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1. INTRODUCTION. Mathematical studies of the dynamic stability of
aircraft began essentially with Bryan's "Stability in Aviation" pub-
lished in 1911 [1]. This analysis appeared at the very beginning of
heavier-than-air flight itself, and has remained the foundation for
practically all subsequent studies of the subject. B. Melville Jones
[2] reported on progress 25 years later and succinctly stated the sub-
ject's principal task:

> "Given the shape of the aeroplane and the properties of the
>
> air through which it moves, the air reactions X, Y, Z, L,
>
> M, N depend on the motion of the aeroplane relative to the
>
> air; that is to say upon the six variables U, V, W, P, Q, R
>
> and their rates of change with respect to time. In prac-
>
> tice, the principal difficulty lies in determining the
>
> relationships between X, Y, . . . and U, V, . . ."

The establishment of these relationships with sufficient realism is
what we now recognize as the province of mathematical modeling.

Bryan's formulation, which originated the subject, has at its core
the assumption that the aerodynamic forces and moments developed at a
given instant are functions only of the instantaneous values of the
variables that determine forces and moments in a steady flow. When in
addition a _linear_ dependence of the forces and moments on these

variables is assumed, the equations governing the motions of aircraft reduce to a set of ordinary differential equations having constant coefficients. With the form of the equations thus established, the mathematical modeling problem is completed, and the study of stability becomes synonymous with the study of the coefficients. The nature and the determination of these coefficients, the "stability derivatives," have been the central concerns of experimenters and analysts alike through the ensuing years.

Later investigations into the transient behavior of the forces and moments in response to sudden changes in the flow around the aircraft led researchers to recognize that the forces and moments at an instant were dependent not only on the instantaneous values of the flow variables but also on their past values (cf. [3] for a comprehensive summary and bibliography). The concept of transient aerodynamic force and moment responses to step changes in the flow variables, i.e., of "indicial functions," coupled with the notion of superposition, led to a new formulation of the equations of motion [4]. This formulation is exact in principle within the assumption of linearity and the equations of motion take the form of integro-differential equations. However, reduction of the equations to equations correct to within a first-order dependence on time-rates-of-change of the variables restores the _form_ of the original ordinary differential equations, which now include terms that account for the past within the order of the approximation [5].

From the standpoint of mathematical modeling, exploiting the concept of a linear indicial response was an important step, since it described the process of generating an aerodynamic response to an arbitrary

motion from a few aerodynamic indicial responses by using superposition
integrals. It is from this idea that the modeling approach derives its
economy in the treatment of time-history effects. Further, use of the
superposition integral overcame in a concise way the first and most
objectionable assumption of Bryan's formulation: the aerodynamic forces
and moments could now depend on the history of the motion. Mathemati-
cally speaking, expressing the aerodynamic forces and moments in the
form of linear superposition integrals replaced Bryan's functions by
linear functionals.

As time passed, the continual expansion of the aircraft performance
envelope brought new aerodynamic phenomena, such as shock waves and
concentrated vortex flows, into play. It became apparent that the ways
in which these phenomena influenced aircraft motion could not all be
encompassed by even the exact linear formulation of the aerodynamic
forces and moments. A reformulation of the aerodynamic system that was
free of the remaining assumption basic to the original formulation was
now in order. It could be anticipated that the reformulation would
result in the replacement of the superposition integrals, which were
linear functionals, by suitable nonlinear functionals. As it turned
out, this task could be accomplished easily by the adoption of Volterra's
original conception of a functional [6]. Functional analysis was used
to construct a framework within which the indicial function could be
reformulated as a nonlinear functional [7]; the result was a new defini-
tion for the indicial function that did not depend on a linearity
assumption. This definition led naturally to the derivation of integral
forms for the aerodynamic forces and moments which were the anticip ted

generalizations of the superposition integrals. As in the case for the
exact linear formulation, it was found possible to reduce the results
to simpler, more practicable forms by virtue of the low angular rates-
of-change characteristic of aircraft motions. Subsequent papers [8-16]
revealed the implications of these results in regard to general motions,
experiments, and a variety of modeling questions. A comprehensive
summary of the work to date is available [17].

In recent years, we have become increasingly aware that our extensive
use of concepts from functional analysis in the modeling of systems
governing the dynamical motions of aircraft has been matched or sur-
passed by workers in a wide variety of fields. Noteworthy in this par-
ticular setting has been the body of work which has developed in several
branches of electrical engineering (for a comprehensive survey, see the
entire contents of [18,19]). We consider it important that the common
features of the efforts in these various fields be brought to the fore
for everyone's benefit. To that end, we shall take this opportunity to
recast the basic ideas of the approach we have taken in the modeling of
nonlinear aerodynamic responses in a way that we hope will be compatible
with some of the approaches taken in electrical engineering. We shall
emphasize the physical aspects of our approach to clarify its relation,
in particular, to the body of ideas underlying the use of nonlinear
functional expansions (cf. [20] for an excellent exposition of the latter
work). Finally, we shall try to show how our analysis can be extended
through its natural connection with ideas from bifurcation theory.

2. MODELING INFLUENCE OF PAST MOTION BY PULSES. For clarity of exposition, it is convenient to adopt the two-dimensional wing as illustration. Results will have more general bearing, however.

Let the wing move away from a coordinate system whose origin is fixed in space at the center of gravity at a time $\xi = 0$. The distance traveled by the center of gravity along the flightpath is measured by a coordinate s. Let the center of gravity move at constant velocity V_0, so that the trace of its path, plotted against time ξ, is a straight line. This is shown on Fig. 1. The wing is allowed to undergo changes only in the angle of attack α, where α is the angle between the velocity vector and the wing chord line. Projections of the leading and trailing edges of the wing onto the plane containing the velocity vector are maximum when $\alpha = 0$. These maximum projections also trace out straight lines on Fig. 1, parallel to the trace of the center of gravity.

As illustrated in Fig. 1, let the angle of attack α be zero for all time ξ except at $\xi = \xi_1$, where a pulse occurs of amplitude $\alpha(\xi_1)$ and infinitesimal width $\Delta\xi_1$. Consider a measuring point s on the wing at a time t subsequent to ξ_1. The loading at the point is influenced by all disturbances that originate in the past and are able to reach the point at the same time t. Each disturbance is propagated at the local speed of sound, and hence, in a plot such as Fig. 1, the zone of its influence is bounded by projections of the rays of an approximately conic surface whose origin is the point of the disturbance. Only disturbances whose cones include the point (s,t) in question can influence the loading at the point. Thus, a certain conic surface,

directed backward in time from the point (s,t) will include within it all points in past time whose disturbances are able to influence the loading at s at time t. The projection of such a conic surface is shown in Fig. 1, where, in the present case, the only disturbances that exist are those originating from the pulse at $\xi = \xi_1$. It will be seen that only disturbances originating from the shaded part of the pulse can influence the measuring point. Also note that if the elapsed time $t - \xi_1$ between pulse and measuring point is held fixed and if the same pulse and the measuring point are translated together to new positions on the wing in such a way that the trace of the measuring point remains parallel to that of the center of gravity, then nothing changes in the form of the loading at the measuring point. This behavior is captured by writing the loading at the measuring point in the form of a Taylor series:

$$\Delta C_p(s,t) = \Delta C_p(x, t - \xi_1, \alpha(\xi_1))_{dir} = \sum_n a_n(x, t - \xi_1)[\alpha(\xi_1)]^n \Delta \xi_1$$

(2.1)

where loading ΔC_p is the difference between pressure coefficients on the lower and upper surfaces, and x is the distance of the measuring point from the maximum projection of the leading edge (Fig. 1). It will be noted that, as required, ΔC_p remains constant when x, $t - \xi_1$, and $\alpha(\xi_1)$ are held fixed. The second of the forms in (2.1) will be used in the subsequent analysis to distinguish between direct (subscript dir) and interference (subscript int) effects. Equation (2.1) holds under the principal assumption that at least a limited range of α will exist in which the loading will depend analytically on α. Further, in more general circumstances, such as in accelerated motion where the

trace of the center of gravity will be a curved line, the dependence on elapsed time $t - \xi_1$ alone will not hold; the a_n will then depend on t and ξ_1 separately.

Now let us consider the response at the measuring point to a pair of pulses located at ξ_1 and ξ_2 with $\xi_1, \xi_2 < t$. Here, in addition to the direct influence of each of the pulses acting as if in isolation, the interference between the pulses will also influence the loading. The interference effect can be written in a form resembling a product of responses to single pulses

$$\Delta C_p(s,t)_{int,2} = \sum_{m,n} b_{mn}(x, t - \xi_1, t - \xi_2)[\alpha(\xi_1)]^m [\alpha(\xi_2)]^n \Delta\xi_1 \, \Delta\xi_2$$

(2.2)

where the subscript (int,2) is meant to be read as "interference between a pair of pulses." With the addition of the direct influence of the two pulses, the loading at (s,t) takes the form

$$\Delta C_p(s,t) = \Delta C_p(x, t - \xi_1, \alpha(\xi_1))_{dir} + \Delta C_p(x, t - \xi_2, \alpha(\xi_2))_{dir}$$

$$+ \Delta C_p(s,t)_{int,2}$$

(2.3)

The process of adding pulses can be continued indefinitely in the same way. At the next stage, e.g., the interference between triplets of pulses must be considered as well as that between pairs. Going to the limit of a continuous distribution of pulses starting at time $\xi = 0$ yields a summation of multiple integrals having the form

$$\Delta C_p(s,t) = \Delta C_{p_{dir}} + \Delta C_{p_{int,2}} + \Delta C_{p_{int,3}} + \ldots$$

(2.4)

8

with

$$\Delta C_{P_{dir}} = \sum_{n} \int_{0}^{t} a_{n}(x, t - \xi_1)[\alpha(\xi_1)]^n d\xi_1 \tag{2.5}$$

$$\Delta C_{P_{int,2}} = \sum_{m,n} \int_{0}^{t} [\alpha(\xi_2)]^m d\xi_2 \int_{0}^{\xi_2} b_{mn}(x, t - \xi_1, t - \xi_2)[\alpha(\xi_1)]^n d\xi_1 \tag{2.6}$$

$$\Delta C_{P_{int,3}} = \sum_{m,n,p} \int_{0}^{t} [\alpha(\xi_3)]^m d\xi_3 \int_{0}^{\xi_3} [\alpha(\xi_2)]^n d\xi_2$$

$$\times \int_{0}^{\xi_2} c_{mnp}(x, t - \xi_1, t - \xi_2, t - \xi_3)[\alpha(\xi_1)]^p d\xi_1 \tag{2.7}$$

. . .

Written as a nonlinear functional expansion, (2.4) represents the loading at the point (s,t) in response to an arbitrary variation of α over the time interval zero to t. The form of (2.4) confirms an important point made in [20]. It will be noted that a summation of the leading terms in (2.5,2.6,...) forms a Volterra series [6,20]. The fact that there are terms left over confirms that the à priori adoption of a Volterra series to represent the loading would have been insufficiently general to accommodate the Taylor series form of the dependence on angle of attack.

3. FORMATION OF INDICIAL RESPONSE. Given (2.4), one can now use it to form the indicial response as we have defined it in [7]. To indicate the form of the result, it will be sufficient to consider terms in (2.4) only through the series representing $\Delta C_{P_{int,2}}$. Two motions need to be considered. In the first, the wing undergoes the motion under study

$\alpha(\xi)$ from time zero up to a time $\xi = \tau$, where $\tau < t$. Subsequent to

τ, α is held constant at $\alpha(\tau)$. Thus, in (2.4,2.5,2.6)

$$\alpha(\xi) = \alpha(\xi) \quad ; \quad 0 < \xi < \tau$$
$$= \alpha(\tau) \quad ; \quad \xi > \tau \tag{3.1}$$

The direct and interference contributions take the forms

$$\Delta C_{p_{dir}} = \sum_n \int_0^\tau a_n(x, t - \xi_1)[\alpha(\xi_1)]^n d\xi_1$$

$$+ \sum_n [\alpha(\tau)]^n \int_\tau^t a_n(x, t - \xi_1) d\xi_1$$

$$\Delta C_{p_{int,2}} = \sum_{m,n} \int_0^\tau [\alpha(\xi_2)]^m d\xi_2 \int_0^{\xi_2} b_{mn}[\alpha(\xi_1)]^n d\xi_1 \tag{3.2}$$

$$+ \sum_{m,n} [\alpha(\tau)]^m \int_\tau^t d\xi_2 \int_0^\tau b_{mn}[\alpha(\xi_1)]^n d\xi_1$$

$$+ \sum_{m,n} [\alpha(\tau)]^{m+n} \int_\tau^t d\xi_2 \int_\tau^{\xi_2} b_{mn} d\xi_1$$

In the second motion, the wing undergoes the same angle of attack

history $\alpha(\xi)$ up to time τ. Subsequent to τ, the angle of attack is

again held constant, but is given an incremental step change $\Delta\alpha$ over

its previous value of $\alpha(\tau)$. Thus, in the second motion,

$$\left. \begin{aligned} \alpha(\xi) &= \alpha(\xi) &&; \quad 0 < \xi < \tau \\ &= \alpha(\tau) + \Delta\alpha &&; \quad \xi > \tau \end{aligned} \right\} \tag{3.3}$$

The direct and interference contributions become

$$
\begin{aligned}
\left. \Delta C_{P_{dir}} \right. &= \sum_n \int_o^\tau a_n [\alpha(\xi_1)]^n d\xi_1 + \sum_n [\alpha(\tau) + \Delta\alpha]^n \int_\tau^t a_n \, d\xi_1 \\[2mm]
\left. \Delta C_{P_{int,2}} \right. &= \sum_{m,n} \int_o^\tau [\alpha(\xi_2)]^m d\xi_2 \int_o^{\xi_2} b_{mn} [\alpha(\xi_1)]^n d\xi_1 \\[2mm]
&+ \sum_{m,n} [\alpha(\tau) + \Delta\alpha]^m \int_\tau^t d\xi_2 \int_o^\tau b_{mn} [\alpha(\xi_1)]^n \, d\xi_1 \\[2mm]
&+ \sum_{m,n} [\alpha(\tau) + \Delta\alpha]^{m+n} \int_\tau^t d\xi_2 \int_\tau^{\xi_2} b_{mn} \, d\xi_1
\end{aligned}
\right\} (3.4)
$$

The difference between the two loadings, with terms retained only of $O(\Delta\alpha)$, yields the incremental change in loading in response to the incremental step change in angle of attack:

$$
\begin{aligned}
\frac{\Delta C_{P_2} - \Delta C_{P_1}}{\Delta\alpha} &= \sum_n n[\alpha(\tau)]^{n-1} \int_\tau^t a_n \, d\xi_1 \\[2mm]
&+ \sum_{m,n} m[\alpha(\tau)]^{m-1} \int_\tau^t d\xi_2 \int_o^\tau b_{mn}[\alpha(\xi_1)]^n \, d\xi_1 \\[2mm]
&+ \sum_{m,n} (m+n)[\alpha(\tau)]^{m+n-1} \int_\tau^t d\xi_2 \int_o^{\xi_2} b_{mn} \, d\xi_1 \qquad (3.5)
\end{aligned}
$$

Equation (3.5) reveals the form of the indicial loading response to a step change in angle of attack in terms of functional expansions. It will be seen that the first and third terms on the right-hand side of (3.5) do not depend on the past motion but only on the level of the

angle of attack $\alpha(\tau)$ at which the step was made. The second term depends on the past motion, however, since $\alpha(\xi_1)$ with $0 < \xi_1 < \tau$ appears within the integral. The leading term of this past dependence has the form $\int_\tau^t d\xi_2 \int_0^\tau \alpha(\xi_1) b_{11}(x, t - \xi_1, t - \xi_2) d\xi_1$. Dependence on the past thus arises from interference effects between pulses prior to τ, the origin of the step, and perturbation pulses of $0(\Delta\alpha)$ originating subsequent to τ. In the general nonlinear case, then, and just as before [7], the indicial response is itself a functional. In the limit as $\Delta\alpha \rightarrow 0$, then (3.5) can be cast in the functional form

$$\lim_{\Delta\alpha \rightarrow 0} \frac{\Delta C_{p_2} - \Delta C_{p_1}}{\Delta\alpha} = \frac{\partial}{\partial\alpha} \Delta C_p(s, t) = \Delta C_{p_\alpha}[\alpha(\xi); x, t, \tau] \qquad (3.6)$$

where it will be noted that, within the functional, dependence on t and τ is indicated separately rather than on elapsed time $t - \tau$ alone. It can be easily verified that the first and third terms in (3.5), which depend only on the level $\alpha(\tau)$, indicate a dependence on $t - \tau$ alone; however, as a consequence of its dependence on the past motion, the second term cannot be cast as a function of $t - \tau$ alone.

4. GENERALIZED SUPERPOSITION INTEGRAL. Just as before, (3.6) can be used to form a generalized superposition integral for the response in ΔC_p to an arbitrary angle-of-attack variation. The result is

$$\Delta C_p(s, t) = \Delta C_p(s, 0) + \int_0^t \Delta C_{p_\alpha}[\alpha(\xi); x, t, \tau] \frac{d\alpha}{d\tau} d\tau \qquad (4.1)$$

By substituting (3.5) for $\Delta C_{p_\alpha}[\alpha(\xi); x, t, \tau]$ in the integral term in (4.1) and carrying out the integration, one will verify that the form of

(4.1) is restored through terms of the series representing $\Delta C_{P_{int,2}}$. This should suffice to demonstrate that our use of indicial responses and generalized superposition integrals is in fact compatible with the approach based on nonlinear functional expansions.

5. PRINCIPAL SIMPLIFICATION. The simplification most instrumental in making our use of nonlinear functionals in the analysis of aircraft dynamics practicable has proved to be the reduction of the general aerodynamic force and moment responses to forms correct to within a first-order dependence on angular rates. This approximation is justified in its application to studies of aircraft dynamics in view of the generally low reduced frequencies characteristic of aircraft motions. In the following, the steps involved in the reduction are reviewed in order to highlight the several advantages of the reduction as well as to set the stage for a discussion of the amendments required to accommodate the occurrence of bifurcation phenomena.

Integrating the expression for loading in (4.1) across the chord eliminates the dependence on x and yields a form for the response in lift coefficient to an arbitrary angle-of-attack variation

$$C_L(t) = C_L(0) + \int_0^t C_{L_\alpha}[t,\tau;\alpha(\xi)] \frac{d\alpha}{d\tau} d\tau \qquad (5.1)$$

Note that the order of the dependencies has been reversed within the functional C_{L_α} to anticipate the enhanced role played by t and τ in the ensuing analysis.

Consider first the behavior of the functional as the elapsed time $t - \tau$ increases. It is clear physically that, with increasing $t - \tau$,

the dependence of C_{L_α} on the past motion $\alpha(\xi)$; $\xi \leq \tau$ must fade away.
Thus, as $t - \tau \to \infty$, C_{L_α} approaches a function which is dependent only
on $t - \tau$ and $\alpha(\tau)$, the level of the angle of attack at which the step
was made. Additionally, we assume here that as $t - \tau \to \infty$, C_{L_α} will
approach a unique, <u>constant</u> value corresponding to the lift-curve slope
which would be measured in a steady flow. More precisely, we assume
that fluctuations, which always exist, are small enough to be neglected
in comparison to mean values resulting from ensemble averaging. It is
this assumption that will require amendment in the consideration of
bifurcation phenomena. Here, however, we invoke it explicitly by making
the substitution

$$C_{L_\alpha}[t,\tau;\alpha(\xi)] = C_{L_\alpha}(\infty;\alpha(\tau)) - F[t,\tau;\alpha(\xi)] \tag{5.2}$$

The quantity F is called the deficiency function(al). Provided the
steady-state term $C_{L_\alpha}(\infty;\alpha(\tau))$ is in fact constant, F must approach
zero as $t - \tau \to \infty$. When (5.2) is substituted in (5.1), the product of
$C_{L_\alpha}(\infty;\alpha(\tau))$ and $\dot\alpha(\tau)$ forms a perfect differential which can be integrated, yielding

$$C_L(t) = C_L(\infty;\alpha(t)) - \int_0^t F[t,\tau;\alpha(\xi)] \frac{d\alpha}{d\tau} \, d\tau \tag{5.3}$$

where $C_L(\infty;\alpha(t))$ represents the (mean) lift coefficient that would be
measured in a steady flow with α held fixed at the instantaneous value
$\alpha(t)$. Now we examine the dependence of F on the past motion $\alpha(\xi)$.
If $\alpha(\xi)$ can be considered an analytic function in a neighborhood of
$\xi = \tau$ (corresponding to the most recent past for an indicial response
with origin at $\xi = \tau$), then its history can be reconstructed, in

principle, from a knowledge of all of the coefficients of its Taylor series expansion about $\xi = \tau$. Since $\alpha(\xi)$ is equally represented by the coefficients of its expansion, it follows that the functional F, with its dependence on $\alpha(\xi)$, can be replaced without approximation by a function which depends on all of the coefficients of the expansion of $\alpha(\xi)$ at $\xi = \tau$. Thus, F can be expressed as

$$F[t,\tau;\alpha(\xi)] = F(t,\tau;\alpha(\tau),\dot{\alpha}(\tau), \ldots) \qquad (5.4)$$

Assuming now that $\alpha(\tau)$ is potentially large but that the rates $\dot{\alpha}(\tau), \ddot{\alpha}(\tau), \ldots$ are all always small, we are permitted to expand (5.4) around the zero values of the rates, so that

$$F[t,\tau;\alpha(\xi)] = F(t,\tau;\alpha(\tau),0,0, \ldots) + \dot{\alpha}(\tau)F_{\dot{\alpha}}(t,\tau,\alpha(\tau),0,0, \ldots)$$

$$+ \frac{\dot{\alpha}^2(\tau)}{2} F_{\dot{\alpha}\dot{\alpha}}(t,\tau;\alpha(\tau),0,0, \ldots) + \ldots$$

$$+ \ddot{\alpha}(\tau)F_{\ddot{\alpha}}(t,\tau;\alpha(\tau),0,0, \ldots) + \ldots + \ldots \qquad (5.5)$$

Returning to (5.3) and retaining terms only to within a linear dependence on $\dot{\alpha}(\tau)$, we get for the integral term

$$I(t) = \int_0^t F(t,\tau;\alpha(\tau),0,0, \ldots) \frac{d\alpha}{d\tau} \, d\tau \qquad (5.6)$$

To the order of the approximation, only the first term of the expansion of F survives in (5.6). As a consequence, and as the notation shows, as far as the functional F is concerned, the past motion is simply the constant motion $\alpha(\xi) = \alpha(\tau)$. On this basis, it is consistent now to say as well that F will depend only on elapsed time $t - \tau$, rather than on t and τ separately. Expansion of $\dot{\alpha}(\tau)$ about $\tau = t$ and the

dependence of F on $\alpha(\tau)$ about $\alpha(t)$ yield, to within a linear dependence on $\dot{\alpha}(t)$,

$$I(t) = \dot{\alpha}(t) \int_0^\infty F(u;\alpha(t))du \qquad (5.7)$$

where we have assumed $F(u;\alpha(t)) \approx 0$ for all $u = t - \tau \geq t_a$ and t large enough so that $t > t_a$. Substituting this result for the integral term in (5.3) yields as the formulation for the lift coefficient

$$C_L(t) = C_L(\infty;\alpha(t)) + \dot{\alpha}(t) \frac{\ell}{V_o} C_{L_{\dot{\alpha}}}(\alpha(t)) \qquad (5.8)$$

where

$$C_{L_{\dot{\alpha}}}(\alpha(t)) = - \frac{V_o}{\ell} \int_0^\infty F(u;\alpha(t))du \qquad (5.9)$$

and ℓ is a characteristic length (e.g., chord length).

The simple form of (5.8) contains a number of important advantages. Since $C_{L_{\dot{\alpha}}}$ depends only on $\alpha(t)$, within the approximation the response to _any_ of a class of sufficiently slowly varying motions which arrive at the same value of α at time t will yield the same value of $C_{L_{\dot{\alpha}}}$. In particular, a harmonic motion of infinitesimal amplitude about a mean value of α equal to the instantaneous value $\alpha(t)$ suffices to obtain $C_{L_{\dot{\alpha}}}$. This motion is the preferable one from both the experimental and the computational standpoint. On the other hand, a _finite_ amplitude harmonic oscillation about a mean value of α, say α_m, can be used as well to obtain $C_{L_{\dot{\alpha}}}$ at the particular points in the motion where the instantaneous amplitude $\hat{\alpha}(t)$ is such that $\hat{\alpha}(t) = \alpha(t) - \alpha_m$. This fact enables one to deduce the results for finite-amplitude oscillatory

motions from those of infinitesimal-amplitude oscillations, a considerable advantage from the computational standpoint. Finally, and most important, as will be seen in the contribution to this collection by Hui and Tobak, the simplification provides the rationale that allows decoupling flow-field equations from the inertial equations of motion and using results for the aerodynamic terms in the mathematical model from known harmonic motions.

6. CONNECTION WITH BIFURCATION THEORY. We intend to show that our use of indicial responses as a basis for arriving at the form of the aerodynamic force and moment leads naturally to the consideration of bifurcation phenomena. To fix ideas, we restrict attention initially to laminar flows and assume that the flows are governed by the Navier-Stokes equations. Further, we assume that the maneuvers and measurements can be carried out with sufficient precision so that they are effectively er.orless and repeatable, eliminating the necessity of accounting for the presence of random fluctuations. The extent to which the analysis must be deepened to accommodate turbulent flows and random fluctuations will be addressed later.

We begin by reconsidering the maneuvers required to form an indicial response. As before, two maneuvers are involved, both beginning at $\xi = 0$, and constrained at $\xi = \tau$, and differing only in the step imposed on the second maneuver at $\xi = \tau$. For each maneuver, the lift coefficient is measured at a time t subsequent to τ. If we assume that the difference between measurements at time t, $\Delta C_L(t)$, divided by the magnitude of the step, $\Delta\alpha$, <u>exists and is unique</u> in the limit as $\Delta\alpha \to 0$ for all

values of $\xi = t > \tau$, then we define this limit as the indicial response in lift coefficient per unit step change in α. Hence, the basic assumption on which the definition of the indicial response rests will <u>fail</u> when the indicial response ceases to exist or to be unique. A very natural way of invalidating the assumption is through the mechanism of instability, which we have not considered in this context before. It is here that the possibility exists of extending our analysis by incorporating ideas from bifurcation theory.

Consider the first of the two maneuvers involved in the formation of the indicial response. The angle of attack attains a constant value $\alpha(\tau)$ subsequent to τ and it is reasonable to expect that the flow field at the subsequent time t will approach an equilibrium state that corresponds to this fixed boundary condition, as the elapsed time $t - \tau \to \infty$. In all of our previous analyses, we have assumed that as the flow field approached the equilibrium state it became time-invariant, which meant that the corresponding lift coefficient $C_L(t)$ approached a unique constant value $C_L(\infty;\alpha(\tau))$ as $t - \tau \to \infty$. As long as this was true, it was reasonable likewise to expect that an incremental change in $\alpha(\tau)$ of $O(\Delta\alpha)$ would result in an incremental change in $C_L(\infty;\alpha(\tau))$, likewise of $O(\Delta\alpha)$. We now recognize that this <u>will</u> be true as long as the time-invariant equilibrium state represented by $C_L(\infty;\alpha(\tau))$ is (asymptotically) stable to small perturbations in α. It can happen, however, that as $\alpha(\tau)$ is increased in small increments, a critical value of $\alpha(\tau)$ can be reached at which the stationary equilibrium state represented by $C_L(\infty;\alpha(\tau))$ will no longer be stable to small perturbations in α. Of the new equilibrium states that are possible, the

system will seek one which can remain stable to small perturbations.
Now this is precisely the situation that bifurcation theory is designed
to address. Bifurcation theory tries to classify and characterize the
properties of the new equilibrium states that can arise when the given
equilibrium state becomes unstable. Of the types of bifurcation phe-
nomena that are possible, perhaps the most typical in aerodynamics is
the "Hopf" type, which is characterized as follows: A previously stable
time-invariant equilibrium state is replaced by a time-varying oscilla-
tory equilibrium state. Physically, the usual origin of such a large-
scale oscillatory state is the onset of vortex-shedding. Of the many
examples, we cite here stall on airfoils when the angle of attack
exceeds a critical angle [21,22] and the wake of the flow past a cylin-
der when the Reynolds number exceeds 50 [23]. Bifurcation theory gives
us the means to incorporate these phenomena within a rational frame-
work, consequently, the possibility of accounting for those critical
points in maneuvers where sudden and dramatic changes in flow structure
may occur.

To conclude, we indicate our current thoughts on the directions in
which the analysis must be extended to acknowledge the important effects
of random fluctuations and turbulent flows. The issue has separate
experimental and computational components.

From the experimental standpoint, the presence of random fluctuations
in the maneuvers and the measurements is a practical question which
arises even with strictly laminar flows. As we have already indicated,
however, (cf., in particular, [12]) adoption of ensemble-averaging allows
us, in principle, to acknowledge the presence of random fluctuations

without otherwise having to alter the analysis. On this basis, one can recognize the probable existence of bifurcation phenomena by the examination of the $\underline{\text{mean}}$ value of the lift coefficient in the equilibrium state, say $\overline{C_L(\infty;\alpha(\tau))}$. Values of α at which $\overline{C_L(\infty;\alpha(\tau))}$ is double-valued, or at which the slope of $C_L(\infty;\alpha(\tau))$ with α is discontinuous, are the signs of probable bifurcation phenomena. We have already noted the existence of the latter symptoms in several of our previous studies (cf. [15,16,17]) and, in [17], we have devised a special scheme for treating them in the particular instance where they reflect the presence of hysteresis in the equilibrium flow. We now recognize the possibility of incorporating such special treatments within the more general framework that an analysis based on ideas from bifurcation theory will provide.

Consideration of the question of random fluctuations and turbulent flows from the computational standpoint brings to the fore an additional issue. While laminar flows could be said to be governed by the Navier-Stokes equations, even in the presence of random fluctuations, accounting for the presence of turbulent flow in computations centers around the problem of having to define the equations governing the flow. This is the turbulence modeling problem. In current practice, it is generally agreed that any particular realization of a turbulent flow could be modeled, in principle with sufficient accuracy, by a solution of the Navier-Stokes equations. However, the existence of rapid and apparently random fluctuations in the flow makes it mandatory that the equations be averaged to suppress the appearance of the rapid fluctuations. The averaged equations, called Reynolds-averaged Navier-Stokes equations, are then closed by the installation of a suitably chosen closure model.

This effectively casts the equations in a form similar to that of the original Navier-Stokes equations. Therefore, in application to modeling the equilibrium flows associated with the formation of indicial responses, we should expect the modeled turbulent flow equations to mirror behavior previously captured by the Navier-Stokes equations in application to laminar flows. Thus, typically, when they are applied at low values of $\alpha(\tau)$, modeled turbulent flow equations should yield solutions for equilibrium flows that are invariant with time. But, just as before, time-invariant equilibrium flow solutions that had been stable for values of $\alpha(\tau)$ below a critical value should be expected to become unstable upon exceeding the critical value, and to seek a new branch of stable solutions. One possibility is a branch consisting of time-varying oscillatory solutions. Encouraging evidence is available that modeled turbulent flow equations in fact can be sufficiently general to exhibit such instability and bifurcation phenomena. In particular, the results of Levy ([24-26]; see also the discussion in [27]) for transonic flow past a biconvex airfoil show the typical Hopf-type bifurcation that reflects the onset of vortex-shedding in the wake, when either Reynolds number, Mach number, or lift coefficient exceed critical values. An analogous problem involving the occurrence of aileron buzz at transonic speeds when either Mach number or angle of attack exceeds critical values, has been treated by Steger and Bailey in [28].

The concerns of turbulence modeling research and those of research in the modeling of aerodynamic responses converge on the issue of the bifurcation behavior of equilibrium flows. In judging the importance of this issue, one should note that bifurcation phenomena reflect the

occurrence of sudden and potentially dangerous changes in flow structure during dynamic maneuvers. A resolution of the issue hinges in great part on the correct identification of these occurrences as the relevant parameters (e.g., angle of attack, sideslip angle, Reynolds number, Mach number) range over their respective envelopes, and their capture by means of computations based on modeled equations of turbulent flow. In application to two-dimensional flows past airfoils, as we have seen, modeled equations of turbulent flow have given evidence of their ability to capture the Hopf-type bifurcations typical of the onset of vortex-shedding. In application to the inescapably three-dimensional flows typical of modern slender aircraft, modeled equations of turbulent flow will be called upon additionally to capture bifurcation phenomena such as the asymmetric vortex flows and vortex breakdowns that will appear as the parameters are varied over their extensive ranges.

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FIGURE CAPTION

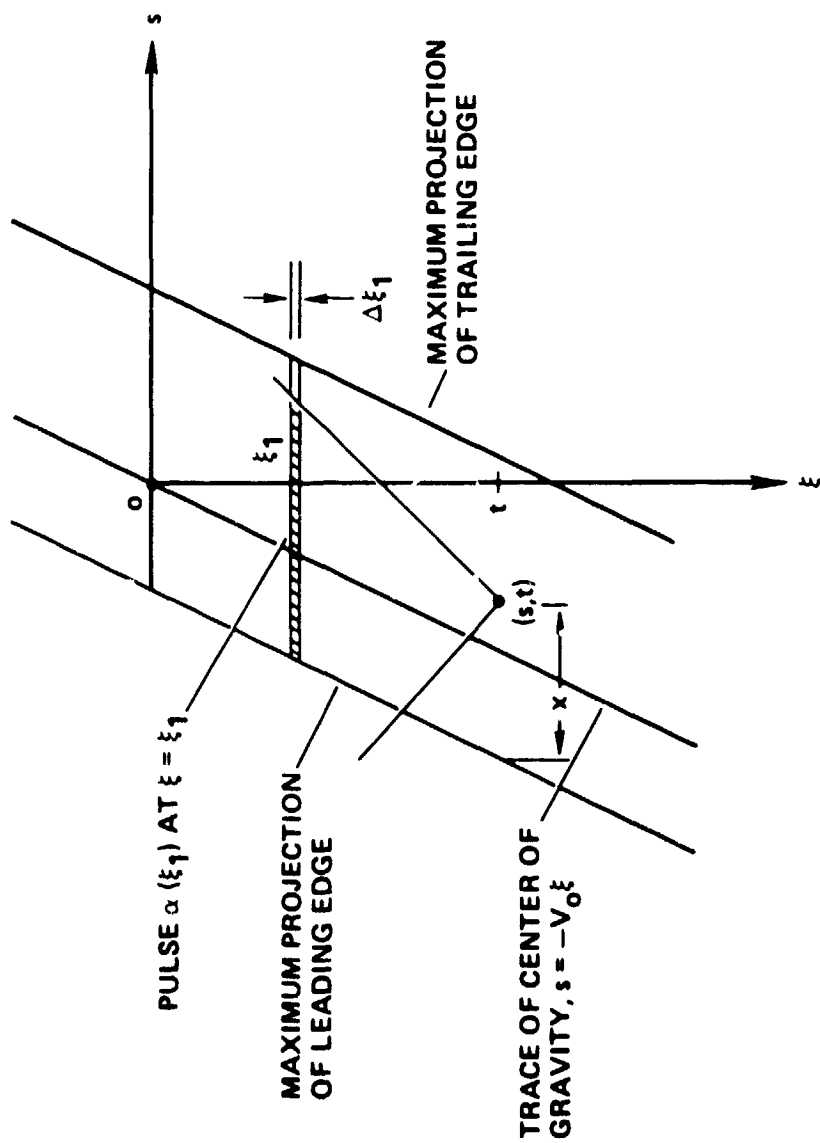Fig. 1 Boundary conditions for loading due to ..... in angle of attack.

Fig. 1