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SELF-SUSTAINED OSCILLATIONS OF A SHOCK WAVE INTERACTING WITH A BOUNDARY LAYER ON A SUPERCRITICAL AIRFOIL

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Summary

A theory is proposed of the self-sustaining oscillations of a weak shock on an airfoil in steady, transonic flow. The interaction of the shock with the boundary layer on the airfoil produces displacement thickness fluctuations which convect downstream and generate sound by interaction with the trailing edge. A feed-back loop is established when this sound impinges on the shock wave, resulting in the production of further fluctuations in the displacement thickness. The details are worked out for an idealized mean boundary layer velocity profile, but strong support for the basic hypotheses of the theory is provided by a comparison with recent experiments involving the generation of acoustic 'tone bursts' by a supercritical airfoil section.

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§1. Introduction

The production of sound by helicopter rotor blades having transonic tip speeds is strongly influenced by the presence of shock waves near the blade tips (see, e.g., refs. [1-3]). Acoustic waves are generated by (i) the interaction of the shock with inflow turbulence and/or the trailing vortices of the other rotor blades, and (ii) the intensification of the nonlinear quadrupole volume sources which occur in the Lighthill theory of aerodynamic sound [4]. In this paper we discuss a possible additional mechanism associated with an instability of the shock caused by its interaction with the boundary layer on the blade. This interaction is strongly dependent on the Reynolds number when the flow is transonic [5-7], the position of the shock being extremely sensitive to temporal variations in the properties of the boundary layer.

Succi et al have reported [8] the occurence of intense, high frequency and highly directional tone bursts during acoustic tests of a scale model of a general aviation propeller operating at high subsonic tip speeds. The amplitude of the burst was of the same order as the propeller noise, and observation strongly suggested that the tones were produced by oscillating shock waves on the blades. The phenomenon was absent if flow separation occured at the blade tips and if the boundary layers ahead of the shocks were made turbulent; this is presumably a particular manifestation of transonic buffeting [9]. The extreme sensitivity of the shock to perturbations in the ambient flow is also illustrated by the experiments of Tijdeman [10] using an airfoil with an oscillating flap. These demonstrate that under certain conditions the shock leaves the airfoil and propogates as an intense acoustic wave to the far field, an effect which is in qualitative agreement with theoretical work of Williams [11]. Similarly, Magnus and Yoshihara [12] argue that discrepancies between experiment and their numerical predictions of motion caused by a pitching airfoil in transonic flow are probably a result of the strong interaction of a shock wave with the boundary layer.

An idealized transonic flow problem is investigated below in order to examine a mechanism of shock wave/boundary layer interaction which is possibly responsible for the intense acoustic fields observed by Succi et al [8]. The analysis proceeds from the unsteady, transonic flow equations which are linearized about a steady mean flow with account taken of the displacement of the shock, in the manner described by Williams [11]. The model problem is illustrated schematically in Figure 1. A two-dimensional flat-plate airfoil is placed at zero angle of attack to a mean flow. Above the airfoil the mean flow is uniform and supersonic ahead of a weak shock wave which in the undisturbed state is assumed to be normal to the airfoil and of unlimited extent. Time harmonic oscillations of the shock about its mean position are considered. At the root of the shock the motion induces fluctuations in the displacement thickness of the boundary layer which propagate downstream as a surface wave on the airfoil. At the trailing edge this wave feeds into an unsteady wake and is responsible for the production of edge-generated sound which subsequently interacts with the shock thereby inducing further fluctuations in the boundary layer. Self-sustaining oscillations are possible provided the returning sound waves are in an appropriate phase relation with the motion of the shock, and are of sufficient amplitude. These conditions determine a preferred wavenumber for the boundary layer waves, and our objective is to determine the frequency of the oscillations in terms of the characteristics of the mean boundary layer.

The analytical problem has similarities with that investigated recently by Goldstein et al [13], concerning the instability of shocks of arbitrary strength in cascades, although no account was taken of shock wave/boundary layer is assumed to be sufficiently weak that the generation of vorticity and entropy by its motion can be neglected. The unsteady flow in the main stream behind the shock may accordingly be expressed in terms of a velocity potential ϕ . At the outer edge of the boundary layer the normal derivative of ϕ equals the displacement velocity of the unsteady boundary layer. Application of this condition is simplified by means of an approximation analogous to that used in thin airfoil theory [14], which permits it to be imposed on the upper surface of the airfoil (c.f. [15-17]).

The principal difficulty in formulating the problem is the modelling of the production of the displacement thickness waves. The approach adopted here is based on the hypothesis that the functional form for the transition in the boudary layer structure across the shock is invariant in a frame of reference fixed with respect to the root of the shock. This assumption of quasi-static behaviour at the root permits the unsteady motion in the boundary layer to be determined in a linearized approximation provided the mean properties of the boundary layer are known. Results are given here only for the highly simplified case in which the mean velocity profile is approximated by a step function, a procedure which has been successfully exploited by Ffowcs Williams and Purshouse [18] and Goldstein [19] in analytical studies involving unsteady boundary layers. Evidently this approach is also applicable to other shock wave/boundary layer interaction problems. For example, it provides an excellent starting point for investigating the interaction of blade-tip generated shocks of a ducted rotor with the boundary layer on the walls of the duct.

The basis of the model problem is discussed, formulated analytically and solved in §§2,3. In §4 the equation for the characteristic frequency of the self-sustaining oscillations is obtained; numerical results given in §5 are examined in relation to the experiment of Succi et al [8].

§2. Formulation of the model problem

A two-dimensional rigid airfoil occupies the portion $-C < x_1 < 0$ of the x_1-axis of a rectangular coordinate system (x_1, x_2), in the presence of a mean flow in the positive x_1-direction. In the undisturbed state a normal shock wave extends from the upper surface of the airfoil at $x_1 = -\ell$ to $x_2 = +\infty$ (Figure 2(a)). Attention is confined to the weak-shock/transonic regime in which the speed U of the main flow downstream of the shock is constant and

$$\frac{U}{c} \equiv M = 1 - \beta^2/2 \quad , \tag{1}$$

where c is the speed of sound, and $\beta \approx \sqrt{1-M^2} << 1$. The Rankine-Hugoniot relations applied to a weak shock imply that the main flow Mach number M_-, say, upstream of the shock is given by

$$M_- = 1 + \beta^2/2 \ . \tag{2}$$

Motion of the shock wave produces fluctuations in the boundary layer and wake which in turn react back on the shock. It is assumed that all such back-reactions occur via acoustic paths intersecting the shock from behind, i.e., possible interaction channels involving with the passage of sound around the leading edge of the airfoil, after propagation through the subsonic flow below the airfoil, are ignored. Similarly, the transmission of acoustic disturbance into the supersonic region via the subsonic boundary layer is neglected.

Consider a time-harmonic perturbation proportional to $e^{-i\omega t}$. Let the instantaneous position of the shock be represented by

$$x_1 = -\ell + z(x_2)e^{-i\omega t} \ , \tag{3}$$

where $z(x_2)$ is assumed to be sufficiently small that the perturbation equations may be linearized. For a weak shock the motion in the main flow above the airfoil is described by a velocity potential $\phi\, e^{-i\omega t}$, in terms of which the perturbation velocity $\underset{\sim}{v} = \nabla\phi$ (the exponential time factor is here and henceforth suppressed). In a linearized approximation ϕ satisfies

$$\left\{\left(-ik_0 + M\frac{\partial}{\partial x_1}\right)^2 - \frac{\partial^2}{\partial x_1{}^2} - \frac{\partial^2}{\partial x_2{}^2}\right\}\phi = 0\,, \tag{4}$$

in the subsonic region $x_1 > -\ell + z(x_2)$, where $k_0 = \omega/c$. In the supersonic flow ahead of the shock the motion is steady, and

$$\phi = 0 \qquad (x_1 < -\ell + z(x_2)) \tag{5}$$

The potential must also satisfy the following conditions at the undisturbed location $x_1 = -\ell$ of the shock:

$$\frac{\partial\phi}{\partial x_1} + \frac{2i\omega\phi}{U\beta^2} = 0\;; \tag{6}$$

$$z(x_2) = \frac{-\phi}{c\beta^2}\,, \tag{7}$$

where it is understood that these conditions are to be satisfied by ϕ and $\partial\phi/\partial x_1$ as $x_1 \to -\ell$ from the downstream region. The derivations of these formulae are given by Williams [11,20].

In the (shock free) main flow region below the airfoil the motion is assumed to be subsonic everywhere and ϕ is required to satisfy equation (4).

Boundary conditions at the airfoil and wake

Let $v_n(x_1)$ denote the normal (i.e. x_2 -) component of velocity on the surface $x_2 = \delta$ lying just outside the boundary layer on the upper surface of the airfoil, δ being the boundary layer thickness; this will be referred to as the boundary layer displacement velocity. On $x_2 = \delta$ ϕ must satisfy

$$\frac{\partial\phi}{\partial x_2} = v_n(x_1)\,, \qquad (x_1 < 0)\,. \tag{8}$$

In order to make the subsequent analysis tractable, this condition is imposed on $x_2 = +0$, an approximation which is expected to be valid provided the length scale $\sim U/\omega$ of boundary layer disturbances is large relative to δ. Boundary layer displacement fluctuations are assumed to be absent on the lower surface $(x_2 = -0)$ of the airfoil, where it is accordingly required that

$$\frac{\partial \phi}{\partial x_2} = 0 \quad , \quad -C < x_1 < 0 \; , \; x_2 = -0 \; . \tag{9}$$

We shall actually apply this condition over the semi-infinite interval $-\infty < x_1 < 0$. This will ensure that acoustic disturbances cannot impinge on the shock from the supersonic region, and avoids difficulties (which cannot be incorporated into the present idealized model) arising from the fact that the mean flow ahead of the shock must actually vary with position.

Similarly, if $\omega h/U << 1$, where h characterises the mean thickness of the wake downstream of the trailing edge, the perturbation pressure

$$p = \rho_0 \left(i\omega - U \frac{\partial}{\partial x_1}\right) \phi \; , \tag{10}$$

(ρ_0 being the mean density, taken to be constant), may be assumed to be continuous across the wake. In the usual approximation of thin airfoil theory [14], this condition may be imposed on the centre-line $x_2 = 0$, $x_1 > 0$ of the wake, and implies that

$$[\phi] = Ae^{i\omega x_1/U} \quad (x_1 > 0) \; , \tag{11}$$

where $[\phi] = \phi(x_1, +0) - \phi(x_1, -0)$ defines the discontinuity in ϕ across the x_1-axis. The value of the constant A is a measure of the strength of vorticity shed from the trailing edge, and is determined by the requirement that the perturbation pressure and velocity should remain finite at the edge (Kutta condition).

A non-zero value of A is associated with the presence of an asymmetric (sinuous) disturbance in the wake. Symmetric "breathing" modes are also possible, however. When $\omega h/U$ is small the pressure is continuous for such modes, but the normal velocity exhibits a simple discontinuity across $x_2 = 0$, $x_1 > 0$. In this case the phase velocity is equal to the minimum mean velocity U_{min}, say, in the wake [21], and if $\kappa_s = U_{min}/\omega$, the breathing mode satisfies

$$\left[\frac{\partial\phi}{\partial x_2}\right] = Be^{i\kappa_s x_1} \qquad (x_1 > 0) , \tag{12}$$

where B is a constant. The value of B is obtained by requiring that there be no net flux of fluid from the wake and boundary layer. This condition is derived from the equation of continuity on the basis that the characteristic acoustic wavelength is large relative to the thickness of both the wake and the boundary layer. Thus in the absence of displacement velocity fluctuations on the lower surface of the airfoil, B is related to the displacement velocity v_n by

$$\int_{-C}^{0} v_n(x_1)dx_1 + \int_{0}^{\infty} Be^{i\kappa_s x_1}\, dx_1 = 0 . \tag{13}$$

Convergence of the second of these integrals is assured by assigning to $\omega = U\kappa_s$ a small positive imaginary component which is subsequently allowed to vanish. The validity of this procedure is a consequence of the causality condition which requires all field variables to be regular for sufficiently large and positive $\text{Im}\,\omega$. The disturbances in the wake arise as a result of the displacement velocity fluctuations in the boundary layer and of the motion of the shock. Hence

$$B = i\kappa_s \int_{-C}^{0} v_n(x_1)dx_1 . \tag{14}$$

Modelling the interaction of the shock and boundary layer

Oscillatory motion of the shock at frequency ω generates displacement velocity fluctuations whose specification according to linear theory depends on a knowledge of the mean boundary layer velocity profile. We consider here only the highly idealized velocity profile illustrated in Figure 2(b), in which the mean shear in the outer region of the boundary layer is concentrated into a vortex sheet at a stand-off distance δ from the airfoil. If the flow in the inner region $0 < x_2 < \delta$ has speed V in the x_1-direction, the vortex sheet can support displacement velocity waves proportional to $e^{i\kappa_{\pm}x_1}$, where

$$\kappa_{\pm} = \frac{\omega}{V}\left\{1 \pm \frac{i\left(\frac{U}{V} - 1\right)\left(\frac{\omega\delta}{V}\right)^{\frac{1}{2}}}{\left[1 - M^2\left(1 - \frac{V}{U}\right)^2\right]^{\frac{1}{2}}}\right\} \tag{15}$$

provided $\omega\delta/V$ is small, i.e., that the wavelength is large relative to δ (c.f. ref. [16], §3) .

In the undisturbed state there will exist a length scale d determining the distance in the x_1-direction over which the mean properties of the boundary layer change across the shock. When oscillations occur at frequency ω it is anticipated that at distances exceeding d downstream of the shock the unsteady motion in the boundary layer consists of a linear combination of the two displacement velocity waves determined by the dispersion equation (15). This notion leads to the quasi-static representation of the interaction of the boundary layer and shock described below.

In the steady state the boundary layer thickness δ increases rapidly with x_1 across the shock, and we write

$$\delta = \delta_0 F(x_1 + \ell) \tag{16}$$

and assume that

$$\left.\begin{aligned} F(x_1+\ell) &\approx 1 \qquad\quad \text{for} \quad x_1+\ell \le -d \\ &\approx 1+\delta_0'/\delta_0 \quad \text{for} \quad x_1+\ell \ge d \end{aligned}\right\}, \tag{17}$$

so that $\delta = \delta_0, \delta_0 + \delta_0'$ respectively in the regions upstream and downstream of the shock. The generally much slower variations of δ_0, δ_0' with x due to the natural growth of the boundary layer are ignored.

In generalising equation (16) to unsteady motion, observe that in practice "fanning" of the shock must occur in the vicinity of the boundary layer, and therefore that the shock-location equation (3) becomes ill-defined as $x_2 \to 0$. This fanning will be neglected to the extent that when $x_2 \to 0$ it will be assumed that a representative value of $z_0 = z(0)$ can be defined to determine the instantaneous position of the root of the shock just outside the boundary layer.

Let $\zeta(x_1,t)$ denote the unsteady boundary layer thickness, and introduce the representation

$$\zeta = \delta_0\left\{F(x+\ell) - z_0\int_{-\infty}^{\infty}\frac{k^2 f(kd)e^{ik(x_1+\ell)}}{(k-\kappa_I)(k-\kappa_I^*)}\,dk\right\}, \tag{18}$$

where κ_I, κ_I^* respectively denote the wavenumber $\kappa_\pm$ defined by equation (15) when the $-, +$ sign is taken. The path of integration in (18) runs below both of the poles at $k = \kappa_I, \kappa_I^*$ (this can always be ensured by taking ω to lie in a suitable region of the upper half-plane). The function $f(kd)$ is assumed to be independent of ω, to be regular on the real k-axis, and to vary significantly only for variations in kd which are $O(1)$. In practice d and δ_0 are of comparable magnitudes, and both are here required to be small relative to the wavelength $\sim 1/Re(\kappa_I)$ of the displacement velocity waves. This implies that the singularities of $f(kd)$ in the complex plane lie further from the real axis than the simple poles at $k = \kappa_I, \kappa_I^*$. Accordingly, when $x_1 + \ell >> d > 0$ the value of the integral in (18) is dominated by the residue contributions from these poles, and the motion thereby described is entirely associated with the corresponding displacement velocity waves. When ω is real, κ_I, κ_I^* respectively determine boundary layer disturbances which grow and decay exponentially with increasing x_1.

In the opposite extreme in which $|x_1+\ell| \lesssim O(d)$, the displacement velocity poles make an insignificant contribution to the integral, and (18) reduces to

$$\zeta = \delta_0 \left\{ F(x+\ell) - z_0 \int_{-\infty}^{\infty} f(kd) e^{ik(x_1+\ell)} dk \right\} . \qquad (19)$$

The hypothesis that the structure of the boundary layer in the neighbourhood of the shock is invariant in a reference frame fixed relative to the shock requires that $f(kd)$ be interpreted as the Fourier transform of $F'(x_1) = dF/dx_1$, i.e.,

$$f(kd) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F'(x_1) e^{-ikx_1} dx_1 , \qquad (20)$$

in which case (19) becomes

$$\zeta = \delta_0 \left\{ F(x_1+\ell-z_0) + O\,(z_0^2) \right\} . \qquad (21)$$

In the following discussion the functional form of $F(x+\ell)$ is assumed to be known.

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The displacement velocity $v_n(x_1)'$ may now be expressed in terms of the displacement z_0 of the root of the shock by substituting the representation (18) into the linear theory formula

$$v_n = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta , \quad (x_1 > -\ell) \tag{22}$$

giving

$$v_n(x_1) = i\, z_0\delta_0 \int_{-\infty}^{\infty} \frac{k^2(\omega - Uk)f(kd)e^{ik(x_1+\ell)}}{(k-\kappa_I)(k-\kappa_I^*)}\, dk . \tag{23}$$

Similarly, since $F'(x)$ vanishes for $x_1 + \ell \leq -d$, upstream of the shock, equation (14) becomes

$$B = i\, z_0\delta_0\kappa_s \int_{-\infty}^{\infty} \frac{kf(kd)e^{ik\ell}}{(k-\kappa_I)(k-\kappa_I^*)}\, dk . \tag{24}$$

Boundary value problem for ϕ

Collecting together the principal strands of the above discussion, the boundary value problem for the potential ϕ may be stated thus:

For $(x_1 > -\ell,\ x_2 > 0)$ and $(-\infty < x_1 < \infty,\ x_2 < 0)$, find admissible solutions ϕ of the equation

$$\left\{\left(-ik_0 + M\frac{\partial}{\partial x_1}\right)^2 - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right\}\phi = 0 \tag{25}$$

which satisfy the radiation condition at large distances from the airfoil together with the following:

I. **Conditions on the Airfoil:**

$$\frac{\partial\phi}{\partial x_2} = i\, z_0 \delta_0 \int_{-\infty}^{\infty} \frac{k^2(\omega - Uk) f(kd) e^{ik(x_1+\ell)}}{(k-\kappa_I)(k-\kappa_I^*)}\, dk \, ,$$

$$(-\ell < x_1 < 0,\ x_2 = +0); \qquad (26a)$$

$$\frac{\partial\phi}{\partial x_2} = 0\, , \quad (-\infty < x_1 < 0,\ x_2 = -0)\, . \qquad (26b)$$

II. **Conditions in the wake ($x_2 = 0$, $x_1 > 0$) ;**

$$\left.\begin{aligned} [\phi] &= A\, e^{i\omega x_1/U} \; ; \\ \left[\frac{\partial\phi}{\partial x_2}\right] &= B\, e^{i\kappa_s x_1} \; . \end{aligned}\right\} \qquad (27a,b)$$

III. **Conditions at the shock:**

$$\frac{\partial\phi}{\partial x_1} + \frac{2i\omega}{U\beta^2}\phi = 0 \qquad (x_1 \to -\ell,\ 0 < x_2 < \infty)\ ; \qquad (28a)$$

$$z_0 = -\frac{\phi}{c\beta^2}\,(x_1 \to -\ell,\ x_2 = 0)\ . \qquad (28b)$$

The coefficient A is to be determined by application of the Kutta condition at the trailing edge of the airfoil, and B is given in terms of z_0 by equation (24).

When used in conjunction with the dispersion equation (15), the solution of this boundary value problem provides the characteristic equation for the admissible frequencies ω for which self-sustaining oscillations of the shock wave are possible.

§3. Solution of the boundary value problem

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The Prandtl-Glauert transformation

$$\left.\begin{aligned} \psi &= \phi\, e^{iKMX} \\ K &= k_0/\beta \\ X &= x_1/\beta \end{aligned}\right\} \tag{29}$$

reduces equation (25) to the standard form

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial x_2^2} + K^2\right)\psi = 0 , \tag{30}$$

and the boundary conditions (26) - (28) become:

I. Conditions on the airfoil:

$$\frac{\partial\psi}{\partial x_2} = iz_0\delta_0 \int_{-\infty}^{\infty} \frac{\nu^2(\omega-U\nu)f(\nu d)e^{i[\sigma X+\nu\ell]}}{(\nu-\kappa_I)(\nu-\kappa_I^*)}\, d\nu$$

$$(-L < X < 0,\ x_2 = +0) , \tag{31a}$$

where $L = \ell/\beta$, $\sigma = \nu\beta + KM$;

$$\frac{\partial\psi}{\partial x_2} = 0 \qquad (-\infty < X < 0,\ x_2 = -0) . \tag{31b}$$

II. Conditions in the wake $(x_2 = 0,\ X > 0)$:

$$[\psi] = A\, e^{i\kappa X} , \quad (\kappa = \omega/\beta U) ; \tag{32a}$$

$$\left[\frac{\partial\psi}{\partial x_2}\right] = B\, e^{i\bar{\kappa}_s X} , \quad (\bar{\kappa}_s = \kappa_s\beta + KM) . \tag{32b}$$

III. **Conditions at the shock:**

$$\frac{\partial\psi}{\partial X} + i\kappa\psi = 0 \qquad (X \to -L, \;\; x_2 > 0) \; ; \tag{33a}$$

$$z_0 = -\frac{1}{c\beta^2}\,\psi\, e^{iKMX} \; (X \to -L, \; x_2 = 0). \tag{33b}$$

Equation (33a) has been simplified by making use of the condition $\beta^2 << 1$.

Since the problem is linear we can set

$$\psi = \psi_1 + \psi_2 \quad , \tag{34}$$

where ψ_1 represents the flow induced by the boundary layer displacement velocity fluctuations, and ψ_2 is that produced by the oscillating motion of the shock. The functions ψ_1, ψ_2 must satisfy reduced forms of conditions (31), (32), and their sum, ψ, must satisfy (33).

Boundary value problem for ψ_1

The perturbation due to the displacement velocity fluctuations is determined by ignoring the presence of the shock, i.e., by assuming the flow to be subsonic everywhere and by imposing (31a) over the semi-infinite interval $(-\infty < X < 0)$. This procedure is permissible provided that the composite potential ψ satisfies conditions (33) on the shock.

A Fourier integral representation for ψ_1 is obtained in Appendix 1 by the Wiener-Hopf technique [22]. In $x_2 > 0$ we find

$$\psi_1 = \frac{i\; z_0\delta_0}{4\pi} \int\!\!\int_{-\infty}^{\infty} \frac{\nu^2(\omega - U\nu) f(\nu d)}{(\nu-\kappa_I)(\nu-\kappa_I^*)} \left\{ \frac{1}{\gamma(k)} \left(\frac{1}{k - \sigma + i0} - \frac{\kappa_s/\nu}{(k - \bar{\kappa}_s - i0)} \right) + \frac{1}{\sqrt{K+k}\,\sqrt{K-\sigma}} \left(\frac{1}{k-\sigma+i0} - \frac{1}{k-\kappa-i0} \right) \right\}$$

$$\times\, e^{i\{\nu\ell + kX + \gamma(k)x_2\}}\, dk\, d\nu \quad , \tag{35}$$

in which the notation $\pm i0$ indicates that the contour of integration in the k-plane passes above the pole at $k = \sigma$ and below the poles at $k = \bar{\kappa}_s, \kappa$. The function $\gamma(k)$ is defined by

$$\gamma(k) = \sqrt{K-k} \cdot \sqrt{K+k} \tag{36}$$

where branch cuts for the radicals on the right-hand side are taken respectively in the upper and lower halves of the k-plane, such that when K is real and positive, $\gamma(k)$ is positive on the real axis for $|k| < K$ and positive imaginary for $|k| > K$.

Boundary value problem for ψ_2

Motion of the shock causes sound to be radiated in the downstream direction. Since the influence of the displacement velocity is included in ψ_1 , the shock associated sound must satisfy

$$\frac{\partial \psi_2}{\partial x_2} = 0 \qquad (-\infty < X < 0 \, , \;\; x_2 = 0) \tag{37}$$

rather than (31), where again the condition is imposed on the half-line $(-\infty < X < 0)$. In the wake there is no need to account for breathing modes generated by the displacement effect upstream of the edge, and ψ_2 is therefore required to satisfy conditions (32) with $B = 0$.

In the absence of the edge an appropriate representation of the motion in $x_2 > 0$ would be

$$\psi_2^I = \frac{2}{\pi} \int_0^\infty \Psi(\lambda) \cos(\lambda x_2) \; e^{i\gamma(\lambda)(X+L)} d\lambda \tag{38}$$

If

$$\psi_2 = \psi_2^I + \psi_2^D \; , \tag{39}$$

ψ_2^D is the diffracted field produced by the edge and wake, and may again be determined by the Wiener-Hopf procedure. In Appendix 2 it is shown that, for $x_2 > 0$

$$\psi_2^D = -\frac{i}{2\pi^2}\int_0^\infty d\lambda \int_{-\infty}^{\infty} \frac{\Psi(\lambda)\sqrt{K+\gamma(\lambda)}}{\sqrt{K+k}} \left\{ \frac{1}{k-\kappa-i0} - \frac{1}{k-\gamma(\lambda)-i0} \right\} e^{i\{kX+\gamma(k)x_2+\gamma(\lambda)L\}} dk \qquad (40)$$

Equations (38)-(40) determine ψ_2 in terms of the as yet unknown function $\Psi(\lambda)$.

The integral equation for $\Psi(\lambda)$

Condition (33a), to be imposed at the undisturbed location of the shock, provides an integral equation which determines $\Psi(\lambda)$ in terms of the displacement z_0 of the root of the shock. To obtain this equation we introduce the Fourier cosine transform $\bar{g}(\mu)$ of a function $g(x_2)$ defined by the reciprocal relations

$$\left.\begin{aligned} \bar{g}(\mu) &= \int_0^\infty g(x_2)\cos(\mu x_2)dx_2 \\ g(x_2) &= \frac{2}{\pi}\int_0^\infty \bar{g}(\mu)\cos(\mu x_2)d\mu \end{aligned}\right\} \qquad (41a,b)$$

(see, e.g. Erdelyi et al [23]) .

Substituting $\psi = \psi_1 + \psi_2$ into (33a), using the representations (35), (38), (40), and taking the cosine transform, we find that Ψ satisfies

$$\Psi(\mu) - \frac{i}{2\pi}\left(\frac{\kappa-\gamma(\mu)}{\kappa+\gamma(\mu)}\right)\int_0^\infty \Psi(\lambda)\,\frac{\sqrt{K+\gamma(\mu)}\cdot\sqrt{K+\gamma(\lambda)}}{\gamma(\mu)} \times \left(\frac{1}{\gamma(\lambda)+\gamma(\mu)} - \frac{1}{\kappa+\gamma(\mu)}\right) e^{iL(\gamma(\mu)+\gamma(\lambda))}d\lambda$$

$$= \frac{i}{(\kappa+\gamma(\mu))}\left[\left(\frac{\partial}{\partial X} + i\kappa\right)\bar{\psi}_1\right]_{X=-L} , \qquad (42)$$

where the cosine transform $\bar{\psi}_1$ is given by

$$\bar{\psi}_1 = i\,\delta_0 z_0 \int_{-\infty}^{\infty} \frac{\nu^2(\omega-U\nu)f(\nu d)e^{i(\nu\ell+\sigma X)}d\nu}{(\nu-\kappa_I)(\nu-\kappa_s^*)(K^2-\mu^2-\sigma^2)}$$

$$+ i\delta_0 z \int_{-\infty}^{\infty} \frac{\nu^2(\omega-U\nu)f(\nu d)}{(\nu-\kappa_I)(\nu-\kappa_I^*)\gamma(\mu)} \left\{ \frac{\kappa_s/\nu}{\gamma(\mu)+\bar{\kappa}_s} - \frac{1}{\gamma(\mu)+\sigma} \right.$$

$$\left. + \frac{\sqrt{K+\gamma(\mu)}}{\sqrt{K-\sigma}} \left[\frac{1}{\gamma(\mu)+\kappa} - \frac{1}{\gamma(\mu)+\sigma}\right] \right\} e^{i(\nu\ell-\gamma(\mu)X)}d\nu \qquad (43)$$

The various terms in equations (42), (43) have the following interpretation:

$\Psi(\mu)$ is the contribution to (33a) from Ψ_2^I, the direct field generated by the shock; the integral involving $\Psi(\lambda)$ on the left of (42) represents the contribution Ψ_2^D of sound waves generated at the trailing edge by diffraction of the primary field Ψ_2^I. The first term on the right of equation (43) denotes the influence of the boundary layer displacement velocity fluctuations when the edge and wake are ignored; the second term represents the effect of sound generated at the edge due to its interaction with the displacement velocity wave.

§4. The characteristic frequencies of self-sustained oscillations

A second relation between z_0 and $\Psi(\lambda)$ can be obtained from condition (33b) imposed at the root of the shock, viz:

$$z_0 = -\frac{e^{iKML}}{c\beta^2}\left\{\psi_1(-L,0) + \Psi_2^I(-L,0) + \Psi_2^D(-L,0)\right\} \tag{44}$$

The elimination of z_0, $\Psi(\lambda)$ between (42), (44) provides an equation relating the frequency ω and the conjugate displacement velocity wavenumbers κ_I, κ_I^* . The roots of this equation define those admissible values of κ_I, κ_I^* for which self-sustained oscillations are possible, and when substituted into the dispersion equation (15) yield the characteristic equation determining ω in terms of the boundary layer thickness $\delta = \delta_0 + \delta_0'$ downstream of the shock.

Self-sustained oscillations are envisaged to occur as a result of the following feed-back loop of mechanical processes (see Figure 3):

1. The displacement z_0 of the root of the shock generates displacement velocity waves in the boundary layer;
2. Sound waves are produced by the subsequent interaction of the displacement velocity waves with the trailing edge;
3. The impingement of the sound on the shock closes the loop by generating further disturbances in the boundary layer.

These observations lead to the following simplifications of equations (42), (43) and (44):

In equation (42) the second term on the left hand side is discarded. This expresses the influence on the shock of sound waves originally generated by the shock (ψ_2^I) and subsequently diffracted at the trailing edge. The amplitude of these waves is expected to be small compared with the sound produced by the boundary layer/edge interaction because of the exponential growth of the displacement velocity waves as they propagate towards the edge.

When the second integral on the right of equation (43) is inserted into the right hand side of (42), it describes the effect on the shock of sound generated by the boundary layer/edge interaction. This interaction will be dominated by the exponentially growing displacement velocity wave (proportional to $e^{i\kappa_I x_1}$) , whose effect is determined by the residue contribution to the integral from the pole at $\nu = \kappa_I$. This will be taken to represent the principal component of the edge generated sound.

Taking account of these approximations we accordingly reduce equation (42) is reduced to

$$\Psi(\mu) = z_0\{\chi_1(\mu) + \chi_2(\mu)\} , \tag{45}$$

where χ_1 χ_2, respectively denote the local and edge-diffracted influences on the shock of the boundary layer motion, and have the explicit representations:

$$\chi_1(\mu) = -\frac{i\delta_0 e^{-iKML}}{[\kappa+\gamma(\mu)]} \int_{-\infty}^{\infty} \frac{\nu^2(\omega-U\nu)(\sigma+\kappa)f(\nu d)\,d\nu}{(\nu-\kappa_I)(\nu-\kappa_I^*)(K^2-\mu^2-\sigma^2)} \tag{46}$$

$$\chi_2(\mu) = \frac{\pi\delta_0}{2}\left(\frac{\kappa-\gamma(\mu)}{\kappa+\gamma(\mu)}\right)\frac{\kappa_I^2(\omega-U\kappa_I)f(\kappa_I d)e^{i(\kappa_I \ell+\gamma(\mu)L)}}{(\kappa_I-\kappa_I^*)}$$

$$\times\left\{\frac{\kappa_S/\kappa_I}{\gamma(\mu)+\bar{\kappa}_S} - \frac{1}{\gamma(\mu)+\sigma_I} + \frac{\sqrt{K+\gamma(\mu)}}{\sqrt{K-\sigma_I}}\left[\frac{1}{\gamma(\mu)+\kappa} - \frac{1}{\gamma(\mu)-\sigma_I}\right]\right\}; \tag{47}$$

where

$$\sigma_I = \kappa_I\beta + KM . \tag{48}$$

Turn attention now to equation (44). The first term in the brace brackets on the right hand side accounts for the backreaction of the boundary layer on the motion at the root of the shock. As above, this decomposes into two components, the first describing the local influence of the boundary layer (when edge effects are ignored), and the second the effect of acoustic waves produced by the boundary layer/edge interaction. Both effects are contained in the double integral representation (35). When $X \to -L$ on $x_2 = +0$ the integration contour in the k-plane may be displaced to $-i\infty$. In so doing the integral along the real axis is transformed into the sum of two terms $z_0\Theta_1$, $z_0\Theta_2$, say, respectively equal to the residue of the simple pole at $k = \sigma$ (which characterises the local effect of the continuum of boundary layer displacement velocity waves specified by equation (23)) and an integral around a contour enclosing the branch cut of $\sqrt{K+k}$, which extends from $k = -K$ to $-K - i\infty$. The latter represents the edge generated sound.

Thus we find:

$$\Theta_1 = \delta_0 e^{-iKML} \int_{-\infty}^{\infty} \frac{\nu^2(\omega-U\nu)f(\nu d)d\nu}{\gamma(\sigma)(\nu-\kappa_I)(\nu-\kappa^*_I)} , \tag{49}$$

and the branch-cut integral can be reduced to the form

$$\Theta_2 = \int_{-\infty}^{\infty} d\nu \int_0^{\infty} \frac{\mathcal{F}(\nu,\xi)e^{-KL\xi}}{\sqrt{\xi}} d\xi , \tag{50}$$

where $\mathcal{F}(\nu_1\xi)$ is regular on the ξ-axis and $\sim O(1/\xi)$ as $\xi \to \infty$.

Now $\beta^2 << 1$, so that $KL \equiv k_0\ell/\beta^2$ is a large parameter which characterises the foreshortening of the wavelengths of sound waves generated at the trailing edge and propagating upstream against a near sonic mean flow. To leading order, we therefore have

$$\theta_2 \approx \int_{-\infty}^{\infty} \mathcal{F}(\nu,0)\,d\nu \int_0^{\infty} \frac{e^{-KL\xi}}{\sqrt{\xi}}\,d\xi$$

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$$= \sqrt{\frac{\pi}{KL}} \int_{-\infty}^{\infty} \mathcal{F}(\nu,0)\,d\nu \quad . \tag{51}$$

The remaining integral with respect to ν is approximated by the residue contribution from the pole at $\nu = \kappa_I$ (see equation (35)) on the basis, discussed above, that the dominant edge-generated sound is produced by the exponentially growing displacement velocity wave proportional to $e^{i\kappa_I x_1}$. In this way equation (51) is found to have the explicit form:

$$\theta_2 \approx -\delta_0 \left(\frac{\pi}{2KL}\right)^{\frac{1}{2}} \frac{\kappa_I^2(\omega - U\kappa_I)}{(\kappa_I - \kappa_I^*)} f(\kappa_I d) \left\{ \frac{\kappa_s/\kappa_I}{(K+\bar{\kappa}_s)} - \right.$$

$$\left. - \frac{1}{(K+\sigma_I)} + \frac{\sqrt{2K}}{\sqrt{K-\sigma_I}} \left(\frac{1}{K+\kappa} - \frac{1}{K+\sigma_I}\right) \right\} e^{i\left[\kappa_I \ell + KL - \pi/4\right]} \tag{52}$$

This and equation (49) determine

$$\psi_1(-L,0) = z_0\left\{\theta_1 + \theta_2\right\} \quad . \tag{53}$$

The second term in the brace brackets of equation (44) is the contribution of the primary field due to the motion of the shock, and it follows immediately from (38) that

$$\psi_2^I(-L,0) = \frac{2}{\pi} \int_0^{\infty} \Psi(\lambda)\,d\lambda \tag{54}$$

The final term in (44) is discarded for reasons discussed previously, since it represents the influence of sound produced by the diffraction of ψ_2^I at the trailing edge. Hence substituting from (53), (54), equation (44) may finally be cast into the approximate form

$$z_0 = \frac{-\frac{-2e^{iKML}}{\pi c\beta^2}\int_0^\infty \Psi(\lambda)d\lambda}{\left\{1 + (\theta_1+\theta_2)\frac{e^{iKML}}{c\beta^2}\right\}} . \tag{55}$$

When this expression for z_0 is inserted into equation (45), the following integral equation is obtained:

$$\Psi(\mu) + \frac{\left[2e^{iKML}/\pi c\beta^2\right]\left[\chi_1(\mu)+\chi_2(\mu)\right]}{\left(1 + (\theta_1+\theta_2)\frac{e^{iKML}}{c\beta^2}\right)}\int_0^\infty \Psi(\lambda)d\lambda = 0 ,$$

$$(0 < \mu < \infty) . \tag{56}$$

The characteristic equation relating admissible values of ω, κ_I, κ_I^* is now deduced by integrating this result over $0 < \mu < \infty$:

$$1 + \frac{2e^{iKML}}{\pi c\beta^2}\left\{\frac{\pi}{2}\left(\theta_1+\theta_2\right) + \int_0^\infty (\chi_1(\mu) + \chi_2(\mu))d\mu\right\} = 0 . \tag{57}$$

A more convenient form of this equation is obtained as follows. Define

$$A(\kappa_I,\kappa_I^*,\omega) = \frac{2e^{iKML}}{\pi c\beta^2}\left\{\frac{\pi\theta_1}{2} + \int_0^\infty \chi_1(\mu)d\mu\right\}$$

$$\equiv \frac{2\delta_0}{\pi c\ \beta^2}\left\{\frac{\pi}{2}\int_{-\infty}^\infty \frac{\nu^2(\omega-U\nu)f(\nu d)d\nu}{\gamma(\sigma)(\nu-\kappa_I)(\nu-\kappa_I^*)}\right.$$

$$\left. - i\int_0^\infty d\mu\int_{-\infty}^\infty \frac{\nu^2(\omega-U\nu)(\sigma+\kappa)f(\nu d)d\nu}{(\nu-\kappa_I)(\nu-\kappa_I^*)(K^2-\mu^2-\sigma^2)(\kappa+\gamma(\mu))}\right\} , \tag{58}$$

where use has been made of equations (46), (49) .

The presence of the factor $e^{i\gamma(\mu)L}$ in the representation (47) of $\chi_2(\mu)$ indicates that, when $KL = k_{0_1}\ell/\beta^2$ is large, as assumed here, the value of $\int_0^\infty \chi_2(\mu)d\mu$ may be approximated by the method of stationary phase [22]. Taking account of this remark we define

$$B\ (\kappa_I,\kappa_I^*,\omega)e^{i\theta} = \frac{2e^{iKML}}{\pi c\beta^2}\left\{\frac{\pi\theta_2}{2} + \int_0^\infty \chi_2(\mu)\,d\mu\right\}\ , \tag{59}$$

where (from (47) and (52)) in a leading approximation

$$B = -\frac{\delta_0}{c\beta^2}\left(\frac{2\pi}{KL}\right)^{\frac{1}{2}}\left(\frac{K}{\kappa+K}\right)\frac{\kappa_I^2(\omega-U\kappa_I)}{(\kappa_I-\kappa_I^*)}\,f(\kappa_I d)$$

$$\times\left\{\frac{\kappa_s/\kappa_I}{K+K_s} - \frac{1}{K+\sigma_I} + \frac{\sqrt{2K}}{\sqrt{K-\sigma_I}}\left(\frac{1}{K+\kappa} - \frac{1}{K+\sigma_I}\right)\right\}\ ; \tag{60a}$$

and

$$\theta = \kappa_I\ell + \frac{k_0\ell}{1-M} - \frac{\pi}{4}\ . \tag{60b}$$

The characteristic equation (57) may accordingly be set in the form

$$1 + A\ (\kappa_I,\kappa_I^*,\omega) + B\ (\kappa_I,\kappa_I^*,\omega)\ e^{i\theta} = 0\ . \tag{61}$$

§5. Discussion of the Characteristic Equation

The characteristic equation (61) gives the condition which must be fulfilled if self-sustaining oscillations are to occur at frequency ω for given values of the displacement velocity wavenumbers κ_I, κ_I^*. The first two terms on the lefthand side represent the influence of the motion at the root of the shock when diffraction at the trailing edge is ignored. The term in $\mathcal{B}$ gives the backreaction at the root of sound produced during the passage of the unstable boundary layer wave over the trailing edge. This interpretation is evident from the structure of

$$\mathrm{Re}\ \theta = \mathrm{Re}(\kappa_I \ell) + \frac{k_o \ell}{1-M} - \frac{\pi}{4} , \qquad (62)$$

wherein (i) $\mathrm{Re}(\kappa_I \ell)$ is the phase change associated with the convection of a boundary layer disturbance from the root of the shock to the trailing edge, and (ii) $k_o \ell/1-M) - \pi/4$ is the phase change experienced by sound waves radiating upstream from the edge to the root together with a correction $(-\pi/4)$ due to the cylindrical spreading of the waves.

In order to derive quantitative predictions from equation (61) it is necessary to introduce an explicit representation of the function $f(\nu d)$ given by equation (20). Consider first the value of $f(\kappa_I d)$ which occurs in the definition (60a) of $\mathcal{B}$. By hypothesis, both the transition length d and the boundary layer thickness $\delta = \delta_o + \delta_o'$ are small relative to the wavelengths of the displacement velocity waves. Since $F'(x_1)$ is significantly different from zero only for $|x_1| \leq O(d)$, it follows from (17) and (20) that, to leading order,

$$f(\kappa_I d) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} F'(x_1) dx_1 = \frac{\delta_0'}{2\pi\delta} , \qquad (63)$$

and this estimate will be employed in calculating $\mathcal{B}$.

To estimate the corresponding values of the coefficient A from its definition (58), it will be assumed for simplicity that $F'(x_1+\ell)$ is an even function with respect to the undisturbed location $x_1 = -\ell$ of the root of the shock. When $\kappa_I d \ll 1$ the leading order asymptotic approximation to equation (58) reduces to

$$A\ (\kappa_I,\kappa_I^*,\omega) = -\frac{i\delta_o M}{\pi\beta^3 d}\int_{-\infty}^{\infty} f(\lambda)\left\{\pi - i\left[2\ln\left(\frac{2\beta|\lambda|}{Kd}\right) - \frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]\right\}d\lambda \qquad (64)$$

Numerical results are presented below for the case

$$f(\lambda) = \frac{(\delta_o'/\delta_o)}{2\pi(1+\lambda^2)}\ , \qquad (65)$$

for which,

$$A\ (\kappa_I,\kappa_I^*,\omega) = -\frac{M}{2\pi\beta^3}\left(\frac{\delta_o'}{d}\right)\left\{2\ \ln\left(\frac{2\beta}{Kd}\right) - \frac{1}{\beta}\ \ln\left(\frac{1+\beta}{1-\beta}\right) + \pi i\right\} \qquad (66)$$

This and other expressions appearing in the characteristic equation (61) will be expressed in non-dimensional form in terms of the reduced frequency S and wavenumber W defined by

$$\left.\begin{aligned} S &= k_o\ell \equiv \frac{\omega\ell}{c} \\ W &\equiv W_R + iW_I = \frac{\kappa_I}{k_o}\ , \end{aligned}\right\} \qquad (67a,b)$$

where W_R, W_I are the real and imaginary parts of W. If the mean velocity in the wake is assumed to relax rapidly to that of the mean stream U, the wavenumber κ_s of the breathing mode in the wake is equal to ω/U, and this value is used below.

Equation (61) may now be expressed in the form

$$\left.\begin{aligned} S\left(W_R + \frac{1}{1-M}\right) - \frac{\pi}{4} + \arg G &= 2n\pi\ , \\ SW_I &= \ln|G|\ , \end{aligned}\right\} \qquad (68a,b)$$

where

$$G\left(\frac{\delta_o'}{\ell}, \frac{d}{\delta_o}, M, S, W\right) \equiv \frac{-B}{1+A}$$

$$= \frac{\left(\frac{\delta_o'}{\ell}\right)\left(\frac{S}{2\pi}\right)^{\frac{1}{2}} \frac{M^2}{(1+M)} \left(W - \frac{1}{M}\right)\left\{\frac{1}{(1+M)} - \frac{W}{(1+M+\beta^2 W)} - \frac{iW\sqrt{2}}{\left[\beta^2 W - 1 + M\right]^{\frac{1}{2}}}\left(\frac{M}{1+M} - \frac{\quad}{(1+M+\beta^2 W)}\right)\right\}}{\left[\frac{M}{2\pi\beta^3}\left(\frac{\delta_o'}{d}\right)\left\{2 \ln\left(\frac{2\beta^2}{S(d/\ell)}\right) - \frac{1}{\beta}\ln\left[\frac{1+\beta}{1-\beta}\right]\right\} - 1\right]} \quad (69$$

and n is an integer. Equation (68a) is the feedback loop condition that the total phase change around the loop should be a multiple of 2π. The term $\arg(G)$ is the change in phase introduced into the loop during the production of displacement velocity and sound waves; the remaining terms on the left of (68c) account for the effects of propagation discussed above. Equation (68b) determines the growth rate of the instability wave in the boundary layer, which must be such as to ensure that the amplitudes of the sound waves returning to the shock are sufficiently great to sustain the loop.

Equations (68) are to be solved simultaneously with equation (15) relating the displacement velocity wavenumber to the characteristics of the assumed boundary layer velocity profile. In terms of dimensionless variables equation (13) gives for the exponentially growing wave

$$W = \frac{1}{M\hat{V}}\left\{1 - \frac{i(1-\hat{V})\left(\frac{S}{M}\frac{\delta}{\ell}\right)^{\frac{1}{2}}}{\hat{V}^{3/2}\left[1-M^2(1-\hat{V})^2\right]^{\frac{1}{2}}}\right\}, \quad (70)$$

where

$$\hat{V} = V/U$$

is the ratio of the inner velocity of the model boundary layer (Figure 2) to the main flow velocity, and $\delta = \delta_o + \delta_o'$ is the boundary layer thickness. Equation (70) characterises a disturbance which grows exponentially as it propogates at phase velocity V downstream from the shock.,

Numerical Results

Equations (68), (70) are used to determine the dependence of the reduced frequency S of the self-sustained oscillations on the boundary layer thickness ratio δ/ℓ. The calculation is performed for fixed values of δ_o'/ℓ, d/δ_o', M and $\hat{V}$. The values of S and δ/ℓ determined by this procedure are actually discrete, and parameterized by the integer n (S increasing with n) . Their variations with n turns out to be sufficiently smooth, however, for it to be convenient to present the results graphically as continuous plots in the $(S,\delta/\ell)$-plane. The results illustrated in Figures 4,5 correspond to the case $\delta_o'/\ell = 0.0005$ and $M = 0.99$. The range of variation of δ/ℓ is set by the constraints (i) that $\delta \geq \delta_o > o$, the boundary layer thickness upstream of the shock, and (ii) the upper limits of validity of the approximate dispersion equation (70) (wherein the second term in the brace brackets is required to be small relative to unity). The solid and dashed curves are for $d/\delta_o' = 1$ and 10 respectively; $\hat{V}$ is equal to 0.2 in Figure 4 and 0.6 in Figure 5. The lower of these velocities is intended to model the order of magnitude of the phase velocity of the long wavelength disturbances in the boundary layer [24]. Prominent, large amplitude boundary layer structures tend to convect at about 60% of the main flow velocity, and this is the situation modelled in Figure 5. In both cases it is evident that the predicted variation of S with δ/ℓ is not significantly dependent on the transition width d/δ_o'.

In the experiments reported by Burdges [7] the increase in the displacement thickness δ^* of the boundary layer across the shock is typically of order 0.001ℓ. If, following Ffowcs Williams and Purshouse [18], we identify the boundary layer thickness δ of the vortex sheet model with the displacement thickness of the real boundary layer, i.e., we take $\delta_o'/\ell = 0.001$, the results depicted in Figure 6 are obtained when $V = 0.2$. It is of interest to consider this case in relation to the experimental findings of Succi et al [8]. In their experiment $\ell \leq 0.8$ cms and acoustic tone bursts were observed at a blade referenced frequency of 12.6 KHz. This is consistent with the present theory in as much as in Figure 6 the <u>resonance</u> frequency $f_R = Sc/2\pi\ell$ varies between 10.2 - 21.6 KHz when $d/\delta_o' = 1$, and between 8.1 - 18.3 KHz at the higher value $d/\delta_o' = 10$.

§6. Conclusion

A linearized theoretical model of the interaction of a normal shock wave with the boundary layer on a supercritical airfoil has been discussed. The system can oscillate at certain discrete resonance frequencies provided an appropriate feedback-loop condition is fulfilled. This is dependent on the convection velocity and growth rate of disturbances generated in the boundary layer by motion of the shock and on the subsequent interaction of those disturbances with the trailing edge of the airfoil. Resonance frequency predictions are consistent with measured "tone burst" frequencies observed in the study of the noise produced by a transonic propeller.

The analysis of the coupling between the shock and the boundary layer is based on the hypothesis that the motion in the neighbourhood of the root of the shock is steady in a reference frame moving with the root. There are obviously many differences between the model and real flows: shocks are not usually weak nor of infinite extent, are often accompanied by a separation bubble and never enter the boundary layer in one front. In addition, disturbances generated downstream of the shock leak through the boundary layer and can also propagate around the leading edge of the airfoil to modify conditions ahead of the shock. However, the theory provides a useful first step in the understanding of unsteady shock/boundary layer interactions, and would also be applicable to other aero-acoustic problems, such as the interaction of blade-tip shocks of a ducted, transonic rotor with wall boundary layers.

APPENDIX 1

To solve equation (30) subject to conditions (31) (with (31a) extended to the interval $-\infty < X < 0$), (32) and the Kutta condition at the trailing edge. In order that the solution be causal, i.e., that the displacement velocity waves arise as a consequence of the motion of the shock (and are therefore present only for $X > -L$) it is assumed initially that ω is appropriately situated in the upper half-plane to ensure that κ_I, κ_I^* both have positive imaginary parts. The final result is obtained by analytically continuing the solution on to the real ω-axis.

Set

$$\psi_1 = \int_{-\infty}^{\infty} C_\pm(k) e^{i(kX \pm \gamma(k) x_2)} dk \, , \tag{A1}$$

$\pm$ according as $x_2 \gtrless 0$. Wiener-Hopf functional equations for $C_\pm(k)$ are obtained by the usual procedure [22, Chapter 2]. To simplify the argument condition (31a) can be replaced by

$$\frac{\partial\psi}{\partial x_2} = e^{i\sigma X} \, , \quad X < 0 \, , \; x_2 = +0 \, ,$$

$$(\sigma = \beta\nu + KM) \, . \tag{A2}$$

Having obtained the solution $\psi_o(\nu)$, say, in this case, the complete solution is given by

$$\psi_1 = i\delta_\sigma z_\sigma \int_{-\infty}^{\infty} \frac{\nu^2(\omega - U\nu) f(\nu d)}{(\nu-\kappa_I)(\nu-\kappa_I^*)} \psi_o(\nu) e^{i\nu\ell} d\nu \tag{A3}$$

Conditions (A2), (31b) lead to the functional equations

$$\gamma(k) C_+(k) \; - \; \frac{1}{2\pi(k - \sigma + i0)} \; = \; L_1(k) \; , \tag{A4}$$

$$\gamma(k) C_-(k) \; = \; L_2(k) \tag{A5}$$

where L_1, L_2 are regular in $\mathrm{Im}\, k < 0$ and vanish as $k \to -i\infty$.

Conditions (32) respectively yield

$$C_+(k) - C_-(k) - \frac{A}{2\pi i(k-\kappa-i0)} = U_1(k) , \tag{A6}$$

$$\gamma(k)\left[C_+(k) + C_-(k)\right] + \frac{\kappa_s/\nu}{2\pi(k-\kappa_s-i0)} = U_2(k) , \tag{A7}$$

where U_1, U_2 are regular in $\mathrm{Im}\, k > 0$ and vanish as $k \to +i\infty$.

Using the method described by Noble [22] to eliminate L_1, L_2, U_1, U_2 one finds ultimately that

$$\gamma(k)C_+(k) = \frac{1}{4\pi}\left(\frac{1}{k-\sigma+i0} - \frac{\kappa_s/\nu}{k-\kappa_s-i0}\right)$$

$$+ \frac{\sqrt{K-k}}{4\pi}\left[\frac{1}{\sqrt{K-\gamma}(k-\sigma+i0)} - \frac{iA\sqrt{K+\kappa}}{k-\kappa-i0}\right] \tag{A8}$$

The perturbation velocity and pressure will remain finite at the trailing edge (Kutta condition) provided $\gamma(k)C_+(k)$ vanishes at least as fast as $1/k$ as $k \to \infty$. This will be so if

$$A = \frac{-i}{\sqrt{K-\sigma}\,\sqrt{K+k}} \tag{A9}$$

Inserting this into (A8), substituting into (A1) and applying the integral operator (A3) gives the result (35) of the main text.

APPENDIX 2

To solve equation (30) subject to condition (32) (with $B=0$),

$$\frac{\partial\psi_2}{\partial x_2} = 0 \quad \text{for} \quad X < 0 \, , \; x_2 = 0 \, , \tag{A10}$$

and the Kutta condition at the trailing edge.

Set

$$\psi_2 = \psi_2^I + \psi_2^D \, , \tag{A11}$$

where ψ_2^I is given by equation (38). ψ_2^D is the edge diffracted field when displacement velocity waves are ignored. First solve the reduced problem for ψ_{20}^D, say, for which ψ_2^I is replaced by

$$\psi_{20}^I = \cos(\lambda x_2)e^{i\gamma(\lambda)X} \, . \tag{A12}$$

The total diffracted field is then given by

$$\psi_2^D = \int_0^\infty \psi_{20}^D(\lambda)\Psi(\lambda)e^{i\gamma(\lambda)L}d\lambda \, . \tag{A13}$$

Define

$$\psi_{20}^D = \pm \int_{-\infty}^{\infty} C(k)e^{i\{kX\pm\gamma(k)x_2\}}dk \, , \tag{A14}$$

$\pm$ according as $x_2 \gtrless 0$. This form automatically satisfies (32b) when $B=0$. On $x_2 = 0$, $X < 0$ equation (A10) implies that

$$\gamma(k)C(k) = L(k) \, , \tag{A15}$$

where $L(k)$ is regular in $\mathrm{Im}k < 0$ and vanishes as $k \to -i\infty$.

Condition (32a) gives

$$2C(k) + \frac{1}{2\pi i(k-\gamma(\lambda)-i0)} - \frac{A}{2\pi i(k-\kappa-i0)} = U(k) , \tag{A16}$$

where $U(k)$ is regular in $\mathrm{Im}k > 0$ and vanishes as $k \to + i\infty$.

Solving (A15), (A16) by the Wiener-Hopf procedure and choosing the value of A to satisfy the Kutta condition, we find

$$C(k) = \frac{\sqrt{K+\gamma(\lambda)}}{4\pi i\sqrt{K+k}}\left\{\frac{1}{(k-\kappa-i0)} - \frac{1}{(k-\gamma(\kappa)-i0)}\right\} . \tag{A17}$$

Substitution into (A14) and application of the integral operator (A13) yields the result (40) of the main text.

REFERENCES

ORIGINAL PAGE IS OF POOR QUALITY

1. A.R. George, 1978. AIAA J. 15, 707-715. Helicopter Noise: State-of-the-Art.

2. C. Kitaplioglu and A.R. George, 1977. AIAA Paper 77-1360. A study of the far-field sound due to unsteady shocks on rotors.

3. F.H. Schmitz and Y.H. Yu, 1980. Paper No.22 presented at Sixth European Rotorcraft and Powered Lift Aircraft Forum, University of Bristol, September 1980. Transonic Rotor Noise-Theoretical and Experimental Comparisons.

4. M.J. Lighthill, 1952. Proc. R. Soc. Lond. A211, 564-587. On sound generated aerodynamically. I. General Theory.

5. G.L. Loving, 1966. NASA TND 3580. Wind-tunnel-flight correlation of shock induced separated flow.

6. T.C. Adamson and A.F. Messiter 1980. Ann.Rev.Fluid.Mech. 12, 103-138. Analysis of two-dimensional interactions between shock waves and boundary layers.

7. K.P. Burdges, 1981. J. Aircraft 18, 431-437. Experimental measurements of shock/boundary-layer interaction on a supercritical airfoil.

8. G.P. Succi, D.H. Munro and K.U. Ingard, 1982. To appear in J. Sound Vib. Propeller Tone Bursts.

9. C.J. Wood, 1960. J. Roy.Aero. Soc. 64, 683-686. Transonic buffeting.

10. H. Tijdeman, 1975. NLR Report TR-75038U. On the motion of shock waves on an airfoil with oscillating flap in two dimensional transonic flow.

11. M.H. Williams, 1980. AIAA J. 18, 615-624. Unsteady thin airfoil theory for transonic flows with embedded shocks.

12. R. Magnus and H. Yoshihara, 1975. AIAA J. 13, 1622-1628. Unsteady transonic flows over an airfoil.

13. M.E. Goldstein, W. Braun and J.J. Adamczyk, 1977. J. Fluid Mech. 83, 569-604. Unsteady flow in a supersonic cascade with strong in-passage shocks.

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14. H. Ashley and M. Landahl, 1965. Aerodynamics of Wings and Bodies. Reading, MA: Addison-Wesley.

15. M.S. Howe 1981, J. Sound Vib. 75, 239-250. The displacement thickness theory of trailing edge noise.

16 M.S. Howe, 1981. Proc. R. Soc. Lond. A374, 543-568. The role of displacement thickness fluctuations in hydroacoustics, and the jet-drive mechanism of the flue organ pipe.

17. M.S. Howe. 1981. Phil. Trans. Roy. Soc. Lond. A303, 151-180. On the theory of unsteady shearing flow over a slot.

18. J.E. Ffowcs Williams and M. Purshouse, 1981. J. Fluid Mech. 113, 187-220. A vortex sheet modelling of boundary layer noise.

19. M.E. Goldstein, 1981. J. Fluid Mech. 104, 217-246. The coupling between flow instabilities and incident disturbances at a leading edge.

20. M.H. Williams, 1979. AIAA J. 17, 394-397. Linearization of unsteady transonic flows containing shocks.

21. R. Betchov and W.O. Criminale, 1967. Stability of Parallel Flows. New York: Academic Press.

22. B. Noble, 1958. Methods based on the Wiener-Hopf technique. Oxford: Pergamon Press.

23. A. Erdelyi, ed. 1954. Tables of Integral Transforms. (Bateman Manuscript Project). Vol.1. New York: McGraw-Hill.

24. C.J. Gedney, 1979. Rept. No. 84618-1 of Acoustics and Vibration Laboratory, Mass. Inst. Tech. Wall pressure fluctuations during transition on a flat plate.

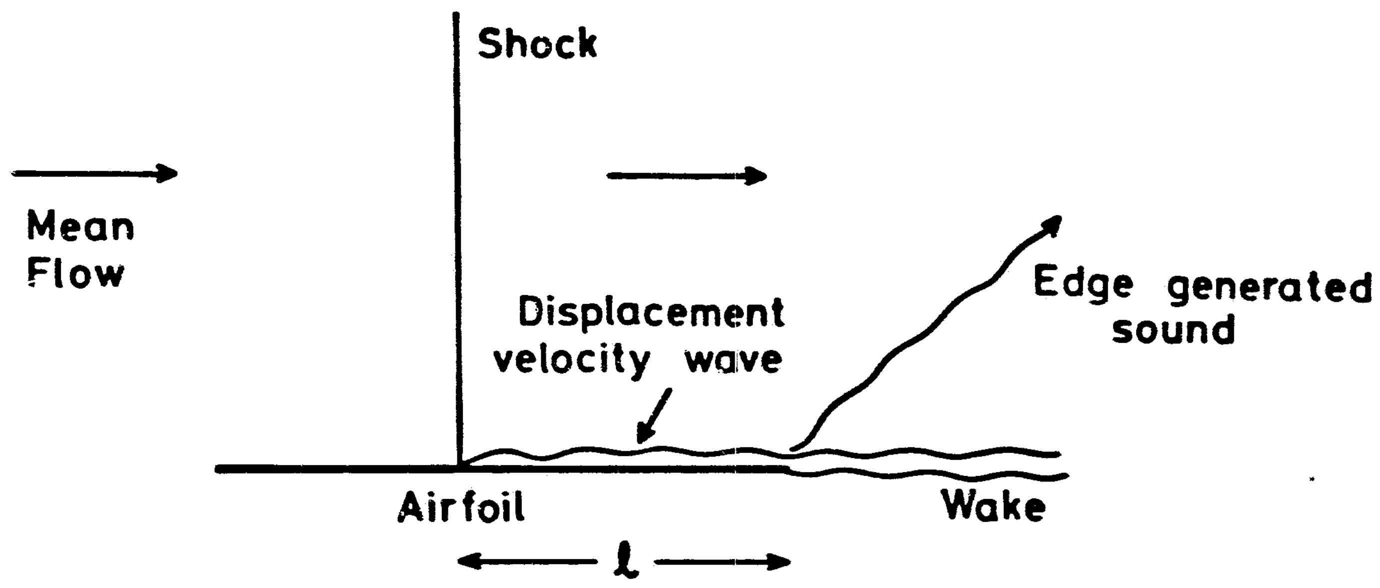

FIG. 1. Schematic illustration of the model problem

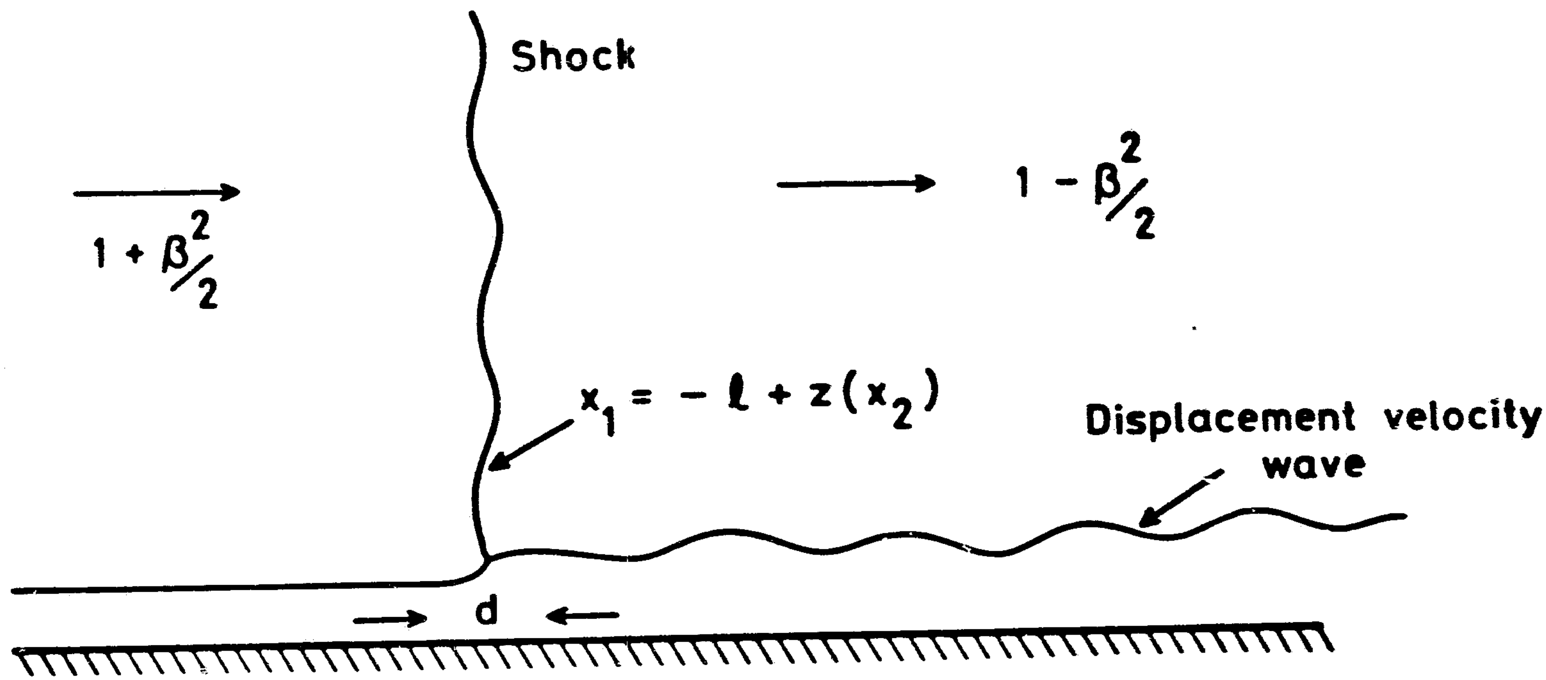

FIG. 2(a). Illustrating the production of displacement velocity waves by the shock wave

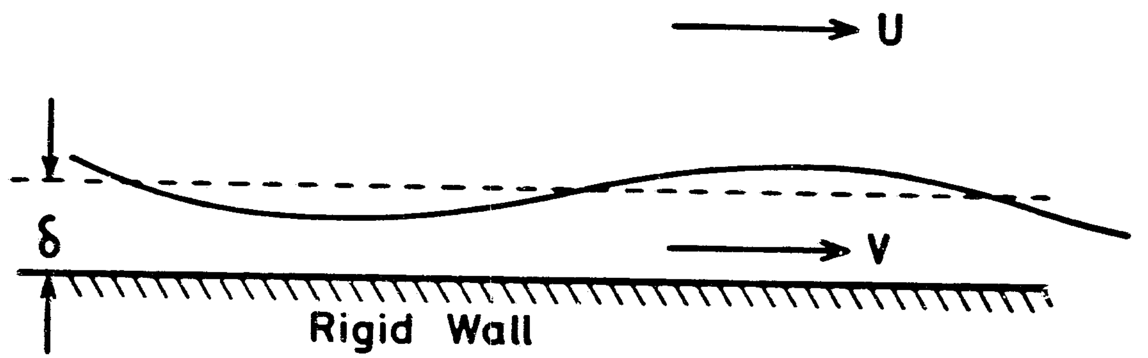

FIG. 2(b). Idealized mean boundary layer velocity profile

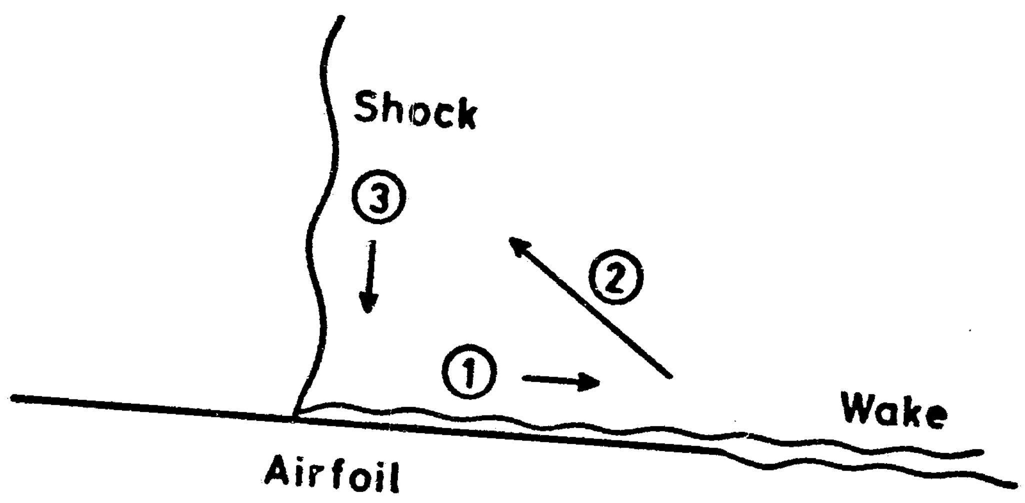

FIG. 3. Principal components of the feed-back loop

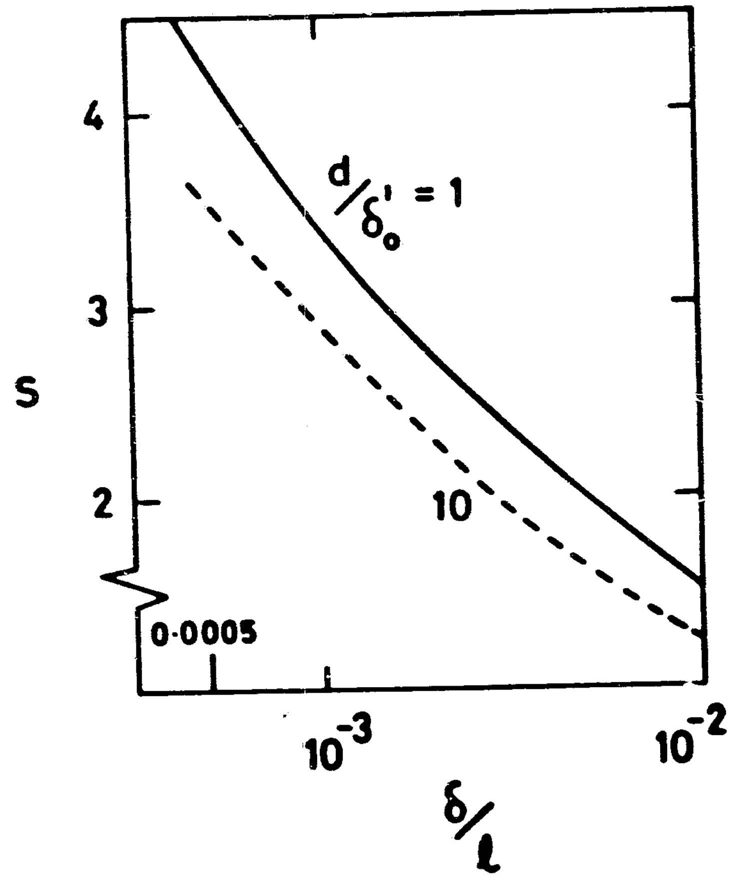

FIG. 4. Variation of the reduced frequency S with the boundary layer thickness downstream of the shock wave, for $\delta_o/\ell = 0.0005$, $M = 0.99$, $V = 0.2U$.

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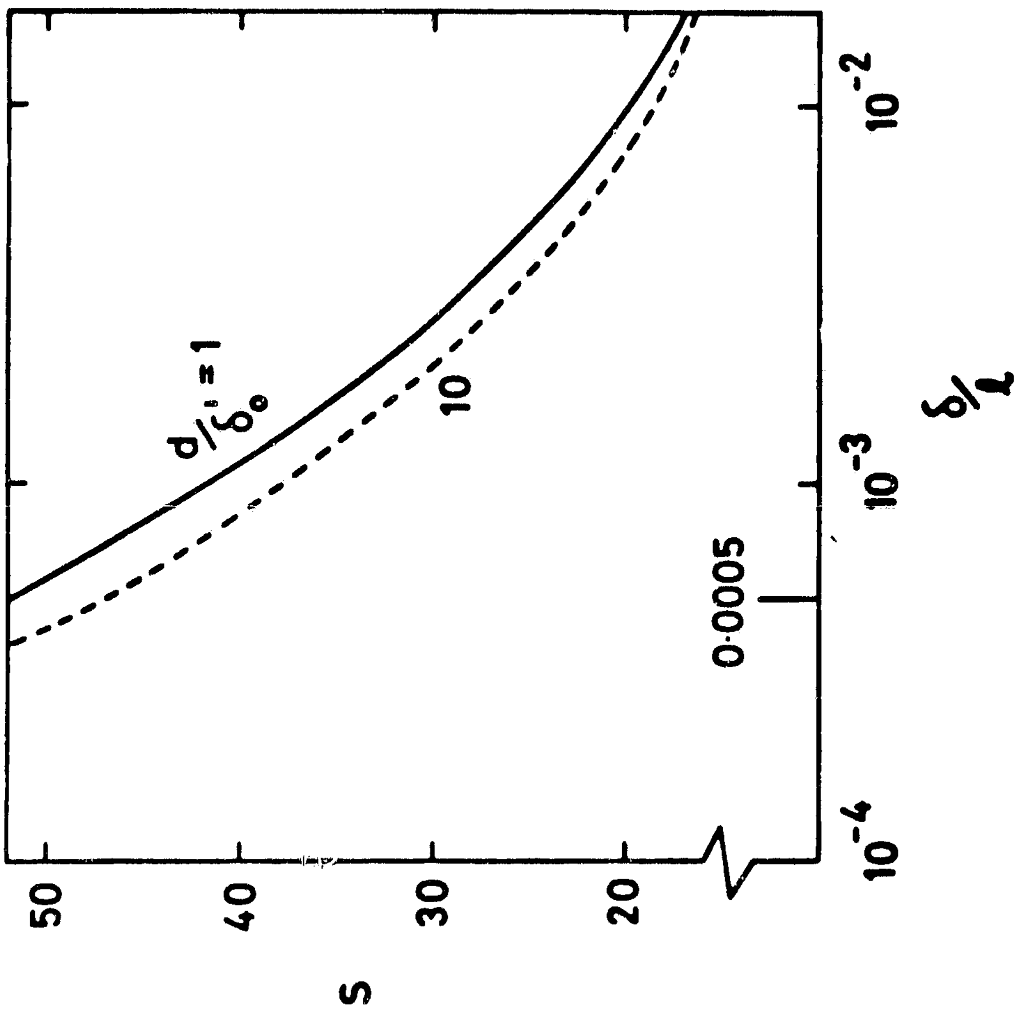

FIG. 5. As for Figure 4 with V = 0.6U

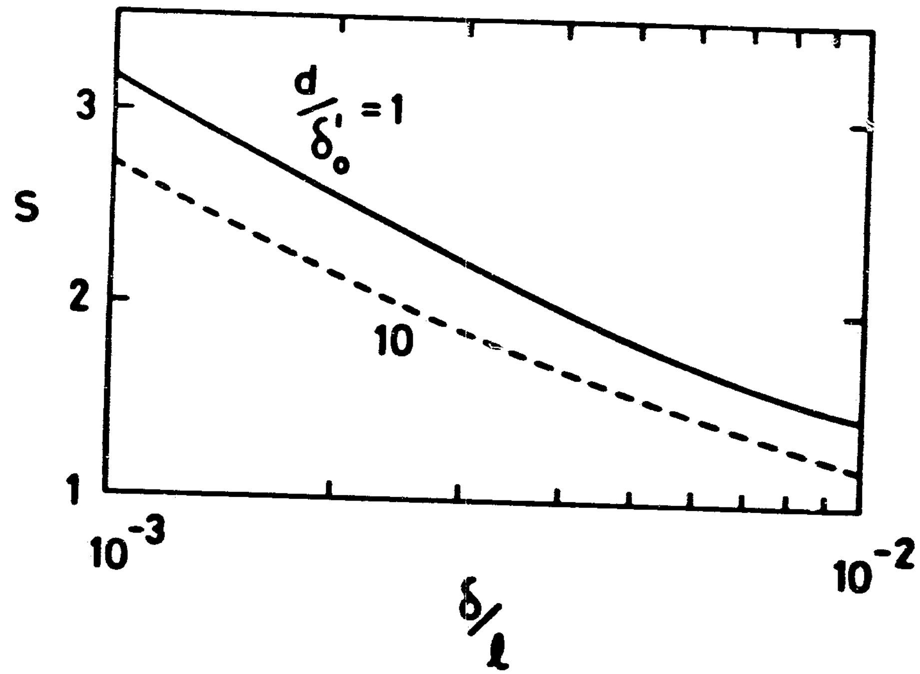

FIG. 6. As for Figure 4 with $\delta_o'/\ell = 0.001$