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DEPARTMENT OF MATHEMATICAL SCIENCES
SCHOOL OF SCIENCES AND HEALTH PROFESSIONS
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA

STABILITY OF THE LAMINAR BOUNDARY LAYER IN A STREAMWISE CORNER

By

William D. Lakin, Principal Investigator

Final Report
For the period ending September 30, 1983

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia

Under
Research Grant NAG1-297
John R. Dagenhart, Technical Monitor
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William D. Lakin*

SUMMARY

Viscous incompressible flow along a streamwise rectangular corner formed by the intersection of two semi-infinite flat plates is often called the corner boundary layer problem. Theoretical work on the mean flow in this problem has primarily been asymptotic and numerical in nature. Unfortunately, experimental results do not recover the flow patterns predicted by theory and, indeed, tend to differ from study to study. Zamir (1981) has suggested that one reason for the differences is an instability of the mean flow. The present work is the first theoretical investigation of the stability of the corner boundary layer problem.

By symmetry, it is sufficient to consider the flow region $y < z$ in which x, y, and z are respectively the streamwise, vertical, and spanwise coordinates, and the plate is at $y = 0$. Rubin (1966) has shown that for x large and fixed and z/x small, the presence of the corner induces a secondary spanwise velocity $\bar{w}$ which reverses direction within the boundary layer. Further, it can be shown that with z/x small, there is a wide range of z such that, to suitable order, all three velocity components $\bar{u}$, $\bar{v}$, and $\bar{w}$ are functions of y alone, so the mean flow is quasi-parallel. This region will be termed the blending boundary layer.

In the present work, three-dimensional, time dependent, small amplitude

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perturbations u , v , w , and p are superimposed on the mean velocity and pressure fields in the blending boundary layer region. As weak streamwise vorticity perturbations may produce large defects in the mean streamwise flow, two different velocity scales are used for the velocity perturbations. This is similar to the treatment of Görtler vortices by Floryan and Saric (1982). Partial differential equations for disturbance quantities are obtained by substituting total velocity and pressure fields into the Navier-Stokes equations and consistently linearizing about the mean flow. In accord with the quasi-parallel nature of the mean flow, dependence on x, z, and t is separated out by assuming normal mode forms for the perturbations. In the Görtler context, Hall (1983) has shown that there is a wave number regime for which parallel flow gives a self-consistent approximation to the linear stability equations.

After some manipulation, the coupled ordinary differential equations for the perturbation amplitude functions $\hat{u}(y)$, $\hat{v}(y)$, $\hat{w}(y)$, and $\hat{p}(y)$ can be reduced to a main uncoupled differential equation for $\hat{v}$, a second differential equation which couples $\hat{w}$ to $\hat{v}$, and algebraic equations for $\hat{u}$ and $\hat{p}$. Some care must be taken in formulating associated 'outer' boundary conditions for $\hat{v}$ and $\hat{w}$ as the line of symmetry $y = z$ may not be crossed.

For large Reynolds number, the asymptotic character of the main stability equation is set by a simple turning point and the resulting critical layer. Uniformly valid approximations to solutions of the main stability equation are derived using generalized Airy functions. Key steps to achieving uniformity on the full semi-infinite domain include definition of an appropriate Langer variable and reformulation of the equation in terms of a a proper large parameter which suitably incorporates the limiting behavior of the mean velocity components. Approximations to the balanced solution are found to have a phase shift across the critical layer which is an indication of instability.

A paper describing this research has been accepted by the Royal Society of London for publication in the Proceedings. It is co-authored by Dr. M. Y. Hussaini of the Institute for Computer Applications in Science and Engineering (ICASE). Publication date will be late 1984. The paper was communicated to the Royal Society by Prof. Keith Stewartson shortly before his death. A copy of the manuscript is included in this report as an Appendix.

The major portion of this research was carried out under the present grant. However, initial work on the corner layer stability problem was done at ICASE prior to the starting date of the grant, and appeared in an ICASE report. Final details of the work, including the final revisions of the manuscript, were done on an unfunded basis after the expiration date of the grant.

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APPENDIX

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STABILITY OF THE LAMINAR BOUNDARY LAYER IN A STREAMWISE CORNER

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Abstract

This work examines the stability of viscous, incompressible flow along a streamwise corner, often called the corner boundary layer problem. The semi-infinite boundary value problem satisfied by small-amplitude disturbances in the "blending boundary layer" region is obtained. The mean secondary flow induced by the corner exhibits a flow reversal in this region. Uniformly valid "first approximations" to solutions of the governing differential equations are derived. Uniformity at infinity is achieved by a suitable choice of the large parameter and use of an appropriate Langer variable. Approximations to solutions of balanced type have a phase shift across the critical layer which is associated with instabilities in the case of two-dimensional boundary layer profiles.

INTRODUCTION

Viscous, incompressible laminar flow along a streamwise corner formed by the intersection of two solid surfaces is often referred to as the corner boundary layer problem. In the theoretical model of this problem, two flat plates, which are infinite in the streamwise and spanwise directions, intersect such that the line of intersection (the corner line) is normal to the plane containing their leading edges. The basic flow is along the line of intersection. Most theoretical work assumes a rectangular corner. This situation is shown in Figure 1.

Solution of the exact corner flow velocity profile from the full Navier-Stokes equations is impractical, and the only exact solution produced to date [10] requires an exponentially increasing suction through the walls. Consequently, most theoretical work on the corner layer problem has been numerical or asymptotic in nature with the main focus being on the boundary layer regions close to the solid walls. An excellent summary of existing work has been given by Zamir [11]. The qualitative features of the flow field which emerge are shown in Figure 2. Region I, away from the solid boundaries, is a basically inviscid region. To lowest order, the flow in this region is irrotational flow. Region II, which lies "close" to the corner line $y = z = 0$, is a viscous corner layer dominated by the contraints of no-slip on the walls and symmetry on the line $y = z$. Regions III and IV are the boundary layers on the solid walls "away" from the corner. By symmetry, it is only necessary to consider region III (say) which is the boundary layer on the plate $y = 0$, $x > 0$, $z > 0$. If the mean velocity field is $(\bar{u},\bar{v},\bar{w})$, then in region III, $\bar{u}$ is the streamwise component, $\bar{v}$ is the outflow from the boundary layer, and $\bar{w}$ is the secondary flow component due to the presence of the corner and the opposite wall. There has been some disagreement over the nature of the secondary flow. However, recent experimental results support the picture derived by Rubin [8], Rubin and Grossman [9] and Ghia [2] which involves a reversal of the secondary flow component for $0 < z/x \ll 1$ in region III. In this regard, the corner boundary layer problem is similar to the boundary layer on a wing along a wing-body junction. For large z in region III (far away from the corner), the usual Blasius profile must be recovered. However, the decay of the secondary flow with increasing z appears relatively slow, algebraic rather than exponential. There will thus be a significant portion of region III where corner effects are important. It will be convenient to denote this region where the corner layer blends into the Blasius-type layer as the "blending boundary layer."

Theoretical work on the corner layer problem has primarily considered the ideal situation where the incoming uniform stream is aligned with the corner line. Hence, the pressure gradient along the corner line is zero. However, it is difficult to produce experimentally a laminar corner flow with a nonzero angle of incidence between the incoming uniform stream and the corner line. This fact, as well as apparent differences between the various experimental studies, has recently been analyzed by Zamir [11]. Based on his experimental results, which indicate a loss of profile similarity, he concludes that differences are the result of instability of the mean corner flow velocity profile. By extrapolating his data, Zamir predicts flow instability at zero angle of incidence when the Reynolds number $R_x = \frac{U x^*}{\nu}$ is order 10^4, where U is the local free stream velocity, x^* is the distance from the leading edge, and ν is the kinematic viscosity.

The present work is the first part of an asymptotic and numerical investigation of the stability of the corner boundary layer problem to small amplitude disturbances. Three-dimensional velocity perturbations are imposed on the mean flow in the portion of region III where z/x is small. The differential equations and boundary conditions governing the amplitude of disturbance qualities are derived, and, for high Reynolds number, the resulting eigenvalue problem is found to be of singular perturbation-type. From an asymptotic point of view, the character of this singular perturbation problem is determined by the simple turning point of one of the differential equations and the resulting critical layer in the blending boundary layer region. Both heuristic and uniformly valid asymptotic approximations to solutions of this disturbance equation are derived. One of the two inviscid-type solutions is found to behave logarithmically away from the turning point leading to a phase shift in the general solution across the critical layer. Phase shifts of this type are characteristic of the critical layer in the case of two-dimensional boundary layer profiles, such as the Blasius boundary layer.

2. THE DISTURBANCE EQUATIONS

It is convenient to work in terms of dimensionless variables to derive the boundary value problem satisfied by small amplitude disturbances to the mean corner layer flow. Accordingly, velocities will be scaled relative to the local free stream velocity U. A characteristic length L may be based on either x^* or the boundary layer thickness δ^*. The choice adopted here is

$$L = x^*/R_x^{1/2}, \tag{2.1}$$

in which case the usual Reynolds number becomes

$$R = \frac{UL}{\nu} = R_x^{1/2}. \tag{2.2}$$

We wish to impose perturbations on the mean velocity and pressure fields in the portion of region III closest to the corner where $0 < z/x << 1$. This subregion will be referred to as IIIa. To lowest order, the principal difference between the mean velocity profile in region IIIa and the usual flat plate Blasius profile is the secondary velocity component $\overline{w}$ which changes direction within the boundary layer. As noted, for z/x large, $\overline{w}$ decays algebraically so for fixed x the Blasius profile is recovered as $z \to \infty$. For z/x small, however, the algebraically decaying portion of $\overline{w}$ enters at higher order.

Asymptotic approximations to the mean velocity profile in region IIIa have been derived by Rubin [8]. The expansions have the forms

$$\left.\begin{aligned} \overline{u} &= \overline{u}_0 + R^{-1}\,\overline{u}_1 + O(R^{-2}), \\ v &= R^{-1}\,\overline{v}_1 + O(R^{-2}), \\ \text{and} \quad \overline{w} &= R^{-1}\,\overline{w}_1 + O\left(R^{-2}, \frac{R^{-1}\,z^2}{x^2}\right) \end{aligned}\right\} \tag{2.3}$$

where $\overline{u}_0(s)$ and $\overline{v}_1(s)$ are the usual Blasius quantities

$$\overline{u}_0(s) = f'(s), \qquad \overline{v}_1(s) = 2^{-1/2}\left(sf'(s) - f(s)\right). \tag{2.4}$$

In the current scaling, the similarty variable $s = 2^{1/2}\, y$,

$$\left.\begin{aligned} &f''' + ff'' = 0, \\ &f(0) = f'(0) = 0, \quad f'(s) \to 1 \quad \text{as} \quad s \to \infty, \end{aligned}\right\} \tag{2.5}$$

and the mean secondary flow is

$$\overline{w}_1(s) = 2^{-1/2}\, \beta_0\, H_0'(s), \qquad \beta_0 \sim 1.2168 \cdots \tag{2.6}$$

where

$$\left.\begin{aligned} &H_0''' + (f\, H_0')' = 1 \\ &H_0'(0) = 0, \quad H_0''(0) = -\beta_0, \quad H_0' \to 1 \quad \text{as} \quad s \to \infty. \end{aligned}\right\} \tag{2.7}$$

The behavior H_0' is shown in Figure 3. In particular, $H_0'(s)$ is negative for $0 < s < 2.15$ and positive above this level.

Consider now the order of magnitude of z in region IIIa. As in our scaling $x = x^*/L = R$, if $z = O(R^p)$ then z/x small requires $p < 1$. Also, we must have $y < z$ to avoid crossing the line of symmetry, and as y may get large this requires $p > 0$. Rubin [8] has shown that in region IIIa, the function $\overline{u}_1$ is z/x times an order one function of s. Consequently, $R^{-1}\, u_1 = O\left(R^{-2+p}\right)$. Similarly, the second term in $\overline{w}$ is $O\left(R^{-2}, R^{-3+2p}\right)$. In particular, if p is the range $0 < p < 1/3$, then the mean velocity profile in region IIIa is

$$(\overline{u}, \overline{v}, \overline{w}) = (\overline{u}_0, R^{-1}\,\overline{v}_1, R^{-1}\,\overline{w}_1) + O(R^{-q}) \tag{2.8}$$

with $q > 4/3$. Thus, to the order required by the present theory for uniform "first approximations", the mean velocity in region IIIa involves only y and the stability problem for small amplitude disturbances can be treated using a quasi-parallel flow approximation.

Dropping the subscripts on mean flow quantities, consider now total (mean plus perturbation) velocity and pressure fields in region IIIa of the form

$$\left.\begin{aligned} u &= \overline{u}(y) + u'(x,y,z,t) \\ v &= R^{-1}\,\overline{v}(y) + R^{-1}\,v'(x,y,z,t) \\ w &= R^{-1}\,\overline{w}(y) + R^{-1}\,w'(x,y,z,t) \\ \text{and}\quad p &= \overline{p}(y) + R^{-2}\,p'(x,y,z,t). \end{aligned}\right\} \tag{2.9}$$

Relations (2.9) involve different velocity scales for the perturbations u', v', and w'. This is similar to the treatment of Görtler vortices by Floryan and Saric [1] and reflects the fact that weak streamwise vorticity perturbations may produce large defects in the mean streamwise flow. The linear theory for the Görtler problem has recently been placed on a firm footing by Hall [3], [4]. In particular, for the Görtler problem it was shown that there is a wavenumber regime for which parallel flow gives a self-consistent approximation to the stability equations.

To obtain the differential equations for the small amplitude disturbances, the expressions (2.9) for the velocity and pressure may be substituted into the Navier-Stokes equations and the resulting equations are consistently linearized about the mean flow. This gives

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \frac{1}{R}\bar{v}\frac{\partial u'}{\partial y} + \frac{v'}{R}\frac{d\bar{u}}{dy} + \frac{1}{R}\bar{w}\frac{\partial u'}{\partial z} = -\frac{1}{R^2}\frac{\partial p'}{\partial x} + \frac{1}{R}\nabla^2 u', \tag{2.10}$$

$$\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} + \frac{1}{R}\bar{v}\frac{\partial v'}{\partial y} + \frac{1}{R}v'\frac{d\bar{v}}{dy} + \frac{1}{R}\bar{w}\frac{\partial v'}{\partial z} = -\frac{1}{R}\frac{\partial p'}{\partial y} + \frac{1}{R}\nabla^2 v', \tag{2.11}$$

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} + \frac{1}{R}\bar{v}\frac{\partial w'}{\partial y} + \frac{1}{R}v'\frac{d\bar{w}}{dy} + \frac{1}{R}\bar{w}\frac{\partial w'}{\partial z} = -\frac{1}{R}\frac{\partial p'}{\partial z} + \frac{1}{R}\nabla^2 w', \tag{2.12}$$

and

$$\frac{\partial u'}{\partial x} + \frac{1}{R}\frac{\partial v'}{\partial y} + \frac{1}{R}\frac{\partial w'}{\partial z} = 0, \tag{2.13}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the usual Laplacian and $\frac{d\bar{u}}{dy} = 2^{-1/2}\frac{d\bar{u}}{ds}$, etc.

This is a set of partial differential equations. However, in the quasi-parallel flow approximation, dependence on x, z, and t can be separated out by assuming normal mode solutions of the form

$$(u', v', w', p') = \left(\hat{u}(y), \hat{v}(y), \hat{w}(y), \hat{p}(y)\right)e^{i(\alpha x+\beta z-\omega t)}. \tag{2.14}$$

Equations (2.10) - (2.13) now become the set of four coupled ordinary differential equations

$$L_2\,\hat{u} = \frac{i\alpha}{R}\hat{p} + \bar{u}'\,\hat{v}\ , \tag{2.15}$$

$$L_2 \hat{v} - \bar{v}' \hat{v} = D \hat{p} \quad , \tag{2.16}$$

$$L_2 \hat{w} = i\beta \hat{p} + \bar{w}' \hat{v} \,, \tag{2.17}$$

and

$$\alpha \hat{u} - \frac{1}{R} (i D\hat{v} - \beta \hat{w}) = 0 \tag{2.18}$$

where $D = d/dy$, L_2 is the differential operator

$$L_2 = \left(D^2 - \alpha^2 - \beta^2\right) - \bar{v}D - iR\left(\alpha\bar{u} + \frac{\beta}{R} \bar{w} - \omega\right), \tag{2.19}$$

and a prime on a mean flow quantity denotes a y-derivative, e.g., $\bar{u}' = d\bar{u}/dy$.

Equations (2.15) - (2.19) are usually either left in their coupled form or rewritten in terms of vorticity components. For the present purposes, however, we wish to retain primitive variables. After some manipulation, $\hat{u}$, $\hat{w}$, and $\hat{p}$ can be eliminated from (2.15) - (2.18) to give a fourth-order uncoupled equation for $\hat{v}(y)$ of the form

$$\left[DL_2 D - (\alpha^2 + \beta^2)L_2\right]\hat{v} + iR(\alpha\bar{u}' + \frac{\beta}{R} \bar{w}')D\hat{v}$$

$$+ iR\left[\alpha\bar{u}'' + \frac{\beta}{R} \bar{w}'' - \frac{i}{R} (\alpha^2 + \beta^2)\bar{v}'\right]\hat{v} = 0. \tag{2.20}$$

The perturbation velocity $\hat{w}$ satisfies the inhomogeneous second-order (in $\hat{w}$) differential equation involving $\hat{v}$ of the form

$$(\alpha^2 + \beta^2)L_2 \hat{w} = i\beta L_2 D\hat{v} - \alpha R\left(\beta\bar{u}' - \frac{\alpha}{R} \bar{w}'\right)\hat{v} \,, \tag{2.21}$$

and from (2.18)

$$\hat{u} = \frac{1}{\alpha R}\left(iD\hat{v} - \beta\hat{w}\right). \tag{2.22}$$

The scenario for determining the perturbation fields is now clear. Equation (2.20), with appropriate boundary conditions on $\hat{v}$, is a fourth-order eigenvalue problem. For spatial stability, the frequency ω is real and fixed and the eigenvalue is the wave number vector (α,β). For temporal stability, α and β are real and fixed and the eigenvalues is the wave speed $c = \omega/(\alpha^2 + \beta^2)^{1/2}$. Having solved (2.20) and obtained $\hat{v}(y)$, $\hat{w}(y)$ is now found by solving the second-order inhomogeneous problem (2.21). Then $\hat{u}(y)$ can be obtained algebraically from (2.22), and $\hat{p}(y)$ is obtained algebraically from either (2.15) or (2.17).

The basic equation to be solved in the current stability problem is thus (2.20). Some simplification of this equation is possible as

$$DL_2 = L_2 D - \left[\bar{v}' D + iR(\alpha\bar{u}' + \frac{\beta}{R}\bar{w}')\right]. \tag{2.23}$$

In particular, (2.20) can be written in the form

$$\left\{(D^2 - \alpha^2 - \beta^2) - \bar{v}\,D - (i\beta\bar{w} + \bar{v}')\right\}(D^2 - \alpha^2 - \beta^2)\hat{v} + i\beta\bar{w}''\,\hat{v}$$

$$= i\alpha R\left\{(\bar{u} - \frac{\omega}{\alpha})\,(D^2 - \alpha^2 - \beta^2)\hat{v} - \bar{u}''\,\hat{v}\right\}. \tag{2.24}$$

Equations (2.24) and (2.21) must be complimented by four boundary conditions on $\hat{v}(y)$ and two boundary conditions on $\hat{w}(y)$. No-slip conditions on the solid wall $y = 0$ require

$$\hat{u}(0) = \hat{v}(0) = \hat{w}(0) = 0. \qquad (2.25a)$$

The continuity equation (2.18) now gives the additional condition

$$D\hat{v}(0) = 0. \qquad (2.25b)$$

Equations (2.25a,b) provide three of the required six boundary conditions for $\hat{v}(y)$ and $\hat{w}(y)$.

For modes in the discrete spectrum, the perturbation quantities u', v', and w' must tend to zero in the free stream, i.e., as y and z tend to infinity. However, the line of symmetry $y \equiv z$ may not be crossed. Care must thus be taken in translating this 'outer' behavior of u', v', and w' into conditions on $\hat{u}$, $\hat{v}$, and $\hat{w}$ as y tends to infinity. Fortunately, the restrictions that $y < z$ to remain below the line of symmetry and $z = O(R^p)$ with $0 < p < 1/3$ for the quasi-parallel flow approximation imply that z will tend to infinity automatically as y tends to infinity. Appropriate 'outer' boundary conditions for (2.24) and (2.21) are thus

$$\hat{v}, D\hat{v}, \hat{w} \to 0 \quad \text{as} \quad y \to +\infty. \qquad (2.26)$$

Consider now equation (2.24), which is the basic equation in the stability problem. This equation has a simple turning point at $y = y_c$ where $\alpha\bar{u}(y_c) - \omega$ vanishes. As the wave speed $c = \omega/(\alpha^2 + \beta^2)^{1/2}$, this turning point is subtly different from the usual turning point in the Blasius problem. Nevertheless, the critical layer associated with y_c sets the asymptotic character of equation (2.24) for large R. Further, as

$\bar{u}_c' = \bar{u}'(y_c) > 0$, the reduced equation

$$R_2 \hat{v} = (\bar{u} - \theta)(D^2 - \alpha^2 - \beta^2)\hat{v} - \bar{u}'' \hat{v} = 0 \qquad (2.27)$$

obtained by formally letting $R \to \infty$ in (2.24) with

$$\theta = \frac{\omega}{\alpha} = \left(1 + \frac{\beta^2}{\alpha^2}\right)^{1/2} c \qquad (2.28)$$

will have a regular singular point at y_c whereas y_c is a regular point of the full fourth-order equation for $\hat{v}(y)$. Dealing with this spurious singularity provides one of the main challenges in deriving uniform approximations to solutions of (2.24). In this regard, (2.24) is similar to the usual Orr-Sommerfeld equation for the stability of two-dimensional boundary layer profiles.

In treating equation (2.24), it is convenient to consider y to be a complex variable. Accordingly, the aim of this study is to seek "first approximations" to solutions of (2.24) which are uniformly valid in sectors of the complex plane containing the non-negative real axis.

3. THE PROPER LARGE PARAMETER AND HEURISTIC APPROXIMATIONS

For large Reynolds number, the most obvious large parameter in equation (2.24) is $i\alpha R$. However, if approximations are required to remain valid as $y \to \infty$, $i\alpha R$ is not an appropriate expansion parameter. This may be most easily seen in connection with the WKB approximations to viscous-type solutions of (2.24). It has been appreciated for some time that the WKB approximations are only valid away from the turning point in certain restricted sectors of the complex y-plane. However, examination of the $O(\alpha R)^{-1/2}$ term in the Poincaré series portion of the usual WKB approximations shows that, in the present context, these approximations also lose validity for $y > O(\alpha R)^{1/2}$. This breakdown at infinity may be traced to the use of $i\alpha R$ as the large parameter and may be eliminated through choice of the proper large parameter.

As $y \to +\infty$, equation (2.24) becomes a constant coefficient fourth-order equation which can be written in the factored form

$$\{D^2 - \alpha^2 - \beta^2\}\{(D - \frac{\bar{v}_\infty}{2})^2 - \gamma^2\}\hat{v} = 0 \tag{3.1}$$

where

$$\gamma^2 = i\alpha R(1 - \theta) + \alpha^2 + \beta^2 + \frac{\bar{v}_\infty^2}{4} + i\beta\bar{w}_\infty \tag{3.2}$$

with

$$\text{re}\,\gamma > 0 \tag{3.3}$$

and

$$\bar{v}_\infty = \bar{w}_\infty = 2^{-1/2}\beta_0 . \tag{3.4}$$

Thus, as $y \to +\infty$, inviscid type solutions of (2.24) will have the asymptotic behavior

$$\text{constant} \times \exp\{\pm(\alpha^2 + \beta^2)^{1/2} y\}, \tag{3.5}$$

and viscous type solutions will have the asymptotic behavior

$$\text{constant} \times \exp\left\{\pm \gamma y + \frac{\bar{v}_\infty}{2}(y - y_c)\right\}. \tag{3.6}$$

Equation (3.6) suggests that the proper large parameter for the corner layer problem is γ^2, not $i\alpha R$. Accordingly, as the starting point of the present asymptotic theory, equation (2.24) will be rewritten in the form

$$\left\{(D^2 - \alpha^2 - \beta^2) - \bar{v}D - (i\beta\bar{w} + \bar{v}')\right\}(D^2 - \alpha^2 - \beta^2)\hat{v} + i\beta\bar{w}''\hat{v}$$

$$= (\gamma^2 - \kappa)\left\{\frac{\bar{u} - \theta}{1 - \theta}(D^2 - \alpha^2 - \beta^2)\hat{v} - \frac{\bar{u}''}{1 - \theta}\hat{v}\right\} \tag{3.7}$$

with

$$\kappa = \alpha^2 + \beta^2 + \frac{\beta_0}{8}(\beta_0 + 8\, i\beta), \tag{3.8}$$

and β_0 as in (2.6).

The so-called heuristic approximations to solutions of equation (3.7) are not uniformly valid in full neighborhoods of the turning point y_c and have a number of other limitations. However, they will be considered here as they contain essential elements which will be needed later to construct the fully uniform approximations. Let these approximations be denoted by $\bar{\phi}(y)$. Consider first the modified WKB approximations for equation (3.7) based on the large parameter γ^2. Application of the usual WKB technique to (3.7) gives the approximations

$$\bar{\phi}_3(y) = c_0(\gamma)\,|\bar{u}-\theta|^{-5/4}\, e^{-\gamma h_0(y) + h_1(y)}\left\{1 - \gamma^{-1} H_1(y) + O(\gamma^{-2})\right\}, \tag{3.9}$$

and

$$\overline{\phi}_4(y) = c_0(\gamma)\ |\overline{u}-\theta|^{-5/4}\ e^{+\gamma h_0(y) + h_1(y)}\ \{1 + \gamma^{-1} H_1(y) + O(\gamma^{-2})\}, \qquad (3.10)$$

where $c_0(\gamma)$ is a constant which may depend on γ,

$$h_0(y) = \int_{y_c}^{y} \{\frac{\overline{u} - \theta}{1 - \theta}\}^{1/2}\ dy \qquad (3.11)$$

and

$$h_1(y) = \frac{1}{2} \int_{y_c}^{y} \overline{v}\ dy. \qquad (3.12)$$

In many applications, an explicit expression for the term $H_1(y)$ in (3.9) and (3.10) is not required. In the present context however, $H_1(y)$ is an essential component in the uniform approximations and is also needed to check that $\overline{\phi}_3(y)$ and $\overline{\phi}_4(y)$ remain valid as $y \to +\infty$. In the usual WKB procedure, $H_1(y)$ is determined algebraically after considerable manipulation. An alternate procedure is to take $\overline{\phi}_4$ (say) in the form

$$\overline{\phi}_4(y) = c_0(\gamma)\ |\overline{u}-\theta|^{-5/4}\ e^{\gamma h_0(y) + h_1(y)}\ H(y,\gamma) \qquad (3.13)$$

with

$$H(y,\gamma) = H(y) + \gamma^{-1} H_1(y) + O(\gamma^{-2}). \qquad (3.14)$$

Substitution of (3.13) and (3.14) into (3.7) now gives explicit expressions for $H'(y)$ and $H_1'(y)$. In particular, $H'(y) = 0$ so $H(y) \equiv 1$. The equation for $H_1'(y)$ is

$$H_1'(y) = -\left[\frac{5}{2} h_0'^{-1} h_1' - \frac{17}{4} h_0'^{-2} h_0'' + \frac{\kappa}{2} h_0'\right]'$$

$$+ \frac{1}{2}\left[i\beta\overline{w} + \overline{v}' + \frac{\overline{v}^2}{4} + \alpha^2 + \beta^2\right] h_0'^{-1} - \left[\frac{1}{2} h_0'' h_1 + h_0'''\right] h_0'^{-2}$$

$$- \frac{7}{8} h_0'^{-3} h_0''^{2} . \qquad (3.15)$$

This equation immediately shows $H_1'(y) \to 0$ as $y \to \infty$ so $H_1(y) \sim$ constant for large y and uniformity at infinity is preserved. Solving for $H_1(y)$ gives

$$H_1(y) = \frac{17}{8} \frac{\overline{u}'}{1-\theta} \left(\frac{\overline{u}-\theta}{1-\theta}\right)^{-3/2} - \frac{5}{4} \overline{v} \left(\frac{\overline{u}-\theta}{1-\theta}\right)^{-1/2} - \frac{\kappa}{2} \left(\frac{\overline{u}-\theta}{1-\theta}\right)^{1/2} + I_1(y), \qquad (3.16)$$

with

$$I_1(y) = \int_{y_c}^{y} \left\{\frac{1}{32} \left(\frac{\overline{u}'}{1-\theta}\right)^2 \left(\frac{\overline{u}-\theta}{1-\theta}\right)^{-5/2} - \frac{1}{2}\left[\frac{\overline{u}''}{1-\theta} + \frac{1}{4} \frac{\overline{u}' \ \overline{v}}{1-\theta}\right]\left(\frac{\overline{u}-\theta}{1-\theta}\right)^{-3/2}\right.$$

$$\left. + \frac{1}{2}\left[i\beta\overline{w} + \overline{v}' + \frac{\overline{v}^2}{4} + \alpha^2 + \beta^2\right]\left(\frac{\overline{u}-\theta}{1-\theta}\right)^{-1/2}\right\} dy. \qquad (3.17)$$

Following the usual practice, the lower limit of integration in (3.17) has been taken to be y_c. Also, $H_1(y)$ has been normalized so that its expansion relative to y_c contains no constant term.

The WKB approximations (3.9) and (3.10) are valid for $0 < |y-y_c| < \infty$ in restricted sectors of the complex plane. For example, in the complete sense of Watson [6], the approximation $\overline{\phi}_3$ must be restricted to the range $|\mathrm{ph}\ \gamma\, h_0(y)| < \pi$.

To approximate solutions of (3.7) of inviscid type, $\overline{\phi}(y)$ may be expanded in inverse powers of γ^2, e.g.,

$$\overline{\phi} = \overline{\phi}^{(0)}(y) + \gamma^{-2}\,\overline{\phi}^{(1)}(y) + O(\gamma^{-4}). \tag{3.18}$$

First outer approximations to two solutions $\overline{\phi}_1$ and $\overline{\phi}_2$ of (3.7) then satisfy the reduced equation

$$R_2\,\overline{\phi}^{(0)} = 0. \tag{3.19}$$

To express the solution of (3.19) which remains bounded as $y \to +\infty$, it is convenient to define the inviscid combination

$$\phi^{(0)}(y) = A^{(0)}\,\overline{\phi}_1^{(0)}(y) + \overline{\phi}_2^{(0)}(y). \tag{3.20}$$

The constant $A^{(0)}$ is determined by the condition that $\phi^{(0)}$ behaves asymptotically like $\exp\{-(\alpha^2+\beta^2)^{1/2}y\}$ as $y \to \infty$. For bounded $|y-y_c|$, expansions for $\overline{\phi}_1^{(0)}$ and $\overline{\phi}_2^{(0)}$ may be obtained by the Frobenius method. They have the forms

$$\overline{\phi}_1^{(0)}(y) = (y-y_c)\,P_1(y) \tag{3.21}$$

and

$$\overline{\phi}_2^{(0)}(y) = P_2(y) + \frac{\overline{u}_c''}{\overline{u}_c'}\,\overline{\phi}_1^{(0)}(y)\,\ln(y-y_c), \tag{3.22}$$

where $P_1(y)$ and $P_2(y)$ are power series in $(y-y_c)$, $P_1(y_c) = P_2(y_c) = 1$, and $P_2'(y_c) = 0$. The subscript "c" above denotes evaluation at y_c, e.g., $\bar{u}_c'' = \bar{u}''(y_c)$ and $\bar{u}_c''' = \bar{u}'''(y_c)$. The power series $P_1(y)$ and $P_2(y)$ will not be given explicitly as they are similar to the series for the solutions of the Rayleigh equation in the usual Orr-Sommerfeld problem. However, it is again worth commenting that $\bar{u}_c'$, $\bar{u}_c''$, and $\bar{u}_c'''$ are different here than in the Orr-Sommerfeld case as the definition of the turning point y_c involves θ rather than the wave speed c.

The approximation (3.21) for the "regular inviscid" solution $\bar{\phi}_1(y)$ is valid in the entire complex plane. However, because of its logarithmic behavior, the approximation (3.22) for the "singular inviscid solution" $\bar{\phi}_2(y)$ is only valid away from the turning point in the restricted sector $-\pi < \text{ph}(\gamma h_0(y)) < -\pi/3$. It should be noted that the logarithm in $\bar{\phi}_2^{(0)}(y)$ induces a phase shift in $\phi^{(0)}(y)$ across the critical layer. This behavior is associated with instability in problems involving two-dimensional boundary layer profiles.

4. A PRELIMINARY TRANSFORMATION

As a first step in deriving uniformily valid approximations to solutions of the disturbance equation (3.7), it is convenient to explicitly bring out the turning point nature of the equation through a preliminary transformation

$$\hat{v}(y) = \chi(\eta), \quad \text{where} \quad \eta = \left\{\frac{3}{2}\int_{y_c}^{y} \left(\frac{\bar{u}-\theta}{1-\theta}\right)^{1/2} dy\right\}^{2/3}. \qquad (4.1)$$

The Langer variable defined in (4.1) is appropriate for the present case of an unbounded domain, and differs from the usual bounded domain variable which is normalized by $\bar{u}_c'$ rather than $1-\theta$. In the present case

$$\eta(y) = \eta_c'(y-y_c) + \frac{1}{2!}\eta_c''(y-y_c)^2 + \cdots \tag{4.2}$$

for bounded $|y-y_c|$ where

$$\eta_c' = \left(\frac{\bar{u}_c'}{1-\theta}\right)^{1/3} \quad \text{and} \quad \eta_c'' = \frac{1}{5}\frac{\bar{u}_c''}{\bar{u}_c'} . \tag{4.3}$$

Also,

$$\eta(y) \sim \left(\frac{3}{2}y\right)^{2/3} \quad \text{as} \quad y \to +\infty. \tag{4.4}$$

If $\delta = \gamma^{-2/3}$, equation (3.7) now becomes

$$\delta^3(\chi^{iv} + f_0\,\chi''') - (\eta + \delta^3 f_1)\chi'' - (g_1 + \delta^3 \hat{g}_1)\chi' - (g_0 + \delta^3 \hat{g}_0)\chi = 0, \tag{4.5}$$

where

$$\left.\begin{aligned}
f_0(n) &= 6\mu - \bar{v}\, n'^{-1},\\
f_1(n) &= -(4\mu' + 11\mu^2 + \kappa n) + 3\mu\,\bar{v}\, n'^{-1} + [2\alpha^2 + 2\beta^2 + i\beta\bar{w} + \bar{v}']n'^{-2},\\
g_1(n) &= n\,\mu,\\
\hat{g}_1(n) &= -(\mu'' + 7\mu\mu' + 6\mu^3 - \kappa\, n\, \mu) + (2\alpha^2 + 2\beta^2 + i\beta\bar{w} + \bar{v}')\mu\, n'^{-2}\\
&\quad + \bar{v}(\mu' + 2\mu^2 - (\alpha^2 + \beta^2)n'^{-2})n'^{-1},\\
g_0(n) &= -\{5\mu + 6n\mu^2 + 2n\mu' + (\alpha^2 + \beta^2)nn'^{-2}\},\\
\hat{g}_0(0) &= -n'^{-4}\{(\alpha^2 + \beta^2)(\alpha^2 + \beta^2 + i\beta\bar{w} + \bar{v}') + i\beta\bar{w}''\} - \kappa\, g_0(n).
\end{aligned}\right\} \quad (4.6)$$

and

$$\mu(n) = n''(y)/[n'(y)]^2. \quad (4.7)$$

The function $\mu(n)$ has the behavior

$$\mu_0 = \mu(0) = \frac{1}{5}\frac{\bar{u}''_c}{\bar{u}'_c}\left(\frac{\bar{u}_c}{1-\theta}\right)^{-1/3} \quad \text{and} \quad \mu(n) \sim -\frac{1}{2}n^{-1} \quad \text{as} \quad n \to \infty. \quad (4.8)$$

Consider next the various heuristic approximations to solutions of equation (4.5). For outer approximations to solutions of inviscid type, $\chi(n)$ may be expanded in the form

$$\chi(n) = \bar{\chi}^{(0)}(n) + \delta^3\,\bar{\chi}^{(1)}(n) + O(\delta^6). \quad (4.9)$$

Then, $\overline{X}^{(0)}$ satisfies the transformed reduced equation

$$R_2 \overline{X}^{(0)} = 0 \quad \text{where} \quad R_2 = \eta D^2 - g_1(\eta)D - g_0(\eta) \tag{4.10}$$

with $D = d/d\eta$. Solutions of (4.10) may now be expressed as

$$\overline{X}_1^{(0)}(\eta) = \eta Q_1(\eta) \tag{4.11}$$

and

$$\overline{X}_2^{(0)}(\eta) = Q_2(\eta) + 5\mu_0 \overline{X}_1^{(0)}(\eta) \ln \eta \tag{4.12}$$

where $Q_1(\eta)$ and $Q_2(\eta)$ are power series in η, $Q_1(0) = Q_2(0) = 1$ and $Q_2'(0) = 0$. For numerical purposes it is easier to compute $\overline{\Phi}_1^{(0)}(y)$ and $\overline{\Phi}_2^{(0)}(y)$ rather than $\overline{X}_1^{(0)}(\eta)$ and $\overline{X}_2^{(0)}(\eta)$. This can easily be done as

$$\left.\begin{aligned} &\overline{X}_1^{(0)}(\eta) = \eta_c' \overline{\Phi}_1^{(0)}(y) \\ &\text{and} \\ &\overline{X}_2^{(0)}(\eta) = \overline{\Phi}_2^{(0)}(y) + \frac{\overline{u}_c''}{\overline{u}_c'} \overline{\Phi}_1^{(0)}(y) \ln \eta_c' \quad . \end{aligned}\right\} \tag{4.13}$$

For later reference, $Q_2(\eta)$ also satisfies the inhomogeneous equation

$$R_2 Q_2(\eta) = -5\mu_0\left(2\eta Q_1' + (g_0(\eta) + 1)Q_1\right). \tag{4.14}$$

The WKB approximations to viscous type solutions of equation (4.5) are

$$\bar{X}_3(\eta) = d_0(\delta)\ |\eta\eta^{-2}|^{-5/4}\ e^{-2/3\ \gamma\eta^{3/2} + h_1(y)}\ \{1 - F_1(\eta)\delta^{3/2} + O(\delta^3)\}$$

and (4.15)

$$\bar{X}_4(\eta) = d_0(\delta)\ |\eta\eta^{-2}|^{-5/4}\ e^{+2/3\ \gamma\eta^{3/2} + h_1(y)}\ \{1 + F_1(\eta)\delta^{3/2} + O(\delta^3)\}$$

where $F_1(\eta) = H_1(y)$. For later use, these approximations will be normalized by choosing the constant $d_0(\delta)$ to be

$$d_0(\delta) = \frac{1}{2}\pi^{-1/2}\ \delta^{5/4}\ \eta_c^{-5/2}. \qquad (4.16)$$

In discussing solutions of equation (4.5), we wish to exploit as much as possible certain symmetries which they exhibit in the complex plane, and for this purpose we consider the Stokes and anti-Stokes lines associated with the WKB approximations (4.15). They are defined by the conditions

$$\text{Im}\ \gamma\ \eta^{3/2} = 0 \quad \text{and} \quad \text{Re}\ \gamma\ \eta^{3/2} = 0, \qquad (4.17)$$

respectively. There are thus, three distinct Stokes and anit-Stokes lines which radiate out from the turning point $\eta = 0$ with equal angular spacings of $2\pi/3$. For marginal stability, c is real, and for modes in discrete spectrum $\kappa << \alpha R(1-\theta)$. In this case, ph $\gamma = \pi/4$ and the Stokes and anti-Stokes lines in the η-plane have the arrangement in Figure 4. In the y-plane, the behavior of the Stokes and anti-Stokes lines is more complicated. As $h_0(y) \equiv \frac{2}{3}\eta^{3/2}$, the definitions corresponding to (4.17) are

$$\text{Im}\ \gamma\ h_0(y) = 0 \quad \text{and} \quad \text{Re}\ \gamma\ h_0(y) = 0. \qquad (4.18)$$

These are again straight lines with equal angular spacing of $2\pi/3$ close to the turning point y_c. However, away from y_c, the global behavior of the Stokes and anti-Stokes lines in the y-plane depends on both the details of the Blasius velocity profile and the specific value of θ chosen. These difficulties in the y-plane provide a compelling reason for use of the appropriate Langer variable.

Consider now seven exact solutions of equation (4.5) which will be denoted by $U_0(\eta)$, $U_k(\eta)$, and $V_k(\eta)$, k=1,2,3. These solutions can be uniquely defined (to within multiplicative factors and modulo an arbitrary additive multiple of U_0 in the case of U_1) in terms of their asymptotic properties. Thus, we require that U_0 be well-balanced, that U_k be (purely) balanced in T_k, and that V_k be recessive in S_k, where S_k and T_k are sectors of the η-plane shown in Figure 4. Associated with these seven exact solutions will be three exact connection formulae which will not be needed for the present purposes.

Uniformly valid asymptotic approximations for these seven solutions will involve slowly-varying coefficients times rapidly-varying generalized Airy functions of a stretched variable ([5], [6])

$$\xi = \eta/\delta. \tag{4.19}$$

Proper indices p and q for the Airy functions may be inferred by expanding coefficients in (4.5) about $\eta = 0$ and deriving inner approximations $\tilde{\chi}(\xi)$. First inner approximations satisfy the equation

$$AD^2 \tilde{\chi}^{(0)}(\xi) = 0, \tag{4.20}$$

where $D = d/d\xi$ and $A = D^2 - \xi$. Solutions of (4.20) are $B_0(\xi,2)$, $B_k(\xi,2,1)$, and $A_k(\xi,2)$, $k=1,2,3$. Higher inner approximations are solutions of inhomogeneous versions of (4.20) and involve the Airy functions $B_0(\xi,p)$, $B_k(\xi,p,1)$, and $A_k(\xi,p)$ with $p=0,1$.

5. SOLUTIONS OF DOMINANT-RECESSIVE TYPE

Uniformly valid "first approximations" to the dominant-recessive solutions $V_k(\eta)$ of equation (4.5) involve the Airy functions $A_k(\xi,p)$, $k=1,2,3$, with $p=0,1,2$. They have the forms

$$V_k(\eta) \sim a(\eta)A_k(\xi,2) + \delta^2 b(\eta)A_k(\xi,1) + \delta c(\eta)A_k(\xi,0), \tag{5.1}$$

where a, b, and c are regular at the turning point. On substituting (5.1) into equation (4.5), the slowly varying coefficients $a(\eta)$ and $c(\eta)$ are found to satisfy the equations

$$R_2(\eta a) = 0 \tag{5.2}$$

and

$$\{2\eta D + (\eta f_0 - g_1)\}\ (a + \eta c) = 0, \tag{5.3}$$

where R_2 is given in (4.10), $D = d/d\eta$. These equations may immediately be integrated to give the regular solutions

$$a(\eta) = a(0)\ \eta^{-1}\ \overline{X}_1^{(0)}(\eta) = a(0)\ Q_1(\eta) \tag{5.4}$$

and

$$a(\eta) + \eta c(\eta) = a(0)\left(\frac{\eta}{\eta_c}\right)^{-5/2} \exp\{h_1(y)\}. \tag{5.5}$$

The differential equation satisfied by $b(\eta)$ is quite involved and will not be given explicitly. A more illuminating way of obtaining this coefficient is to require that the outer expansion of $V_1(\eta)$ (say) be the WKB approximation $\overline{X}_3(\eta)$ away from the turning point in the sector S_1 where this solution is purely recessive. With $\overline{X}_3$ normalized as in (4.15) and (4.16), this condition gives

$$a(0) = 1 \tag{5.6}$$

and

$$b(\eta) = 2\eta^{-1} c(\eta) + \eta^{-1/2}\big(a(\eta) + \eta c(\eta)\big)\left(F_1(\eta) - \frac{101}{48}\eta^{-3/2}\right). \tag{5.7}$$

Thus, $b(\eta)$ is essentially a regularized form of $F_1(\eta) = H_1(y)$.

6. SOLUTIONS OF BALANCED AND WELL-BALANCED TYPE

The uniform first approximation to the well-balanced solution is simply

$$U_0(\eta) = \overline{X}_1^{(0)}(\eta) + O(\delta^3). \tag{6.1}$$

First approximations to the three solutions of balanced type have the forms

$$U_k(\eta) \sim G(\eta) + \lambda\delta\{A(\eta)\, B_k(\xi,2,1) + \delta^2\, B(\eta)B_k(\xi,1,1)$$

$$+ \delta C(\eta)\, B_k(\xi,0,1)\}, \qquad (6.2)$$

where G, A, B, and C are regular at the turning point and λ is a constant to be determined.

Substitution of (6.2) into (4.5) shows that A, B, and C satisfy the same differential equations as a, b, and c. Consequently

$$\{A(\eta),B(\eta),C(\eta)\} = A(0)\{a(\eta),b(\eta),c(\eta)\}. \qquad (6.3)$$

The well-balanced part $G(\eta)$ of (6.2) is found to satisfy the equation

$$R_2\, G = \lambda\{4\eta(A + \eta C)' + A - 2\eta(\eta C)' + f_0(A + \eta C) - g_1\, \eta C\} \qquad (6.4)$$

which, on simplification, becomes

$$R_2\, G = \lambda A(0)\{2\eta Q_1' + (1 + g_0)Q_1\}. \qquad (6.5)$$

A comparison of this equation with (4.14) now shows that the solution of (6.5) which is regular at $\eta = 0$ is of the form

$$G(\eta) = Q_2(\eta) + b_0\, \bar{X}_1^{(0)}(\eta), \qquad (6.6)$$

where b_0 is an arbitrary constant, provided

$$\lambda = -5\mu_0 \quad \text{and} \quad A(0) = 1. \tag{6.7}$$

The constant b_0 in (6.6) may be chosen in a variety of ways. A particularly convenient way is to choose b_0 so that the outer expansion of $U_2(\eta)$ (say) to $O(\delta^2)$ in the sector T_2 contains no multiple of $\overline{\phi}_1^{(0)}(y)$. Using the asymptotic expansions of the B-type Airy functions in T_2 and the usual outer expansion operator E_2 to $O(\delta^2)$,

$$E_2\, U_2\,(\eta) = Q_2(\eta) + \overline{\chi}_1^{(0)}\{b_0 + 5\gamma_0[\log \eta - \log \delta + \hat{\gamma} - 1 - 2\pi i]\} \tag{6.8}$$

where $\hat{\gamma} = 0.577 \cdots$ is Euler's constant. This result can now be written in the form

$$E_2\, U_3(\eta) = \overline{\phi}_2^{(0)}(y) + \overline{\chi}_1^{(0)}(y)\{b_0 - 5\mu_0[\log \delta - \log \eta_c' + 1 - \hat{\gamma} + 2\pi i]\}. \tag{6.9}$$

Thus, $E_2\, U_2(\eta)$ contains no multiple of $\overline{\phi}_1^{(0)}(y)$ provided

$$b_0 = 5\mu_0[\log \delta - \log \eta_c' - \hat{\gamma} + 1 + 2\pi i]. \tag{6.10}$$

It is worth noting that the slowly varying coefficients in both (5.1) and (6.2) are expressible in terms of known quantities in the heuristic approximations: the regular solution and the regular part of the singular solution of the reduced equation, and the terms $\exp(h_1(y))$ and $H_1(y)$ in the WKB approximations.

7. CONCLUDING REMARKS

The major aims of this paper have been to obtain the boundary value problem associated with small amplitude disturbances in the blending boundary layer region, and to derive "first approximations" to solutions of the governing equation for $\hat{v}(y)$ which are uniformly valid on a semi-infinite interval. This requires the introduction of a new large parameter and an appropriate Langer variable. The uniform "first approximations" involve slowly varying, analytic coefficients times rapidly varying generalized Airy functions of a stretched Langer variable. Because of the important first and third derivative terms in the fourth-order disturbance equation, as well as the presence of $\bar{v}$ and $\bar{w}$ in other terms, these analytic coefficients differ from their counterparts in the usual stability problem for two dimensional mean profiles. Approximations to solutions of balanced type display a phase shift across the critical layer that is an indication of instability.

A derivation of the eigenvalue relation associated with the boundary conditions (2.25) and (2.26), numerical results for the solution of equation (2.21) for $\hat{w}$, and stability characteristics of the basic corner flow will be given in a subsequent paper. The characteristic equation for this problem is of the form $\Delta(0) = 0$ where

$$\Delta(y) = \eta_c' \, W(\Phi, \delta V_1)(\eta), \tag{7.1}$$

W is the usual Wronskian,

$$\Phi(\eta) = A^{(0)} U_0(\eta) + U_2(\eta) = \Phi(y), \tag{7.2}$$

and $A^{(0)}$ is again determined by the condition that $\Phi(\eta)$ remains bounded as $\eta \to +\infty$. The need to use uniformly valid approximations for U_0, U_2, and V_1 in analyzing (7.1) may be easily seen. If, following the usual heuristic procedure, Frobenius-type approximations are used for $\phi^{(0)}$ and either first WKB approximations or first inner approximations are used for V_1, the resulting crude approximation to (7.1) is independent of both the mean outflow velocity $\bar{v}$ and the mean secondary flow component $\bar{w}$.

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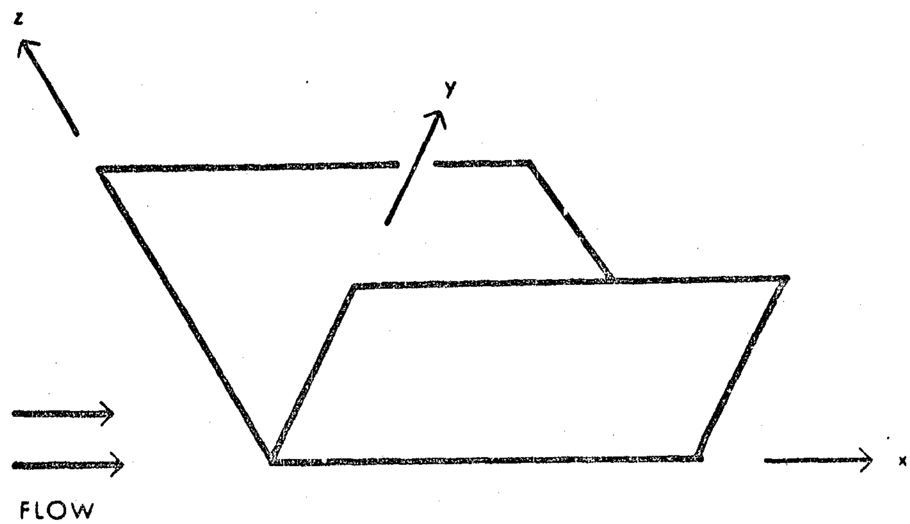

Figure 1. Geometry for flow along a rectangular streamwise corner. The incoming uniform stream is aligned with the corner line $y = z = 0$.

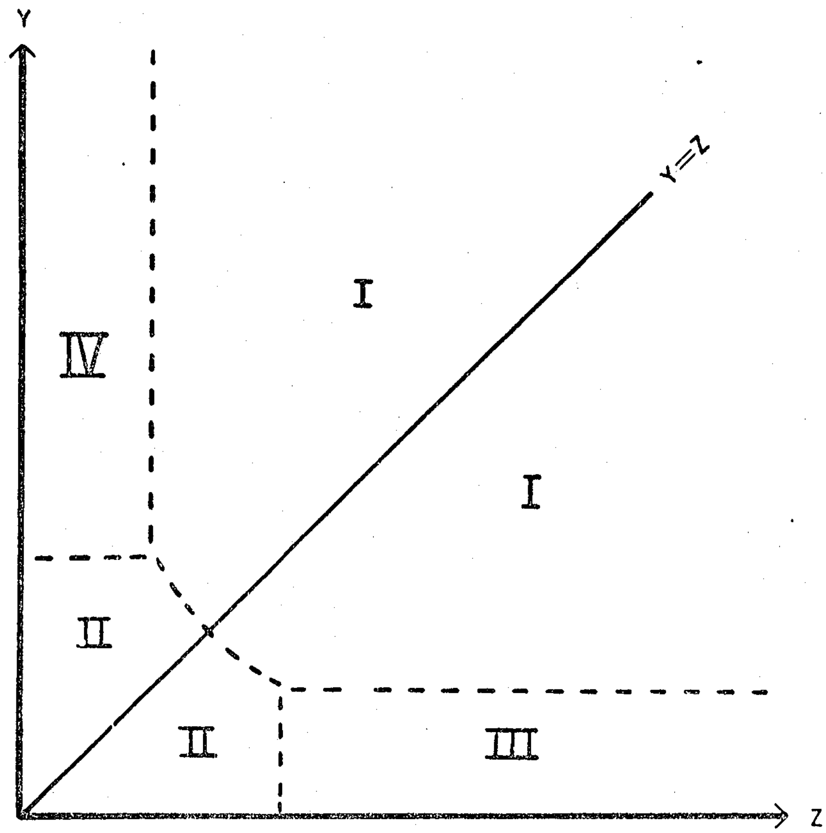

Figure 2. Qualitative regions of the mean flow velocity field in the plane perpendicular to the corner.

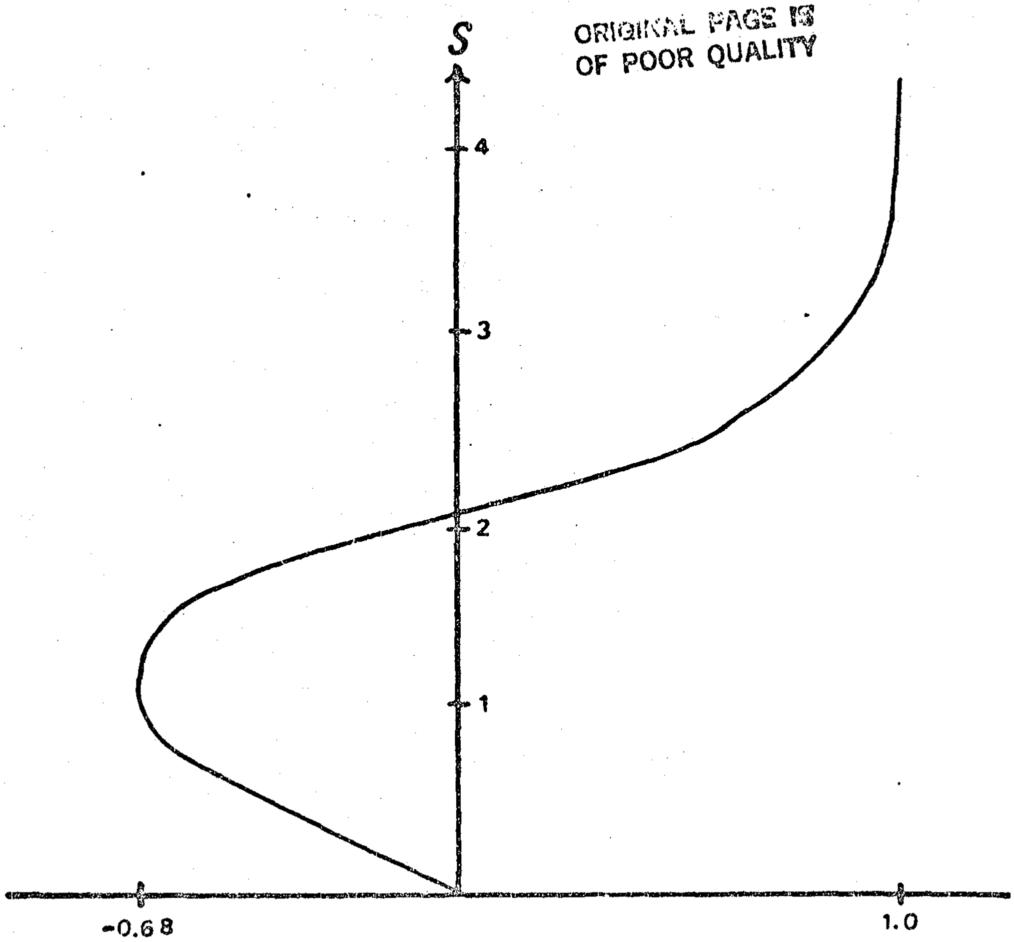

Figure 3. The function $H'_D(s)$ from the secondary velocity $R^{-1}\bar{w}_1$.

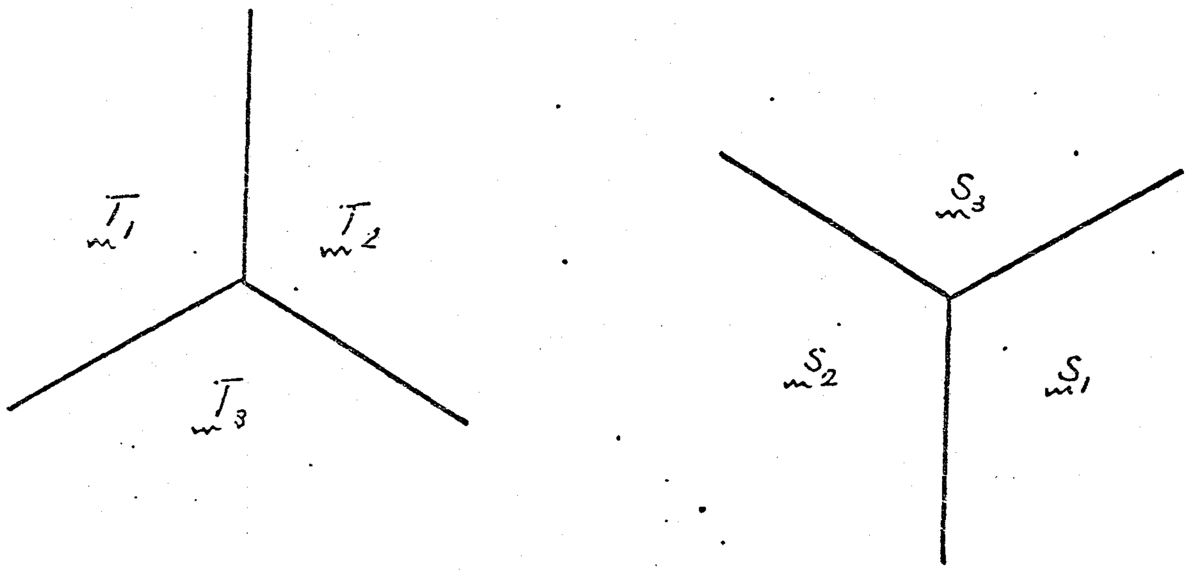

Fig. 4. The Stokes lines (left) and the anti-Stokes lines (right) in the η-plane with θ real and ph $\gamma = \frac{1}{4}\pi$.

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