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Effect of Nonideal Square-Law Detection on Static Calibration in Noise-Injection Radiometers

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SUMMARY

In this paper, the effect of nonideal square-law detection on static calibration for a class of Dicke radiometers is examined. It is shown that fourth-order curvature in the detection characteristic adds a nonlinear term to the linear calibration relationship normally ascribed to noise-injection, balanced Dicke radiometers. The minimum error, based on an optimum straight-line fit to the calibration curve, is derived in terms of the power series coefficients describing the input-output characteristic of the detector. These coefficients can be determined by simple measurements, and detection nonlinearity is, therefore, quantitatively related to radiometric measurement error.

INTRODUCTION

A microwave radiometer is a receiver specifically designed to measure microwave-radiated power P and thereby determine the radiometric brightness temperature T_b, defined by

$$T_b = \frac{P}{kB} \tag{1}$$

where k is Boltzmann's constant, and B is the bandwidth. Radiometers have been implemented in a variety of configurations (ref. 1) and are still in an evolutionary phase in which both theoretical and hardware innovations are being incorporated. This work concerns an error-producing mechanism in pulsed-noise, balanced Dicke radiometers and is a theoretical examination of the measurement error resulting from nonideal square-law detection in such a system.

The square-law detector (S.L.D.) is present in all noise-injection radiometers and can be realized in many forms utilizing a variety of nonlinear elements. In this analysis, the S.L.D. was modeled as a memoryless, nonlinear "black box" having an input-output relationship given by

$$V_o = k_1'V_{in} + k_2'V_{in}^2 + k_3'V_{in}^3 + k_4'V_{in}^4 \tag{2}$$

where V_o and V_{in} are the detector output and input random-noise voltages, respectively, and k_i' represents the coefficients in the power series. This investigation was limited to the important case where the S.L.D. is very nearly square-law because of the values of the coefficients in equation (2) and a restriction on $|V_{in}|$. An earlier investigation of this subject (ref. 2) was part of a broader discussion of a particular radiometer and was highly condensed and less general than this analysis.

SYMBOLS

B	bandwidth
dBR	$= 10 \log_{10} \frac{P_x}{P_r}$
$E\{ \}$	expected value operator
k	Boltzmann's constant
k_2	coefficient of second-order term in power series relating V_o and P_{in}
k_4	coefficient of fourth-order term in power series relating V_o and P_{in}
k_1'	coefficient of first-order term in power series relating V_o and V_{in}
k_2'	coefficient of second-order term in power series relating V_o and V_{in}
k_3'	coefficient of third-order term in power series relating V_o and V_{in}
k_4'	coefficient of fourth-order term in power series relating V_o and V_{in}
L	loss factor
N	number of decades on logarithmic scale
P	power
P_a	antenna (input) power
ΔP_a	error associated with straight-line approximation to $P_a(\beta)$
$\hat{P}_a$	approximation to P_a
P_{al}	optimum linear approximation to $\hat{P}_a(\beta)$
P_{in}	detector input power, radiometer mode
P_n	injected-noise source power
P_r	reference power at detector input
P_s	system noise power at detector input
P_x	detector input power, sinusoidal excitation
R	detector input resistance
T	Dicke-cycle period
T_a	antenna (input) temperature
ΔT_a	temperature error corresponding to ΔP_a

T_n	injected-noise source temperature
T_r	reference termination temperature
T_s	system equivalent input noise temperature
t	time
V_a	antenna (input) random-noise voltage
V_{in}	detector (input) random-noise voltage
V_n	injected random-noise voltage
V_o	detector output random-noise voltage
V_r	reference-noise voltage
V_s	system noise voltage
V_{o1}	detector output during first interval of Dicke cycle
V_{o2}	detector output during second interval of Dicke cycle
V_{o3}	detector output during third interval of Dicke cycle
V_{or}	value of V_{ox} designated as "reference level"
V_{ox}	value of detector output voltage, sinusoidal input
V_x	magnitude of sinusoidal test signal
z	logarithmic transformation of V_{ox}, referenced to V_{or}
Δz	deviation from straight line
α	ratio of k_4 to k_2
β	injected-noise duty cycle
β_o	value of β for which $\hat{P}_a = 0$
δ_1	normalized departure from linearity, reference-noise excitation
δ_2	normalized departure from linearity, sinusoidal excitation
ϵ_i	power-balance error on the ith Dicke cycle
σ_a	standard deviation of antenna noise voltage
σ_n	standard deviation of injected-noise voltage
σ_s	standard deviation of system noise voltage

Abbreviations:

F.S. full scale

max maximum

min minimum

S.L.D. square-law detector

Mathematical notation:

$\doteq$ very nearly equal to

THEORY

Null-Balance Static-Error Derivation

In this section, an expression is derived to relate the input (antenna) power and the pulsed-noise duty cycle in a pulsed noise-injection, balanced Dicke radiometer, which is shown in simplified form in figure 1. The system is a closed-loop, nulling servomechanism, which adds noise to the antenna noise such that their sum power, as measured over the antenna half-Dicke cycle, equals the power measured over the reference half-Dicke cycle. This analysis deals specifically with the case of a single noise pulse per Dicke cycle; however, the results are applicable to a pulse-rate modulation format. Furthermore, the target is assumed fixed, such that the input noise process can be considered as stationary and ergodic.

Detector waveforms typifying the radiometer model of figure 1 in an equilibrium state are shown in figure 2. The system is designed to force $\hat{\varepsilon}_i$, the finite-time average value of a sequence of random variables, to approach zero. The random variable ε_i is defined as follows:

$$\varepsilon_i \triangleq \frac{2}{T}\int_{(i-1)T}^{(i-1)T+T/2} V_o(t)\,dt - \frac{2}{T}\int_{(i-1)T+T/2}^{iT} V_o(t)\,dt \tag{3}$$

This is accomplished by the error-nulling nature of the servomechanism, which drives the duty cycle of the injected noise β to be that value necessary to make $\hat{\varepsilon}_i$ approach zero. Equation (3) can be rewritten so that each integral is restricted to a time interval in which the detector input consists of only one combination of the four relevant noise processes: input noise $V_a(t)$, injected noise $V_n(t)$, system noise $V_s(t)$, and reference noise $V_r(t)$. Thus,

$$\varepsilon_i = \frac{2}{T}\int_{(i-1)T}^{(i-1)T+\beta T/2} V_{o1}(t)\,dt + \frac{2}{T}\int_{(i-1)T+\beta T/2}^{(i-1/2)T} V_{o2}(t)\,dt - \frac{2}{T}\int_{(i-1/2)T}^{iT} V_{o3}(t)\,dt \tag{4}$$

where the subscripts on $V_o(t)$ distinguish the three distinct intervals of a Dicke cycle, as depicted in figure (2). To the extent that the servo system drives $\hat{\varepsilon}_i$ to zero and, as previously stated, the input noise processes are all stationary such that time and ensemble averages can be interchanged, the "power-balance condition" requires that

$$E\{\varepsilon_i\} \equiv 0 \tag{5}$$

Applying this constraint to equation (4) and integrating yields

$$\beta\ E\{V_{o1}(t)\} + (1 - \beta)\ E\{V_{o2}(t)\} - E\{V_{o3}(t)\} = 0 \tag{6}$$

During the first time interval, the detector input consists of the sum of the antenna noise, injected noise, and system noise, and the first term of equation (6) may be written as

$$\beta\ E\{V_{o1}\} = \beta\ E\{k_1'(V_a + V_n + V_s) + k_2'(V_a + V_n + V_s)^2 + k_3'(V_a + V_n + V_s)^3 + k_4'(V_a + V_n + V_s)^4\} \tag{7}$$

following equation (2). The random processes V_a, V_n, and V_s are assumed to be Gaussian, zero-mean, and statistically independent. Further reduction of equation (7) is based on the application of

$$E\{V^j\} = \begin{cases} 1 \cdot 3 \cdot 5 \cdot \ldots (j - 1)\sigma^j & \text{(for } j \text{ even)} \\ 0 & \text{(for } j \text{ odd)} \end{cases} \tag{8}$$

from reference 3, and the fact that $E\{pq\} = E\{p\} \cdot E\{q\}$ if p and q are independent random variables. Expanding equation (7) and applying these results leads to

$$\beta\ E\{V_{o1}\} = \beta\big(k_2'\sigma_a^2 + k_2'\sigma_n^2 + k_2'\sigma_s^2 + 3k_4'\sigma_a^4 + 3k_4'\sigma_n^4 + 3k_4'\sigma_s^4 + 6k_4'\sigma_a^2\sigma_n^2 + 6k_4'\sigma_a^2\sigma_s^2 + 6k_4'\sigma_n^2\sigma_s^2\big) \tag{9}$$

This can be expressed as

$$\beta\ E\{V_{o1}\} = \beta\big(k_2P_a + k_2P_n + k_2P_s + 3k_4P_a^2 + 3k_4P_n^2 + 3k_4P_s^2 + 6k_4P_aP_n + 6k_4P_aP_s + 6k_4P_nP_s\big) \tag{10}$$

since $k_2' \triangleq k_2/R$ and $k_4' \triangleq k_4/R^2$, where R is the input resistance of the detector. The application of similar arguments to the remaining two expected-value operations yields

$$E\{V_{o2}\} = k_2P_a + k_2P_s + 3k_4P_a^2 + 3k_4P_s^2 + 6k_4P_aP_s \tag{11}$$

$$E\{V_{o3}\} = k_2P_r + k_2P_s + 3k_4P_r^2 + 3k_4P_s^2 + 6k_4P_rP_s \tag{12}$$

and the power-balance equation (eq. (4)) can be written as

$$\begin{aligned} &\beta\left(k_2P_a + k_2P_n + k_2P_s + 3k_4P_a^2 + 3k_4P_n^2 + 3k_4P_s^2 + 6k_4P_aP_n + 6k_4P_aP_s\right. \\ &\left.+ 6k_4P_nP_s\right) + (1 - \beta)\left(k_2P_a + k_2P_s + 3k_4P_a^2 + 3k_4P_s^2 + 6k_4P_aP_s\right) \\ &- k_2P_r - k_2P_s - 3k_4P_r^2 - 3k_4P_s^2 - 6k_4P_rP_s = 0 \end{aligned} \tag{13}$$

Upon normalizing by k_2 and defining $\alpha = k_4/k_2$, equation (13) reduces to

$$\begin{aligned} &3\alpha P_a^2 + (1 + 6\alpha\beta P_n + 6\alpha P_s)P_a + \beta P_n + 3\alpha\beta P_n^2 + 6\alpha\beta P_nP_s \\ &- P_r - 3\alpha P_r^2 - 6\alpha P_rP_s = 0 \end{aligned} \tag{14}$$

This quadratic equation in P_a defines the relationship between P_a and β resulting from the "power-balance" condition. In a perfect square-law detector, α equals zero and equation (14) simplifies to a linear relationship between P_a and β

$$P_a = P_r - \beta P_n \tag{15}$$

The major question to be answered by this paper is: What error in measuring P_a results when a linear relationship between P_a and β is assumed, as in equation (15), when in fact, α is actually nonzero, albeit very small, and equation (14) more accurately applies to the situation? This is not a trivial question, as the usual assumption in the radiometry field is that the detector is ideal, and therefore, a linear calibration relationship is justified (ref. 1, p. 402).

A representation for $P_a(\beta)$ is sought which explicitly expresses any nonlinearity between P_a and β when α is nonzero. The most direct approach to obtaining such an expression for $P_a(\beta)$ is to solve equation (14) for P_a with the understanding that the solution will be a function of β. Application of the quadratic formula to equation (14) yields (see appendix A)

$$P_a = \frac{1}{6\alpha}\left\{-(1 + 6\alpha\beta P_n + 6\alpha P_s) + (1 + 6\alpha P_r + 6\alpha P_s)\left[1 + \frac{36\alpha^2\beta(\beta - 1)P_n^2}{(1 + 6\alpha P_r + 6\alpha P_s)^2}\right]^{1/2}\right\} \qquad (16)$$

which is an exact solution only if β equals zero or unity, in which cases the radical reduces to unity, and equation (16) yields exactly the same value for P_a as equation (15). The range of β is dependent on the value of P_n. For $\alpha = 0$, the range of β is

$$0 < \beta < P_r/P_n \qquad (\text{for } P_r > P_a > 0) \qquad (17)$$

A simplified form of equation (16) that is good for all values within the range of β can be obtained by approximation. Since α has been restricted to very small values, it is justifiable to approximate equation (16) as

$$\hat{P}_a = \frac{1}{6\alpha}\left\{-(1 + 6\alpha\beta P_n + 6\alpha P_s) + (1 + 6\alpha P_r + 6\alpha P_s)\left[1 + \frac{18\alpha^2\beta(\beta - 1)P_n^2}{(1 + 6\alpha P_r + 6\alpha P_s)^2}\right]\right\} \qquad (18)$$

which simplifies to

$$\hat{P}_a = P_r - \beta\left(P_n + \frac{3\alpha P_n^2}{1 + 6\alpha P_r + 6\alpha P_s}\right) + \frac{3\alpha P_n^2}{1 + 6\alpha P_r + 6\alpha P_s}\,\beta^2 \qquad (19)$$

Equation (19) shows that the fourth-order term in equation (2) has resulted in a slope change relative to the ideal situation where $\alpha = 0$ given by equation (15). This can be accommodated in the calibration process; however, nonzero values of α also produced a nonlinear β^2 term which cannot be removed by a straight-line calibration.

Some feeling for the physical significance of these results can be obtained by examining the behavior of equation (19) for three ranges of P_n: $P_n < P_r$, $P_n = P_r$, and $P_n > P_r$, with σ either zero or nonzero. The first case, $P_n < P_r$, has limited practical importance in radiometry, but it is included here for completeness.

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Figure 3 shows $\hat{P}_a$ versus β for the above-mentioned conditions. The following points should be noted:

1. The linear ($\alpha = 0$) and nonlinear ($\alpha > 0$ and $\alpha < 0$) curves intersect at $\beta = 0$ and $\beta = 1$. (See eq. (18).)

2. The practical range of β is bounded by zero and the lesser of approximately P_r/P_n or 1, since P_a cannot be negative, and β cannot exceed unity.

3. In the practical operating region where $P_n > P_r$, increasingly positive values of α make β_o, the value of β for which $\hat{P}_a = 0$, increasingly less than P_r/P_n, and increasingly negative values of α make β_o increasingly greater than P_r/P_n. (See eq. (C7).)

4. Increasing α increases the spread between the linear and nonlinear cases.

Now that an expression is available which explicitly expresses $\hat{P}_a$ as a function of β, it is possible to determine the error arising from a straight-line calibration. The "best" straight-line fit will be defined as that which results in the smallest peak error between itself and equation (19) over the region from $\beta = 0$ to $\beta = \beta_o$, which corresponds to a $\hat{P}_a$ range of P_r to zero. This optimum linear fit to $\hat{P}_a(\beta)$ is designated as P_{a1}, and the quadratic error function

$$\Delta P_a \triangleq \hat{P}_a(\beta) - P_{a1} \tag{20}$$

is to be minimized in the sense stated above. It is shown in appendix B that this optimum linear fit to $\hat{P}_a(\beta)$ results in a peak value for ΔP_a that is one-eighth of the value attained by the β^2 term in equation (19) when β attains its maximum value of β_o, at which point $\hat{P}_a$ equals zero. It is necessary to determine β_o in order to evaluate ΔP_a. This is accomplished in appendix C by solving for the root of equation (19), which is shown in equation (C7) to be

$$\beta_o \doteq \frac{P_r}{P_n}\left[1 - \frac{3\alpha(P_n - P_r)}{1 + 3\alpha P_n + 6\alpha P_r}\right]$$

The minimum value of ΔP_a as described above and in appendix B can now be evaluated using equations (19), (B14), and (C7), and is

$$\Delta P_{a,\text{peak}}\Big|_{\min} \doteq \frac{3\alpha P_r^2}{8(1 + 6\alpha P_r + 6\alpha P_s)}\left[1 - \frac{3\alpha(P_n - P_r)}{1 + 3\alpha P_n + 6\alpha P_s}\right]^2 \tag{21}$$

which essentially equals $3\alpha P_r^2/8$ when the degree of nonlinearity is slight. This is the principal result of this section and establishes a lower bound on the radiometric measurement error caused by nonzero values of α. It should be emphasized that the actual error based on the usual "two-point" calibration technique will almost certainly exceed this minimum value.

Detector Nonlinearity Characterization

Evaluation of equation (21) requires a knowledge of P_r and α; P_r is a known or easily measured parameter. This section discusses two methods for determining α. The first method is an "in-circuit" method, suitable for use with an existing radiometer; the second method is based on sinusoidal excitation.

In the in-circuit method, the Dicke switch is "locked" in the reference mode, and a calibrated step attenuator of loss factor L is placed at the input of the detector. The input-output characteristic is then determined by using the random signals $V_r + V_s$ as excitation. The average value of the detector output voltage is given by equation (12) as

$$V_{o3} \triangleq E\{V_{o3}(t)\} = k_2 L(P_r + P_s) + 3k_4 L^2 (P_r + P_s)^2 \tag{22}$$

In figure 4, V_{o3} is plotted as a function of the input power $L(P_r + P_s)$. The normalized departure from linearity is defined as the ratio of the deviation of V_{o3} from the linear term $k_2 L(P_r + P_s)|_{L=1}$ to the linear term, and can be expressed as

$$\delta_1 \triangleq \frac{\Delta V_{o3}}{k_2(P_r + P_s)} = \frac{3k_4(P_r + P_s)^2}{k_2(P_r + P_s)} = 3\alpha(P_r + P_s) \tag{23}$$

from which α is found to be

$$\alpha = \frac{\delta_1}{3(P_r + P_s)} \tag{24}$$

Thus, α can be defined using the measured quantity δ_1 and the known or easily measured total input power in the reference mode.

Determination of α using a sinusoidal input signal follows the same general procedure. The mean output voltage of the detector, described by equation (2), when driven by $V_{in} = V_x \sin \omega t$ (ω is the radian frequency), can be found to be

$$\bar{V}_{ox} = \frac{k_2' V_x^2}{2} + \frac{3k_4' V_x^4}{8} \tag{25}$$

which can be written in terms of power, since $k_2' \triangleq k_2/R$, $k_4' \triangleq k_4/R^2$, and $P_x = V_x^2/2R$. Thus,

$$\bar{V}_{ox} = k_2 P_x + \frac{3k_4 P_x^2}{2} \tag{26}$$

from which the normalized departure from linearity (see fig. 5) can be written as

$$\delta_2 = \frac{3k_4 P_x^2}{2k_2 P_x} = \frac{3\alpha P_x}{2} \tag{27}$$

and α found to be

$$\alpha = \frac{2\delta_2}{3P_r} \tag{28}$$

It is now possible to numerically evaluate ΔP_a, as given by equation (21), for some typical radiometer parameter values. Suppose, for example, that $T_r = T_s = 308$ K, that equation (28) is evaluated at a sinusoidal power level of $P_x = P_s + P_r$, and that δ_2 is 0.01, which approaches the limits of analog observability. Then, from the approximation to equation (21),

$$\Delta P_{a,\text{peak}}\Big|_{\min} \doteq \frac{3}{8}\left[\frac{2(0.01)}{3(P_r + P_s)}\right]P_r^2 = 0.00125P_r \tag{29}$$

The ΔT_a associated with this ΔP_a is, from equations (1) and (29),

$$\Delta T_a\Big|_{\min} = 0.00125T_r = 0.385\ \text{K} \tag{30}$$

and it is clear that the detector must be extremely true square-law if radiometric errors of less than 1 K are demanded and a linear relationship between P_a and β is assumed. Alternately, since the effect is predictable, it could be included by using a nonlinear calibration relationship such as equation (19). This approach becomes more attractive if the possibility exists that other nonideal aspects of the system hardware could produce nonlinearity between P_a and β. Admitting the presence of a β^2 term in $\hat{P}_a(\beta)$ could significantly improve calibration accuracy.

As a final consideration, the consequences of following the usual laboratory procedure of logarithmically plotting the data used to define α (ref. 4) will be examined. The resulting compression of the observed variable significantly masks nonlinearity. To show this, equation (26) is written such that the log reference is the value of $\bar{V}_{ox}$ when $P_x \triangleq P_r$, or

$$\frac{\bar{V}_{ox}}{\bar{V}_{or}} = \frac{k_2 P_x + (3/2)k_4 P_x^2}{k_2 P_r + (3/2)k_4 P_r^2} = \frac{(P_x/P_r)\left[1 + (3\alpha P_x/2)\right]}{1 + (3\alpha P_r/2)} \tag{31}$$

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Taking $10 \log_{10}$ of both sides of equation (31) yields

$$z \triangleq 10 \log_{10} \frac{\bar{V}_{ox}}{\bar{V}_{or}} = 10 \log_{10} \frac{P_x}{P_r} + 10 \log_{10}\left(1 + \frac{3\alpha P_x}{2}\right) - 10 \log_{10}\left(1 + \frac{3\alpha P_r}{2}\right) \tag{32}$$

or

$$z = \mathrm{dBR} - 10 \log_{10}\left(1 + \frac{3\alpha P_r}{2}\right) + 10 \log_{10}\left(1 + \frac{3\alpha P_x}{2}\right) \tag{33}$$

which is sketched in figure 6. Now, the coefficient ratio α is evaluated in terms of the transformed variable z. From figure 6, the normalized deviation from a straight line at full scale (F.S.) is

$$\frac{\Delta z}{\mathrm{F.S.}} \triangleq \left.\frac{10 \log_{10}[1 + (3\alpha P_x/2)]}{10N}\right|_{P_x = P_r} = \frac{1}{N} \log_{10}\left(1 + \frac{3\alpha P_r}{2}\right) \tag{34}$$

when N is the number of decades on the vertical scale. Writing a two-term Maclaurin's series approximation for the right-hand side of equation (34) in terms of αP_r yields

$$\frac{\Delta z}{\mathrm{F.S.}} \doteq \frac{3 \log_{10} e}{2N} \alpha P_r = \frac{3\alpha P_r}{2N \ln 10} \tag{35}$$

For comparison purposes, consider the same data plotted on a linear scale. When $P_x = P_r$, the normalized deviation from a straight line was shown to be

$$\delta_2 = \frac{3\alpha P_r}{2} \tag{36}$$

Comparison of equations (35) and (36) reveals that logarithmic plotting has reduced the observable deviation from a straight line by the factor $N \ln 10$. In other words, the radiometer error associated with a given $\Delta z/\mathrm{F.S.}$ has increased by the factor $N \ln 10$. For the example cited earlier, based on a δ_2 of 0.01, the ΔT_a of 0.385 K would increase to 2.66 K! Thus, logarithmic plotting should be avoided when attempting to define α.

CONCLUSIONS

The results of the examination of the effect of nonideal square-law detection on the static calibration of noise-injection, balanced Dicke radiometers may be summarized as follows:

1. Fourth-order curvature in the square-law detector was shown to produce a nonlinear relationship between the pulsed duty cycle β and the measured antenna temperature.

2. The minimum peak-measurement error with respect to an optimum straight-line fit was related to a ratio of coefficients in a power series describing the input-output characteristic of the square-law detector.

3. The coefficient ratio was shown to be obtainable from simple laboratory measurements.

4. The minimum calibration nonlinearity resulting from square-law detection which deviates from ideal by only 1 percent was shown to be on the order of 0.5 K.

5. The accuracy penalty incurred by the use of logarithmically plotted data to find the coefficient ratio was assessed and shown to be significant.

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APPENDIX A

DERIVATION OF NONLINEAR CALIBRATION EQUATION

This appendix includes the steps followed in the derivation of equation (19), for which the quadratic solution roots are, by inspection,

$$P_a = \frac{1}{6\alpha}\left[-(1 + 6\alpha\beta P_n + 6\alpha P_s) \pm (1 + 36\alpha^2\beta^2 P_n^2 + 36\alpha^2 P_s^2 + 12\alpha\beta P_n + 12\alpha P_s\right.$$

$$+ 72\alpha^2\beta P_n P_s - 12\alpha\beta P_n - 36\alpha^2\beta P_n^2 - 72\alpha^2\beta P_n P_s + 12\alpha P_r$$

$$\left. + 36\alpha^2 P_r^2 + 72\alpha^2 P_r P_s)^{1/2}\right] \quad \text{(A1)}$$

After simplification and regrouping, equation (A1) may be written as

$$P_a = \frac{1}{6\alpha}\left\{-(1 + 6\alpha\beta P_n + 6\alpha P_s) + \left[1 + 12\alpha(P_r + P_s) + 36\alpha^2(P_r + P_s)^2\right.\right.$$

$$\left.\left. + 36\alpha^2\beta(\beta - 1)P_n^2\right]^{1/2}\right\} \quad \text{(A2)}$$

and simplified to

$$P_a = \frac{1}{6\alpha}\left\{-(1 + 6\alpha\beta P_n + 6\alpha P_s) \pm (1 + 6\alpha P_r + 6\alpha P_s)\left[1 + \frac{36\alpha^2\beta(\beta - 1)P_n^2}{(1 + 6\alpha P_r + 6\alpha P_s)^2}\right]^{1/2}\right\}$$

Because of the stipulated smallness of α (corresponding to only slight deviation from true square-law in the detector), the term in brackets is approximated by the first two terms of a Maclaurin's series to yield

$$\hat{P}_a = \frac{1}{6\alpha}\left\{-(1 + 6\alpha\beta P_n + 6\alpha P_s) \pm (1 + 6\alpha P_r + 6\alpha P_s)\left[1 + \frac{18\alpha^2\beta(\beta - 1)P_n^2}{(1 + 6\alpha P_r + 6\alpha P_s)^2}\right]\right\} \quad \text{(A3)}$$

Only the positive square root is significant and produces realistic values for $\hat{P}_a$. Thus, equation (A3) simplifies to

$$\hat{P}_a = P_r - \beta P_n\left(\frac{1 + 3\alpha P_n + 6\alpha P_r + 6\alpha P_s}{1 + 6\alpha P_r + 6\alpha P_s}\right) + \frac{3\alpha P_n^2}{1 + 6\alpha P_r + 6\alpha P_s}\beta^2 \quad \text{(A4)}$$

which is equation (19) in the text in a slightly different form.

APPENDIX A

An approximation differing only slightly from equation (A4) can be obtained by writing a Maclaurin's series for equation (14) with the coefficients obtained by implicitly differentiating equation (14) and evaluating the derivatives at $\beta = 0$. This procedure yields an approximation for P_a that differs from equation (A4) only in the coefficient of the β^2 term, which is

$$\frac{1}{2!}\left.\frac{d^2P_a}{d\beta^2}\right|_{\beta=0} = \frac{3\alpha P_n^2(1 + 3\alpha P_n + 6\alpha P_r + 6\alpha P_s)(1 - 3\alpha P_n + 6\alpha P_r + 6\alpha P_s)}{1 + 6\alpha P_r + 6\alpha P_s} \tag{A5}$$

and which can be expressed as

$$\frac{1}{2!}\left.\frac{d^2P_a}{d\beta^2}\right|_{\beta=0} = \left[1 + \left(\frac{3\alpha P_n}{1 + 6\alpha P_r + 6\alpha P_s}\right)^2\right]\frac{3\alpha P_n^2}{1 + 6\alpha P_r + 6\alpha P_s} \tag{A6}$$

to emphasize the extreme closeness of the two approximations.

APPENDIX B

DETERMINATION OF BEST STRAIGHT-LINE FIT TO A QUADRATIC FUNCTION ON A FINITE INTERVAL[1]

For a quadratic equation

$$y = ax^2 + bx + c \tag{B1}$$

the conditions are derived that cause the linear approximation to y,

$$Y_L \triangleq Ax + B \tag{B2}$$

to be the best fit to equation (B1) in the sense that

$$\Delta y\big|_{max} = \big|y - Y_L\big|_{max} \tag{B3}$$

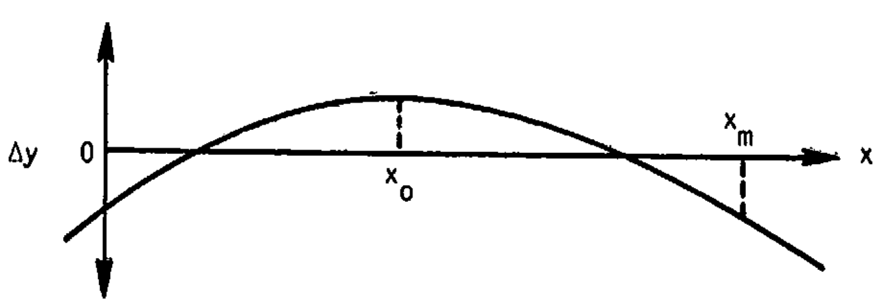
is minimized over the interval from zero to x_m. The quadratic equation

$$\Delta y = y - Y_L = ax^2 + (b - A)x + (c - B) \tag{B4}$$

can have a maximum of two real roots for which $\Delta y = 0$. Thus, over the interval from zero to x_m, the roots are positioned so that

$$\Delta y(0) = \Delta y(x_m) = -\Delta y(x_o) \tag{B5}$$

as shown in the sketch below.



[1]This development could be presented in terms of Tchebychev polynomials by making a suitable change of variables.

APPENDIX B

Applying the first set of equalities in equation (B5) to equation (B4) yields

$$c - B = ax_m^2 + (b - A)x_m + c - B \tag{B6}$$

which reduces to

$$A = b + ax_m \tag{B7}$$

and defines the slope of Y_L.

In order to find the intercept of Y_L, it is necessary to first find x_o, the flex point over the interval from zero to x_m. From equation (B4), the derivative

$$\frac{d}{dx}\Delta y = 2ax + (b - A) \tag{B8}$$

must equal zero at x_o. Thus,

$$x_o = \frac{A - b}{2a} \tag{B9}$$

Combining this with the result stated in equation (B7) yields

$$x_o = \frac{x_m}{2} \tag{B10}$$

Thus, the value of Δy at x_o is

$$\Delta y(x_o) = a\frac{x_m^2}{4} + (b - b - ax_m)\frac{x_m}{2} + (c - B) \tag{B11}$$

which by specification of equation (B5) must equal $-y(0)$, or $-(c - B)$. Solving for B yields

$$B = c - \frac{ax_m^2}{8} \tag{B12}$$

APPENDIX B

The minimum peak deviation from a straight line defined by equation (B2) can be found by evaluating equation (B4) at either zero, x_o, or x_m. Evaluating at zero yields

$$\Delta y_{min} \triangleq |\Delta y(0)| = c - B \tag{B13}$$

Combining this with equation (B12) gives the desired result,

$$\Delta y_{min} = \frac{ax_m^2}{8} \tag{B14}$$

APPENDIX C

DETERMINATION OF β_o

In this appendix, the value of β for which $\hat{P}_a$, equation (19), is zero is determined. Equation (19) is first put in standard quadratic form,

$$\beta_o^2 - \left(\frac{1 + 3\alpha P_n + 6\alpha P_r + 6\alpha P_s}{3\alpha P_n}\right)\beta_o + \left(\frac{1 + 6\alpha P_r + 6\alpha P_s}{3\alpha P_n^2}\right)P_r = 0 \qquad \text{(C1)}$$

and the quadratic formula applied to yield

$$\beta_o = \frac{1}{2}\left\{\frac{1 + 3\alpha P_n + 6\alpha P_r + 6\alpha P_s}{3\alpha P_n} \pm \left[\left(\frac{1 + 3\alpha P_n + 6\alpha P_r + 6\alpha P_s}{3\alpha P_n}\right)^2 - \frac{12\alpha(1 + 6\alpha P_r + 6\alpha P_s)}{(3\alpha P_n)^2}\right]^{1/2}\right\} \qquad \text{(C2)}$$

Removing common factors and expanding gives

$$\beta_o = \frac{1 + 3\alpha P_n + 6\alpha P_r + 6\alpha P_s}{6\alpha P_n} \pm \frac{1}{6\alpha P_n}\left(1 + 9\alpha^2 P_n^2 + 36\alpha^2 P_r^2 + 36\alpha^2 P_s^2 + 6\alpha P_n + 12\alpha P_r + 12\alpha P_s + 36\alpha^2 P_n P_r + 36\alpha^2 P_n P_s + 72\alpha^2 P_r P_s - 12\alpha P_r - 72\alpha^2 P_r^2 - 72\alpha^2 P_r P_s\right)^{1/2} \qquad \text{(C3)}$$

which can be simplified and written as

$$\beta_o = \frac{1 + 3\alpha P_n + 6\alpha P_r + 6\alpha P_s}{6\alpha P_n} \pm \frac{1}{6\alpha P_n}\left[(1 + 3\alpha P_n + 6\alpha P_s)^2 + 36\alpha^2 P_r(P_n - P_r)\right]^{1/2} \qquad \text{(C4)}$$

This can be put in a form suitable for approximating by removing the large factor from the term in brackets; thus,

$$\beta_o = \frac{1 + 3\alpha P_n + 6\alpha P_r + 6\alpha P_s}{6\alpha P_n} \pm \frac{1 + 3\alpha P_n + 6\alpha P_s}{6\alpha P_n}\left[1 + \frac{36\alpha^2 P_r(P_n - P_r)}{(1 + 3\alpha P_n + 6\alpha P_s)^2}\right]^{1/2} \qquad \text{(C5)}$$

Taking the negative square root and approximating the term in brackets as the first two terms of a Maclaurin's series gives

$$\beta_o \doteq \frac{P_r}{P_n}\left(\frac{1 + 3\alpha P_r + 6\alpha P_s}{1 + 3\alpha P_n + 6\alpha P_s}\right) \tag{C6}$$

which is the desired result. This can be written as

$$\beta_o \doteq \frac{P_r}{P_n}\left[1 - \frac{3\alpha(P_n - P_r)}{1 + 3\alpha P_n + 6\alpha P_s}\right] \tag{C7}$$

to emphasize the fact that for small values of α, β_o differs only slightly from P_r/P_n.

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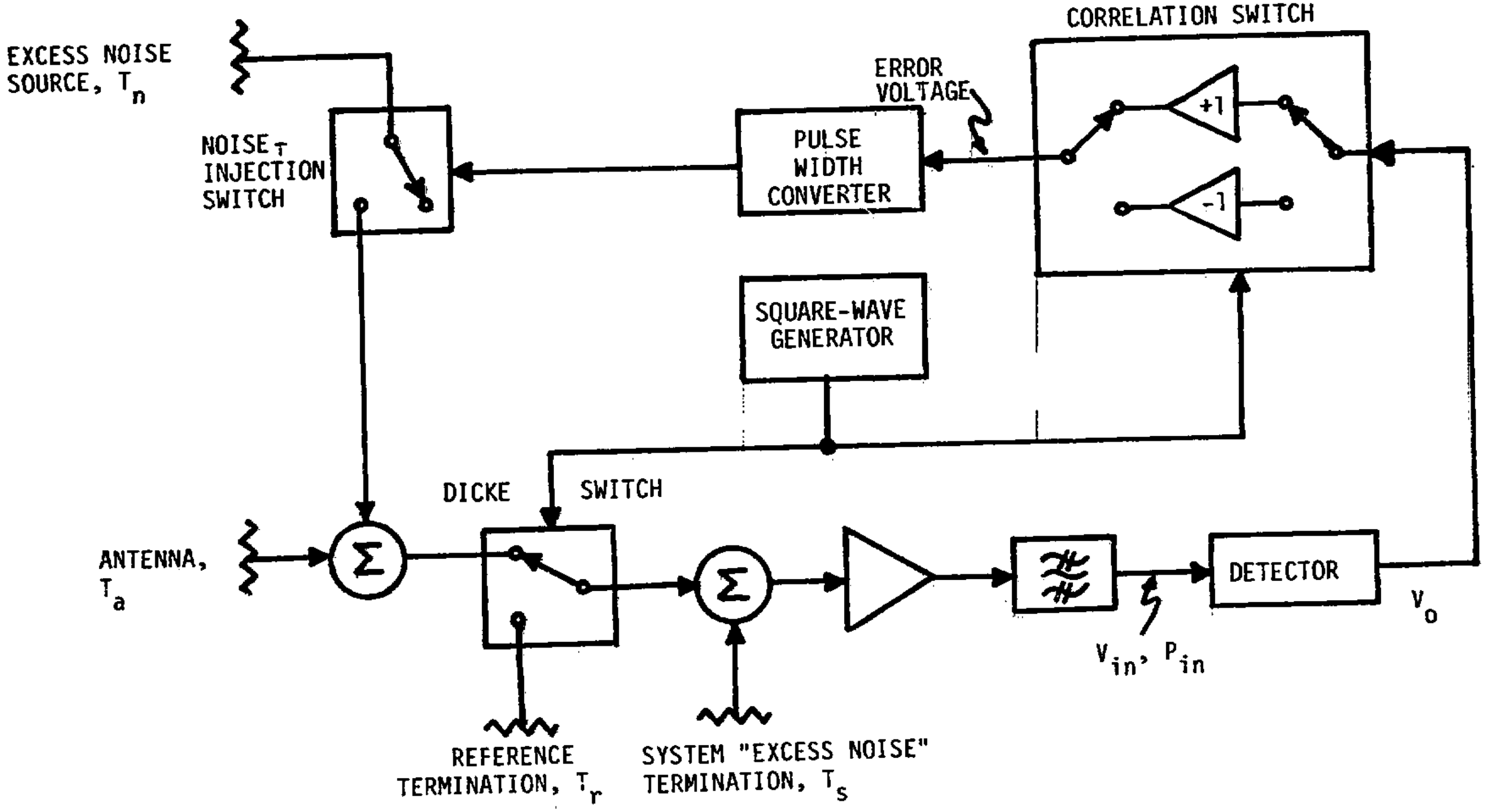

Figure 1.- Idealized noise-injection Dicke radiometer.

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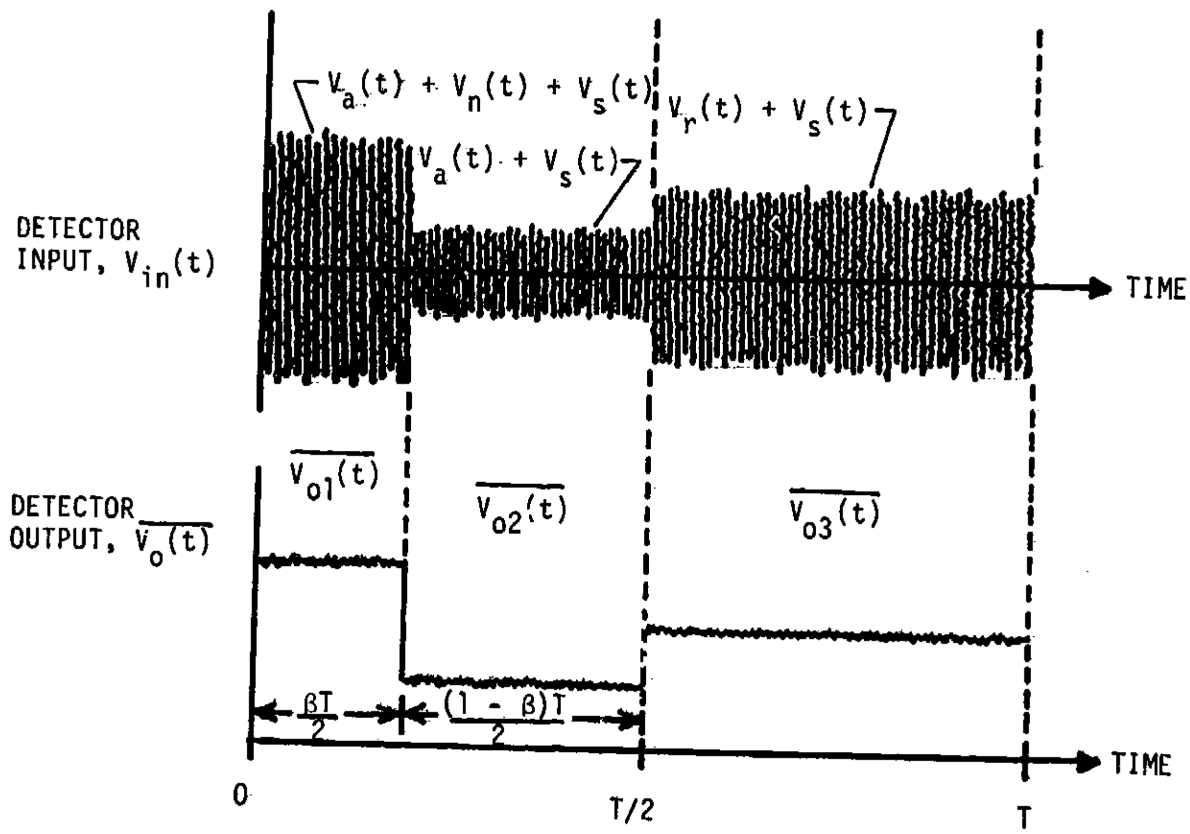

Figure 2.- Input and output waveforms at detector for full Dicke cycle.

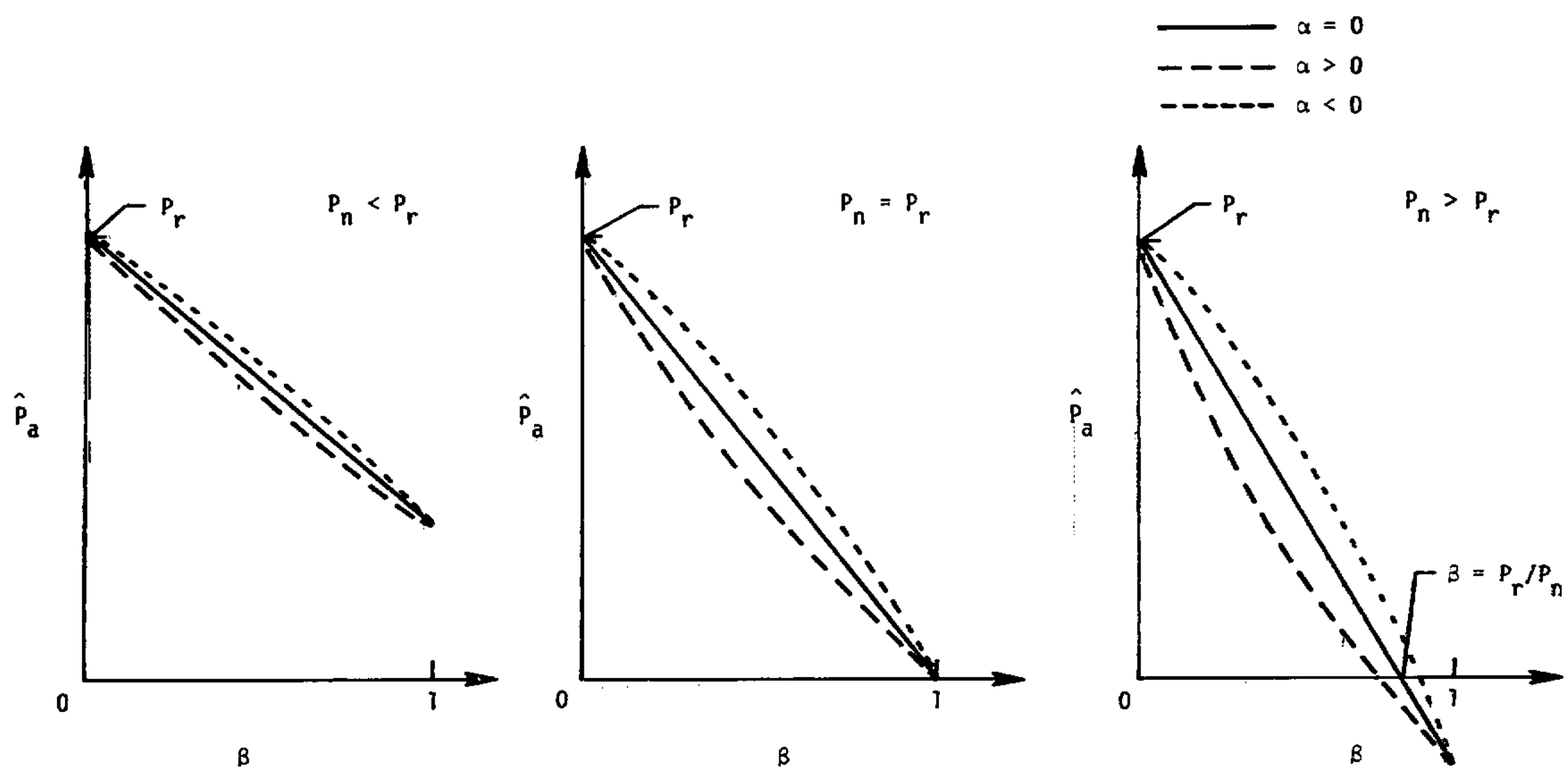

Figure 3.- $\hat{P}_a$ versus β with P_n and α as parameters.

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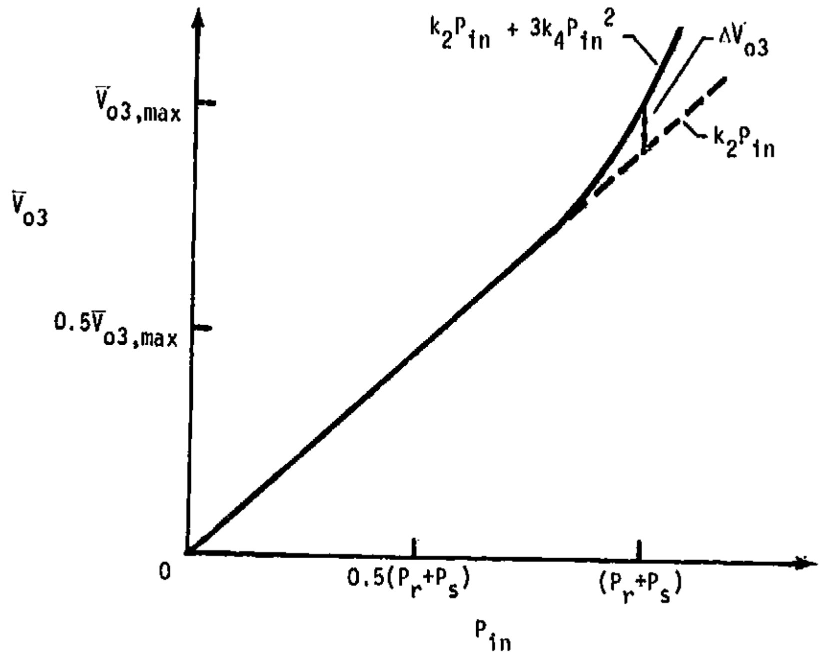

Figure 4.- Graphical determination of nonlinearity coefficient using reference-noise excitation.

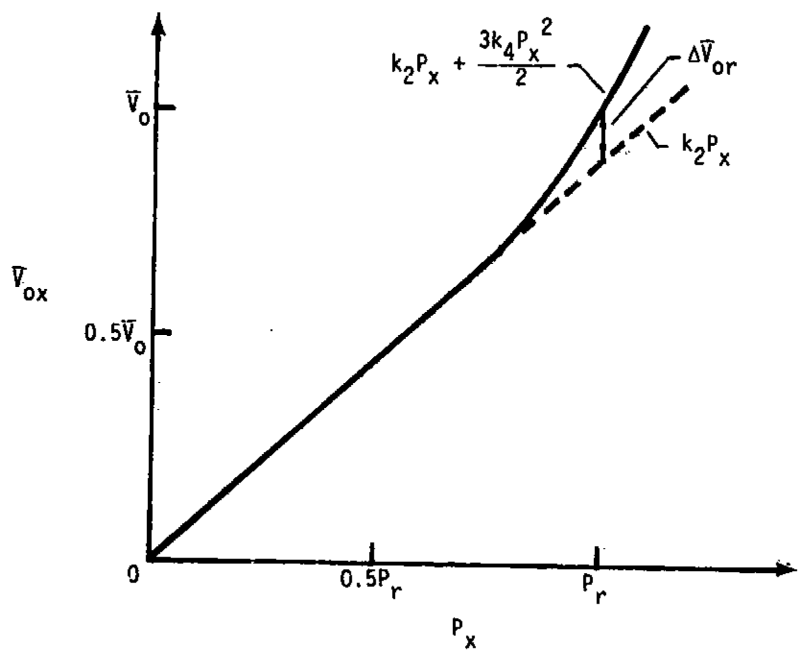

Figure 5.- Graphical determination of nonlinearity coefficient using sinusoidal excitation.

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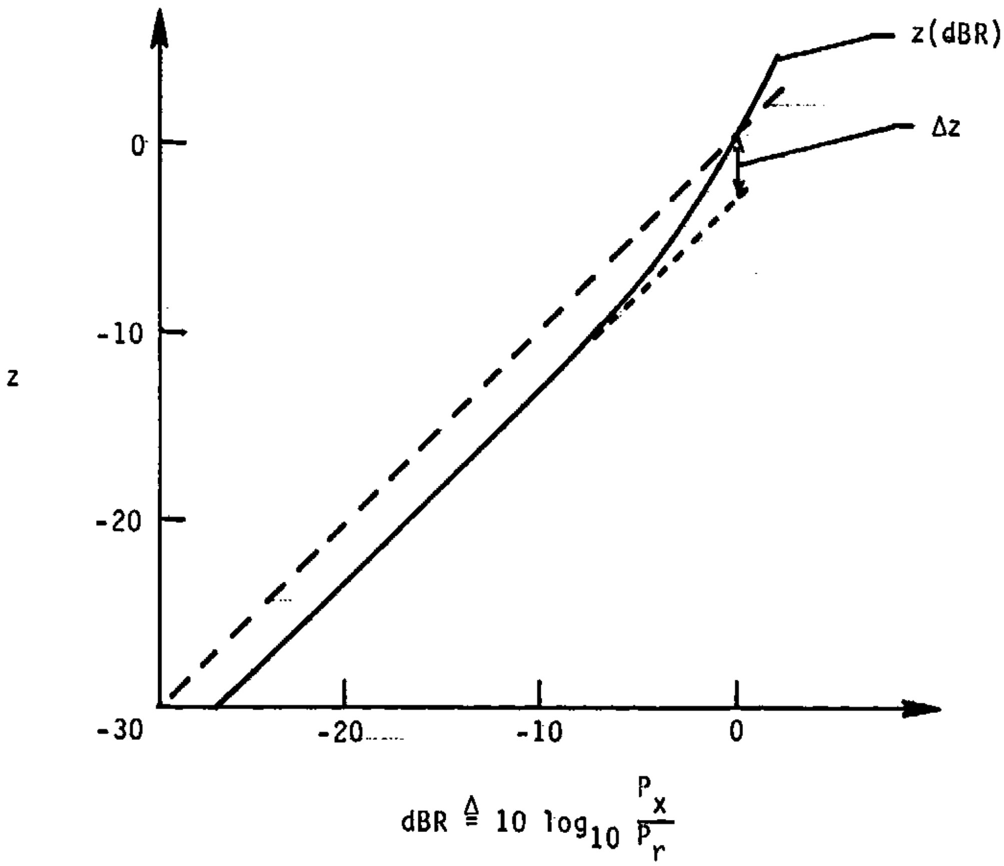

Figure 6.- Nonlinearity characterization using logarithmic plotting.