nasa CR-172,290

NASA Contractor Report 172290

AN APPROXIMATION FORMULA FOR A CLASS OF MARKOV RELIABILITY MODELS

Allan L. White

KENTRON INTERNATIONAL, INC.
Kentron Technical Center
Hampton, Virginia 23666

Contract NAS1-16000
January 1984

NASA
National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

TABLE OF CONTENTS

1.0 INTRODUCTION

For a small class of reliability models, this report shows how to just look at the model and write down a convenient approximation for the answer.

The models considered are appropriate for redundant reconfigurable digital equipment that operates for a short period of time without maintenance and that collects only permanent faults. The models are pure death Markov processes where all the fault occurrence rates are low and all the system recovery rates are high. A discussion of the numerical bounds for the parameters is in the text. For such a model the method gives a formula in terms of fault rates, recovery rates, and operating time. The approximation formulas are simple enough that a pocket calculator yields reliability estimates and an examination shows the relative influence on reliability of each of the parameters. The simple formulas have easy partial derivatives that give the change in reliability with respect to a change in any parameter.

The first half of the report describes the approximation procedure and presents several examples. The interested reader can sample the illustrative computations until he feels comfortable with the methods. Because there is no analysis of error bounds, the examples also compare the approximate solutions to exact numerical solutions to establish confidence in the approximation.

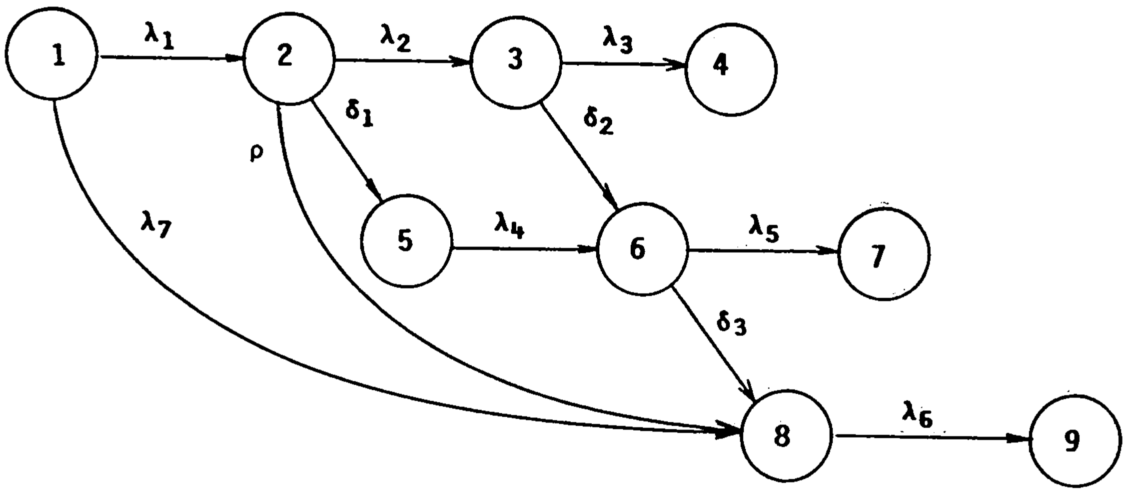
To avoid the formulas appearing completely mysterious, the section on paths in pure death processes derives the approximations for two short paths. Unfortunately, the inductive procedure suggested by these derivations proved too hard for the author. After several induction steps to guess the formula, material from the theory of matrix differential equations is used to show that the obvious answer is the correct answer. The explanatory material in the sections on paths shows the first two induction steps while the section on matrix series in the second part presents the material needed to tackle the general case.

The second half of the report contains all the interesting material. The approximation is actually derived from an exact solution and the first two sections in the second half give the algebraic and analytical parts of the derivation for this exact solution. Two of the three elements that generate the approximation formulas from the exact solution are easy: the exponential of a large negative number is nearly zero, and a very large number plus or minus a very small number is nearly equal to the very large number. The third element is the approximation of a multinomial Taylor series, where all the terms are small, by the first nonzero term in the series. A separate section presents the matrix Taylor series solution and identifies its first nonzero term. The last section derives the approximation formula from the exact solution in a bookkeeping proof -- the only challenge is keeping the notation straight.

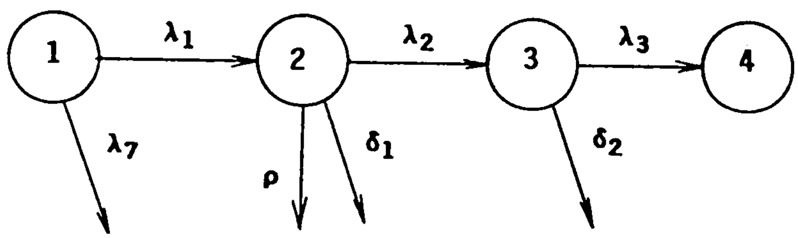
PART I

2.0 PATHS IN A PURE DEATH PROCESS

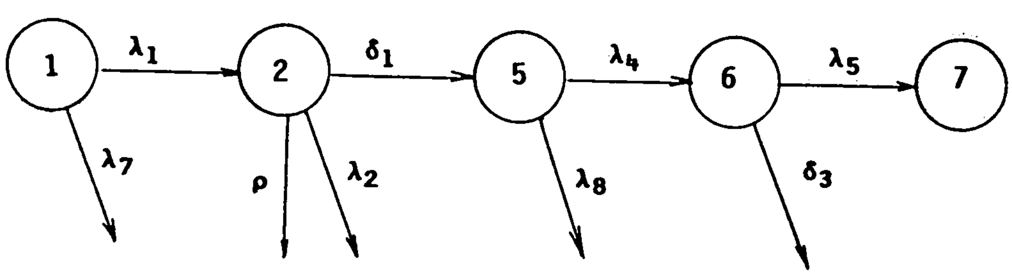
Events in a pure death Markov process move from the initial state to an absorbing state along paths. In the Markov process

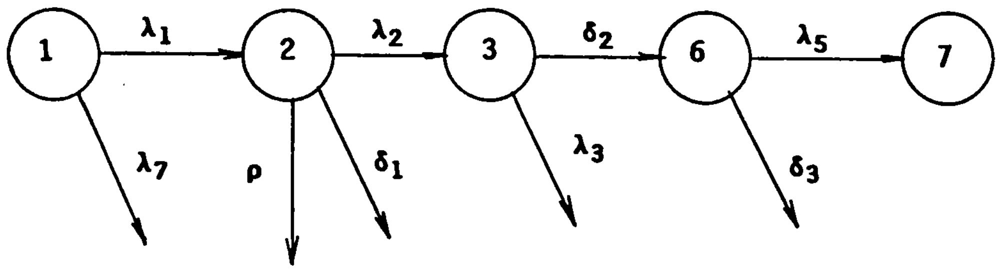


there is one path from state 1 to state 4



and two paths from state 1 to state 7



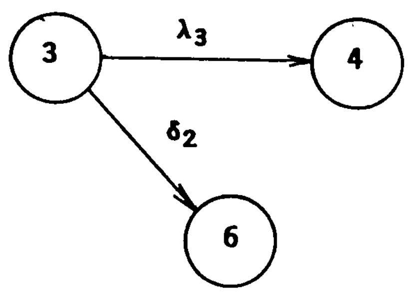


The probability of being in state 4 at time T is the probability of traversing the first path by time T. Even though the second and third paths have some states in common, they are distinct paths. The probability of being in state 7 by time T is the probability of traversing the first path to state 7 by time T plus the probability of traversing the second path to state 7 by time T. In all cases, the probability of traversing a path by a certain time includes the probability of events following that path.

Without attaching any meaning to the Markov process, the probabilities of traversing the path to state 4 and the first path to state 7 will be estimated. The estimations involve computing the probability of going from one state to the next on the path and computing the probability of the transition occurring within the specified time.

The λ's are low fault occurrence rates, the ρ and δ's fast system recovery rates.

To begin, consider



Given the system is in state 3 at time t_0 the probabilities of being in state 4 and state 6 by time t_1 are, respectively,

$$P_{4|3}(t_1|t_0) = \frac{\lambda_3}{\lambda_3 + \delta_2} \left(1 - e^{-(\lambda_3 + \delta_2)(t_1 - t_0)}\right)$$

$$P_{6|3}(t_1|t_0) = \frac{\delta_2}{\lambda_3 + \delta_2} \left(1 - e^{-(\lambda_3 + \delta_2)(t_1 - t_0)}\right) \quad .$$

Of course events will go from state 3 to one of the states 4 or 6, but not to both. In these equations the probability of going to state 4 or 6 is given by the first factor in the expression on the right. The probability of having made the transition by time $t_1 - t_0$ is given by the second factor in the expression, which is the same in both equations. Since δ_2 is large, the second factor in both expressions rapidly approaches 1 as t_1 increases. Hence, the holding time in state 3 is negligible, and

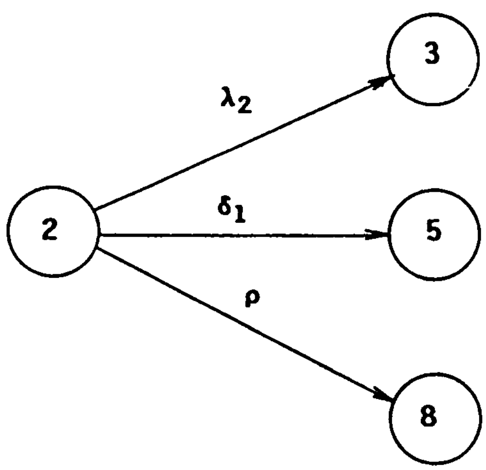
$$P_{4|3} \approx \frac{\lambda_3}{\lambda_3 + \delta_2} \approx \frac{\lambda_3}{\delta_2}$$

$$P_{6|3} \approx \frac{\delta_2}{\lambda_3 + \delta_2} \approx 1$$

where the second approximation comes from

$$\lambda_3 + \delta_2 \approx \delta_2 \quad .$$

To continue working backwards on the path from state 1 to state 4, consider



Similar reasoning gives

$$P_{3|2} \approx \frac{\lambda_2}{\rho + \delta_1}$$

$$P_{5|2} \approx \frac{\lambda_1}{\rho + \delta_1}$$

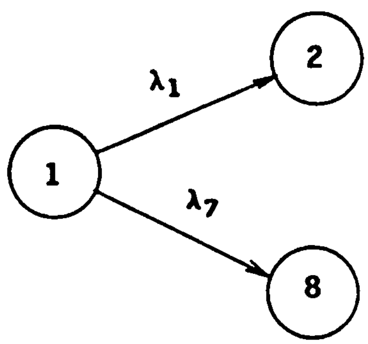
$$P_{8|2} \approx \frac{\rho}{\rho + \delta_1}$$

and the holding time in state 2 is negligible.

By independence, the probability of going from state 2 to state 4 is

$$\left(\frac{\lambda_2}{\rho + \delta_1}\right) \left(\frac{\lambda_3}{\delta_2}\right) .$$

Because of the large transition rates, the transition time from state 2 to state 4 is negligible. Hence, the computation of the probability of going from state 1 to state 4 by time T can be finished by multiplying the expression above and the probability of going from state 1 to state 2 by time T. To this end, consider



and get

$$P_{2|1}(T|0) = \frac{\lambda_1}{\lambda_1 + \lambda_7} \left(1 - e^{-(\lambda_1 + \lambda_7)T}\right) .$$

Since $(\lambda_1 + \lambda_7)T$ is small, the approximation

$$1 - e^{-(\lambda_1 + \lambda_7)T} \approx (\lambda_1 + \lambda_7)T$$

gives

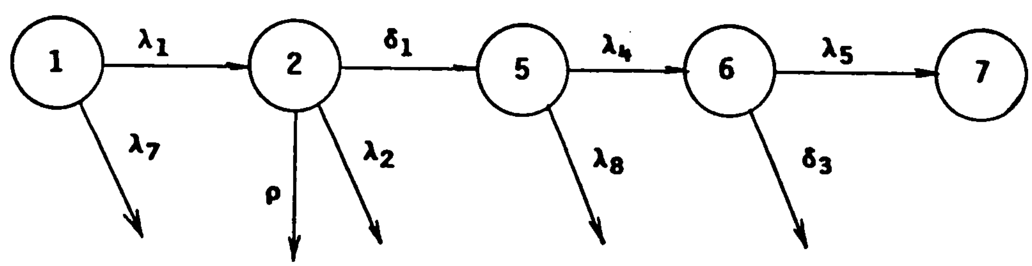
$$P_{2|1}(T|0) \approx \left(\frac{\lambda_1}{\lambda_1 + \lambda_7}\right) (\lambda_1 + \lambda_7)T = \lambda_1 T \ .$$

Notice that the approximation does not involve the low rate leading off the path.

Combining all the material above gives the probability of being in state 4 by time T as

$$P_4(T) \approx (\lambda_1 T) \left(\frac{\lambda_2}{\rho + \delta_1}\right) \left(\frac{\lambda_3}{\delta_2}\right) = \frac{\lambda_1\lambda_2\lambda_3 T}{\delta_2(\rho + \delta_1)} \ .$$

Next, the probability of going from state 1 to state 7 by time T along the path

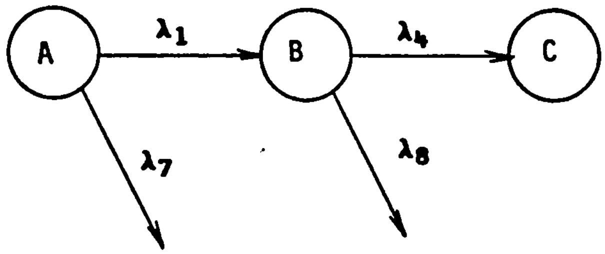


will be estimated to show what happens when there is a sequence of low rate transitions.

The transitions from state 2 to state 5 and from state 6 to state 7 take a negligible amount of time and contribute a factor of

$$\left(\frac{\delta_1}{\rho + \delta_1}\right) \left(\frac{\lambda_5}{\delta_3}\right) \ .$$

Collect the transitions from state 1 to state 2 and from state 5 to state 6 to form the diagram



and get

$$P_{C|A}(T|0) = \lambda_1\lambda_4 \left\{ \frac{e^{-(\lambda_1 + \lambda_7)T}}{(\lambda_4 + \lambda_8 - \lambda_1 - \lambda_7)(-\lambda_1 - \lambda_7)} + \frac{e^{-(\lambda_4 + \lambda_8)T}}{(\lambda_1 + \lambda_7 - \lambda_4 - \lambda_8)(-\lambda_4 - \lambda_8)} + \frac{1}{(\lambda_1 + \lambda_7)(\lambda_4 + \lambda_8)} \right\}$$

$$= \frac{\lambda_1 \lambda_4}{(\lambda_1 + \lambda_7)(\lambda_4 + \lambda_8)(\lambda_1 + \lambda_7 - \lambda_4 - \lambda_8)} \times \left\{(\lambda_4 + \lambda_8)e^{-(\lambda_1 + \lambda_7)T} - (\lambda_1 + \lambda_7)e^{-(\lambda_4 + \lambda_8)T} + (\lambda_1 + \lambda_7 - \lambda_4 - \lambda_8)\right\} .$$

Since $(\lambda_1 + \lambda_7)T$ and $(\lambda_4 + \lambda_8)T$ are small, the approximations

$$1 - e^{-(\lambda_1 + \lambda_7)T} \approx (\lambda_1 + \lambda_7)T - \frac{(\lambda_1 + \lambda_7)^2 T^2}{2}$$

$$1 - e^{-(\lambda_4 + \lambda_8)T} \approx (\lambda_4 + \lambda_8)T - \frac{(\lambda_4 + \lambda_8)^2 T^2}{2}$$

can be used which gives

$$P_{C|A}(T|0) \approx \frac{\lambda_1\ \lambda_4}{(\lambda_1 + \lambda_7)(\lambda_4 + \lambda_8)(\lambda_1 + \lambda_7 - \lambda_4 - \lambda_8)}$$

$$\times \left\{(\lambda_1 + \lambda_7)\left[(\lambda_4 + \lambda_8)T - \frac{(\lambda_1 + \lambda_7)^2\ T^2}{2}\right]\right\}$$

$$-(\lambda_4 + \lambda_8)\left[(\lambda_1 + \lambda_7)T - (\lambda_1 + \lambda_7)^2\ T^2\right]\}$$

$$= \frac{\lambda_1\ \lambda_4\ T^2}{2(\lambda_1 + \lambda_7)(\lambda_4 + \lambda_8)(\lambda_1 + \lambda_7 - \lambda_4 - \lambda_8)}$$

$$\times \left\{-(\lambda_1 + \lambda_7)(\lambda_4 + \lambda_8)^2 + (\lambda_4 + \lambda_8)(\lambda_1 + \lambda_7)^2\right\}$$

$$= \frac{\lambda_1\ \lambda_4\ T^2}{2} \ .$$

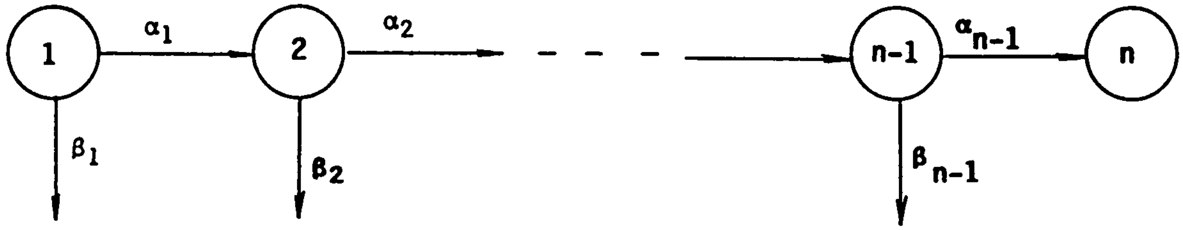
Notice that the terms linear in T vanish, and that once again the low rates leading off the path do not appear in the approximation. The expressions above combine to give the probability of traversing the path by time T as approximately

$$\left(\frac{\lambda_1\ \lambda_4\ T^2}{2}\right)\left(\frac{\delta_1}{\rho + \delta_1}\right)\left(\frac{\lambda_5}{\delta_3}\right) \ .$$

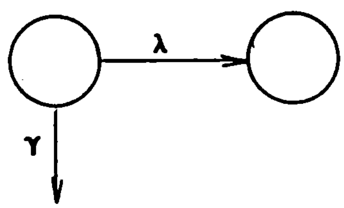
The last derivation ends the illustrative development, and it's now possible to guess the general formulas presented in the next section although two comments are in order before proceeding. First, since the low rates that lead off the path do not appear in the approximation, these low rates can be adjusted to insure that no zeros appear in the denominators of the exact solution. Second, the approximation formulas are given for binary nodes with one rate for the transition on the path and one rate for the transition off the path. Any complex node can be changed to a binary node by adding the rates that lead off the path.

3.0 DESCRIPTION OF THE APPROXIMATION METHOD

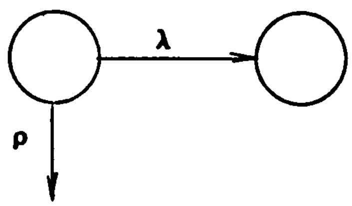
A Markov process model of redundant reconfigurable electronic equipment with permanent faults consists of two types of transition rates -- low rates for component failure and high rates for system recovery. Any path through the Markov model from the initial state to an absorbing state such as



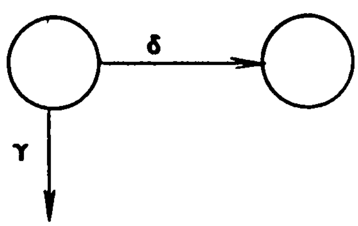
has four classes of transitions:



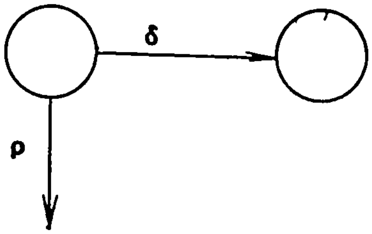
$\lambda << 1$, $\gamma << 1$



$\lambda << 1$, $\rho >> 1$



$\gamma << 1$, $\delta >> 1$

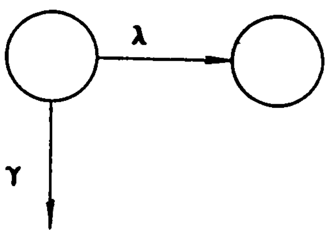


$\delta \gg 1 \quad , \quad \rho \gg 1$

In the diagrams above, λ and δ are special cases of the α's. They represent transitions that stay in the chosen path. The transitions γ and ρ are special cases of the β's. They represent transitions that lead off the path.

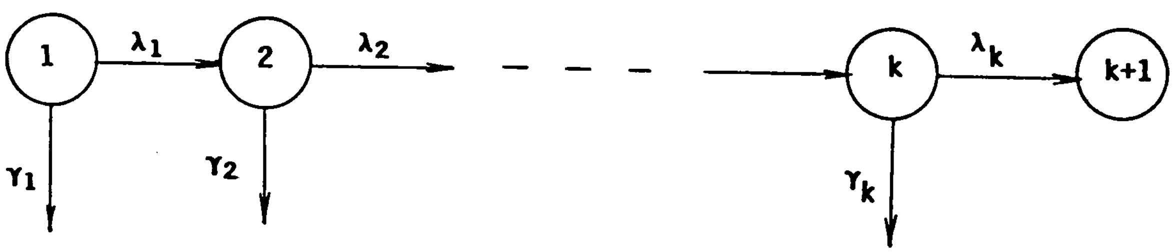
Since the arrival time at the end of the path is the sum of waiting time for independent processes, the path can be rearranged.

To get an approximation for the probability of traversing the path by time T, first collect all transitions of the class



$\lambda \ll 1 \quad , \quad \gamma \ll 1$

to get, supposing there are k of them,

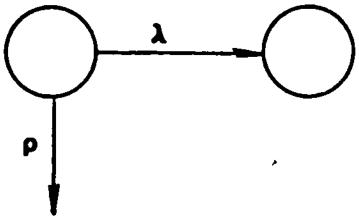


This group contributes a factor of

$$\frac{\lambda_1 \lambda_2 \ldots \lambda_k T^k}{k!}$$

to the probability of traversing the path by time T.

Any transition of the class

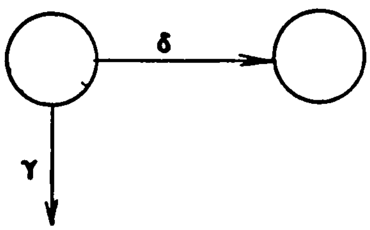


$\lambda \ll 1 \quad , \quad \rho \gg 1$

contributes a factor of

$$\frac{\lambda}{\rho}$$

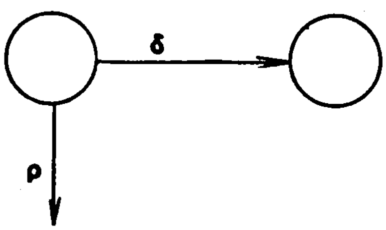
Any transition of the class



$\delta \gg 1 \quad , \quad \gamma \ll 1$

contributes a factor of 1.

Any transition of the class



$\delta \gg 1 \quad , \quad \rho \gg 1$

contributes a factor of

$$\frac{\delta}{\delta + \rho} .$$

At the moment there is no error analysis, but the approximations appear to be accurate for the parameter bounds

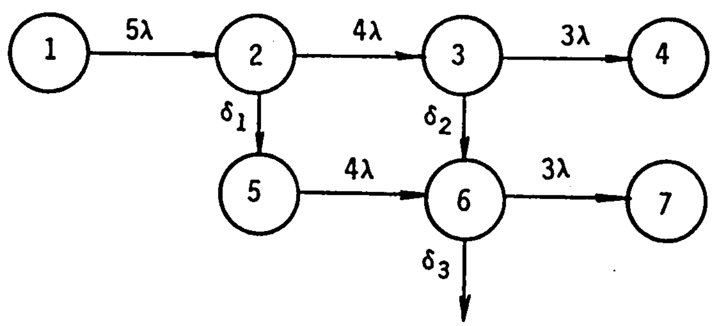
$1 \leq T \leq 10$	
$\lambda T \leq 10^{-2}$	all λ's
$\gamma T \leq 10^{-2}$	all γ's
$\delta \geq 10^{2}$	all δ's
$\rho \geq 10^{2}$	all ρ's

The next sections compare the approximation to a numerical solution for a variety of reliability models. These sections assume familiarity with Markov models of fault tolerant computers. The reference on reliable system design gives an engineering presentation of this material.

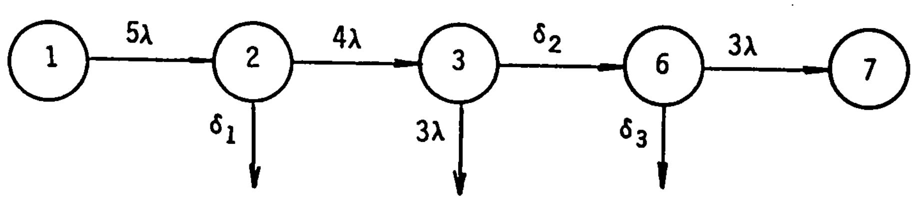
4.0 APPLYING THE FORMULA

To apply the approximation method, it is necessary to identify the paths to an absorbing state and then separate the states on a path into two classes--the states with at least one high exiting rate and the states with all low exiting rates. The system spends a negligible amount of time in states that have at least one high exiting rate, and these states can be considered individually. The system spends an appreciable amount of time in states that have only low exiting rates, and these states must be considered as a group.

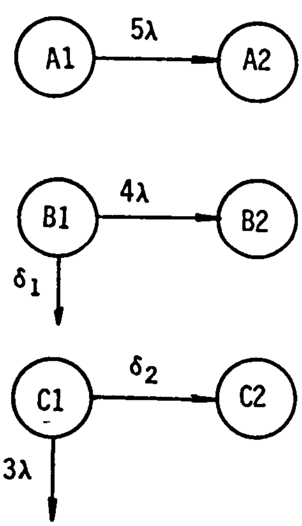
As an example consider state 7 in the diagram below

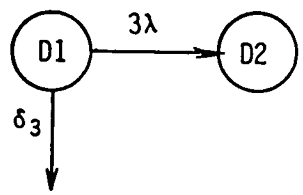


Rearrange the first path to state 7 which is



as





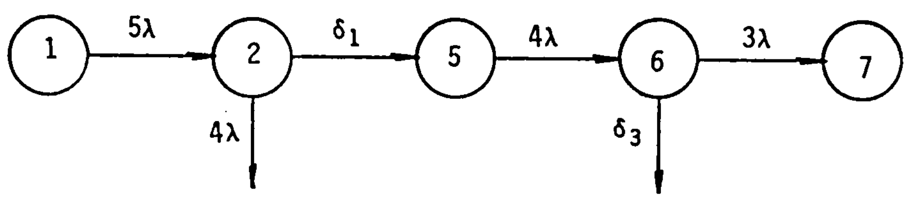
where the transitions, respectively, contribute factors of

$$5\lambda T,\ \frac{4\lambda}{\delta_1}\ ,\ 1,\ \frac{3\lambda}{\delta_3}$$

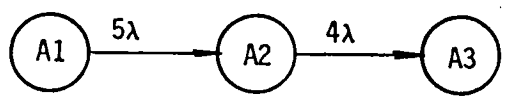
which gives the contributions from the first path as

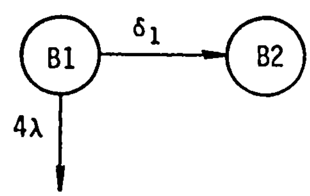
$$\frac{60\ \lambda^3\ T}{\delta_1\ \delta_3}.$$

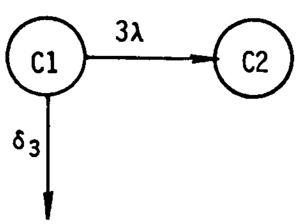
Rearrange the second path to state 7 which is



as







where the transitions, respectively, contribute factors of

$$\frac{20\ \lambda^2\ T^2}{2!}\ ,\ 1,\ \frac{3\lambda}{\delta_3}$$

which gives the contribution from the second path as

$$\frac{30\ \lambda^3\ T^2}{\delta_3}.$$

Since the paths are distinct, they are disjoint events. The probability of being in state 7 by time T given the system is in state 1 at time 0 is

$$\frac{60\ \lambda^3\ T}{\delta_1\ \delta_3} + \frac{30\ \lambda^3\ T^2}{\delta_3}.$$

$$P_9(T) = \frac{9\lambda^2\ T^2}{2} \quad .$$

In the numerical comparisons note that if

$$\lambda = 10^{-2}$$

$$T = 1$$

then

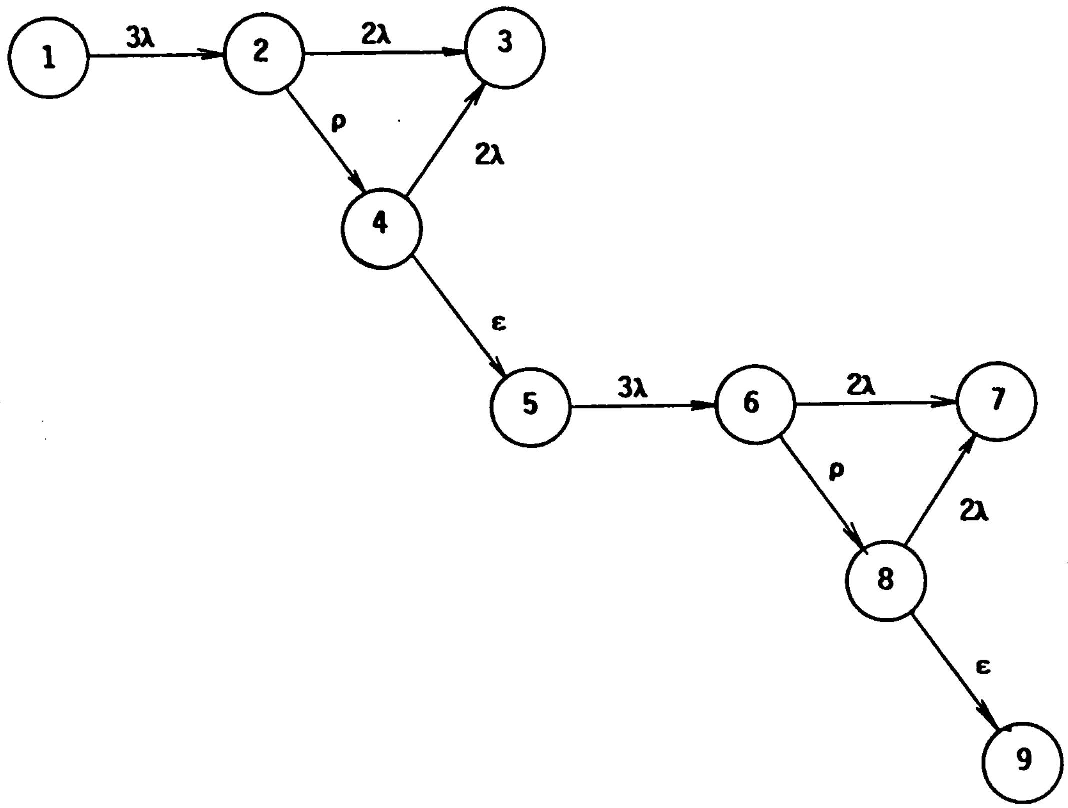
$$3\lambda T > 10^{-2} \ .$$

The numerical comparisons for this example, and for later examples, do not adhere strictly to the parameter bounds mentioned in the section that described the formulas.

For the examples in this and the next three sections, the model is followed by the approximation formulas and then tables comparing the approximate and exact solutions for parameter values that are close to the parameter bounds.

5.0 EXAMPLE: THREE-PLEX WITH A TWO STEP RECOVERY

The Markov reliability model is



with $\lambda \ll 1$ and $\rho, \varepsilon \gg 1$.

The approximations are

$$P_3(T) = (3\lambda T)\left(\frac{2\lambda}{\rho}\right) + (3\lambda T)\left(\frac{2\lambda}{\varepsilon}\right)$$

$$= 6\lambda^2 T \left(\frac{1}{\rho} + \frac{1}{\varepsilon}\right)$$

$$P_7(T) = \left(\frac{9\lambda^2\ T^2}{2}\right)\left(\frac{2\lambda}{\rho}\right) + \left(\frac{9\lambda^2\ T^2}{2}\right)\left(\frac{2\lambda}{\varepsilon}\right)$$

$$= 9\lambda^3\ T^2 \left(\frac{1}{\rho} + \frac{1}{\varepsilon}\right)$$

TABLE I

$T = 1$

	STATE	EXACT	FORMULA
$\lambda = 10^{-2}$	3	1.16×10^{-5}	1.20×10^{-5}
$\rho = 10^{2}$; $\varepsilon = 10^{2}$	7	1.64×10^{-7}	1.80×10^{-7}
	9	4.07×10^{-4}	4.50×10^{-4}
$\lambda = 10^{-2}$	3	6.44×10^{-6}	6.60×10^{-6}
$\rho = 10^{2}$; $\varepsilon = 10^{3}$	7	9.30×10^{-8}	9.90×10^{-8}
	9	4.22×10^{-4}	4.50×10^{-4}
$\lambda = 10^{-2}$	3	6.44×10^{-6}	6.60×10^{-6}
$\rho = 10^{3}$; $\varepsilon = 10^{2}$	7	9.30×10^{-8}	9.90×10^{-8}
	9	4.22×10^{-4}	4.50×10^{-4}
$\lambda = 10^{-2}$	3	1.18×10^{-6}	1.20×10^{-6}
$\rho = 10^{3}$; $\varepsilon = 10^{3}$	7	1.75×10^{-8}	1.80×10^{-8}
	9	4.38×10^{-4}	4.50×10^{-4}

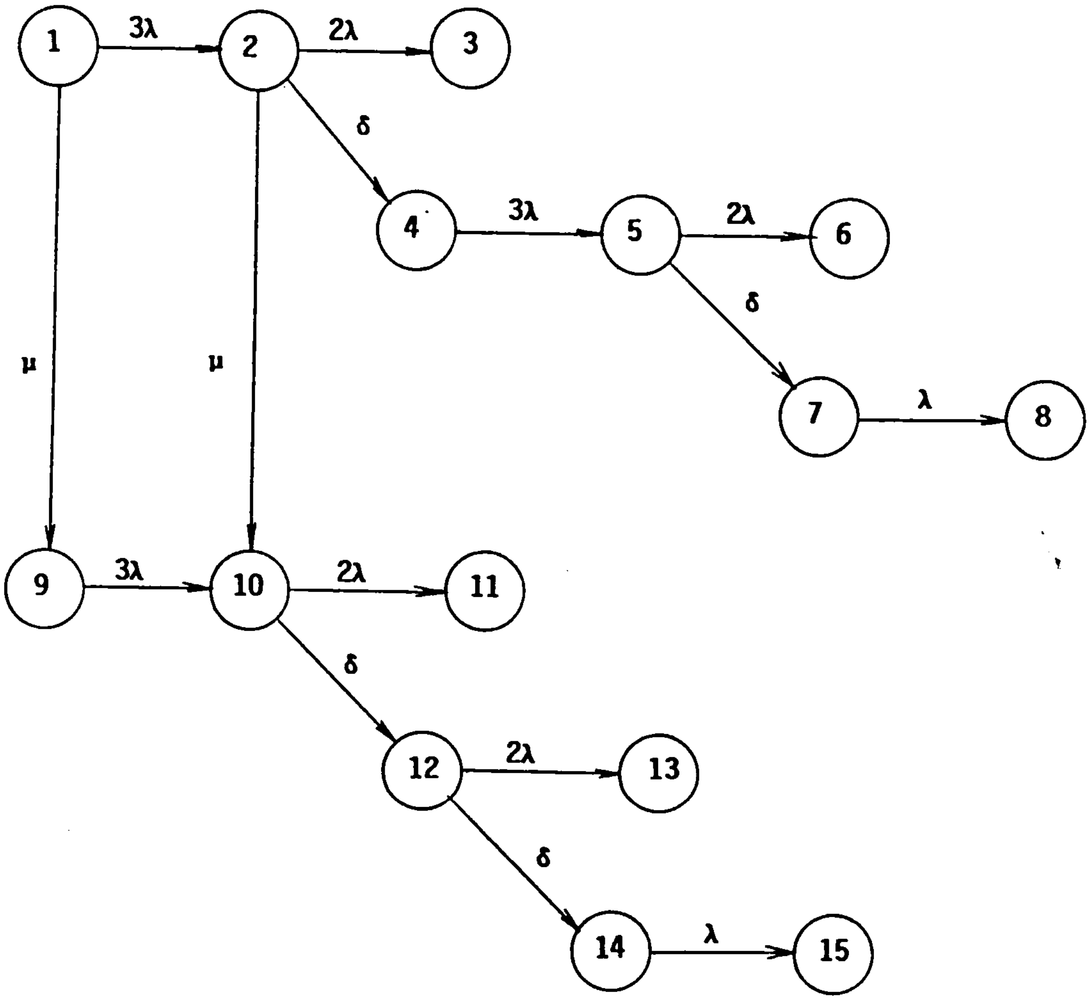
TABLE II

$T = 10$

	STATE	EXACT	FORMULA
$\lambda = 10^{-3}$	3	1.18×10^{-6}	1.20×10^{-6}
$\rho = 10^2$; $\varepsilon = 10^2$	7	1.75×10^{-8}	1.80×10^{-8}
	9	4.38×10^{-4}	4.50×10^{-4}
$\lambda = 10^{-3}$	3	1.18×10^{-7}	1.20×10^{-7}
$\rho = 10^3$; $\varepsilon = 10^3$	7	1.76×10^{-9}	1.80×10^{-9}
	9	4.41×10^{-4}	4.50×10^{-4}
$\lambda = 10^{-4}$	3	1.20×10^{-8}	1.20×10^{-8}
$\rho = 10^2$; $\varepsilon = 10^2$	7	1.78×10^{-11}	1.80×10^{-11}
	9	4.46×10^{-6}	4.50×10^{-6}
$\lambda = 10^{-4}$	3	1.20×10^{-9}	1.20×10^{-9}
$\rho = 10^3$; $\varepsilon = 10^3$	7	1.80×10^{-12}	1.80×10^{-12}
	9	4.49×10^{-6}	4.50×10^{-6}

6.0 EXAMPLE: TRIAD PLUS A COOL SPARE

The Markov reliability model is



with $\delta \gg 1$ and $\mu, \lambda \ll 1$.

The approximations are

$$P_3(T) = \frac{6\lambda^2 \; T}{\delta}$$

$$P_6(T) = \frac{9\lambda^3 \; T^2}{\delta}$$

$$P_8(T) = \frac{3\lambda^3 \; T^3}{2}$$

$$P_{11}(T) = \frac{3\lambda^2 \; \mu \; T^2}{\delta} + \frac{6 \; \lambda^2 \; \mu \; T}{\delta^2}$$

$$P_{13}(T) = \frac{3\lambda^2 \; \mu \; T^2}{2} + \frac{6\lambda^2 \; \mu \; T}{\delta^2}$$

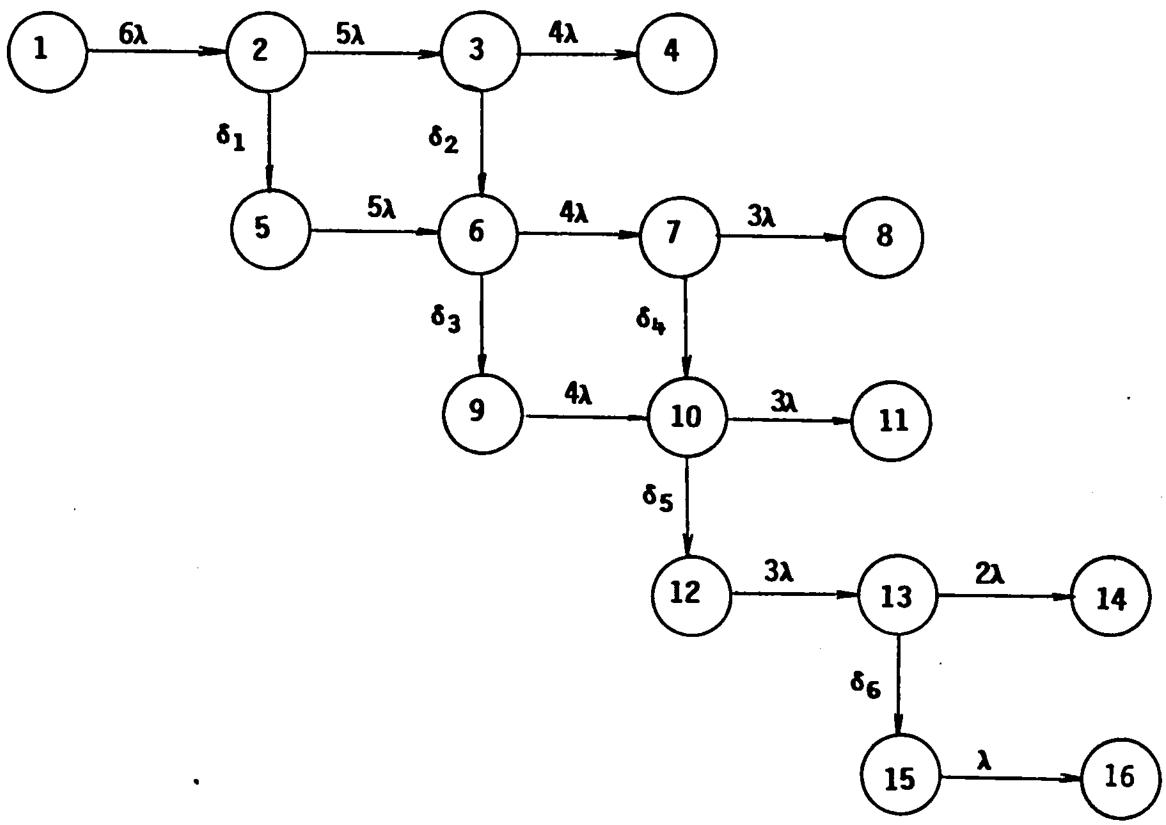
$$P_{15}(T) = \frac{\lambda^2 \; \mu \; T^2}{2} + \frac{3\lambda^2 \; \mu \; T^2}{2\delta}$$

TABLE III

	STATE	EXACT	FORMULA
$T = 1$	3	5.85×10^{-6}	6.00×10^{-6}
	6	8.47×10^{-8}	9.00×10^{-8}
$\lambda = 10^{-2}$	8	1.39×10^{-6}	1.50×10^{-6}
$\mu = 10^{-3}$	11	2.94×10^{-9}	3.06×10^{-9}
$\delta = 10^{2}$	13	2.88×10^{-9}	3.06×10^{-9}
	15	4.77×10^{-8}	5.15×10^{-8}
$T = 1$	3	5.98×10^{-9}	6.00×10^{-9}
	6	8.95×10^{-12}	9.00×10^{-12}
$\lambda = 10^{-3}$	8	1.49×10^{-9}	1.50×10^{-9}
$\mu = 10^{-4}$	11	2.99×10^{-13}	3.01×10^{-13}
$\delta = 10^{3}$	13	2.99×10^{-13}	3.01×10^{-13}
	15	4.98×10^{-11}	5.02×10^{-11}
$T = 10$	3	5.90×10^{-7}	6.00×10^{-7}
	6	8.78×10^{-9}	9.00×10^{-9}
$\lambda = 10^{-3}$	8	1.46×10^{-6}	1.50×10^{-6}
$\mu = 10^{-4}$	11	2.94×10^{-10}	3.01×10^{-10}
$\delta = 10^{2}$	13	2.93×10^{-10}	3.01×10^{-10}
	15	4.90×10^{-8}	5.02×10^{-8}
$T = 10$	3	5.99×10^{-10}	6.00×10^{-10}
	6	8.98×10^{-13}	9.00×10^{-13}
$\lambda = 10^{-4}$	8	1.50×10^{-9}	1.50×10^{-9}
$\mu = 10^{-5}$	11	2.99×10^{-14}	3.00×10^{-14}
$\delta = 10^{3}$	13	2.99×10^{-14}	3.00×10^{-14}
	15	4.99×10^{-11}	5.00×10^{-11}

7.0 EXAMPLE: CRITICAL TRIPLE SIX-PLEX

The Markov reliability model is



with $\lambda \ll 1$ and $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \gg 1$.

TABLE V

$\lambda = 10^{-3}$

$\delta_1 = 10^7$; $\delta_2 = 10^6$; $\delta_3 = 10^5$; $\delta_4 = 10^4$; $\delta_5 = 10^3$; $\delta_6 = 10^2$

	STATE	EXACT	FORMULA
T = 1	4	1.20×10^{-20}	1.20×10^{-20}
	8	1.79×10^{-19}	1.84×10^{-19}
	11	5.96×10^{-14}	6.18×10^{-14}
	14	2.86×10^{-16}	3.12×10^{-16}
	16	2.83×10^{-15}	3.15×10^{-15}
T = 10	4	1.16×10^{-19}	1.20×10^{-19}
	8	1.74×10^{-17}	1.80×10^{-17}
	11	5.78×10^{-11}	6.02×10^{-11}
	14	2.88×10^{-12}	3.01×10^{-12}
	16	2.89×10^{-10}	3.02×10^{-10}

The approximations are

$$P_4(T) = \frac{120\ \lambda^3\ T}{\delta_1\ \delta_2}$$

$$P_8(T) = \frac{180\ \lambda^4\ T^2}{\delta_3\ \delta_4} + \frac{360\ \lambda^4\ T}{\delta_1\ \delta_3\ \delta_4}$$

$$P_{11}(T) = \frac{60\ \lambda^4\ T}{\delta_5} + \frac{180\ \lambda^4\ T^2}{\delta_3\ \delta_5} + \frac{180\ \lambda^4\ T^2}{\delta_1\ \delta_5} + \frac{360\ \lambda^4\ T}{\delta_1\ \delta_3\ \delta_5}$$

$$P_{14}(T) = \frac{30\ \lambda^5\ T^4}{\delta_6} + \frac{120\ \lambda^5\ T^3}{\delta_3\ \delta_6} + \frac{120\ \lambda^5\ T^3}{\delta_1\ \delta_6} + \frac{360\ \lambda^5\ T^2}{\delta_1\ \delta_3\ \delta_6}$$

$$P_{16}(T) = 3\ \lambda^5\ T^5 + \frac{15\ \lambda^5\ T^4}{\delta_3} + \frac{15\ \lambda^5\ T^4}{\delta_1} + \frac{60\ \lambda^5\ T^3}{\delta_1\ \delta_3}$$

TABLE IV

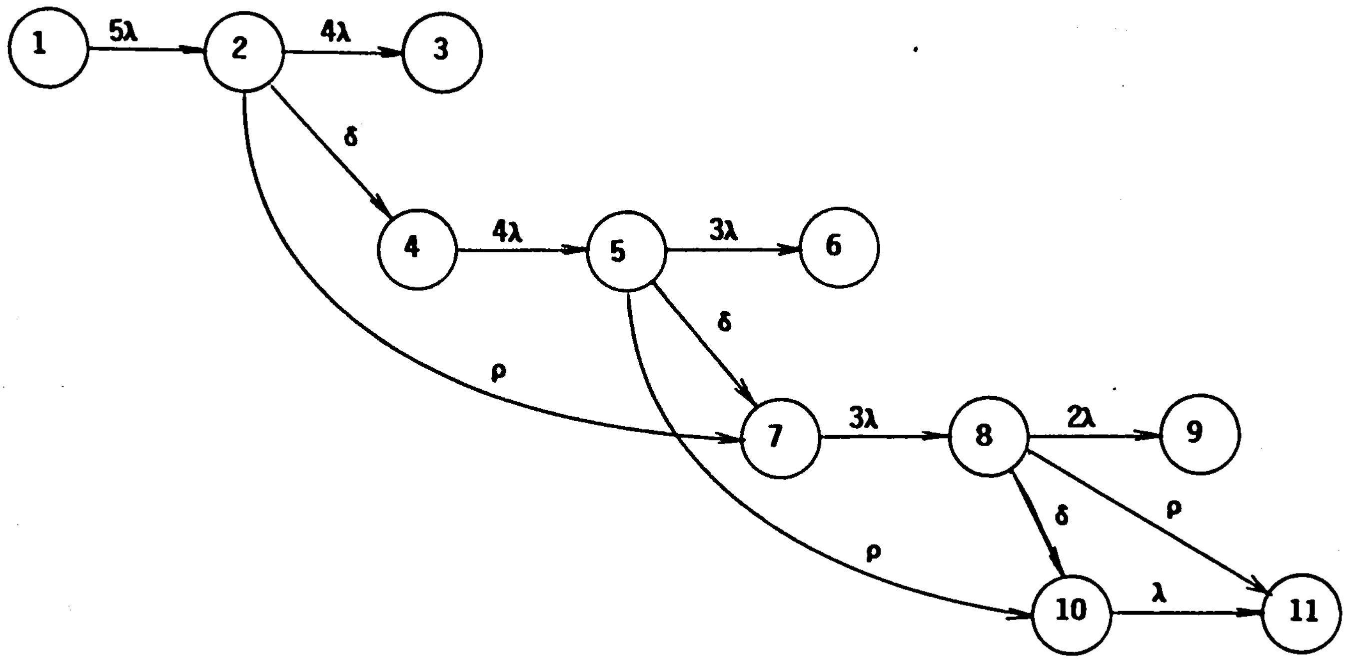
$\lambda = 10^{-3}$

$\delta_1 = 10^2$; $\delta_2 = 10^3$; $\delta_3 = 10^4$; $\delta_4 = 10^5$; $\delta_5 = 10^6$; $\delta_6 = 10^7$

	STATE	EXACT	FORMULA
T = 1	4	1.18×10^{-12}	1.20×10^{-12}
	8	1.79×10^{-19}	1.84×10^{-19}
	11	5.98×10^{-17}	6.18×10^{-17}
	14	2.99×10^{-21}	3.12×10^{-21}
	16	2.99×10^{-15}	3.15×10^{-15}
T = 10	4	1.16×10^{-11}	1.20×10^{-11}
	8	1.74×10^{-17}	1.80×10^{-17}
	11	5.78×10^{-14}	6.02×10^{-14}
	14	2.89×10^{-17}	3.01×10^{-17}
	16	2.91×10^{-10}	3.02×10^{-10}

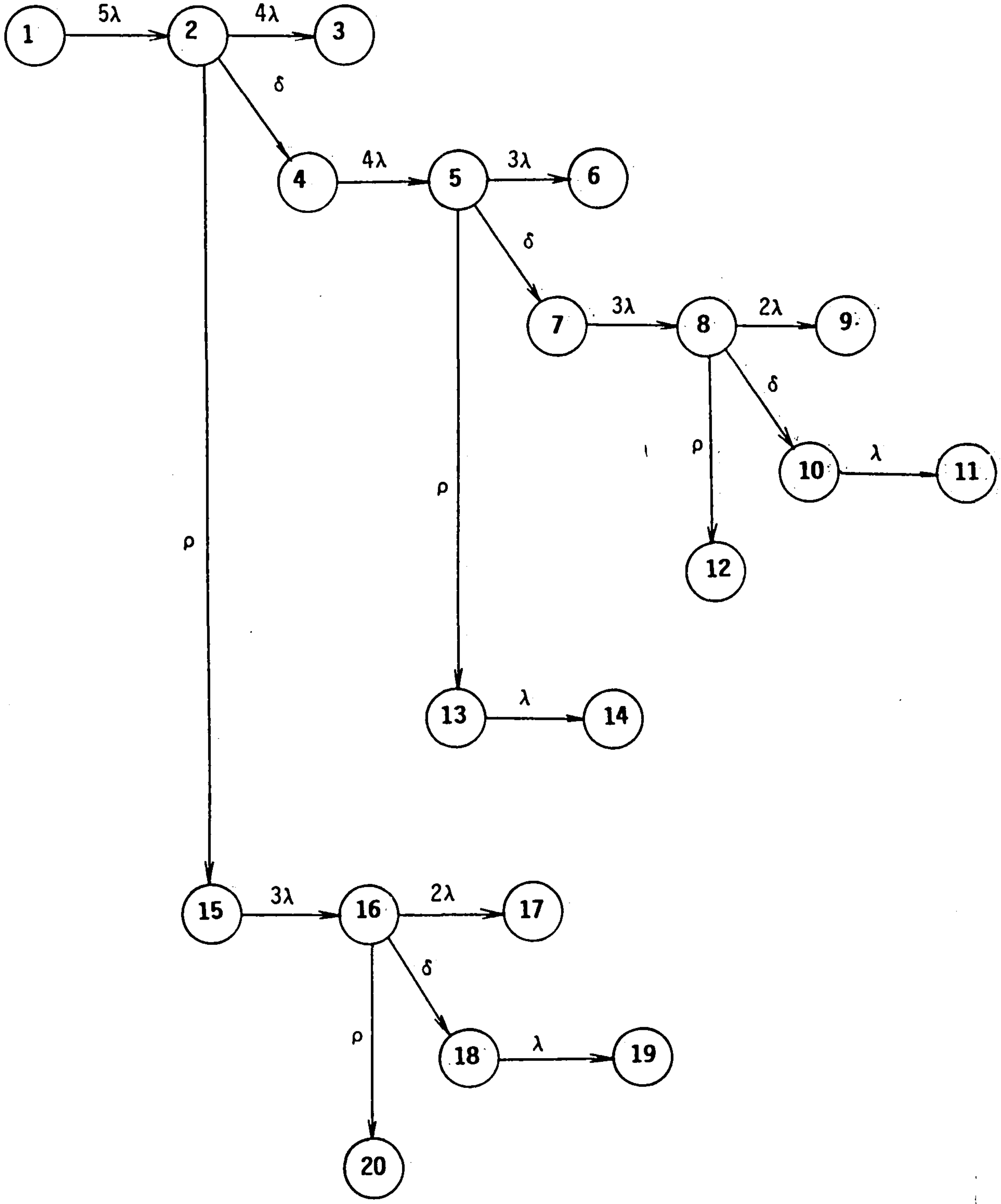
8.0 EXAMPLE: CRITICAL PAIR FIVE-PLEX WITH POOR RECOVERY

This system may excise one too many components during reconfiguration. The original Markov reliability model is



The δ refers to good recovery, the ρ to poor recovery.

An extended model for purposes of comparison with exact numerical solutions is



The approximation formulas for the extended model are

$$P_3(T) = \frac{20\ \lambda^2\ T}{(\delta + \rho)}$$

$$P_6(T) = \frac{30\ \lambda^3\ \delta\ T^2}{(\delta + \rho)^2}$$

$$P_9(T) = \frac{20\ \lambda^4\ \delta^2\ T^3}{(\delta + \rho)^3}$$

$$P_{11}(T) = \frac{5\ \lambda^4\ \delta^3\ T^4}{2(\delta + \rho)^3}$$

$$P_{12}(T) = \frac{10\ \lambda^3\ \delta^2\ \rho\ T^3}{(\delta + \rho)^3}$$

$$P_{14}(T) = \frac{10\ \lambda^3\ \delta\ \rho\ T^3}{3(\delta + \rho)^2}$$

$$P_{17}(T) = \frac{15\ \lambda^3\ \rho\ T^2}{(\delta + \rho)^2}$$

$$P_{19}(T) = \frac{5\ \lambda^3\ \rho\ \delta\ T^3}{2(\delta + \rho)^2}$$

$$P_{20}(T) = \frac{15\ \lambda^2\ \rho^2\ T^2}{2(\delta + \rho)^2}\ .$$

TABLE VI

$\lambda = 10^{-3}$

$\delta = 10^{2}$; $\rho = 10$

	STATE	EXACT	FORMULA
T = 1	3	1.80×10^{-7}	1.82×10^{-7}
	6	2.38×10^{-10}	2.48×10^{-10}
	9	1.38×10^{-13}	1.50×10^{-13}
	11	1.68×10^{-12}	1.88×10^{-12}
	12	6.40×10^{-10}	7.51×10^{-10}
	14	2.60×10^{-10}	2.75×10^{-10}
	17	1.19×10^{-11}	1.24×10^{-11}
	19	1.95×10^{-10}	2.07×10^{-10}
	20	5.96×10^{-8}	6.20×10^{-8}
T = 10	3	1.77×10^{-6}	1.82×10^{-6}
	6	2.40×10^{-8}	2.48×10^{-8}
	9	1.45×10^{-10}	1.50×10^{-10}
	11	1.81×10^{-8}	1.88×10^{-8}
	12	7.23×10^{-7}	7.51×10^{-7}
	14	2.67×10^{-7}	2.75×10^{-7}
	17	1.20×10^{-9}	1.24×10^{-9}
	19	2.01×10^{-7}	2.07×10^{-7}
	20	6.01×10^{-6}	6.20×10^{-6}

TABLE VII

$\lambda = 10^{-3}$

$\delta = 10; \quad \rho = 10^2$

	STATE	EXACT	FORMULA
T = 1	3	1.80×10^{-7}	1.82×10^{-7}
	6	2.38×10^{-11}	2.48×10^{-11}
	9	1.38×10^{-15}	1.50×10^{-15}
	11	1.68×10^{-15}	1.88×10^{-15}
	12	6.40×10^{-11}	7.51×10^{-11}
	14	2.60×10^{-10}	2.75×10^{-10}
	17	1.19×10^{-10}	1.24×10^{-10}
	19	1.95×10^{-10}	2.07×10^{-10}
	20	5.96×10^{-6}	6.20×10^{-6}
T = 10	3	1.77×10^{-6}	1.82×10^{-6}
	6	2.40×10^{-9}	2.48×10^{-9}
	9	1.45×10^{-12}	1.50×10^{-12}
	11	1.81×10^{-11}	1.88×10^{-11}
	12	7.23×10^{-8}	7.51×10^{-8}
	14	2.67×10^{-7}	2.75×10^{-7}
	17	1.20×10^{-8}	1.24×10^{-8}
	19	2.01×10^{-7}	2.07×10^{-7}
	20	6.01×10^{-4}	6.20×10^{-4}

PART II

9.0 ALGEBRAIC RESULT FOR THE EXACT SOLUTION

The following technical result is used in the next section.

<u>Lemma</u> If $a_i \neq a_j$ for $i \neq j$, then

$$- \sum_{i=1}^{k} \frac{1}{\prod_{\substack{j=1 \\ j\neq i}}^{k+1} (a_j - a_i)} = \frac{1}{\prod_{j=1}^{k} (a_j - a_{k+1})} .$$

Proof

The proof is by induction. Only the induction step is shown.

Let

$$D = \sum_{i=1}^{k} \frac{1}{\prod_{\substack{j=i \\ j\neq 1}}^{k+1} (a_j - a_i)} + \frac{1}{\prod_{j=1}^{k} (a_j - a_{k+1})} .$$

By the induction hypothesis,

$$\frac{1}{\prod_{j=1}^{k} (a_j - a_{k+1})} = \frac{1}{(a_1 - a_{k+1})} \frac{1}{\prod_{j=2}^{k} (a_j - a_{k+1})}$$

$$= - \frac{1}{(a_1 - a_{k+1})} \sum_{i=2}^{k} \frac{1}{\prod_{\substack{j=2 \\ j\neq i}}^{k+1} (a_j - a_i)} .$$

Hence,

$$D = \frac{1}{\prod_{j=2}^{k+1} (a_j - a_1)} + \sum_{i=2}^{k} \frac{1}{\prod_{\substack{j=1 \\ j \neq i}}^{k+1} (a_j - a_i)} - \frac{1}{(a_1 - a_{k+1})} \sum_{i=2}^{k} \frac{1}{\prod_{\substack{j=2 \\ j \neq i}}^{k+1} (a_j - a_i)}$$

$$= \frac{1}{\prod_{j=2}^{k+1} (a_j - a_1)}$$

$$+ \sum_{i=2}^{k} \left[\frac{1}{(a_1 - a_i) \prod_{\substack{j=2 \\ j \neq i}}^{k+1} (a_j - a_i)} - \frac{1}{(a_1 - a_{k+1}) \prod_{\substack{j=2 \\ j \neq i}}^{k+1} (a_j - a_i)} \right]$$

$$= \frac{1}{\prod_{j=2}^{k+1} (a_j - a_1)}$$

$$+ \sum_{i=2}^{k} \left[\frac{a_1 - a_{k+1} - a_1 + a_i}{(a_1 - a_i)(a_1 - a_{k+1}) \prod_{\substack{j=2 \\ j \neq i}}^{k+1} (a_j - a_i)} \right]$$

$$= \frac{1}{\prod_{j=2}^{k+1} (a_j - a_1)}$$

$$- \sum_{i=2}^{k} \left[\frac{(a_{k+1} - a_i)}{(a_1 - a_i)(a_1 - a_{k+1})(a_{k+1} - a_i) \prod_{\substack{j=2 \\ j \neq i}}^{k+1} (a_j - a_i)} \right]$$

$$= \frac{1}{\prod_{j=2}^{k+1} (a_j - a_1)} - \sum_{j=2}^{k} \frac{1}{(a_1 - a_{k+1}) \prod_{\substack{j=1 \\ j\neq i}}^{k} (a_j - a_i)}$$

$$= \frac{1}{(a_{k+1} - a_1)} \left[\frac{1}{\prod_{j=2}^{k} (a_j - a_1)} + \sum_{i=2}^{k} \frac{1}{\prod_{\substack{j=1 \\ j\neq i}}^{k} (a_j - a_i)}\right]$$

Apply the induction hypothesis to the second term inside the brackets to get

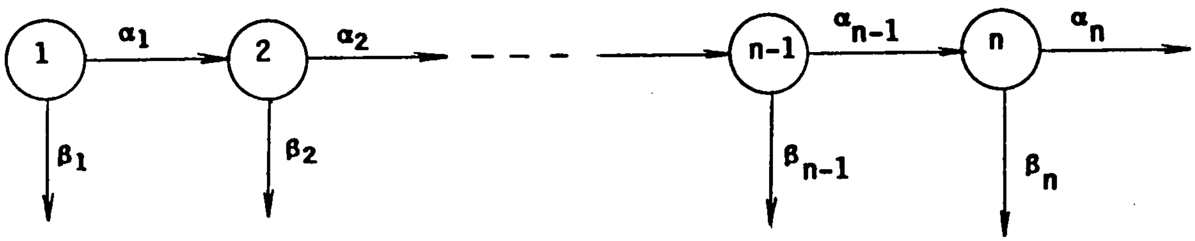
$$D = \frac{1}{(a_{k+1} - a_1)} \left[\frac{1}{\prod_{j=2}^{k} (a_j - a_1)} - \frac{1}{\prod_{j=2}^{k} (a_j - a_1)}\right]$$

$$= 0$$

and the lemma is proved.

10.0 DERIVATION OF THE EXACT SOLUTION

<u>Theorem</u> If $\alpha_i + \beta_i \neq \alpha_j + \beta_j$ for $i \neq j$, then the Markov process



with the initial conditions

$$P_1(0) = 1$$

$$P_1(0) = P_3(0) = \ldots = P_n(0) = 0$$

has the solution

$$P_n(t) = \alpha_1 \ldots \alpha_{n-1} \sum_{i=1}^{n} \frac{e^{-(\alpha_i + \beta_i)t}}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j + \beta_j - \alpha_i - \beta_i)}$$

Proof

The proof is by induction. Only the induction step is shown.

The set of differential equations for a Markov process gives

$$P_n'(t) = \alpha_{n-1} P_{n-1}(t) - (\alpha_n + \beta_n) P_n(t)$$

or

$$[e^{(\alpha_n + \beta_n)t} P_n(t)]' = \alpha_{n-1} e^{(\alpha_n + \beta_n)t} P_{n-1}(t) \quad .$$

By the induction hypothesis,

$$[\, e^{(\alpha_n + \beta_n)t} \; P_n(t)]' = \alpha_1 \cdots \alpha_{n-1} \sum_{i=1}^{n-1} \frac{e^{(\alpha_n + \beta_n - \alpha_i - \beta_i)t}}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n-1} (\alpha_j + \beta_j - \alpha_i - \beta_i)} \, .$$

Hence,

$$P_n(t) = c\, e^{-(\alpha_n + \beta_n)t} + \alpha_1 \cdots \alpha_{n-1} \sum_{i=1}^{n-1} \frac{e^{-(\alpha_i + \beta_i)t}}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (\alpha_j + \beta_j - \alpha_i - \beta_i)} \, .$$

By the initial conditions

$$C = -\alpha_1 \cdots \alpha_{n-1} \sum_{i=1}^{n-1} \frac{1}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (\alpha_j + \beta_j - \alpha_i - \beta_i)} \, .$$

By the previous technical lemma,

$$C = \alpha_1 \cdots \alpha_{n-1} \frac{1}{\prod\limits_{j=1}^{n-1} (\alpha_j + \beta_j - \alpha_n - \beta_n)} \, .$$

Hence,

$$P_n(t) = \alpha_1 \cdots \alpha_n \sum_{i=1}^{n} \frac{e^{-(\alpha_i + \beta_i)t}}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (\alpha_j + \beta_j - \alpha_i - \beta_i)} \, .$$

and the theorem is proved.

11.0 THE MATRIX TAYLOR SERIES

Recall that the set of linear differential equations

$$\begin{bmatrix} P_1'(t) \\ P_2'(t) \\ \vdots \\ P_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n_1} & a_{n_2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_n(t) \end{bmatrix}$$

with the initial conditions

$$\begin{bmatrix} P_1(0) \\ P_2(0) \\ \vdots \\ P_n(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

has the solution

$$\begin{bmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_n(t) \end{bmatrix} = e^{At} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

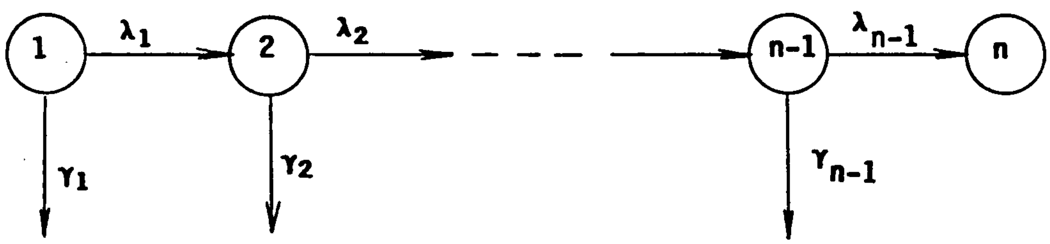
where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n_1} & a_{n_2} & \cdots & a_{nn} \end{bmatrix}$$

and

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \quad .$$

<u>Theorem</u> For the Markov process



with the initial conditions

$$P_1(0) = 1$$

$$P_2(0) = \ldots = P_n(0) = 0 \quad ,$$

the first nonzero term of $P_n(t)$ in the matrix series expansion is

$$\frac{\lambda_1 \lambda_2 \ldots \lambda_{n-1} t^{n-1}}{(n-1)!} \quad .$$

Proof

The coefficient matrix of the set of linear differential equations for the Markov process is

$$A = \begin{bmatrix} -(\lambda_1 + \gamma_1) & & & & \\ \lambda_1 & -(\lambda_2 + \gamma_2) & & & \\ & \lambda_2 & \ddots & & \\ & & & -(\lambda_{n-1} + \gamma_{n-1}) & \\ & & & \lambda_{n-1} & 0 \end{bmatrix}$$

with all the entries not on the diagonal or subdiagonal being zero.

Let

$$\begin{bmatrix} M \end{bmatrix}_{n,1}$$

denote the entry in row n and column 1 of the matrix M. Now

$$\begin{bmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_n(t) \end{bmatrix} = e^{At} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which means

$$P_n(t) = \begin{bmatrix} e^{At} \end{bmatrix}_{n,1} .$$

Since

$$\left[A^k\right]_{n,1} = \begin{cases} 0 & 0 \le k < n-1 \\ \lambda_1 \lambda_2 \ldots \lambda_{n-1} & k = n-1 \end{cases},$$

the theorem is proved.

12.0 DERIVATION OF THE APPROXIMATION FORMULA

Given a pure death Markov process place any path of length n that goes from the initial state to a death state in the form given on the next page where

$$1 \leq n_1 \leq n_2 \leq n_3 \leq n_4 = n$$

and

$$\lambda_1 << 1 \qquad \text{all } i$$

$$\gamma_i << 1 \qquad \text{all } i$$

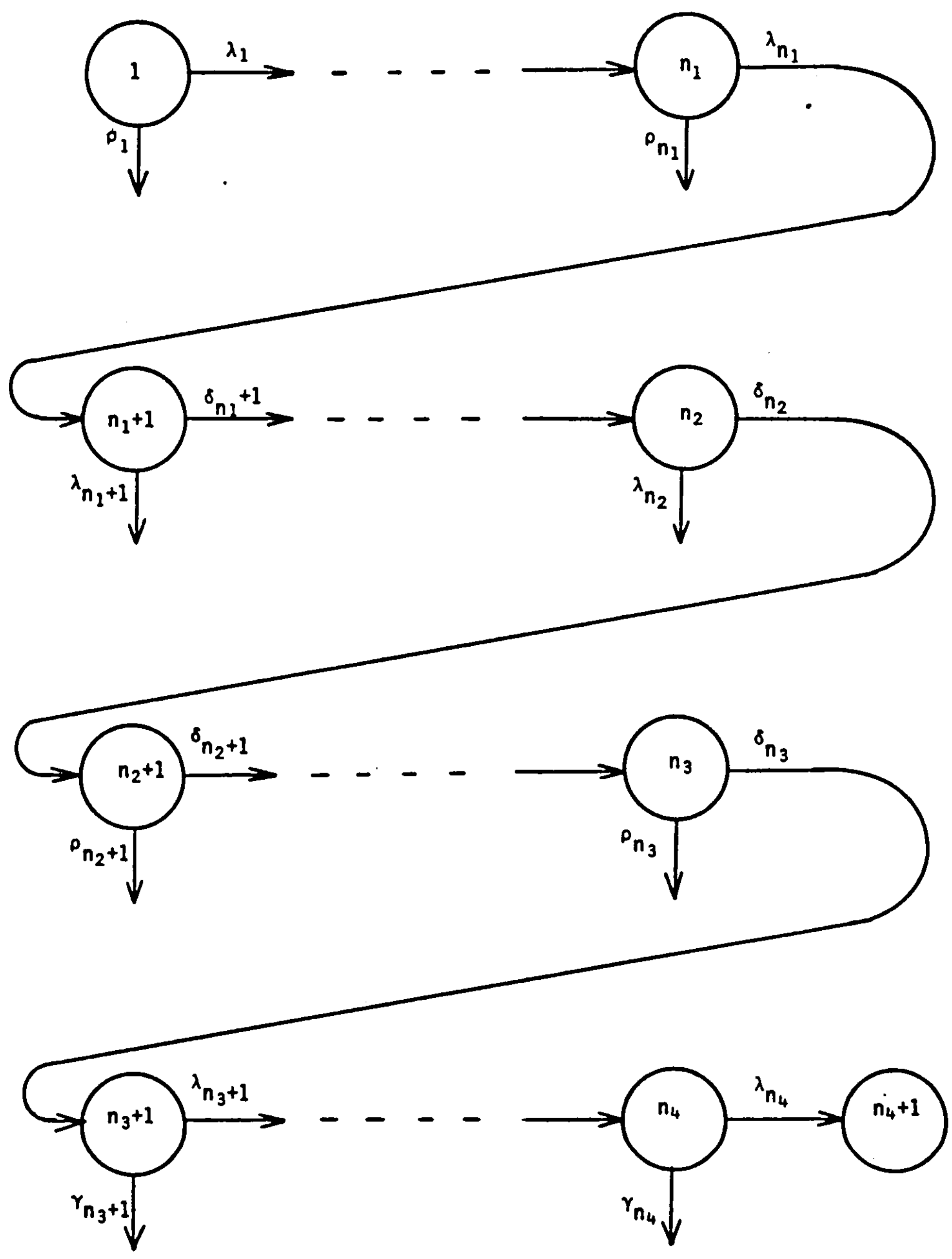
$$\delta_i >> 1 \qquad \text{all } i$$

$$\rho_i >> 1 \qquad \text{all } i$$

Let

$$a_i = \begin{cases} \lambda_i + \rho_i & 1 \leq i \leq n_1 \\ \delta_i + \gamma_i & n_1 + 1 \leq i \leq n_2 \\ \delta_i + \rho_i & n_2 + 1 \leq i \leq n_3 \\ \lambda_i + \gamma_i & n_3 + 1 \leq i \leq n_4 \\ 0 & i = n_4 + 1 \end{cases}$$

Without loss of generality assume $a_i \neq a_j$ if $i \neq j$.


1
λ_1
ρ_1
n_1
λ_{n_1}
ρ_{n_1}
n_1+1
δ_{n_1+1}
λ_{n_1+1}
n_2
δ_{n_2}
λ_{n_2}
n_2+1
δ_{n_2+1}
ρ_{n_2+1}
n_3
δ_{n_3}
ρ_{n_3}
n_3+1
λ_{n_3+1}
γ_{n_3+1}
n_4
λ_{n_4}
γ_{n_4}
n_4+1

The exact solution is

$$P_{n+1}(t) = \prod_{i=1}^{n_1} \lambda_i \prod_{j=n_1+1}^{n_2} \delta_j \prod_{k=n_2+1}^{n_3} \delta_k \prod_{\ell=n_3+1}^{n_4} \lambda_\ell$$

$$\times \left\{ \prod_{m=1}^{n_4+1} \frac{e^{-a_m t}}{\prod_{\substack{q=1 \\ q \neq m}}^{n_4+1} (a_q - a_m)} \right\}$$

Most of the terms inside the braces are set to zero by the approximation that the exponential of a large negative number is nearly zero. That is,

$$\frac{e^{-a_m t}}{\prod_{\substack{q=1 \\ q \neq m}}^{n_4+1} (a_q - a_m)} \approx 0$$

for $1 \leq m \leq n_3$. Still working inside the braces, the denominators of the remaining terms are simplified by the approximation that a very large number plus or minus a very small number is nearly equal to the very large number. That is,

$$a_q - a_m \approx \begin{cases} \rho_q & 1 \leq q \leq n_1 \\ \delta_q & n_1+1 \leq q \leq n_2 \\ \delta_q + \rho_q & n_2+1 \leq q \leq n_3 \end{cases} .$$

By the approximations above

$$P_{n+1}(t) \approx \prod_{i=1}^{n_1} \lambda_i \prod_{j=n_1+1}^{n_2} \delta_j \prod_{k=n_2+1}^{n_3} \delta_k \prod_{\ell=n_3+1}^{n_4} \lambda_\ell$$

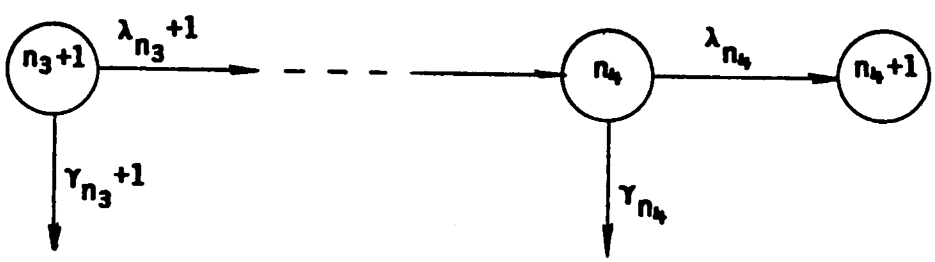
$$\times \left\{ \sum_{m=n_3+1}^{n_4+1} \frac{e^{-a_m t}}{\prod_{i=1}^{n_1} \rho_i \prod_{j=n_1+1}^{n_2} \delta_j \prod_{k=n_2+1}^{n_3} (\delta_k + \rho_k) \prod_{\substack{q=n_3+1 \\ q \neq m}}^{n_4+1} (a_q - a_m)} \right\} .$$

Since the first three products in the denominator do not involve the index of summation m, they may be pulled outside the summation.

$$P_{n+1}(t) \approx \prod_{i=1}^{n_1} \left(\frac{\lambda_i}{\rho_i}\right) \prod_{j=n_1+1}^{n_2} \left(\frac{\delta_j}{\delta_j}\right) \prod_{k=n_2+1}^{n_3} \left(\frac{\delta_k}{\delta_k + \rho_k}\right)$$

$$\times \prod_{\ell=n_3+1}^{n_4} \lambda_\ell \times \left\{ \sum_{m=n_3+1}^{n_4+1} \frac{e^{-a_m t}}{\prod_{\substack{q=n_3+1 \\ q \neq m}}^{n_4+1} (a_q - a_m)} \right\}$$

The last two factors are the exact solution of the Markov model



where all λ's and γ's are small. The first nonzero term of the matrix Taylor series solution is

$$\left(\prod_{\ell=n_3+1}^{n_4} \lambda_\ell\right) \left(\frac{t^{n_4 - n_3 - 1}}{(n_4 - n_3 - 1)!}\right) .$$

Hence, the final approximation is

$$P_{n+1}(t) \approx \left(\prod_{i=1}^{n_1} \frac{\lambda_i}{\rho_i}\right) \left(\prod_{k=n_2+1}^{n_3} \frac{\delta_k}{\delta_k + \rho_k}\right) \left(\prod_{\ell=n_3+1}^{n_4} \lambda_\ell\right) \left(\frac{t^{n_4 - n_3 - 1}}{(n_4 - n_3 - 1)!}\right) .$$

REFERENCES

W. Feller, An Introduction to Probability and its Applications, Vol. 2, 1966, John Wiley, New York.

M. Hirsch and S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra, 1974, Academic Press, New York.

Siewiorek and Swarz, The Theory and Practice of Reliable System Design, 1982, Digital Press, Bedford.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-172290		

4. Title and Subtitle	5. Report Date
AN APPROXIMATION FORMULA FOR A CLASS OF MARKOV RELIABILITY MODELS	January 1984
	6. Performing Organization Code
7. Author(s) Allan L. White	8. Performing Organization Report No.
9. Performing Organization Name and Address Kentron International, Inc. Kentron Technical Center Hampton, VA 23666	10. Work Unit No.
	11. Contract or Grant No. NAS1-16000
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546	13. Type of Report and Period Covered Contractor Report
	14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Charles W. Meissner, Jr.

16. Abstract

For a small, but often used, class of reliability models this report shows how to just look at the model and write down an algebraic approximation of the system's reliability. The models considered are appropriate for redundant reconfigurable digital control systems that operate for a short period of time without maintenance, and for such systems the method gives a formula in terms of component fault rates, system recovery rates, and system operating time.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Reliability Estimation Fault Tolerance Digital Process Control	Unclassified-Unlimited Subject Category 66

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	50	A03

For sale by the National Technical Information Service, Springfield, Virginia 22161

LIBRARY MATERIAL SLIP

DO NOT REMOVE SLIP FROM MATERIAL

Delete your name from this slip when returning material to the library.

NAME	DATE	MS
~~J, Shin~~	~~03/01~~	~~160~~

NASA Langley Form 474 (Rev. Oct. 1999)