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ABSTRACT

The extinction paradox is a contradiction between results of geometrical optics which predict that at high frequencies the scattering cross section of an object should equal its geometrical cross section and rigorous scattering theory which shows that at high frequencies the scattering cross section approaches twice the geometrical cross section of the object. Confusion about the reason for this paradox persists today even though the nature of the paradox was correctly identified many years ago by Brillouin. In this paper the resolution of the paradox will be restated and illustrated with an example, and then the implications to the interpretation of scattering cross sections will be identified.

Introduction

The scattering cross section, σ_s, for a particle is defined to be the ratio of the total power scattered by the object to the power per unit area incident on the object (e.g. Born and Wolf, 1959; Bohren and Huffman, 1983; Ishimaru, 1978; van de Hulst, 1981):

$$\sigma_s = \frac{\text{total power scattered}}{\text{incident power per unit area}} \tag{1}$$

To calculate the scattered power it is standard procedure in electromagnetic theory to resolve the total electric and magnetic fields into "incident" and "scattered" components as follows:

$$\begin{aligned} \bar{E} &= \bar{E}_{inc} + \bar{E}_{scat} \\ \bar{H} &= \bar{H}_{inc} + \bar{H}_{scat} \end{aligned} \tag{2}$$

Then the time average "scattered" power density can be obtained from the Poynting vector for the scattered fields:

$$\bar{P}_{scat} = (1/2) \ \mathrm{Re} \ [\bar{E}_{scat} \times \bar{H}^*_{scat}] \tag{3}$$

and the scattering cross section becomes:

$$\sigma_s = \frac{\oiint \bar{P}_{scat} \cdot d\bar{s}}{\bar{P}_{inc} \cdot \hat{i}} \tag{4}$$

where $\hat{i}$ is a unit vector in the direction of propagation of the incident wave, $\bar{P}_{inc} = (1/2)$ Re $[\bar{E}_{inc} \times \bar{H}^*_{inc}]$ and the integration of Equation 4 is over a closed surface surrounding the particle.

In the case of particles which are large compared to wavelength (i.e. in the limit $k \to \infty$) a simple intuitive solution for σ_s can be obtained from geometrical optics. To do so consider a plane wave with wave number $k = \omega \sqrt{\mu\epsilon}$ incident on an object of finite size as illustrated in Figure 1. In the limit $k \to \infty$, one expects the object to cast a shadow on a screen placed behind the object, and one expects this shadow to be approximately the orthogonal projection of the object onto the screen. Such a shadow is called the geometrical cross section of the object. If the object is completely opaque, then the shadow is a region in which there are no electromagnetic waves and therefore no power. Under these circumstances, the object must have scattered (or possibly absorbed) an amount of power equivalent to that which in the absence of the particle would have been in the shadow. This scattered power can be calculated by multiplying the Poynting vector of the incident wave times the geometrical cross section of the object. Hence, if no power is absorbed by the object, then the scattering cross section should be equal to the geometrical cross section of the object. (If the object is absorbing, then the scattering cross section will be smaller than the geometrical cross section depending on the amount absorbed.)

The preceding derivation of the scattering cross section is based on intuition obtained from geometrical optics. It is correct to the same extent that geometrical optics is correct and therefore ought to apply when $k \to \infty$. However, the expression obtained for the scattering cross section is not consistent with solutions obtained using exact solutions for the scattered fields, $\overline{E}_{scat}$ and $\overline{H}_{scat}$. This was apparently first noticed in connection with the Mie solution for scattering from a perfectly conducting sphere (Brillouin, 1949). The geometrical cross section of a sphere of radius a is πa^2 but the total power scattered from the sphere divided by the incident power density approaches $2\pi a^2$ in the limit $k \to \infty$ (Born and Wolf, 1959; van de Hulst, 1981). The same is true of other "canonical" shapes for which exact solutions to the wave equation can be found. For example, the geometrical cross section per unit length of a perfectly conducting infinite circular cylinder of radius a is 2a, but the scattering cross section per unit length obtained from theory approaches 4a in the limit $k \to \infty$ (Bowman, Senior and Uslenghi, 1969). The scattering cross section also approaches twice the geometrical cross section for scattering from cylinders of elliptical cross section, independent of polarization, and this is also true for acoustic scattering from spheres (Bowman, Senior, and Uslenghi, 1969).

Hence, theory predicts that the scattering cross section approaches twice the geometrical cross section as $k \to \infty$ whereas geometrical optics arguments lead to the conclusion that the scattering cross section is equal to the geometrical cross section. This contradiction is not dependent upon assumptions made about absorption because the results cited above are for lossless (non-absorbing) objects.

Nor is it the result of approximations inherent in geometrical optics (e.g. which neglects diffraction). In fact, both arguments described above are correct in the limit $k \to \infty$. The contradiction is of more subtle origin and has been called the "extinction" paradox (e.g. Ishimaru, 1978; Bohren and Huffman, 1983; van de Hulst, 1981.)

The extinction paradox is a consequence of the definition (Equation 2) adopted for the scattered fields. This was originally pointed out by Brillouin (1949) and is particularly evident in the limit $k \to \infty$. In this case, $\bar{E}_{scat}$ and $\bar{H}_{scat}$ correctly give the fields reflected from the object but as defined in Equation 2 they also include a term which combines with the incident wave to produce the shadow behind the object. For example, consider a plane mirror in the presence of radiation from a high frequency source as illustrated in Figure 2. In the limit $k \to \infty$ one expects to find a reflected wave and also a shadow behind the mirror. Thus, $\bar{E}_{scat}$ as defined in Equation 2 must consist of one term which combines with the incident field to produce the shadow and must also contain a term which gives the wave reflected by the mirror. In the limiting case ($k \to \infty$) there is no power in the region behind the mirror (e.g. a completely dark shadow) and the mirror simply redirects the incident power into the reflected wave. The power in the reflected wave is therefore just the power intercepted by the mirror from the incident wave, and in this case the ratio of the reflected power divided by the incident power per unit area is just the geometrical cross section of the mirror. This is the geometrical optics argument for deriving the scattering cross section. However, when one computes the total scattered power using Equation 3, one is including in $\bar{E}_{scat}$ and $\bar{H}_{scat}$ a contribution which accounts for the reflected wave and also a contribution from the terms needed to create the shadow. Clearly, in the limiting case ($k \to \infty$) with a completely opaque, lossless mirror, these two contributions are equal and each is equal to the power intercepted from the incident wave by the geometrical cross section of the object. Thus, in the high frequency limit one expects the scattering cross section obtained from Equation 4 to yield twice the geometrical cross section, as in fact is found to be true when the calculation is performed for simple shapes using the "exact" solutions for $\bar{E}_{scat}$ and $\bar{H}_{scat}$.

Example

The two component nature of the scattered fields will be illustrated in this section with an example of scattering from dielectric disks. Imagine a plane wave incident on a dielectric disk as illustrated in Figure 3. The unit vector, $\hat{i}$, in the direction of propagation of the incident wave lies in the y–z plane and the unit vectors $\hat{h}$ and $\hat{v}$, indicating the direction of the electric field for horizontally and vertically polarized waves, are perpendicular or parallel to the y–z plane, respectively. The unit vector $\hat{o}$ indicates the direction of propagation of the scattered wave. Calculations of the scattered electric field, $\overline{E}_{scat}$, due to the incident plane wave have been made using the physical optics approximation of Le Vine, et al., (1983). In this approach, a solution for the scattered fields is first written in terms of the sources (currents) induced in the disk by the incident wave. These are unknown, but for linear media can be expressed in terms of the dielectric properties of the medium (complex dielectric constant) and the fields inside the disk. The approximation is to use the fields inside an identically oriented, infinite slab of the same thickness and dielectric properties to approximate the actual fields inside the disk. One expects the results to be reasonable for non-grazing incidence for disks whose minimum characteristic dimension (e.g. diameter of circular disks) is very large compared to wavelength. It has been shown (Le Vine, 1984) that this approximation is consistent with results obtained using other techniques and agrees with experiments at normal incidence.

To perform the required calculations it is convenient to express the scattered fields in terms of a dyadic scattering amplitude, $\overline{\overline{f}}(\hat{o}, \hat{i})$, as follows:

$$\overline{E}_{scat} = \hat{p} \cdot \overline{\overline{f}}(\hat{o}, \hat{i}) \frac{e^{jkr}}{r} E_0 \tag{5}$$

where $\hat{p}$ is a unit vector in the direction of polarization of the incident wave and E_0 is the amplitude of the incident wave. Thus, the component of the scattered wave with polarization in the $\hat{q}$-direction due to an incident wave with polarization $\hat{p}$ is:

$$\overline{E}_{scat} \cdot \hat{q} = \hat{p} \cdot \overline{\overline{f}}(\hat{o}, \hat{i}) \cdot \hat{q} \frac{e^{jkr}}{r} E_0 \tag{6}$$

Calculations have been made of $\hat{v} \cdot \bar{\bar{f}}(\hat{o}, \hat{i}) \cdot \hat{v}$ using the physical optics theory of Le Vine, et al. (1983; Equations 22–23). Examples are shown in Figure 4 for a circular disk of radius 10 cm, thickness 0.5 cm, frequency 9 GHz ($ka = 6\pi$) and relative dielectric constant of $\epsilon_r = 25 + j11$ which is representative of leaves at microwave frequencies. Polar plots of $|\hat{v} \cdot \bar{\bar{f}}(\hat{o}, \hat{i}) \cdot \hat{v}|$ as a function of $\hat{o}$ are presented for fixed $\hat{i}$. Each plot was obtained by calculating $|\hat{v} \cdot \bar{\bar{f}}(\hat{o}, \hat{i}) \cdot \hat{v}|$ as $\hat{o}$ was rotated through 360° in the z–y plane. The z-axis ($\hat{o}=\hat{i}$) is at the top of each figure and the circles are contours of constant amplitude, the outer most circle representing unit amplitude and the inner most circle being 0.2. Four plots are shown in Figure 4. They differ only in the angle of incidence of the plane wave (which is indicated by the arrow). In the example labelled $\Theta = 0°$, the plane wave is normally incident on the disk ($\hat{i} = -\hat{n} = -\hat{z}$) and in successive examples $\hat{i}$ is obliquely incident at increasingly large incidence angles, Θ. Notice that in each example the magnitude of the scattering amplitude has two peaks: one which is always behind the disk in the direction of propagation of the incident wave, and another which is in the direction of specular reflection (angle of incidence equals angle of reflection). The peak in the forward direction is due to the component of the scattered field which combines with the incident wave to form the "shadow" behind the disk, and the peak in the direction of specular reflection represents energy whose direction of propagation has been changed by the disk (i.e. electromagnetic fields reflected from a mirror). The two-peaked nature of $\bar{E}_{scat}$ evident in Figure 4 is not peculiar to disks nor to the method used here to obtain a solution; rather it is characteristic of scattering at high frequencies (e.g. Papayiannakis and Kriezis, 1983, and Kozaki, 1983).

The two peaks in the scattered field both contribute to the "scattered" power and therefore to the scattering cross section of the disk. In particular, the power density with polarization $\hat{v}$ scattered from the disk in the $\hat{o}$ direction by an incident wave with polarization $\hat{v}$ is :

$$\begin{aligned} \bar{P}_{vv} &= \tfrac{1}{2} \operatorname{Re} (\bar{E}_{scat} \times \bar{H}^{*}_{scat} \cdot \hat{o}) \\ &= \tfrac{1}{2} \sqrt{\epsilon/\mu}\, E_0^2\, |\hat{v} \cdot \bar{\bar{f}}(\hat{o}, \hat{i}) \cdot \hat{v}|^2 \end{aligned} \tag{7}$$

where $\frac{1}{2}\sqrt{\epsilon/\mu}\, E_0^2$ is the power density in the incident wave. Thus, the total power scattered in all directions divided by the incident power density is

$$\frac{\text{Total power scattered with polarization } \hat{v}}{\text{Incident power density with polarization } \hat{v}} = \iint |\hat{v} \cdot \bar{f}(\hat{o}, \hat{i}) \cdot \hat{v}|^2 \, d\Omega \tag{8}$$

It is clear from the example in Figure 4 that the integral in Equation 8 includes contributions from both the fields specularly reflected from the disk and also from the fields needed to create the shadow.

IMPLICATIONS

As indicated in the previous paragraphs, the extinction paradox is the result of a contradiction between physical intuition which leads one to understand the term "scattered" to mean "reflected", and the conventional definition of scattered fields which includes both the reflected fields and a term which combines with the incident wave to create a shadow behind the object. Problems occur when one uses the conventional definition but interprets the results using the intuitive notion. Two examples will be considered below.

A. The extinction cross section

The time average power absorbed by a particle, $\overline{P}_a$, is equal to the net flux of the time average Poynting vector into a closed surface surrounding the particle. That is:

$$\begin{aligned} -P_a &= \tfrac{1}{2}\,\mathrm{Re} \oiint [\overline{E}_{inc} + \overline{E}_{scat}] \times [\overline{H}^*_{inc} + \overline{H}^*_{scat}] \cdot ds \\ &= P_s - P_t \end{aligned} \tag{9a}$$

where

$$P_s = \tfrac{1}{2}\,\mathrm{Re} \oiint \overline{E}_{scat} \times \overline{H}^*_{scat} \cdot d\overline{s} \tag{9b}$$

$$-P_t = \tfrac{1}{2}\,\mathrm{Re} \oiint [\overline{E}_{inc} \times \overline{H}^*_{scat} + \overline{E}_{scat} \times \overline{H}^*_{inc}] \cdot d\overline{s} \tag{9c}$$

Dividing both sides by the incident power density (i.e. $P_{inc} = \frac{1}{2}\sqrt{\epsilon/\mu}\, E_0^2$) and using the definitions $\sigma_a = P_a/P_{inc}$, $\sigma_s = P_s/P_{inc}$ and $\sigma_T = P_t/P_{inc}$, one may write Equation 9a in the form:

$$\sigma_T = \sigma_s + \sigma_a \tag{10}$$

where σ_T is called the "total" or "extinction" cross section and σ_a and σ_s are called the "absorption and "scattering" cross sections, respectively. By substituting Equation 5 into Equation 9c one can express the total cross section associated with an incident wave with polarization $\hat{p}$ in terms of the scattering amplitude as follows (e.g. Born and Wolf, 1959; Karam and Fung, 1982):

$$\sigma_T = (2\pi/k)\ \mathrm{Im}\ \hat{p} \cdot \bar{\bar{f}}(\hat{i}, \hat{i}) \cdot \hat{p} \tag{11}$$

This result is called the "optical" or "forward" scatter theorem.

The conventional interpretations of σ_T is as a measure of the power scattered and absorbed by the object (e.g. Bohren and Huffman, 1983; van de Hulst, 1981; Ishimaru, 1978): specifically, that it is an equivalent area which intercepts from the incident wave the same power that is absorbed and scattered by the particle. This interpretation would be correct if σ_s were a measure of the power "reflected" from the object. However, as was pointed out above, σ_s also includes a contribution from the shadow producing fields. Although these terms may not be distinct for finite k, one can write for conceptual purposes:

$$\sigma_s = \sigma_{rs} + \sigma_{ss} \tag{12}$$

where σ_{rs} accounts for the reflected power and σ_{ss} is the contribution due to the shadow producing part of $\overline{E}_{scat}$. Using this notation one can write:

$$\begin{aligned} \sigma_T &= (\sigma_{rs} + \sigma_{ss}) + \sigma_a \\ &= (\sigma_{rs} + \sigma_a) + \sigma_{ss} \\ &= \sigma_t + \sigma_{ss} \end{aligned} \tag{12}$$

where $\sigma_t = \sigma_{rs} + \sigma_a$. Clearly, σ_t is the cross section which measures the power reflected and absorbed by the particle, and σ_T measures the power reflected and absorbed plus an additional term which is present because, by nature of their definition, $\overline{E}_{scat}$ and $\overline{H}_{scat}$ include a term which combines with the incident fields to produce a shadow behind the object. When there is no absorption ($\sigma_a = 0$) and the object is completely opaque, conservation of energy leads one to expect that $\sigma_{rs} = \sigma_{ss}$; and when $k \to \infty$ geometrical optics suggests that $\sigma_{rs} = \sigma_g$ where σ_g is the geometrical cross section of the object. Hence for lossless, opaque objects one expects:

$$\lim_{k\to\infty} \sigma_T = 2\sigma_g \tag{13a}$$

$$\lim_{k\to\infty} \sigma_t = \sigma_g \tag{13b}$$

Thus, the definitions are consistent with both intuition and theory. Paradoxical results occur when one tries to interpret σ_T as a measure of the total power "reflected" and absorbed without consideration of the fact that it includes an additional term which is present as a consequence of the definition adopted for $\overline{E}_{scat}$. In fact, Equations 10 and 11 are correct, but the label "total" or "extinction" cross section given to σ_T is misleading.

B. Radar Cross Section

It is conventional to define the radar cross section of an object as follows (Bowman, Senior and Uslenghi, 1969; Ishimaru, 1978; Ruck, et al., 1970):

$$\sigma = \lim_{R\to\infty} 4\pi R^2 \left| \frac{E_{scat} \cdot \overline{E}^*_{scat}}{E_{inc} \cdot E^*_{inc}} \right| \tag{14}$$

In the case of an incident wave with polarization $\hat{p}$ and a scattered wave with polarization $\hat{q}$ one obtains (e.g. using Equation 5):

$$\sigma_{pq} = 4\pi \left| \hat{p} \cdot \overline{\overline{f}}(\hat{o}, \hat{i}) \cdot \hat{q} \right|^2 \tag{15}$$

The physical interpretation of Equations 14 and 15 is that σ_{pq} is 4π times the power per unit solid angle scattered in the $\hat{o}$ direction normalized by the incident power density (e.g. see Equation 7). This is correct if $\overline{E}_{scat}$ represents the scattered (i.e. reflected) power. But as pointed out above, $\overline{E}_{scat}$ also includes the shadow producing term.

To illustrate the problems that can be encountered, again consider scattering from a disk at high frequencies. The scattering cross section σ_{vv} behaves as a function of $\hat{o}$ as indicated in Figure 4. (See Le Vine, 1984 for examples in a more conventional format). Because the frequency is high (large ka) σ_{vv} has two distinct peaks which are manifestations of the reflection and shadow producing terms in $\overline{E}_{scat}$. At normal incidence it can be seen from Figure 4 that the predominate contribution to the backscatter radar cross section ($\hat{o} = \hat{i}$) is the reflected term in E_{scat}. In this case one expects a reasonable agreement between the definition (Equation 14) and observation (i.e. a measurement) because the peak is, in fact, due to "reflected" power. However, in the case of forward scatter, the radar cross section as given by Equation 14 will also have a peak. However, this peak is due to the shadow producing term in E_{scat} and does not represent "reflected" power. In the laboratory a radar receiver directly behind the disk will not receive the electric field, $\overline{E}_{scat}$; rather it will be in the shadow of the disk where it will receive only a small signal. It is necessary to use $E = E_{inc} + E_{scat}$ in this region to correctly describe the electric field at the receiver.

Summary

As has been pointed out above, the electric field, E_{scat}, as conventionally defined ($E = E_{inc} + \overline{E}_{scat}$) includes both waves "reflected" from the object and also a term which when combined with $\overline{E}_{inc}$ produces the shadow behind the object. Thus, E_{scat} is a measure of more than just the electric fields removed from the incident wave by the object. Failure to recognize this leads to inconsistencies such as the extinction paradox and to misleading interpretations such as that the total scattering cross section, $\sigma_T = \sigma_a + \sigma_s$, is proportional to the power absorbed and scattered by the particle. On the other hand, as long as one uses both the incident and scattered fields (i.e. $E_{inc} + E_{scat}$) to represent the total electric field, then the cross sections which occur in an analysis will be used correctly. For example, consider the propagation of a plane wave with polarization $\hat{p}$ through a layer of random scatterers. Using a Born approximation, one concludes (e.g. Bohren and Huffman, 1983; Oguchi, 1983) that the effective propagation constant in the layer is:

$$k = k_o + (2\pi/k_o)\ N\ (\hat{p} \cdot \overline{\overline{f}}(\hat{i}, \hat{i}) \cdot \hat{p})$$

where N is the number of particles per unit volume, Thus, the effective attenuation coefficient, $\alpha = \mathrm{Im}(k)$, for propagation through the layer is:

$$\begin{aligned} \alpha &= N\ \mathrm{Im}\ [(2\pi/k_o)\ \hat{p} \cdot \overline{\overline{f}}(\hat{i}, \hat{i}) \cdot \hat{p}] \\ &= N\ \sigma_T \end{aligned}$$

This result, which is derived using $\bar{E} = E_{inc} + E_{scat}$ is correct to within the limits set by the Born approximation. What is incorrect (perhaps the more appropriate word is "misleading") is to interpret this result to mean that because $\sigma_T = \sigma_a + \sigma_s$, the attenuation is determined by N times the power "scattered" and absorbed in the medium. The attenuation coefficient is correctly given by N σ_T, but the scattering cross section, σ_s, does not correctly measure the power deflected by the particles from the incident plane wave. In fact, in the high frequency limit it is incorrect by a factor of two ($\sigma_s \approx 2\sigma_g$).

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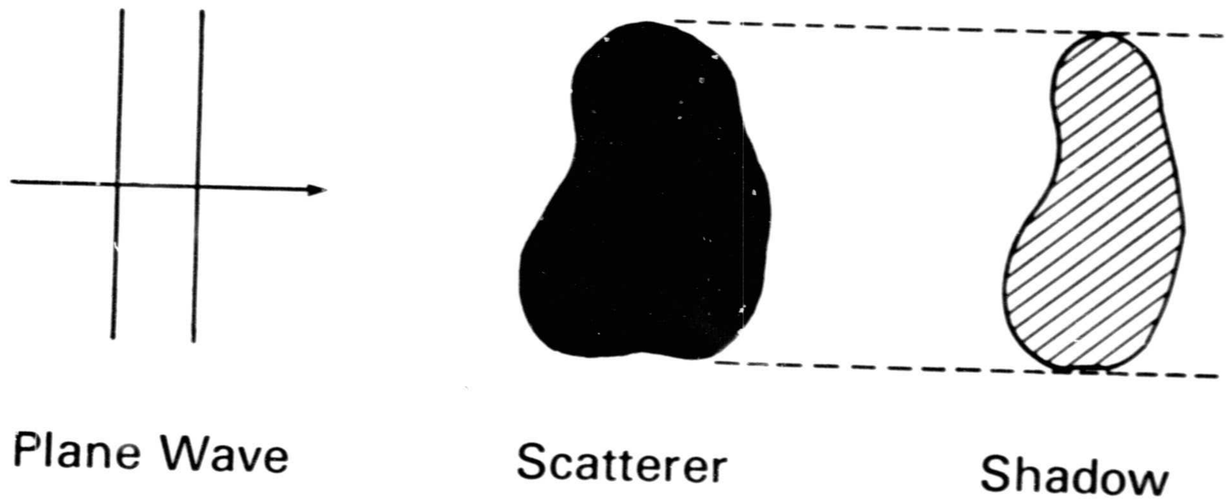

Figure 1. Plane wave incident on a scatterer at high frequencies.

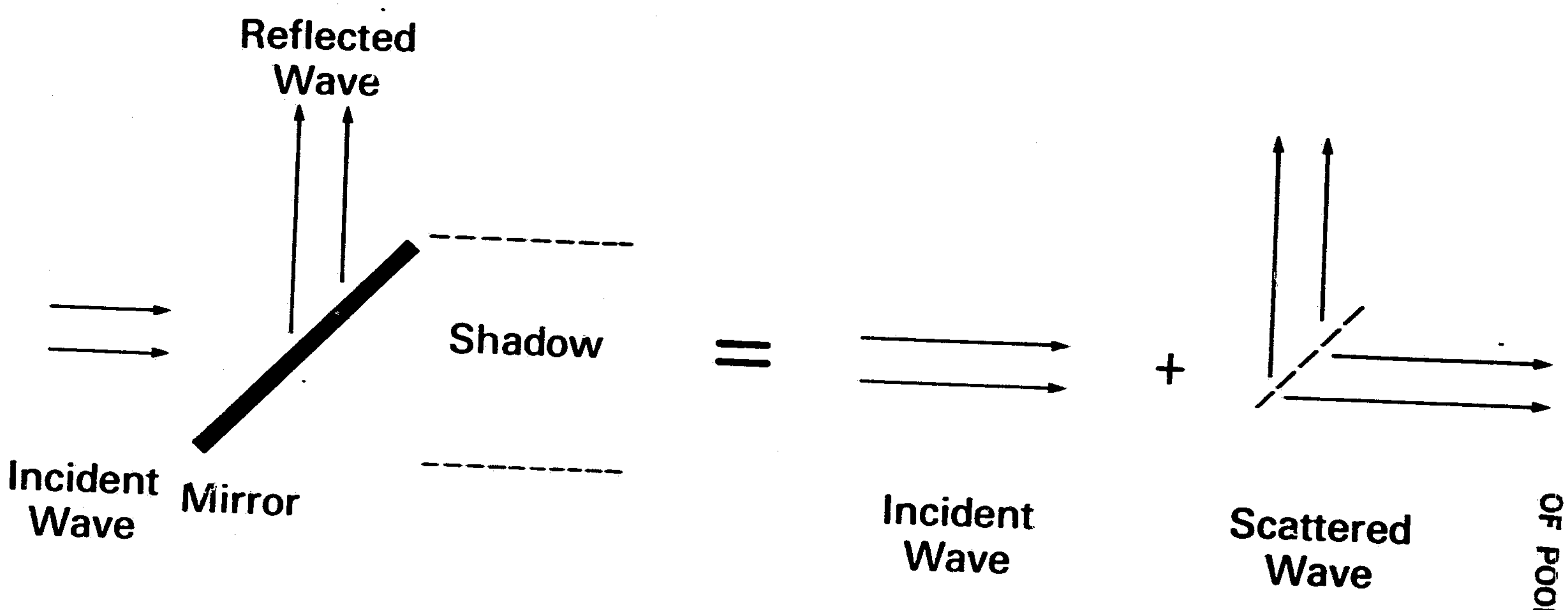

Figure 2. Components of the scattered wave.

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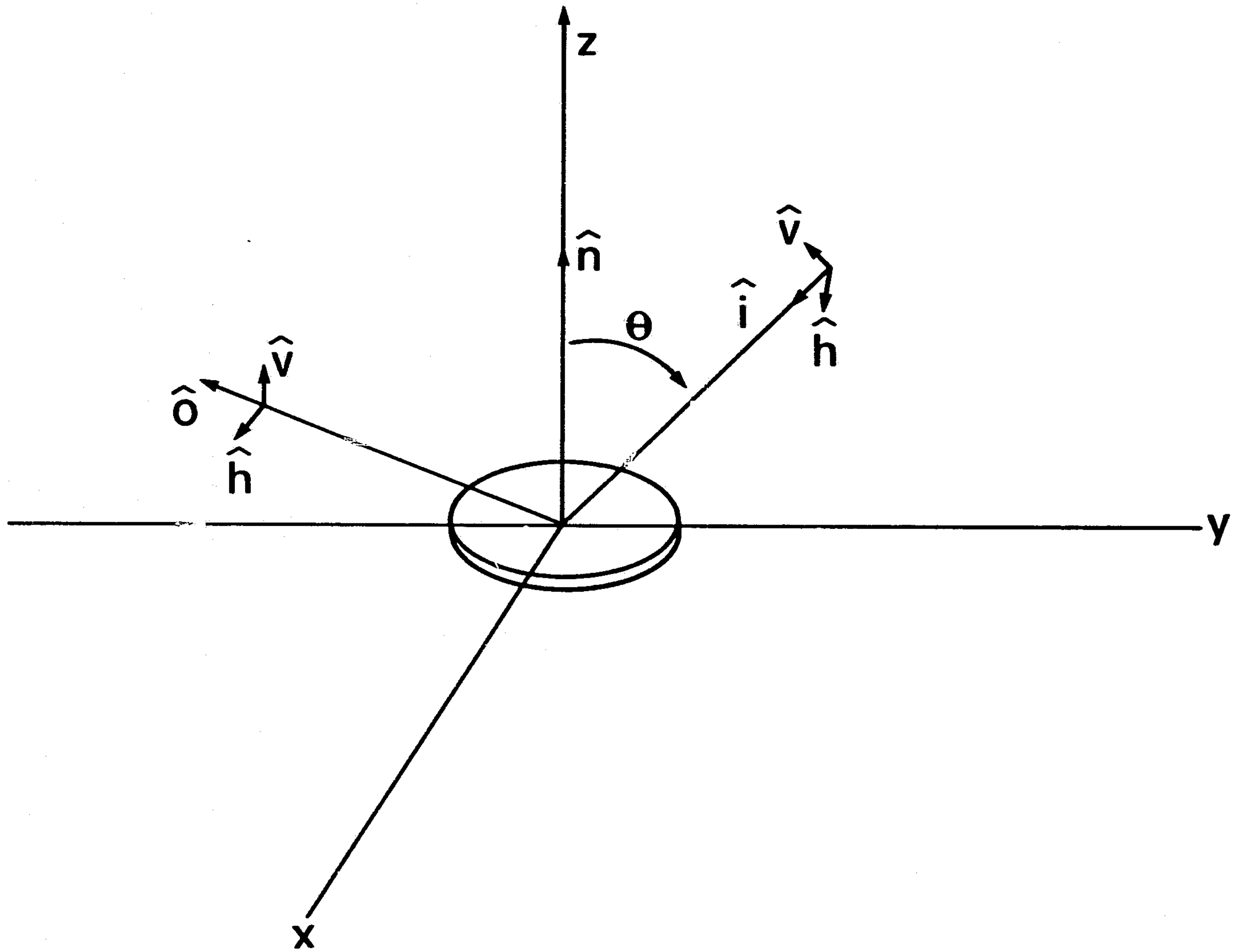

Figure 3. Geometry for scattering from a dielectric disk. In the examples presented in the text $\hat{i}$ and $\hat{o}$ are both in the y–z plane.

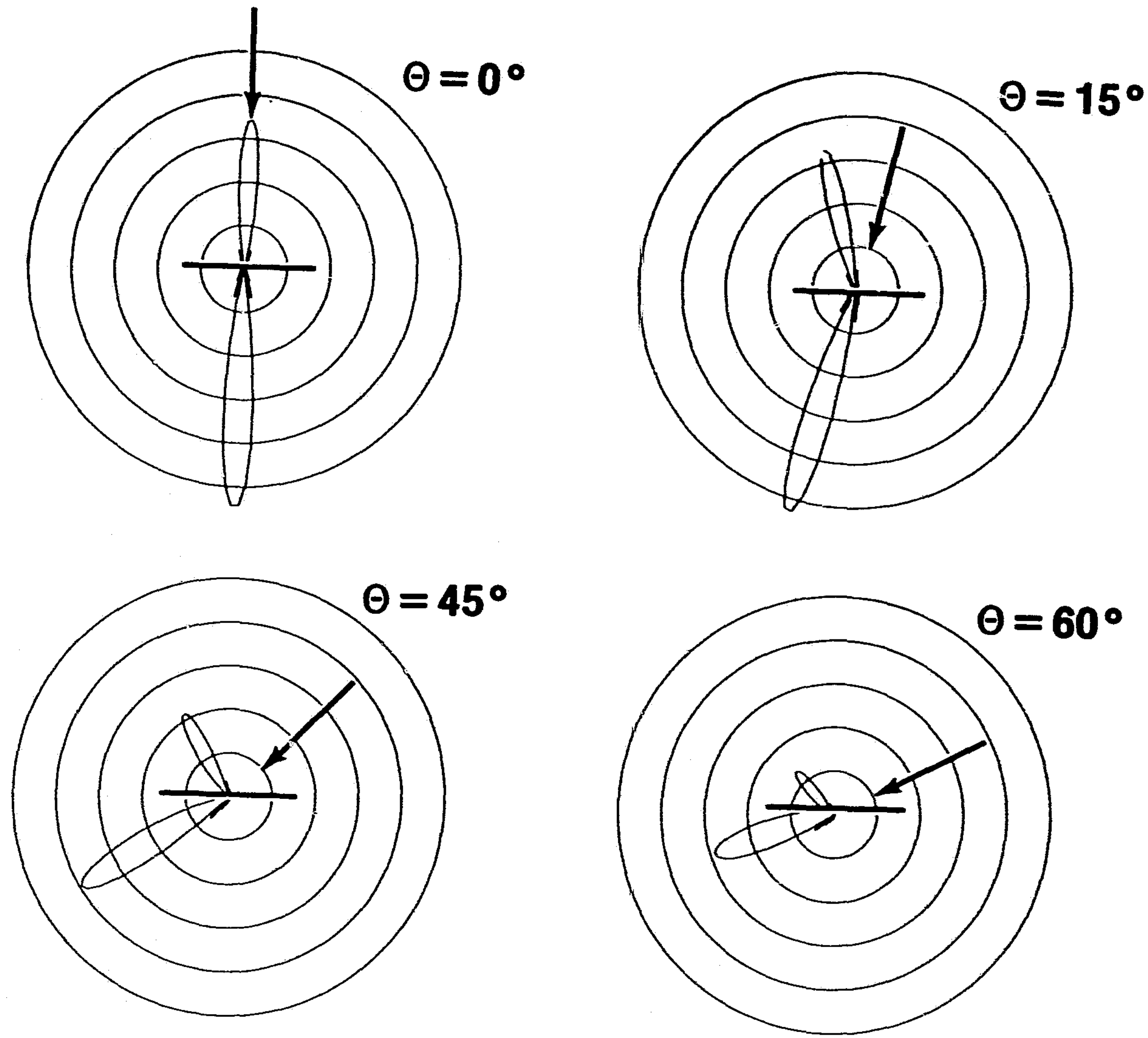

Figure 4. Magnitude of the scattering amplitude for vertically polarized incident and scattered waves. The arrow indicates the direction of propagation of the incident wave. Θ is the angle between the direction of propagation of the incident wave and the normal to the disk.

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