of the displacement vector or of the principal stresses can simultaneously have a strict maximum or minimum at any point in the body which does not lie either on the surface or on a material interface.

Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17070 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, Virginia 23665.

## INTRODUCTION

The conditions for the static equilibrium of an elastic material can often be reduced to a discussion of the biharmonic equation. Because this equation does not satisfy a maximum principle it is commonly assumed that general maximum principles for the displacements and principal stresses do not exist for elastic bodies. This paper reexamines this question.

If $\underline{u} = (u_1, u_2, u_3)^T$ we write $\underline{u} \geqslant 0$ if $u_i \geqslant 0$ and $\underline{u} > 0$ if $u_i > 0$; $i = 1,2,3$. We say that $\underline{u}$ has an isolated restricted maximum (minimum) at a point $P$ if $\underline{u}(P) > \underline{u}(P')$ for every $P'$ in a ball with $P$ as center.

We shall show that neither the displacement vector nor the principal stresses can have an isolated restricted maximum or minimum at a point which is not on the surface or on a material interface of an elastic body which is in static equilibrium in the absence of volume forces. We emphasize that this result does not exclude the possiblility that any individual component of the displacement or of a principal stress can lie interior to the body.

## 1. PRELIMINARY REMARKS

In the following, $D$ is a domain with boundary $\Gamma$ on which $\underline{n}$ is a unit outward drawn normal. For a material occupying $D$, $\underline{u} = (u_1, u_2, u_3)^T$ is the displacement vector, $\tau(\underline{u})$ is the stress tensor arising from $\underline{u}$ and $\varepsilon(\underline{u})$ is the strain tensor

$$\varepsilon(\underline{u}) = \frac{1}{2} \left(\nabla \underline{u} + (\nabla \underline{u})^T\right),$$

where $\nabla \underline{u} = \text{grad } \underline{u}$.

The static equilibrium equations are

$$
\begin{aligned}
\operatorname{div} \tau &= \underline{f} \\
\tau &= C(\varepsilon),
\end{aligned}
\tag{1}
$$

where $\underline{f}$ is a prescribed body force and $C$ is a $9 \times 9$ symmetric matrix function of $\varepsilon$, which express a general Hooke's law. Unless otherwise indicated $\underline{u}$ and $\tau$ are assumed smooth in $D$. Boundary conditions for (1) are that $\underline{u}$ is prescribed on a part of the surface, say $\Gamma_1$, and $\underline{p} = \tau \cdot \underline{n}$ is prescribed on the remaining part $\Gamma_2$.

Let $\gamma$ denote any closed surface lying interior to $D$ and let $\pi(\gamma)$ indicate the volume enclosed by $\gamma$. The work $W$ due to any displacement $\underline{u}$ in $\pi(\gamma)$ is

$$
W = \frac{1}{2} \int_{\pi(\gamma)} \varepsilon(\underline{u})\, \tau(\underline{u})\, d\pi \geqslant 0,
\tag{2}
$$

with equality holding if and only if $\nabla \underline{u} \equiv 0$ in $\pi(\gamma)$. Integration by parts yields

$$
2W = \oint_{\gamma} \underline{u}^T \underline{p}\, d\gamma - \int_{\pi(\gamma)} \underline{u}^T \operatorname{div} \tau\, d\pi \geqslant 0,
$$

so that in view of (1),

$$
\oint_{\gamma} \underline{u}^T \underline{p}\, d\gamma \geqslant \int_{\pi(\gamma)} \underline{u}^T \underline{f}\, d\pi .
\tag{3}
$$

## 2. A RESTRICTED MAXIMUM PRINCIPLE FOR DISPLACEMENTS

Write $\underline{u} = \underline{c}_\gamma$ to indicate that $\underline{u}$ has the constant value $\underline{c}_\gamma$ on a closed surface $\gamma$. It is evident that if $\underline{u}$ has an isolated restricted maximum at a point $P$, then there exist closed level surfaces $\gamma$ given by $\underline{u} = \underline{c}_\gamma$ within which $\underline{u}(P) > \underline{u}(P') > \underline{c}_\gamma$ for $P' \neq P$.

**THEOREM:** If $\underline{f} \geq 0$, then $\underline{u}$ cannot have an isolated restricted maximum in $D$.

Proof: If $\underline{u}$ had an isolated positive restricted maximum at $P$, there would exist a level surface given by $\underline{u} = \underline{c}_\gamma > 0$ within which $\underline{u}(P) > u(P') > c_\gamma$ for $P' \neq P$.

Then

$$\oint_\gamma \underline{u}^T \underline{p}\, d\gamma = \underline{c}_\gamma^T \oint_\gamma p\, d\,\gamma$$

$$= \underline{c}_\gamma^T \int_{\pi(\gamma)} \operatorname{div} \tau\, d\pi$$

$$= \underline{c}_\gamma^T \int_{\pi(\gamma)} \underline{f}\, d\pi .$$

It would then follow from (3) that

$$\underline{c}_\gamma^T \int_{\pi(\gamma)} \underline{f}\, d\pi \geq \int_{\pi(\gamma)} \underline{u}^T \underline{f}\, d\pi ,$$

so that from the mean value theorem, since $f \geq 0$,

$$\underline{c}_\gamma \geq \hat{\underline{u}},$$

where $\hat{\underline{u}} = \left(u_1(P_1'), u_2(P_2'), u_3(P_3')\right)^T$ and $P_i'$, $i=1,2,3$ are points within $\pi(\gamma)$. This implies $\hat{\underline{u}} > \underline{c}_\gamma$, which is a contradiction.

Again, if $\underline{u}(P)$ were an isolated negative restricted maximum we could, by a rigid body displacement ($\underline{u}' = \underline{u} =$ const.), obtain a solution for which $\underline{u}'(P)$ was an isolated positive restricted maximum, which has just been shown not to be possible.

It clearly follows, also, that if $f \leqslant 0$, then $\underline{u}$ cannot have an isolated negative restrictive minimum interior to D. Hence

**COROLLARY:** <u>If</u> $f \equiv 0$ <u>in</u> D, <u>then</u> $\underline{u}$ <u>cannot have either an isolated restricted maximum or an isolated restricted minimum value in</u> D.

It is not difficult to extend these arguments so as to conclude that u cannot have either a non-isolated restricted maximum or a non-isolated restricted minimum in a closed subdomain interior to D when grad $\underline{u} \neq 0$ in D.

These arguments apply to any material body within which $\underline{u}$ and $\tau(\underline{u})$ are smooth. Our conclusions remain valid for a composite material, providing that we exclude material interfaces across which the stresses can be discontinuous; thus

**COROLLARY:** <u>If</u> $f \equiv 0$, <u>any restricted maximum or restricted minimum displacement can occur only at the exterior boundary or internal interfaces of a composite material</u>.

## 3. RESTRICTED MAXIMUM PRINCIPLE FOR PRINCIPAL STRESSES

We state

**LEMMA:** _If_ $f \geq 0$ $(f \leq 0)$, _then_ $\underline{p} = \underline{c}_\gamma$ _on a closed surface_ $\gamma$ _iff_ $\underline{c}_\gamma \geq 0$ $(\underline{c}_\gamma \leq 0)$.

_Proof:_ Since

$$\oint_\gamma \underline{p} \, d\gamma = \int_{\pi(\gamma)} \operatorname{div} \tau \, d\pi = \int_{\pi(\gamma)} \underline{f} \, d\pi \geq 0,$$

if $f \geq 0$, then $\underline{p} = \underline{c}_\gamma$ on $\gamma$ implies $\underline{c}_\gamma \geq 0$.

**THEOREM:** _If_ $\underline{f} \equiv 0$ _in_ D, _the principal stresses cannot form closed level surfaces interior to_ D. _Thus any restricted maximum or restricted minimum values of the principal stresses lie on the boundary or on interior interfaces of a material body_.

_Proof:_ If $\underline{f} = 0$, then $\underline{p} = \underline{c}_\gamma$ on a closed surface $\gamma$ implies $\underline{c}_\gamma = 0$. However,

$$\oint_\gamma \underline{u}^T \underline{p} \, d\gamma > 0$$

if $\nabla u \not\equiv 0$. Hence $\underline{c}_\gamma \neq 0$.

**ACKNOWLEDGMENT**

To I. Babuska, G. Fix, and A. Noor for helpful discussions.

| 1. Report No. | 2. Government Accession No. | 3. Recipient's Catalog No. |
|---|---|---|
| NASA CR-172255 | | |

| 4. Title and Subtitle | 5. Report Date |
|---|---|
| Restricted Maximum Principles for Elastic Bodies | October 1983 |
| | 6. Performing Organization Code |
| 7. Author(s) | 8. Performing Organization Report No. |
| Milton E. Rose | 83-58 |
| 9. Performing Organization Name and Address | 10. Work Unit No. |
| Institute for Computer Applications in Science and Engineering<br>Mail Stop 132C, NASA Langley Research Center<br>Hampton, VA 23665 | 11. Contract or Grant No.<br>NAS1-17070 |
| 12. Sponsoring Agency Name and Address | 13. Type of Report and Period Covered |
| National Aeronautics and Space Administration<br>Washington, D.C. 20546 | contractor report |
| | 14. Sponsoring Agency Code |

15. Supplementary Notes

Langley Technical Monitor: Robert H. Tolson
Final Report

16. Abstract

In an elastic material in static equilibrium which is free of volume forces not all of the components of either the displacement vector or the principal stresses can simultaneously have a strict maximum or minimum at any point in the body which is neither on the surface nor on a material interface.

| 17. Key Words (Suggested by Author(s)) | 18. Distribution Statement |
|---|---|
| elastic materials<br>equilibrium<br>maximum principles | 24 Composite Materials<br>39 Structural Mechanics<br><br>Unclassified-Unlimited |

| 19. Security Classif. (of this report) | 20. Security Classif. (of this page) | 21. No. of Pages | 22. Price |
|---|---|---|---|
| Unclassified | Unclassified | 7 | A02 |