NASA Technical Memorandum 83549

Improved Finite-Difference Vibration Analysis of Pretwisted, Tapered Beams

K. B. Subrahmanyam and K. R. V. Kaza
Lewis Research Center
Cleveland, Ohio

Prepared for
The Southeastern Conference on Theoretical and Applied Mechanics
(SECTAM XII)
sponsored by Auburn University
Pine Mountain, Georgia, May 10–11, 1984

NASA

IMPROVED FINITE-DIFFERENCE VIBRATION ANALYSIS OF PRETWISTED, TAPERED BEAMS

K. B. Subrahmanyam* and K. R. V. Kaza
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

SUMMARY

An improved finite-difference procedure based upon second order central differences is developed in this paper. Several difficulties encountered in earlier works with fictitious stations that arise in using second order central differences, are eliminated by developing certain recursive relations. The need for forward or backward differences at the beam boundaries or other similar procedures is eliminated in the present theory. By using this improved theory, the vibration characteristics of pretwisted and tapered blades are calculated. Results of the second order theory are compared with published theoretical and experimental results and are found to be in good agreement. The present method generally produces close lower bound solutions and shows fast convergence. Thus, extrapolation procedures that are customary with first order finite-difference methods are unnecessary. Furthermore, the computational time and effort needed for this improved method are almost the same as required for the conventional first order finite-difference approach.

INTRODUCTION

Determination of the vibration characteristics of turbomachine blading by theoretical means has become very important in recent years. Blade failures are still not uncommon even with the present day material technology and improved design methodology. These failures are normally attributed to fatigue which occurs when blades vibrate at or near resonant conditions. It thus becomes imperative that the designer be able to determine the natural frequencies and mode shapes of the vibrating blades as accurately as possible at the early stages of design. Several methods of solution have been developed in the past in which only a few of the parameters such as pretwist, taper, asymmetry of cross section, disc and root flexibility, or of the construction complexities such as lacing wires and shrouding, are taken into account together. Inclusion of shear deflection, rotary inertia, warping, thermal and Coriolis effects in the vibration analysis becomes important for short and stubby beams subjected to high centrifugal force field. Recent research in helicopter rotor blade and advanced turboprop blade vibrations reveals the necessity of including the geometric nonlinearities up to varying degrees in the analysis in order to obtain fair prediction of the vibration characteristics and flutter boundaries. Some recent contributions in deriving the equations of motion include those of Kaza and Kvaternik [1], Hodges and Dowell [2] who used geometric-nonlinear theory and that of Subrahmanyam, et.al. [3] who used a linear theory but accounted for shear and rotary inertia effects, suitable for turbomachine blades.

*National Research Council – NASA Research Associate.

The various theoretical methods of solution can be broadly classified as belonging to either the continuum model approach or the discrete model approach. In the continuum model approach, applications of the potential energy, complementary energy or the Galerkin methods are well known. These methods produce upper bound solutions. The Reissner and the Dean and Plass methods which are classified as mixed methods [4, 5] were shown to produce upper bound solutions with relatively quicker convergence as compared to the classical approaches. In certain instances, the Reissner method may produce oscillatory convergence, depending on the choice of shape functions used [6]. While the approximate methods mentioned above have certain advantages, at least one disadvantage is associated with the numerical evaluation of the integrals arising in the formulation of the frequency equation.

In the discrete model approach, the Holzer-Myklestad, Stodola, polynomial frequency equation, transformation, Station function and finite-element methods are well developed. Application of the Galerkin, finite-difference and collection methods to solve the equations of motion has been known for quite some time. In general, the discrete model approaches produce lower-bound solutions due to the discretization of the distributed mass and elasticity. In certain finite-element applications, the convergence may be upper-bound depending upon the mass matrix formulation. Among the methods producing lower bound solutions, the first order finite-difference method is perhaps the single method that has attracted the greatest attention. Almost all the works dealing with the finite-difference method point out that the conventional first-order finite-difference method converges relatively slowly with mesh refinement. The improvement in accuracy expected with refinement of mesh size may be completely overshadowed by round-off and truncation errors that result from the increased matrix sizes. To avoid these difficulties, the Richardson's extrapolation procedure based on two or three solutions with different mesh sizes [7, 8] has been applied. The extrapolated result shows improved accuracy but the results being extrapolated must possess a monotonic convergence. Furthermore, the extrapolated result may not necessarily be a bound.

Relatively few works exist which use higher-order finite-difference techniques. Greenwood [9] used first-order and second-order finite differences to analyze uniform cantilever beams in flexure, and one case with uniform breadth but varying depth. The fourth order differential equation was transformed into four first order equations, and the slopes and shearing forces were evaluated at half integer stations, while the deflections and bending moments were evaluated at integer stations. A central difference approximation of second order was used. One sided approximations were used to satisfy the boundary conditions. As an alternative approach, a complicated, but symmetric, method of applying the boundary conditions was illustrated.

It was concluded that one sided approximation gave more accurate results. It is interesting to note from Greenwood's results that for a tapered beam, the first-order central-difference theory produces more accurate results than the second-order theory. The reason was attributed to the lumping of non-uniform mass in the second-order theory.

In this paper a second-order central difference approach is presented which eliminates most of the shortcomings discussed above. Uniform and tapered beams with or without pretwist are analyzed by using both the firstand second-order finite differences. The fictitious stations encountered in the development of the second-order theory are eliminated by using certain recursive relations derived by extending the central difference expressions for the boundary conditions given by the first-order theory. This logical extension in developing the recursive relations is shown to produce accurate results which converge more rapidly than results from the first-order theory. Furthermore, close lower-bound solutions can be obtained by the present approach, and extrapolations, such as are customary with the first-order theory, are not necessary when uncoupled vibrations are considered.

VIBRATION ANALYSIS OF A PRETWISTED TAPERED BLADE

Equations of Motion

Figure 1 shows a uniformly pretwisted and tapered blade of rectangular cross-section and the coordinate axes. The equations of motion [3, 10] for the coupled bending-bending vibrations of such a cantilever beam can be shown to be of the following form.

$$\frac{d^2}{dz^2}\left\{EI_{xx}\frac{d^2y}{dz^2} + EI_{xy}\frac{d^2x}{dz^2}\right\} - \rho A p_n^2 y = 0 \tag{1}$$

$$\frac{d^2}{dz^2}\left\{EI_{yy}\frac{d^2x}{dz^2} + EI_{xy}\frac{d^2y}{dz^2}\right\} - \rho A p_n^2 x = 0 \tag{2}$$

The sectional properties for a rectangular cross-section blade are

$$I_{xx} = I_{YY}\sin^2\nu\eta + I_{XX}\cos^2\nu\eta,$$

$$I_{yy} = I_{YY}\cos^2\nu\eta + I_{XX}\sin^2\nu\eta \tag{3,4}$$

$$I_{xy} = \frac{I_{YY} - I_{XX}}{2}\sin(2\nu\eta)$$

$$I_{YY} = I_{oy}(1+\delta\eta)(1+\beta\eta)^3 \tag{5,6}$$

$$I_{XX} = I_{ox}(1+\delta\eta)^3(1+\beta\eta),$$

$$A = A_0(1+\delta\eta)(1+\beta\eta) \tag{7,8}$$

In equations (1) to (8), E is the Youngs modulus, ρ the mass density of the blade material, p_n the natural radian frequency, x and y the dynamic displacements of the centroid in xzand yz-planes, A the area at any section, A_0 the area of the root section, ν the angle of pretwist at the blade tip and I_{xx}, I_{oy}, etc., the second moments of area about the axes specified by the subscripts. The breadth and depth taper parameters are given by

$$\beta = (a_0 - b_0)/b_0, \quad \delta = (d_0 - c_0)/c_0 \tag{9,10}$$

Furthermore,

$$\eta = z/L, \quad \frac{d(\ )}{dz} = \frac{1}{L}\frac{d(\ )}{d\eta} \tag{11,12}$$

For an untwisted blade, $\nu = 0$ and thus equations (1) and (2) are uncoupled.

By making use of equations (3) to (12) in equations (1) and (2), performing the differentiations successively and grouping terms, the equations of motion can be written in the following form.

$$a\frac{d^4y}{d\eta^4} + b\frac{d^3y}{d\eta^3} + c\frac{d^2y}{d\eta^2} + d\frac{d^4x}{d\eta^4} + e\frac{d^3x}{d\eta^3} + f\frac{d^2x}{d\eta^2} - \lambda p_n^2 y = 0 \tag{13}$$

$$q\frac{d^4y}{d\eta^4} + r\frac{d^3y}{d\eta^3} + s\frac{d^2y}{d\eta^2} + t\frac{d^4x}{d\eta^4} + u\frac{d^3x}{d\eta^3} + v\frac{d^2x}{d\eta^2} - \lambda p_n^2 x = 0 \tag{14}$$

In the above equations, $\lambda = \rho A_0 L^4/EI_{0x}$ and the coefficients $a, b, \ldots, f, q, r, \ldots, v$ are all functions of η. For brevity these functions are not presented here but can be found in reference [11].

Finite-Difference Equations for Derivatives

First-Order Central Differences - The first order central differences for the derivatives of a function ϕ at any arbitrary station i are given by

$$\phi_i' = \frac{1}{2h}\left\{-\phi_{i-1} + \phi_{i+1}\right\} \tag{15}$$

$$\phi_i'' = \frac{1}{h^2}\left\{\phi_{i-1} - 2\phi_i + \phi_{i+1}\right\} \tag{16}$$

$$\phi_i''' = \frac{1}{2h^3}\left\{-\phi_{i-2} + 2\phi_{i-1} - 2\phi_{i+1} + \phi_{i+2}\right\} \tag{17}$$

$$\phi_i^{iv} = \frac{1}{h^4}\left\{\phi_{i-2} - 4\phi_{i-1} + 6\phi_i - 4\phi_{i+1} + \phi_{i+2}\right\} \tag{18}$$

Second-Order Central Differences - The second order central differences for the derivatives of a function ϕ at any arbitrary station i are given by

$$\phi_i' = \frac{1}{12h}\left\{\phi_{i-2} - 8\phi_{i-1} + 8\phi_{i+1} - \phi_{i+2}\right\} \tag{19}$$

$$\phi_2'' = \frac{1}{12h^2}\left\{-\phi_{i-2} + 16\phi_{i-1} - 30\phi_i + 16\phi_{i+1} - \phi_{i+2}\right\} \tag{20}$$

$$\phi_i''' = \frac{1}{8h^3}\left\{\phi_{i-3} - 8\phi_{i-2} + 13\phi_{i-1} - 13\phi_{i+1} + 8\phi_{i+2} - \phi_{i+3}\right\} \quad (21)$$

$$\phi_i^{iv} = \frac{1}{6h^4}\left\{-\phi_{i-3} + 12\phi_{i-2} - 39\phi_{i-1} + 56\phi_i - 39\phi_{i+1} + 12\phi_{i+2} - \phi_{i+3}\right\} \quad (22)$$

In equations (15) to (22), the subscripts i-3, i-2, i-1, i, i+1, i+2 and i+3 represent stations separated by the interval h , where $h = L/n$, n being the number of segments into which the beam of length L is divided.

Boundary Conditions

The boundary conditions associated with the coupled bending-bending vibrations of a pretwisted cantilever beam fixed at $\eta = 0$ and free at $\eta = 1$ reduce to

$$y = x = y' = x' = 0 \text{ at } \eta = 0;$$

$$y'' = x'' = y''' = x''' = 0 \text{ at } \eta = 1 \quad (23,24)$$

By using the central difference relations for the derivatives of x and y in the forms shown by equations (15) to (18), one can easily show that

$$y_{-1} = y_1,\ x_{-1} = x_1,\ x_0 = y_0 = 0 \quad (25)$$

$$y_{n+1} = 2y_n - y_{n-1},\ x_{n+1} = 2x_n - x_{n-1} \quad (26)$$

$$y_{n+2} = 4y_n - 4y_{n-1} + y_{n-2},\quad x_{n+2} = 4x_n - 4x_{n-1} + x_{n-2} \quad (27)$$

Equations (25) to (27) eliminate all the fictitious stations encountered in the development of the finite-difference formulation with first order central differences. In order to eliminate the fictitious stations y_{-2} and y_{n+3} encountered in the second order central differences, one can assume a symmetry condition around the built-in end so that

$$y_{-2} = y_2,\ x_{-2} = x_2 \quad (28)$$

This assumption satisfies the boundary conditions (23) in terms of second order central differences. Equations (26) to (27) are rewritten in the following form to establish a possible recursive relation:

$$y_{n+1} - y_{n-1} = 2(y_n - y_{n-1});$$

$$x_{n+1} - x_{n-1} = 2(x_n - x_{n-1}) \quad (29)$$

$$y_{n+2} - y_{n-2} = 4(y_n - y_{n-1});$$

$$x_{n+2} - x_{n-2} = 4(x_n - x_{n-1}) \quad (30)$$

and, by recursion

$$y_{n+3} - y_{n-3} = 6(y_n - y_{n-1});$$

$$x_{n+3} - x_{n-3} = 6(x_n - x_{n-1}) \quad (31)$$

Equations (31) can be used to eliminate y_{n+3} and x_{n+3}. The error introduced by these assumptions can be evaluated by introducing the values of y_{n+1}, y_{n+2}, and y_{n+3} obtained from equations (29) to (31) into the second order finite difference expressions for y_n'' and y_n'''. It can be seen that $y_n''' = 0$ while the expression for y_n'' yields

$$y_{n-2} - 2y_{n-1} + y_n = 0 \quad (32)$$

Equation (32) states that the deflection at the $(n-1)^{th}$ station is the average of the deflections at the preceding and the succeeding stations. Thus, the deflection curve near the tip of the cantilever beam assumes a straight line form. Since the bending moment at the free end must be zero, the condition of constant slope near the tip is justified and, thus, the boundary conditions represented by equations (28) to (32) should give accurate results for a suitably large value of n.

Finite-Difference Implementation

By introducing the finite-difference expressions for the derivatives, as given by equations (15) to (18) or (19) to (22) in terms of the functions x and y, into the differential equations (13) and (14), one obtains a coupled set of equations for any arbitrary station. Each of this set can be evaluated for each station with i replaced by 1, 2....n successively. A system of 2n equations can be written in terms of the variables $y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n$. The parameters $y_{-1}, y_{-2}, y_{n+1}, y_{n+2}, y_{+3}$, ..., etc., can be eliminated by using equations (25) to (27) or (28) to (32). The resulting equations can be represented in the familiar form of the eigenvalue problem

$$\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{C} & \underset{\sim}{D} \end{bmatrix} \begin{Bmatrix} y_i \\ x_i \end{Bmatrix} - p_n^2 \lambda \begin{Bmatrix} y_i \\ x_i \end{Bmatrix} = 0 \quad (33)$$

For the special case of zero pretwist, equation (33) reduces to two independent equations representing uncoupled vibrations in the two principal planes as follows:

$$\left[\underset{\sim}{A} - p_y^2 \lambda\right]\{y_i\} = 0,\ \left[\underset{\sim}{D} - p_x^2 \lambda\right]\{x_i\} = 0 \quad (34,35)$$

where p_x and p_y are the uncoupled natural radian frequencies in the stiff xz-plane and flexible yz-plane respectively.

In the preceding equations, $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ and $\underset{\sim}{D}$ are square matrices of order (n x n) each, and $\{y\}_i$ and $\{x_i\}$ are column matrices containing the displacement vectors of the n-stations. The eigenvalues and the associated eigenvectors can be determined by using standard solution procedures. It may be noted here that the first and second order central difference methods generate frequency determinants of the same size for a given number of assumed stations. Both methods produce matrices that are banded. Each submatrix $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ or $\underset{\sim}{D}$ possesses a band width of five for the first-order theory and seven for second-order theory. Furthermore, the matrices are non-symmetric for both methods developed. To save space the elements of the matrices are not presented here.

NUMERICAL RESULTS AND DISCUSSION

A FORTRAN computer program was developed to solve the eigenvalue problems defined by equations (33) to (35). The program was run on an IBM 370 computer at the NASA Lewis Research Center. Use was made of an IMSL routine EIGZF [12]. This library subroutine evaluates all the natural frequencies and the associated mode shapes (eigenvectors). As expected for a conservative system, it was found that all the eigenvalues and the corresponding eigenvectors are real for both the first and second order finite-difference applications. Several configurations of uniform and pretwisted blades were solved with and without taper. The results are presented below.

Untwisted Uniform and Tapered Blades

The following numerical data [13 to 14] were used to study the uncoupled vibrations of cantilever beams of length 0.254 m with various breadth and depth taper ratios. The thickness ratio at the root section was kept as unity (square cross section), and the beams had an aspect ratio (L/b_0) of 40. A convergence study was made for a uniform beam divided into odd and even number segments. The frequency ratios presented in Table 1 indicate that the convergence is monotonic from below for both the first and second order central difference methods and that the convergence is continuous for even or odd values of n. Furthermore, the convergence shown by the second order theory is very rapid, and the lowest five flexural frequencies, though only the first three modes are shown in Table 1, can be obtained to within 0.195 percent error with n = 30. The first order theory shows errors of the order of 2.6 percent with n = 30, but an extrapolation procedure could be used to improve the accuracy.

The natural frequencies and mode shapes are also calculated for tapered cantilever beams with various combinations of breadth and depth taper parameters. The results obtained by using the second order theory with n = 30 are in good agreement with the theoretical and experimental results presented in References [11, 13]. Because of space limitations, these results together with the results for uncoupled axial and torsional vibrations of tapered cantilever beams were presented in Reference [14].

Pretwisted Uniform Blades

Recent investigations of the vibrational characteristics of pretwisted uniform blades include those of Slyper [15] using the Stodola method, Dawson [16] using the Rayleigh-Ritz method, Dawson and Carnegie [17] using a transformation technique, Dokumaci, Thomas and Carnegie [18] using matrix displacement analysis and Subrahmanyam, Kulkarni and Rao [4, 5] using the potential energy and Reissner methods. Numerical data for the present study were chosen from these references, and the coupled bending-bending vibration frequencies and mode shapes were determined by both the first and second order finite-difference methods. Convergence trends remained similar to those presented earlier though the convergence rates were not as fast as observed for the uncoupled case. Using n = 30 in the second order theory, the frequency parameter ratios, (λ/λ_0), [where λ_0 is the fundamental frequency parameter for a blade with zero taper (exact value)], for the coupled bending-bending vibrations of uniform cantilever beams were obtained for typical values of breadth to depth ratios and pretwists. A typical set of such results are presented in Table 2, where a comparison of the present results is made with extrapolated results obtained from the matrix displacement method [18] and the first order finite-difference method. It can be seen from these results that the second order finite-difference method produces fairly accurate natural frequencies without extrapolations. It is worth mentioning that an extrapolated result may have an improved accuracy but can be either above or below the exact value. The present results obtained by the second order central differences show a maximum variation of the order of about ±0.3 percent from the extrapolated results. In order to see whether the mode shapes for the coupled bending vibrations are obtained to reasonable accuracy, numerical data used in references [4, 6] are employed in the second order finite-difference method, and the natural frequencies and mode shapes are obtained. Although not presented here, the agreement between the results obtained in this manner and those presented in reference [6] is found to be extremely close.

Pretwisted Tapered Blades

Among the recent contributions to pretwisted, tapered blade vibration analysis, first order finite-differences were used by Carnegie et.al. [19, 20]; the Galerkin method was used by Rao [21]; finite element method was used by Gupta and Rao [22]; and the Reissner method was employed by Subrahmanyam and Rao [6]. Numerical data used in references [19, 20] are employed in the present investigation, and several cases of tapered and pretwisted blades are solved. A typical set of results, for a 45° pretwisted blade with a breadth taper of -0.25 and a depth taper of 0.75, are presented in Figure 2 in order to represent the relative convergence trends of first and second order theories. It has been observed [14] that the finite-difference method clearly yields a lower bound for the uncoupled vibration cases, but for coupled bending-bending vibrations, the convergence is seen to be from above for certain coupled modes. As can be seen in Figure 2, the first order theory has oscillatory convergence for the first coupled mode while the second order theory produced an upper bound. However, the higher modes are clear lower bound solutions. Extensive convergence studies were made for a variety of blades having various breadth-to-depth ratios and pretwist angle combinations discussed in [6, 19 to 21] by using both the first and second order finite-difference approaches. It appeared that for rectangular cross section blades, when the vibratory mode was such that the maximum component deflection was in the flexible plane, the corresponding coupled frequency converged from above and when the maximum component deflection was in the stiffer plane, the associated coupled frequency yielded a lower bound. For blades with a square cross section at the root, the flexural rigidities are equal in both the planes and pretwist does not bring in any coupling. Coupled vibration can only occur if such a blade

ORIGINAL PAGE IS
OF POOR QUALITY

has unequal breadth and depth tapers and also possesses a non-zero pretwist. In such cases, the convergence yields upper or lower bound depending upon the unequal flexural rigidities due to taper, the degree of coupling, and the mode of vibration. In general, for pretwisted tapered beams having square cross section at the root presented here, the convergence has been observed to be lower bound except for the first coupled mode.

The authors have extended this work more recently to analyze rotating blades of asymmetric cross section, without pretwist, but which perform coupled vibrations having coupling between bending in one plane and torsion. Those results, although not shown here, indicated that the second order finite difference method predicts close lower bound solutions for the coupled bending-torsion modes.

By comparing the trends observed for the uncoupled vibration cases, coupled bending-bending vibrations of pretwisted blades and the coupled bending-torsion vibrations of untwisted asymmetric cross section blades under rotation, it is concluded that the coupling between the bending in two planes, brought forth by pretwist, is responsible for the occurrence of both upper bound and lower bound convergence in the finite-difference approaches depending on the mode of vibration.

Figures 3 to 5 show some typical results produced by the second order central difference method with $n = 30$. For certain values of breadth and depth taper, the effect of varying pretwist is shown in Figure 3. Figure 4 shows the effects of depth taper for a given value of breadth taper and pretwist. Figure 5 shows the coupled bending-bending mode shapes of a 90° pretwisted blade having a breadth taper of -0.25 and various values of depth taper. Further results, obtained for $\beta=-0.5$; $\beta=0.5$ and various values of δ and ν, are not shown here, but all these results are in good agreement with the theoretical and experimental values available in the literature [19, 20]. The effects of pretwist, breadth taper and depth taper in producing coupling of the modes are discussed in detail in references [6, 10, 17, 19].

Relative Computational Efficiency

In order to evaluate the relative computational efficiency of the first and second order finite difference methods, the average CPU time required by each method was determined for $n = 30$. The methods require nearly the same amount of CPU time; the maximum variation between the two is about ± 1.7 percent.

CONCLUDING REMARKS

The second order finite-difference method has been applied to determine the coupled bending-bending frequencies and the mode shapes of pretwisted and tapered cantilever beams. Simple recursive relations have been used to eliminate the fictitious stations outside the beam domain by making logical extensions from the first order theory. The present approach is shown to produce accurate natural frequencies and mode shapes. In the course of this study, several conclusions have emerged.

1. For the same mesh size (step length h), the second order finite difference method produces natural frequencies with greater accuracy than the first order theory. The convergence of the lower mode frequencies is much more rapid for second order central differences than for first.

2. Second order theory yields natural frequencies and mode shapes to accuracies of practical interest with relatively coarse mesh. Further, extrapolation procedures are not necessary to obtain accurate results.

3. Results presented here indicate that the second order theory produces close lower bound solutions for uncoupled mode of vibration. For the modes having coupling between bending in two planes, either close lower or close upper bound solutions are obtained. The probable conditions under which an upper bound solution is obtained varied with the breadth to depth ratio, pretwist, taper ratios and the extent of coupling between the two component modes.

Majority of short comings encountered in the earlier investigations such as the necessities of using integer and half integer stations, of transforming the equations of motion into a set of first order equations or of using backward or forward differences at the boundaries to avoid fictitious stations, etc., are eliminated by the present improved theory. Extension of the method presented here using plate theories may prove beneficial, since the strong convergence characteristics would reduce computational time to a considerable extent in two dimensional cases.

REFERENCES

[1] K. R. V. Kaza and R. G. Kvaternik, "Nonlinear Aeroelastic Equations for Combined Flapwise Bending, Chordwise Bending, Torsion and Extension of Twisted Nonuniform Rotor Blades in Forward Flight," NASA TM 74059, 1977.

[2] D. H. Hodges and E. H. Dowell, "Nonlinear Equations of Motion for the Elastic Bending and Torsion of Twisted Nonuniform Rotor Blades," NASA TN D-7818, 1974.

[3] K. B. Subrahmanyam, S. V. Kulkarni and J. S. Rao, "Application of the Reissner Method to Derive the Coupled Bending-Torsion Equations of Dynamic Motion of Rotating Pretwisted Cantilever Blading with Allowance for Shear Deflection, Rotary Inertia, Warping and Thermal Effects," J. Sound and Vibration, Vol. 84, No. 2, 1982, pp. 223-240.

[4] K. B. Subrahmanyam, S. V. Kulkarni and J. S. Rao, "Coupled Bending-Bending Vibrations of Pretwisted Cantilever Blading Allowing for Shear Deflection and Rotary Inertia by the Reissner Method," Int. J. Mechanical Science, Vol. 23, No. 9, 1981, pp. 517-530.

[5] K. B. Subrahmanyam, S. V. Kulkarni and P. M. Rao, "Dean and Plass Method Calculations of the Flexural Frequencies of Timoshenko Beams," J. Sound and Vibration, Vo. 81, No. 1, 1982, pp. 141-146.

ORIGINAL PAGE IS
OF POOR QUALITY

[6] K. B. Subrahmanyam and J. S. Rao, "Coupled Bending-Bending Vibrations of Pretwisted-Tapered Cantilever Beams Treated by the Reissner Method," J. Sound and Vibration, Vol. 82, No. 4, 1982, pp. 577-592.

[7] L. F. Richardson, "The Approximate Arithmetical Solution by Finite-Differences of Physical Problems Involving Differential Equations, with an Application to the Stresses in a Masonry Dam," Philosophical Transactions, Royal Society of London, Vol. 210, 1911, pp. 307-357.

[8] M. G. Salvadori, "Numerical Computation of Buckling Loads by Finite Differences," Proc. American Soc. Civil Engineers, Vol. 75, No. 10, Dec. 1949, pp. 1441-1475.

[9] D. T. Greenwood, "The Use of Higher-Order Difference Methods in Beam Vibration Analysis," NASA TN D-964, 1961.

[10] W. Carnegie, "Vibrations of Pretwisted Cantilever Blading," Proc. Institution of Mechanical Engineers, London, Vol. 173, No. 12, 1959, pp. 343-374.

[11] W. Carnegie and J. Thomas, "Natural Frequencies of Long Tapered Cantilevers," The Aeronautical Quarterly, Vol. XVIII, Nov. 1967, pp. 309-320.

[12] The International Mathematical and Statistical Library (IMSL), Houston, Texas (1975).

[13] K. B. Subrahmanyam and S. V. Kulkarni, "Reissner Method Analysis of Tapered Cantilever Beams Vibrating in Flexure," J. Sound and Vibration, Vol. 77, No. 4, 1981, pp. 578-582.

[14] K. B. Subrahmanyam and K. R. V. Kaza, "An Improved Finite-Difference Analysis of Uncoupled Vibrations of Tapered Cantilever Beams," NASA TM-83495, 1983.

[15] H. A. Slyper, "Coupled Bending Vibrations of Pretwisted Cantilever Beams," J. Mechanical Engineering Science, Vol. 4, No. 4, Dec. 1962, pp. 365-379.

[16] B. Dawson, "Coupled Bending Vibrations of Pretwisted Cantilever Blading Treated by Rayleigh-Ritz Method," J. Mechanical Engineering Science, Vol. 10, No. 5, Dec. 1968, pp. 381-388.

[17] B. Dawson and W. Carnegie, "Modal Curves of Pretwisted Beams of Rectangular Cross Section," J. Mechanical Engineering Science, Vol. 1, No. 1, Feb. 1969, pp. 1-13.

[18] E. Dokumaci, J. Thomas and W. Carnegie, "Matrix Displacement Analysis of Coupled Bending-Bending Vibrations of Pretwisted Blading," J. Mechanical Engineering Science, Vol 9, No. 4, Oct. 1967, pp. 247-254.

[19] W. Carnegie and J. Thomas, "The Coupled Bending-Bending Vibration of Pretwisted Tapered Blading," J. Engineering for Industry, Trans. ASME, Vol. 94, No. 1, Feb. 1972, pp. 255-266.

[20] W. Carnegie, B. Dawson and J. Thomas, "Vibration Characteristics of Cantilever Blading," Proc. Inst. Mechanical Engineers, London, Vol. 180, Part 3I, 1965-66, pp. 71-89.

[21] J. S. Rao, "Flexural Vibrations of Pretwisted Tapered Cantilever Beams," J. Engineering for Industry, Trans. ASME, Vol. 94, No. 1, Feb. 1972, pp. 343-346.

[22] R. S. Gupta and S. S. Rao, "Finite Element Eigenvalue Analysis of Tapered and Twisted Timoshenko Beams," J. Sound and Vibration, Vol. 56, No. 2, Jan. 22, 1978, pp. 187-200.

BIOGRAPHICAL SKETCHES

Dr. K. B. Subrahmanyam, Professor of Mechanical Engineering, NBKR Institute of Science and Technology, Vidyanagar, India, and currently an NRC-NASA Resident Research Associate at NASA Lewis Research Center. His areas of research activity are turbomachine blade vibration and optimization techniques in machine design. He has published several research papers.

Dr. Krishna Rao V. Kaza is a Research Scientist at the NASA Lewis Research Center. He is involved in research in aeroelasticity and structural dynamics of rotating elastic structures such as advanced turboprops, turbofan jet engines, windturbines, helicopters, rotorcraft, and beams and has published several research papers and reports. He is reviewer for AIAA and ASME Journals and Applied Mechanics Reviews. He is an Associate Fellow of AIAA and listed in Jane's "Who is Who in Aviation and Aerospace".

TABLE I. - CONVERGENCE PATTERN OF FLEXURAL FREQUENCIES OF UNIFORM CANTILEVER BEAM USING FIRST ORDER AND SECOND ORDER CENTRAL DIFFERENCE SCHEMES:

NONDIMENSIONAL FREQUENCY, $p_y\sqrt{\frac{\rho A L^4}{E I_{ox}}}$

n	I Mode		II Mode		III Mode	
	1st order	2nd order	1st order	2nd order	1st order	2nd order
[illegible]	3.4021	3.5062	1[illegible].870	21.204	45.334	54.421
6	3.4359	3.5104	19.107	21.553	49.370	57.320
10	3.4866	3.5148	21.134	21.934	56.603	60.774
11	3.4916	3.5152	21.284	21.960	57.423	61.014
12	3.4955	3.5153	21.340	21.977	58.063	6[illegible].179
15	3.5029	3.5158	21.623	22.006	59.318	61.443
17	3.5057	3.5158	21.713	22.015	59.827	61.526
18	3.5069	3.5158	21.750	22.018	60.024	61.555
20	3.5086	3.5159	21.801	22.023	60.334	61.595
23	3.5104	3.5159	21.857	22.027	60.660	61.632
24	3.5108	3.5159	21.872	22.028	60.744	61.640
25	3.5113	3.5159	21.884	22.029	60.817	61.647
28	3.5125	3.5160	21.923	22.031	61.041	61.666
30	3.5127	3.5160	21.930	22.031	61.083	61.669
Exact solution	3.5160		22.0345		61.6973	
Percent error based on n=30	-0.094	0.000	-0.474	-0.016	-0.996	-0.046

TABLE II. - COMPARISON OF FREQUENCY PARAMETER RATIOS, $\frac{\lambda}{\lambda_0}$, OF PRETWISTED UNIFORM BLADES

b_0/c_0 ratio	ν	Mode number	First order finite-difference method: extrapolated from n = 10, 15, 20 [19]	Matrix displacement method [18] extrapolated from		Second order finite-difference method: n = 30 without extrapolation
				3 and 4 elements	8, 10 and 16 elements	
2	30°	1	1.0059	1.0049	1.0015	1.0053
		2	3.9166	3.9168	3.9132	3.9139
		3	40.3667	40.3834	40.5648	40.3745
		4	148.8397	149.0163	149.3350	148.6643
2	90°	1	1.0425	1.0429	1.0444	1.0434
		2	3.4020	3.4037	3.3935	3.3958
		3	48.6700	48.6587	48.6587	48.7710
		4	113.4711	113.9809	113.3201	113.0779
4	30°	1	1.0044	1.0060	----	1.0081
		2	13.9468	13.9524	----	13.8957
		3	45.3107	45.3285	----	45.4342
		4	278.3168	280.9137	----	277.6319
		5	693.4500	----	----	693.9047

ORIGINAL PAGE IS
OF POOR QUALITY

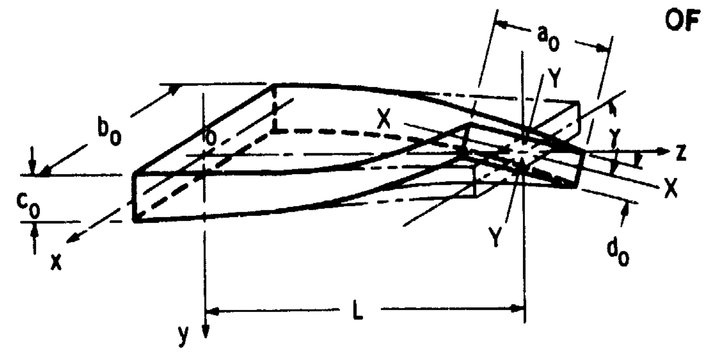

Figure 1. - Pretwisted tapered blade and co-ordinate axes.

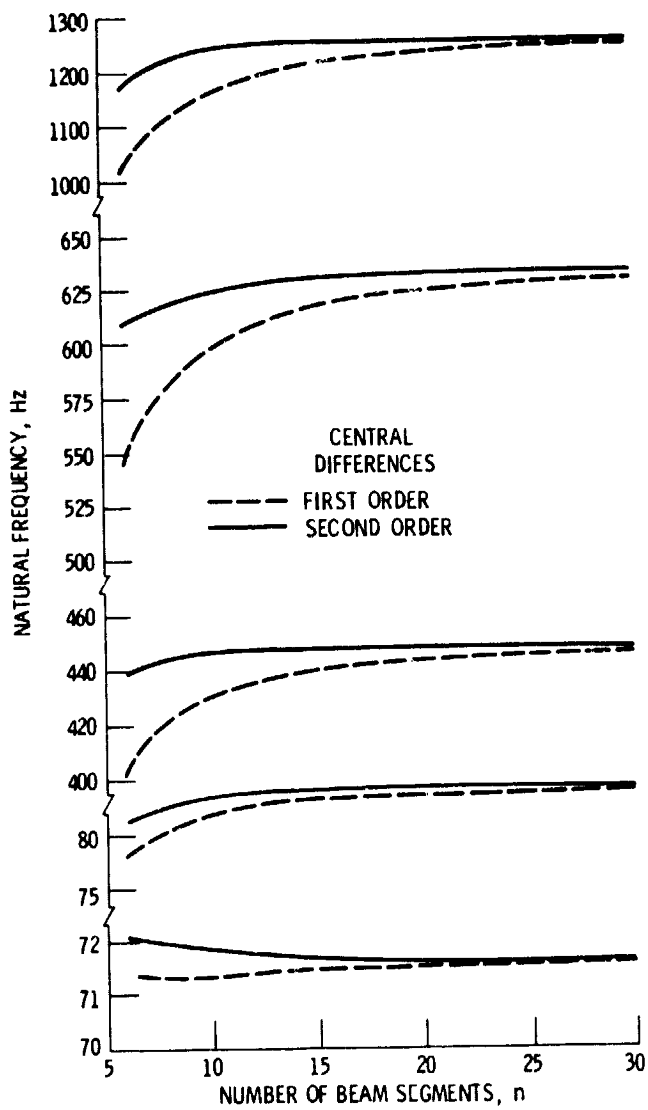

Figure 2. - Convergence pattern of coupled flexwial frequencies. ($b_0/c_0 = 1$, $\beta = -0.25$, $\delta = 0.75$, $\gamma = 45^0$).

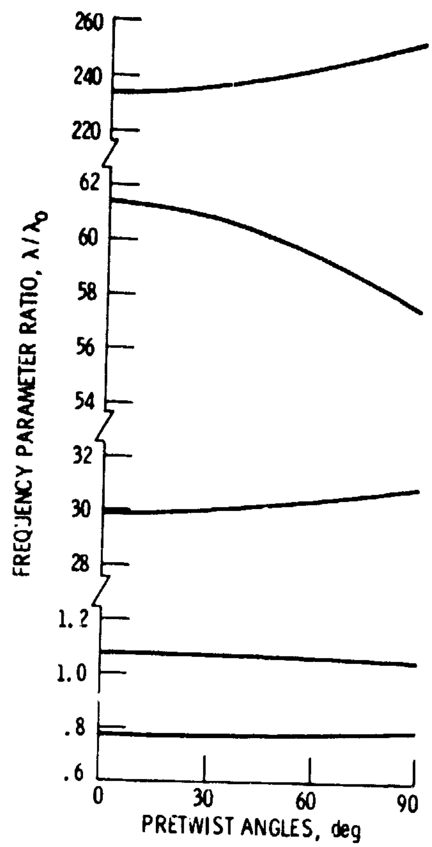

Figure 3. - Frequency parameter ratio against pretwist angle for a blade with $\beta = -0.25$, $\delta = 0.75$ and $b_0/c_0 = 1$.

ORIGINAL PAGE IS
OF POOR QUALITY

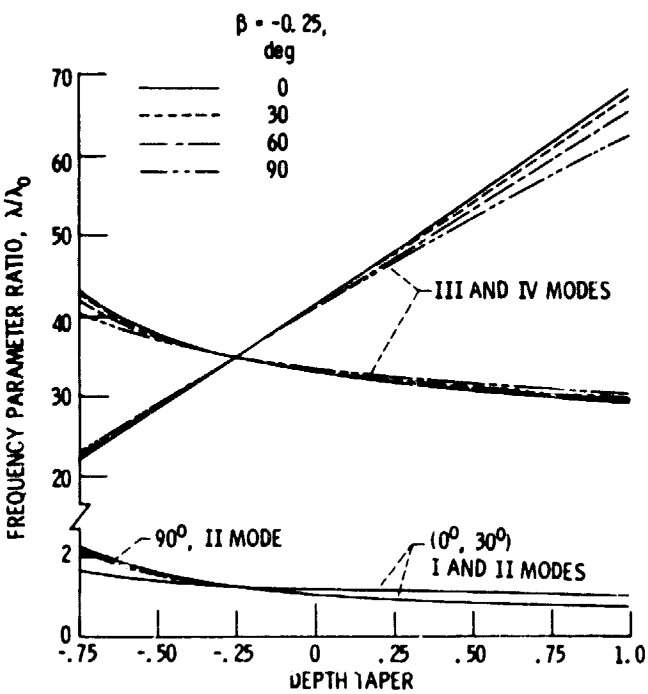

Figure 4. - Frequency parameter ratio for various pretwist angles.

ORIGINAL PAGE IS
OF POOR QUALITY

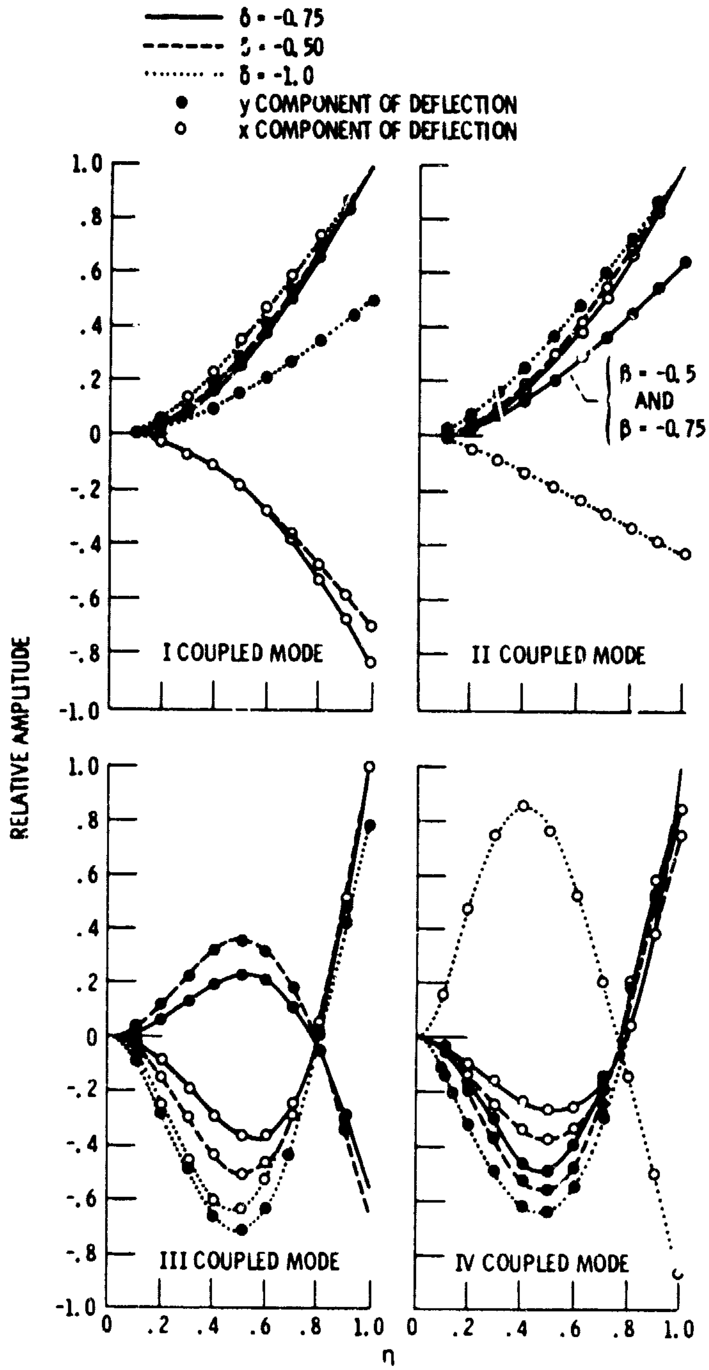

Figure 5. - Coupled bending-bending mode shapes of pretwisted tapered blade: $b_0/c_0 = 1$, $\gamma = 90^0$, $\beta = -0.25$.