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THE GENERALIZED EULER-MASCHERONI CONSTANTS

INTRODUCTION

The generalized Euler-Mascheroni constants are defined by

$$\gamma_n = \lim_{M\to\infty} \sum_{k=1}^{M} \frac{\ln^n k}{k} - \frac{\ln^{n+1} M}{n+1} \quad , \qquad n = 0, 1, 2, \ldots$$

and are the coefficients of the Laurent expansion of

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n (z-1)^n}{n!} \quad .$$

They were first defined by Stieltjes in 1885, discussed by Stieltjes and Hermite [1], and have been periodically reinvented over the years [2,3,4].

The rate of convergence is painfully slow (Table 1) so one is forced to seek some method to speed up convergence, and of course the most common way is by employing the Euler-Maclaurin formula. Following a suggestion made by Edwards [5] in computing the Riemann Zeta function, the sum portion of γ_n is split into two sums so that the first (p −1) terms are directly summed and the p to M sum is approximated by the Euler-Maclaurin formula as M tends to infinity. Hence,

$$\gamma_n = \sum_{k=1}^{p-1} \frac{\ln^n k}{k} + \lim_{M\to\infty} \left\{ \int_p^M \frac{\ln^n x}{x}\, dx + 1/2 \left[\frac{\ln^n M}{M} + \frac{\ln^n p}{p}\right] + \sum_{k=1}^{j-1} \frac{B_{2k}}{(2k)!} \left[\left(\frac{\ln^n x}{x}\right)^{(2k-1)}_{x=M} \right.\right.$$

$$\left.\left. - \left(\frac{\ln^n x}{x}\right)^{(2k-1)}_{x=p} \right] + \frac{B_{2j}}{(2j)!} \sum_{k=1}^{p} \left(\frac{\ln^n x}{x}\right)^{(2j)}_{x=p+k+\theta} - \left(\frac{\ln^n M}{n+1}\right) \right\} \tag{1}$$

where $0 < \theta < 1$ and B_{2k} are the Bernoulli numbers. Letting $M \to \infty$, it follows that

$$\gamma_n = \sum_{k=1}^{p-1} \frac{\ln^n k}{k} - \frac{\ln^{n+1} p}{P} + \frac{1}{2}\frac{\ln^n p}{p} - \sum_{k=1}^{j-1} \frac{B_{2k}}{(2k)!}\left(\frac{\ln^{n+1} x}{x}\right)^{(2k-1)}_{x=p} + \frac{B_{2j}}{(2j)!}\sum_{k=0}^{p-1}\left(\frac{\ln^n x}{x}\right)^{(2j)}_{x=p+k+\theta} \quad . \tag{2}$$

The chief difficulty in using this formula is determining $(\ln^n x/x)^{(j)}$ in a usable form.

Lemma.

$$\frac{d^j}{dx^j}\left(\frac{\ln^n x}{x}\right) = \frac{\ln^{n-j} x}{x^{j+1}} \prod_{k=1}^{j} (y-k \ln x)$$

where, after expanding the product, put $y^k = n!/(n-k)!$.

Proof: Let

$$Q_m = \prod_{k=1}^{m} (y-k \ln x)$$

and set

$$\frac{d^m}{dx^m}\left[\frac{\ln^n x}{x}\right] = \frac{\ln^{n-m} x}{x^{m+1}} Q_m \tag{3}$$

and assume that, for some value of m,

$$D^{m+1}\ \frac{\ln^n x}{x} = D\left[\frac{\ln^{n-m} x}{x^{m+1}}\right] Q_m + \frac{\ln^{n-m} x}{x^{m+1}} Q_m' = \frac{\ln^{m-n-1} x}{x^{m+2}} \Big\{[n-m-(m+1)\ln x]\, Q_m + x \ln x\, Q_m'\Big\} \tag{4}$$

and define

$$H_m = [n-m-(m+1)\ln x]\, Q_m + x \ln x\, Q_m' \quad .$$

Now

$$Q_m = \frac{n!}{(n-m)!} - \frac{p_1 n! \ln x}{(n-m+2)!} + \frac{p_2 n! \ln^2 x}{(n-m+1)!} + \ldots + (-1)^m p_m \ln^m x$$

$$= y_m - y^{m-1} p_1 \ln x + p_2 y^{m-2} \ln^2 x + \ldots + (-1)^m p_m \ln^m x . \quad (5)$$

Since

$$(n-m) \frac{n!}{(n-m)!} = \frac{n!}{(n-m-1)!} = y^{m+1}$$

we obtain, upon substitution of (5) into (4):

$$H_m = y^{m+1} - y^m [p_1 + m+1] \ln x + y^{m-1} [p_2 + (m+1) p_2] \ln^2 x + \ldots + (-1)^m (-1) (m+1) \ln^{m+1} x \ .$$

This is obviously

$$[y - (m+1)] \, Q_m = \prod_{k=1}^{m+1} (y - k \ln x) = Q_{m+1} \ .$$

Inserting this into (3) proves the theorem by induction. The last term in equation (2) is

$$\frac{B_{2j}}{(2j)!} \sum_{k=0}^{p-1} \left(\frac{\ln^n x}{x} \right)^{(2j)}_{x=p+k+\theta} \quad (6)$$

and must now be given bounds. By the Cauchy integral formula,

$$\left[\frac{\ln^n (p+k)}{p+k} \right]^{(2j)} \leqslant \frac{(2j)!}{2\pi} \frac{2\pi R}{R^{2j+1}} \frac{\ln^n (p+k+Re^{\theta i})}{p+k+Re^{\theta i}} \leqslant \frac{(2j)! \, 2^{2j} \ln^n (k+p-2)}{p^{2j} \quad p}$$

where $R = p/2$,

$$| \ln^n (k+p+p/2 \, e^{i\theta}) | \leqslant 2 \ln^n (k+2p)$$

and

$$\frac{1}{p} > \left| \frac{1}{p+k+p/2\, e^{i\theta}} \right|$$

Inserting this bound into (6) yields

$$\frac{B_{2j}(2j)!2^{2j}}{(2j)!p^{2j+1}} \sum_{k=1}^{p-1} \frac{\ln^n(2p+k)}{p} \leqslant \frac{B_{2j}2^{2j}}{p^{2j+2}}\, p\ln^n(3p+1) \leqslant \frac{2^{2j}\ln^n(3p+1)2\,(2j)!}{p^{2j+1}(2\pi)^{2j}(1-2^{1-2j})} \leqslant \frac{2(2j)!\ln^n(3p+1)}{(p\pi)^{2j}(1-2^{1-2j})} \quad . \tag{7}$$

COMPUTATION

The fourth term of equation (2) requires the construction of three matrices A, B, C and two vectors $\underline{u}$, $\underline{v}$. A is a (2j–3)X(2j–2) matrix whose first (L + 1) elements in the Lth row are given by the coefficients of

$$\prod_{k=1}^{L} (y - k \ln p)$$

and the remainder set equal to zero. B is a matrix of the same dimension whose Lth row is given by

$$\left(\frac{n!}{(n-L)!} \;,\; \frac{n!}{(n-L-1)!} \;,\; \ldots \right)$$

The elements are, of course, zero if $L > n$. C is a matrix defined by

$C = (a_{ij}b_{ij})$ for $i = 1, 3, 5, 7 \ldots,$ and $j = 1, 2, 3, 4\ldots$

Next define

$$\underline{u}^T = (1, \ln p, \ln^2 p, \ldots, \ln^{2j-2} p)$$

and

$$\underline{v}^T = \left(\frac{1}{2!} B_2, \frac{1}{4!} B_4, \ldots, \frac{1}{(2j-2)!} B_{2j-2} \right)$$

With these equations, (2) may be written as:

$$\gamma_n = \sum_{k=1}^{p} \frac{\ln^n k}{k} - \frac{\ln^{n+1} p}{p} + \frac{1}{2}\frac{\ln^n p}{n} - \underline{v}^T C \underline{u}$$

with an error bounded by (7). An APL computer program was written to evaluate γ_n (see Appendix).

It was observed that the minimum error in computing γ_n occurred in a neighborhood of p=j and a sensitivity study indicated that p=j=10 was an optimum choice. Table 2 contains the first 32 Euler-Mascheroni constants.

Table 3 shows that the Laurent expansion provides a very effective means of computing the Riemann ζ function in a neighborhood of $z = 1$. However, the expansion is not a useful method for extending the list of zeros of $\zeta(z)$ known today.

The behavior of the Euler-Mascheroni constants themselves have been the subject of investigation. Briggs [6] showed that infinitely many γ_n are negative and infinitely many are positive and Mitrovic extended this result by showing that each of these inequalities $\gamma_n < 0$, $\gamma_{2n-1} < 0$, $\gamma_n > 0$, $\gamma_{2n-1} > 0$ holds for infinitely many n [7].

Good [8] recently conjectured that the lengths of the runs of the same sign of $\Delta\gamma_n$ never decrease.

TABLE 1. γ_n VERSUS NUMBER OF TERMS COMPUTED (M)

	$M = 10^3$	$M = 10^4$	$M = 10^5$	$M = 10^6$
$n = 0$	0.57771558156810	0.57726566406712	0.57722066488224	0.57721616479093
$n = 1$	−0.06936246015836	−0.07235533352434	−0.07275828112056	−0.07280893950511
$n = 2$	0.01416535317172	−0.00544890007751	−0.00902762943042	−0.00959495724544
$n = 5$	7.86463444728723	3.31473675482265	1.01213466958143	0.25241822958923

TABLE 2. GENERALIZED EULER-MASCHERONI CONSTANTS

n	γ_n
0	0.57721566490153
1	-0.07281584548368
2	-0.00969036319287
3	0.00205383442030
4	0.00232537006546
5	0.00079332381728
6	-0.0002387693455
7	-0.0005272895671
8	-0.0003521233539
9	-0.0000343947747
10	0.000205332814
11	0.000270184439
12	0.00016727291
13	-0.00002746381
14	-0.00020920927
15	-0.00028346867
16	-0.0001996969
17	0.0000262769
18	0.0003073682
19	0.000503605
20	0.000466342
21	0.0001044
22	-0.0005416
23	-0.0012439
24	-0.0015885
25	-0.0010746
26	0.000657
27	0.003477
28	0.006399
29	0.00737
30	0.00355
31	-0.00752

TABLE 3.

x	iy	Reζ(z)	Imζ(z)
0.5	0	−1.46035450881	0.00000000000
0.01	0	−0.50929071404	0.00000000000
2	0	1.64493406685	0.00000000000
3	0	1.20205690316	0.00000000000
4	0	1.08232323371	0.00000000000
5	0	1.03692775514	0.00000000000
6	0	1.01734306198	0.00000000000
0.5	1	0.14393642708	−0.72209974353
0.5	3	0.53273667097	−0.07889651343
0.5	5	0.70181237117	0.23103800839

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APPENDIX

```
      ∇EULER[⎕]∇
    ∇ R←NK EULER M;L;LM;Y;T;K;N
[1]   LM←(L←⍟M)*¯1+ι2×K←NK[2];N←NK[1]
[2]   Y←(+/((⍟ιM)*N)÷ιM)-((L*N+1)÷N+1)+0.5×(L*N)÷M
[3]   T←(÷M*1+ι¯1+2×K)×(L*N-ι¯1+2×K)×((GENM 2×K)×(2×K)GENMN N)+.×LM
[4]   R←N,Y-+\(2↓BERNOULLI K+2)×T[K↑¯1+2×ιK]÷!2×ιK
[5]   ⍝  NK[1] IS THE N'TH EULER CONSTANT
[6]   ⍝  NK[2] IS THE NUMBER OF TERMS IN THE BERNOULLI SUM
[7]   ⍝  M IS THE NUMBER OF TERMS IN THE FINITE SUM
    ∇


      ∇GENM[⎕]∇
    ∇ M←GENM K;V;I;T
[1]   M←((K-1),K)ρ0;I←2
[2]   V←(2,K)ρ(Kρ1),-ιK
[3]   M[1;]←V[;1],(K-2)ρ0
[4]   L:T←V[;I]∘.×(M[I-1;]≠0)/M[I-1;]
[5]   T←(T[1;],0),[0.5]0,T[2;]
[6]   M[I;]←(+⌿T),(K-I+1)ρ0
[7]   →(K>I←I+1)/L
[8]   ⍝ GENERATES THE COEF. OF THE DERIVATIVE OF (LOG X)*N/X
    ∇


      ∇GENMN[⎕]∇
    ∇ M←K GENMN N;T
[1]   M←((K-1),K)ρ0;I←2
[2]   M[1;]←(N,1),(K-2)ρ0
[3]   L:M[I;]←(I KR N),¯1↓M[I-1;]
[4]   →(K>I←I+1)/L
[5]   ⍝  GENERATES MATRIX OF       N×(N-1)×...×(N-K+1)
[6]   ⍝  USED WITH FUNCTION EULER
    ∇
```

```
        ∇BERNOULLI[⎕]∇
      ∇ B←BERNOULLI N
[1]     →(N>22)/L
[2]     B←22⍴0
[3]     B[1]←1÷1
[4]     B[2]←¯1÷2
[5]     B[3]←1÷6
[6]     B[4]←¯1÷30
[7]     B[5]←1÷42
[8]     B[6]←¯1÷30
[9]     B[7]←5÷66
[10]    B[8]←¯691÷2730
[11]    B[9]←7÷6
[12]    B[10]←¯3617÷510
[13]    B[11]←43867÷798
[14]    B[12]←¯174611÷330
[15]    B[13]←854513÷138
[16]    B[14]←¯236364091÷2730
[17]    B[15]←8553103÷6
[18]    B[16]←¯2.374946E10÷870
[19]    B[17]←8.615841E12÷14322
[20]    B[18]←¯7.709321E12÷510
[21]    B[19]←2.577688E12÷6
[22]    B[20]←¯2.631527E19÷1919190
[23]    B[21]←2.929994E15÷6
[24]    B[22]←¯1.929658E16
[25]    →0;B←N↑B
[26]    L:'N MUST BE LESS THAN OR EQUAL TO 22'
[27]    ⍝  FIRST 22 BERNOULLI NUMBERS
      ∇
```

APPROVAL

THE GENERALIZED EULER-MASCHERONI CONSTANTS

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The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

G. F. McDonough
Director, Systems Dynamics Laboratory

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16. ABSTRACT

Methods for evaluating the Euler-Mascheroni constants which appear in the Laurent expansion of the Reimann zeta function about Z = 1 are presented. The first 32 of these numbers are listed.

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