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ABSTRACT

In this paper the transient thermal stress problem in a hollow cylinder or a disk containing a radial crack is considered. It is assumed that the cylinder is reinforced on its inner boundary by a membrane which has thermoelastic constants different than those of the base material. The transient temperature, thermal stresses and the crack tip stress intensity factors are calculated in a cylinder which is subjected to a sudden change of temperature on the inside surface. The results are obtained for various dimensionless parameters and material constants. The special cases of the crack terminating at the cylinder-membrane interface and of the broken membrane are separately considered and some examples are given.

## 1. Introduction

Cracking due to thermal stresses arises in many practical applications particularly when sudden changes occur in the environmental temperature. Suddenly cooled glass plates and containers are generally the familiar examples for the phenomenon. The problem may also be quite important in such structural components as the pressure vessels, piping and hollow circular disks subjected to thermal transients. In these structures the actual problems are usually very complicated three-dimensional crack problems. Under thermal shock a part-through crack may initiate or an existing part-through crack may propagate in axial or circumferential as well as in the thickness direction. However, if the crack driving force

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due mechanical stresses is sufficiently small, then because of the self-equilibrating nature of the thermal stresses the growth of the crack in thickness direction is generally arrested. A clear demonstration of this process may be seen in the results reported in [1] where it was observed that in relatively thick-walled hollow circular glass cylinders suddenly cooled from inside an axial initial crack penetrated into the cylinder wall only partially but propagated axially along the entire length of the cylinder. It is clear that in such problems the corresponding plane strain or plane stress solutions may provide very useful bounds for the actual part-through crack problem.

The axisymmetric problem for a thick-walled cylinder containing a part-through circumferential crack and subjected to transient thermal stresses was considered in [2]. In this paper we consider the corresponding plane strain or plane stress problem for a hollow circular cylinder which contains a radial crack. To simulate the cladding (in, for example, nuclear pressure vessels) it is assumed that the cylinder is reinforced on the inside boundary by a (thin) membrane. The results found may also be applicable to other composite cylinders or disks in which the thickness of the inner cylinder or ring is relatively very small.

In solving the problem it is assumed that the material is linear, the thermo-mechanical constants of the cylinder and the reinforcing membrane are independent of temperature, the bending stiffness of the reinforcing membrane and all thermoelastic coupling effects are negligible, and the transient thermal stress problem may be treated as a quasi-static time-dependent problem, that is, all inertia effects may be neglected. Previous studies on dynamic thermoelasticity indicate that this last assumption, which simplifies the analysis of the problem quite considerably, would not cause any significant changes in the results (see, for example, [3] and [4]). Since the problem is linear the solution of the crack problem due to thermal stresses and to all other sources of loading may be considered separately. Also, the solution of the crack problem under thermal stresses may be obtained in two steps. The first would be the calculation of time-dependent thermal stresses in an uncracked reinforced cylinder under given temperature and stress boundary conditions.

Then the second step would be the solution of an isothermal crack problem in which the self-equilibrating crack surface tractions equal and opposite to that obtained in the first solution are the only external loads. The complete solution may be obtained by adding these last two solutions to the other isothermal results. Needless to say, in thermal stress as well as in the isothermal problems the important information which is useful in fracture considerations is contained in the perturbation solutions.

## 2. The Thermal Stresses

Referring to Figure 1 for notation let the temperatures in reinforcing and main cylinders be $T_1$ and $T_2$, respectively. If we assume that the crack surfaces are well-insulated, then $T_1$ and $T_2$ would be functions of radial coordinate r and the time t only. Let $T_0$ be the initial temperature of the composite cylinder and the temperature changes $\theta_1$ and $\theta_2$ be defined by

$$\theta_1(r,t) = T_1(r,t) - T_0 \quad , \quad (a_0<r<a,\ 0\leq t) \quad , \tag{1}$$

$$\theta_2(r,t) = T_2(r,t) - T_0 \quad , \quad (a<r<b,\ 0\leq t) \quad . \tag{2}$$

In the case of uncracked cylinders the problem is axisymmetric, the shear stresses are zero, and the relevant displacement and stress components may be expressed as follows (see, for example, [5]):

$$u^T_{ri}(r,t) = \frac{1+\nu_i}{1-\nu_i}\frac{\alpha_i}{r}\int_{a_i}^{r}\theta_i(\rho,t)\rho d\rho + C_{1i}r + \frac{C_{2i}}{r} \quad , \tag{3}$$

$$\sigma^T_{ri}(r,t) = -\frac{\alpha_i E_i}{1-\nu_i}\frac{1}{r^2}\int_{a_i}^{r}\theta_i(\rho,t)\rho d\rho + \frac{E_i}{1+\nu_i}\left(\frac{C_{1i}}{1-2\nu_i} - \frac{C_{2i}}{r^2}\right) \quad , \tag{4}$$

$$\sigma^T_{\theta i}(r,t) = \frac{\alpha_i E_i}{1-\nu_i}\frac{1}{r^2}\int_{a_i}^{r}\theta_i(\rho,t)\rho d\rho - \frac{E_i\alpha_i}{1-\nu_i}\theta_i(r,t) + \frac{E_i}{1+\nu_i}\left(\frac{C_{1i}}{1-2\nu_i} + \frac{C_{2i}}{r^2}\right) \quad , \tag{5}$$

where $i = 1,2,$

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$$a_i = a_o \;,\quad a_o<r<a \;,\ \text{for } i = 1 \;,$$

$$a_i = a \;,\quad a<r<b \;,\ \text{for } i = 2 \;,$$

$\alpha_i$, $E_i$ and $\nu_i$ are the standard thermoelastic constants of the materials $u_{ri}^T$ is the radial displacement and the unknown constants $C_{1i}$ and $C_{2i}$ are determined from the following boundary and continuity conditions:

$$\sigma_{r1}^T(a_o,t) = 0 \;,\quad \sigma_{r2}^T(b,t) = 0 \;, \tag{6a,b}$$

$$u_{1r}^T(a,t) = u_{2r}^T(a,t) \;,\ \sigma_{r1}^T(a,t) = \sigma_{r2}^T(a,t) \;. \tag{7a,b}$$

Thus, from (3), (4), (6) and (7) it can be shown that

$$C_{11} = \frac{1-2\nu_1}{a_o^2}\,\frac{\Delta_1}{\Delta_o} \;,\quad C_{12} = \frac{\Delta_1}{\Delta_o} \;,$$

$$C_{21} = \frac{\alpha_2(1+\nu_2)(1-2\nu_2)B}{(1-\nu_2)b^2} + \frac{1-2\nu_2}{b^2}\,\frac{\Delta_2}{\Delta_o} \;,\quad C_{22} = \frac{\Delta_2}{\Delta_o} \;, \tag{8a-d}$$

where

$$\Delta_o = -\frac{E_2(a^2-b^2)[(1-2\nu_1)a^2+a_o^2]}{(1+\nu_2)a_o^2a^4b^2} + \frac{E_1(a^2-a_o^2)[(1-2\nu_2)a^2+b^2]}{(1+\nu_1)a_o^2a^4b^2} \;,$$

$$\Delta_1 = -\frac{E_2(a^2-b^2)[-(1+\nu_1)(1-\nu_2)\alpha_1 b^2A + (1-\nu_1)(1+\nu_2)(1-2\nu_2)\alpha_2 a^2B]}{(1-\nu_1)(1-\nu_2^2)a^4b^4}$$

$$+ \frac{[\alpha_2E_2a^2(1-\nu_1)B + \alpha_1E_1b^2(1-\nu_2)A][(1-2\nu_2)a^2+b^2]}{(1-\nu_1)(1-\nu_2)a^4b^4} \;,$$

$$\Delta_2 = \frac{[(1-2\nu_1)a^2+a_o^2][\alpha_2E_2a^2(1-\nu_1)B + \alpha_1E_1b^2(1-\nu_2)A]}{(1-\nu_1)(1-\nu_2)a_o^2a^4b^2}$$

$$+ \frac{E_1(a^2-a_o^2)[(1+\nu_1)(1-\nu_2)\alpha_1b^2A - (1-\nu_1)(1+\nu_2)(1-2\nu_2)\alpha_2a^2B]}{(1-\nu_1^2)(1-\nu_2)a_o^2a^4b^2} ,$$

$$A = \int_{a_o}^{a} \theta_1(r,t)r\,dr \quad , \quad B = \int_{a}^{b} \theta_2(r,t)r\,dr \quad . \tag{9a-e}$$

It should be noted that the expressions (3)-(5), (8) and (9) are valid for the case of plane strain only. For "disk" problems which may be approximated by generalized plane stress $E_i$ and $\nu_i$ should be replaced by $E_i(1+2\nu_i)/(1+\nu_i^2)$ and $\nu_i/(1+\nu_i)$, respectively.

To obtain the temperature distributions $\theta_1$ and $\theta_2$ the following diffusion equations must be solved under given initial and boundary conditions:

$$\nabla^2\theta_i(r,t) = \frac{1}{D_i}\frac{\partial\theta_i(r,t)}{\partial t} \quad , \quad (i=1,2) \tag{10}$$

where $a_o<r<a$ for $i = 1$, $a<r<b$ for $i = 2$ and $D_1$ and $D_2$ are the respective coefficients of diffusivity. A particular problem of interest is the sudden cooling of the composite cylinder from inside which is initially under a homogeneous temperature $T_o$. Thus, if it is assumed that for $t>0$ the cylinder wall $r = a_o$ is maintained at a constant temperature $T_{ao}$ and the outer surface $r = b$ is insulated, (10) must be solved under

$$\theta_1(r,0) = 0 \quad , \quad (a_o<r<a) \quad , \tag{11}$$

$$\theta_2(r,0) = 0 \quad , \quad (a<r<b) \quad , \tag{12}$$

$$\theta_1(a,t) = \theta_2(a,t) \; , \; k_1 \frac{\partial}{\partial r}\theta_1(a,t) = k_2 \frac{\partial}{\partial r}\theta_2(a,t) \; , \; (0<t) \tag{13a,b}$$

$$\theta_1(a_o,t) = \theta_{ao}H(t) \; , \; \frac{\partial}{\partial r}\theta_2(b,t) = 0 \tag{14a,b}$$

where

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$$\theta_{ao} = T_{ao} - T_o \quad , \tag{15}$$

$k_1$ and $k_2$ are the coefficients of heat conduction and $H(t)$ is the Heaviside function. By using the Laplace transforms the solution of (10) subject to (11)-(14) may be obtained as follows:

$$\frac{\theta_1(r^*,t^*)}{\theta_{ao}} = 1 - \sum_1^\infty \frac{2e^{-\lambda_n^2 t^*}}{a^*\lambda_n H^*(\lambda_n)} [\kappa_o \beta Z_{oo}(a^*\lambda_n, r^*\lambda_n) Z_{11}(\beta a^*\lambda_n, \beta\lambda_n)$$

$$+ Z_{o1}(\beta a^*\lambda_n, \beta\lambda_n) Z_{o1}(r^*\lambda_n, a^*\lambda_n)] \quad , \tag{16}$$

$$\frac{\theta_2(r^*,t^*)}{\theta_{ao}} = 1 + \frac{4}{\pi} \sum_1^\infty \frac{Z_{o1}(\beta r^*\lambda_n, \beta\lambda_n) e^{-\lambda_n^2 t^*}}{a^{*2}\lambda_n^2 H^*(\lambda_n)} \tag{17}$$

where the cylinder functions and the dimensionless quantities are defined by

$$Z_{ij}(x,y) = J_i(x)Y_j(y) - Y_i(x)J_j(y) \ , \ (i,j=0,1) \quad , \tag{18}$$

$$r^* = r/b \ , \ a_o^* = a_o/b \ , \ a^* = a/b \ , \ t^* = D_1 t/b^2 \ , \ \beta = (D_1/D_2)^{\frac{1}{2}}, \ \kappa_o = k_2/k_1 \ , \tag{19}$$

$$H^*(\lambda_n) = \kappa_o \beta^2 Z_{oo}(a_o^*\lambda_n, a^*\lambda_n)[Z_{o1}(\beta a^*\lambda_n, \beta\lambda_n) - \frac{1}{a^*} Z_{o1}(\beta\lambda_n, \beta a^*\lambda_n)]$$

$$+ Z_{o1}(\beta a^*\lambda_n, \beta\lambda_n)[-Z_{oo}(a_o^*\lambda_n, a^*\lambda_n) + \frac{a_o^*}{a^*} Z_{11}(a_o^*\lambda_n, a^*\lambda_n)]$$

$$- \beta\kappa_o Z_{11}(\beta\lambda_n, \beta a^*\lambda_n)[-Z_{o1}(a_o^*\lambda_n, a^*\lambda_n) + \frac{a_o^*}{a^*} Z_{o1}(a^*\lambda_n, a_o^*\lambda_n)]$$

$$+ \beta Z_{o1}(a_o^*\lambda_n, a^*\lambda_n)[-Z_{11}(\beta\lambda_n, \beta a^*\lambda_n) + \frac{1}{a^*} Z_{oo}(\beta\lambda_n, \beta a^*\lambda_n)] \ , \tag{20}$$

$\lambda_n$ are the roots of

$$\beta\kappa_0 Z_{00}(a_0^*\lambda_n, a^*\lambda_n) Z_{11}(\beta\lambda_n, \beta a^*\lambda_n) + Z_{01}(\beta a^*\lambda_n, \beta\lambda_n) Z_{01}(a_0^*\lambda_n, a^*\lambda_n) = 0, \tag{21}$$

and $J_n$ and $Y_n$ are the Bessel functions of first and second kind, respectively.

The thermal stresses in the composite cylinder without the crack may be obtained by substituting from (16) and (17) into (4) and (5). For example, the hoop stresses $\sigma_{\theta 1}$ and $\sigma_{\theta 2}$ which are needed in the solution of the crack problem are found to be

$$\sigma_{\theta 1}^T(r^*,t^*) = \frac{\alpha_1 E_1 \theta_{ao}}{1-\nu}\left[\frac{F_1(r^*)}{r^{*2}} - \frac{\theta_1(r^*,t^*)}{\theta_{ao}} + \frac{r^{*2}+a_0^{*2}}{r^{*2}} S_1\right], \tag{22}$$

$$\sigma_{\theta 2}^T(r^*,t^*) = \frac{\alpha_2 E_2 \theta_{ao}}{1-\nu}\left[\frac{F_2(r^*)}{r^{*2}} - \frac{\theta_2(r^*,t^*)}{\theta_{ao}} + F_2(1) + \frac{\alpha_1(1+r^{*2})}{\alpha_2 r^{*2}} S_2\right], \tag{23}$$

where

$$F_1(r^*) = \int_{a_0^*}^{r^*} \frac{\theta_1(r^*,t^*)}{\theta_{ao}} r^* dr^* = \frac{r^{*2}-a_0^{*2}}{2} - \sum_1^\infty \frac{2e^{-\lambda_n^2 t^*}}{a^*\lambda_n^2 H^*(\lambda_n)}$$

$$\cdot\{\kappa_0\beta Z_{11}(\beta a^*\lambda_n,\beta\lambda_n)[r^* Z_{01}(a^*\lambda_n, r^*\lambda_n) - a_0^* Z_{01}(a^*\lambda_n, a_0^*\lambda_n)]$$

$$+ Z_{01}(\beta a^*\lambda_n,\beta\lambda_n)[r^* Z_{11}(r^*\lambda_n, a^*\lambda_n) - a_0^* Z_{11}(a_0^*\lambda_n, a^*\lambda_n)]\},$$

$$F_2(r^*) = \int_{a^*}^{r^*} \frac{\theta_2(r^*,t^*)}{\theta_{ao}} r^* dr^* = \frac{r^{*2}-a^{*2}}{2} + \frac{4}{\pi}\sum_1^\infty \frac{e^{-\lambda_n^2 t^*}}{\beta a^{*2}\lambda_n^3 H^*(\lambda_n)}$$

$$\cdot[r^* Z_{11}(\beta r^*\lambda_n, \beta\lambda_n) - a^* Z_{11}(\beta a^*\lambda_n, \beta\lambda_n)],$$

$$S_1 = \frac{[(1-2\nu)a^{*2}+1]\left[\frac{\alpha_2 E_2}{\alpha_1 E_1} a^{*2} F_2(1) + F_1(a^*)\right] + \frac{E_2}{E_1}(a^{*2}-1)\left[F_1(a^*) - (1-2\nu)\frac{\alpha_2}{\alpha_1} a^{*2} F_2(1)\right]}{(a^{*2}-a_0^{*2})[(1-2\nu)a^{*2}+1] - \frac{E_2}{E_1}(a^{*2}-1)[(1-2\nu)a^{*2}+a_0^{*2}]},$$

$$S_2 = \frac{[(1-2\nu)a^{*2}+a_o^{*2}][\frac{\alpha_2 E_2}{\alpha_1 E_1} a^{*2}F_2(1)+F_1(a^*)]+(a^{*2}-a_o^{*2})[F_1(a^*)-(1-2\nu)\frac{\alpha_2}{\alpha_1} a^{*2}F_2(1)]}{(a^{*2}-a_o^{*2})[(1-2\nu)a^{*2}+1]-\frac{E_2}{E_1}(a^{*2}-1)[(1-2\nu)a^{*2}+a_o^{*2}]}$$

(24 a-d)

and, for simplicity, it is assumed that $\nu_1 = \nu_2 = \nu$.

If one further assumes that $E_1 = E_2 = E$ as well as $\nu_1 = \nu_2 = \nu$ (i.e., that the elastic constants but not the thermal coefficients of the two cylinders are the same), then the expressions for hoop stresses are further simplified and become

$$\sigma^T_{\theta 1}(r,t) = \frac{E}{1-\nu}\frac{1}{r^2}\{\frac{r^2+a_o^2}{b^2-a_o^2}[\alpha_1\int_{a_o}^{a}\theta_1(r,t)rdr + \alpha_2\int_a^b\theta_2(r,t)rdr]$$

$$+\alpha_1\int_{a_o}^{r}\theta_1(r,t)rdr - \alpha_1 r^2\theta_1(r,t)\} \quad , \tag{25}$$

$$\sigma^T_{\theta 2}(r,t) = \frac{E}{1-\nu}\frac{1}{r^2}\{\frac{r^2+a_o^2}{b^2-a_o^2}[\alpha_1\int_{a_o}^{a}\theta_1(r,t)rdr + \alpha_2\int_a^b\theta_2(r,t)rdr]$$

$$+\alpha_1\int_{a_o}^{a}\theta_1(r,t)rdr + \alpha_2\int_a^r\theta_2(r,t)rdr - \alpha_2 r^2\theta_2(r,t)\}. \tag{26}$$

If the thickness $a-a_o$ of the inner cylinder is indeed very small, then the analysis given in this section may be simplified quite considerably by using the identities

$$Z_{oo}(x,x) = 0,\ Z_{11}(x,x) = 0,\ Z_{o1}(x,x) = -\frac{2}{\pi x} \quad . \tag{27}$$

In this case the characteristic equation (21) giving the roots $\lambda_n$ may be approximated by

$$Z_{o1}(\beta a^*\lambda_n, \beta\lambda_n) = 0 \quad . \tag{28}$$

An interesting special case of the thermal shock problem under consideration would be the yielding of the inner cylinder which may occur if the step change $\theta_{ao}$ in the inner wall temperature is sufficiently high. The limiting value $\theta_F = \theta_{ao}$ which corresponds to the yielding of the membrane cladding may be obtained from (22) by letting

$$\sigma^T_{\theta 1}(a^*, +0) = \sigma_F \quad , \tag{29}$$

where $\sigma_F$ is a measure of the yield condition of the material (e.g., the flow stress).

For the very thin inner cylinder mentioned above it can be shown that

$$H^*(\lambda_n) \cong \frac{2\beta}{\pi a^*\lambda_n} [Z_{11}(\beta\lambda_n, \beta a^*\lambda_n) - \frac{1}{a^*} Z_{oo}(\beta\lambda_n, \beta a^*\lambda_n)] \; ,$$

$$F_1(a^*) \cong 0 \quad , \quad \theta_1(a^*, t^*) \cong \theta_{ao} \quad , \tag{30}$$

and from (29) we find

$$\sigma_F = - \frac{(1-\nu)\sigma_F}{\alpha_1 E_1} \quad . \tag{31}$$

## 3. The Crack Problem

The thermal stress problem for a composite cylinder containing an axial crack in a radial plane (Fig. 1) may now be solved by using the equal and opposite of the hoop stress given by (23) as the crack surface traction. As shown in [6], this problem must be solved under the following boundary conditions:

$$\sigma_{rr}(b,\theta,t) = 0, \; \sigma_{r\theta}(b,\theta,t) = 0, \; (0 \le \theta \le 2\pi, \; 0 < t) \quad , \tag{32a,b}$$

$$\sigma_{rr}(a,\theta,t)-\lambda\sigma_{\theta\theta}(a,\theta,t) = 0 \quad , \quad (0\leq\theta<2\pi,\ 0<t) \quad , \tag{33}$$

$$\sigma_{r\theta}(a,\theta,t) + \lambda \frac{\partial}{\partial\theta}\sigma_{\theta\theta}(a,\theta,t) = 0 \ , \ (0\leq\theta<2\pi,\ 0<t) \tag{34}$$

$$\sigma_{\theta\theta}(r,+0,t) = \sigma_{\theta\theta}(r,-0,t) = -\sigma^T_{\theta 2}(r,t) \ , \ (e<r<g,\ 0<t) \ , \tag{35a}$$

$$u_\theta(r,0,t) = 0,\ (a<r<e,\ g<r<b,\ 0<t) \quad . \tag{35b}$$

where $\sigma_{ij}$, $(i,j=r,\theta)$ and $u_\theta$ are the stresses and the circumferential displacement in cylinder 2 for the perturbation problem and

$$\lambda = \begin{cases} (1 - \frac{a_o}{a} E_1/E_2 \ , & \text{for plane stress} \\ (1 - \frac{a_o}{a}) \dfrac{aE_1(1-\nu_2{}^2)}{aE_2(1-\nu_1{}^2)+(a-a_o)E_1\nu_2(1+\nu_2)} \ , & \text{for plane strain.} \end{cases} \tag{36}$$

By using the model to treat a cracked cylinder reinforced by a membrane developed in [6], the present transient thermal stress problem may be reduced to the following integral equation:

$$\int_e^g \frac{f(\rho,t)}{\rho-r}\,d\rho + \int_e^g k(r,\rho)f(\rho,t)\,d\rho = \frac{\pi(1+\kappa)}{2\mu}\,\sigma^T_{\theta 2}(r,t) \ ,$$

$$(e<r<g,\ 0<t) \ , \tag{37}$$

where $\kappa = 3-4\nu_2$ for plane strain, $\kappa = (3-\nu_2)/(1+\nu_2)$ for plane stress, $\mu = E_2/2(1+\nu_2)$, and the unknown function $f$ and the kernel $k$ are defined by

$$f(r,t) = \frac{\partial}{\partial r}\,[u_\theta(r,+0,t) - u_\theta(r,-0,t)],\ (e<r<g) \quad , \tag{38}$$

$$k(r,\rho) = -\frac{1}{2}\{-\frac{\beta_o}{r^2} + 2\gamma_o + \frac{2\gamma_1}{r^3} + 6\delta_1 r$$

$$+ \sum_{n=2}^{\infty}[\alpha_n r^{n-2}+\beta_n(n+2)r^n+\gamma_n r^{-(n+2)}+\delta_n(n-2)r^{-n}]\}, \tag{39}$$

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$$\beta_0 = \frac{2(1-\lambda)a^2}{a^2(1-\lambda)-b^2(1+\lambda)}\left(\rho - \frac{b^2}{\rho^2}\right) ,$$

$$\gamma_0 = \frac{1}{a^2(1-\lambda)-b^2(1+\lambda)}\left[-(1+\lambda)\rho + \frac{a^2(1-\lambda)}{\rho}\right] ,$$

$$\gamma_1 = \frac{a^4(1-3\lambda)}{a^4(1-3\lambda)-b^4(1+\lambda)}\left[-\frac{3\rho^2}{2} + b^2 + \frac{b^4}{2\rho^2}\right] ,$$

$$\delta_1 = \frac{1}{a^4(1-3\lambda)-b^4(1+\lambda)}\left[-\frac{3(1+\lambda)\rho^2}{2} + b^2(1+\lambda) + \frac{a^4(1-3\lambda)}{2\rho^2}\right] ,$$

$$\alpha_n = b^{-2(n-1)}\,\rho^{n-1} - (n-1)b^2\frac{c_n}{c_0} - b^{-2(n-1)}\frac{d_n}{c_0} ,$$

$$\gamma_n = \frac{1}{1+\lambda}\left[-a^{2(n+1)}\rho^{-(n+1)}(1-2\lambda n-\lambda)-a^2(1+n-n\lambda-\lambda)\frac{d_n}{c_0} + a^{2(n+1)}(1-2n\lambda-\lambda)\frac{c_n}{c_0}\right] ,$$

$$c_0 = n^2(1+2\lambda+\lambda^2)a^{-2}b^2-2(n^2-1)(1-\lambda^2)-[1+2n\lambda+(2n-1)\lambda^2]a^{-2n}b^{2n}$$

$$-[1-2n\lambda-(2n+1)\lambda^2]a^{2n}b^{-2n} + n^2(1-2\lambda-3\lambda^2)a^2b^{-2} ,$$

$$c_n = \{(1-n^2)(1-\lambda^2)-[1-2n\lambda-(2n+1)\lambda^2]a^{2n}b^{-2n}+n^2(1-2\lambda-3\lambda^2)a^2b^{-2}\}\rho^{-(n+1)}$$

$$+ \{(2+n-n^2)(1-\lambda^2)b^{-2}-(2+n-n^2)(1+\lambda)^2a^{-2}\}\rho^{-(n-1)}$$

$$+ \{(1+n)[1+2n\lambda-(1-2n)\lambda^2]a^{-2n}-[(1+n)-(n^2+1)\lambda^2]b^{-2n}\}\rho^{n-1}$$

$$+ \{(2+n)(1+\lambda)^2a^{-2}b^{-2n}-(2+n)[1+2n\lambda-(1-2n)\lambda^2]a^{-2n}b^{-2}\}\rho^{n+1} ,$$

$$
\begin{aligned}
d_n = &\{-(1-n)[1-2n\lambda-(1+2n)\lambda^2]a^{2n}+(1-n)(1-\lambda^2)b^{2n}\}\rho^{-(n+1)} \\
&-\{(2-n)(1+\lambda)^2a^{-2}b^{2n}-(2-n)[1-2n\lambda-(1+2n)\lambda^2]a^{2n}b^{-2}\}\rho^{-(n-1)} \\
&+\{n^2(1+\lambda)^2a^{-2}b^2+(1-n^2)(1-\lambda^2)-[1-2n\lambda-(1+2n)\lambda^2]a^{2n}b^{-2n}\}\rho^{n-1} \\
&+\{(2-n-n^2)(1+\lambda)^2a^{-2}-(2-n-n^2)(1-\lambda^2)b^{-2}\}\rho^{n+1} \quad . \qquad (40)
\end{aligned}
$$

From (35b) and the definition of f as given by (38) it is seen that the integral equation (37) must be solved under the following single-valuedness condition:

$$
\int_e^g f(r,t)dr = 0 \quad . \qquad (41)
$$

The singular integral equation may easily be solved by using the technique described, for example, in [7]. If the crack is fully embedded in cylinder 2 (i.e., if $a<e<g<b$), then the solution of (37) is of the form

$$
f(r,t) = \frac{F(r,t)}{[(r-e)(g-r)]^{1/2}} \quad , \quad (e<r<g, \; 0<t) \qquad (42)
$$

where F is a bounded unknown function. In order to have a solution such as (42) it is, of course, assumed that the combination of mechanical and transient thermal hoop stresses in the neighborhood of the crack are predominantly tensile. Otherwise the crack may close at one or both ends and (42) would not be valid. If part of the crack lies in a region of compressive hoop stress, then the crack surfaces may close smoothly and the location of the related crack tip would be unknown. In this case the unknown crack tip location is determined from the cusp condition which requires that the corresponding stress intensity factor be zero and the problem may be solved by using the technique described in [8].

After solving (37) the Mode I stress intensity factors at the crack tips e and g may be determined from

$$k_1(e,t) = \frac{2\mu}{1+\kappa} \lim_{r\to e} \sqrt{2(r-e)}\, f(r,t) = \frac{2\mu}{1+\kappa} \frac{F(e,t)}{\sqrt{(g-e)/2}} \quad , \tag{43}$$

$$k_1(g,t) = -\frac{2\mu}{1+\kappa} \lim_{r\to g} \sqrt{2(g-r)}\, f(r,t) = -\frac{2\mu}{1+\kappa} \frac{F(g,t)}{\sqrt{(g-e)/2}} \quad . \tag{44}$$

In the special case of the inner crack tip touching the reinforcing membrane (i.e., for e=a) it was shown analytically in [6] that at this crack tip the derivative f of the crack surface displacement is bounded. Physically this follows from the fact that since the membrane has no bending stiffness, when the crack touches it, it would form a kink. In this case the solution of (37) is of the form

$$f(r,t) = \frac{F(r,t)}{(g-r)^{\frac{1}{2}}} \quad , \quad (a=e<r<g \ , \ 0<t) \quad , \tag{45}$$

where the bounded function F is determined by following the technique used in [6].

In the other special case in which the external loads are sufficiently high so that the membrane is fully yielded (if, for example, $\theta_{ao}>\theta_F$ and no mechanical loading is present), the problem may be treated as an "edge crack" problem and the yielding membrane may be replaced by constant tensile tractions on the crack surface. In this case, too, the only relevant stress intensity factor is that at the crack tip r=g.

Finally, there is the case of the broken membrane in which the problem may be treated as an edge crack problem.

## 4. Results and Discussion

Some calculated results giving the transient temperature and thermal stress distributions obtained from (16), (17) and (26) are shown in

Figures 2-5. In (16) and (17) fifty terms are used in the series to calculate the temperatures which proved to be quite sufficient. The results are calculated by varying dimensionless quantities defined by (19). In these calculations it is assumed that for the two cylinders the elastic constants are equal ($E_1=E_2=E$, $\nu_1=\nu=\nu$) but the thermal constants are different ($D_1 \neq D_2$, $k_1 \neq k_2$) and the composite cylinder is under a state of plane strain. Figures 2-5 show the temperature and the hoop stress distribution for two different clad thicknesses ($(a-a_0)/(b-a)$ = 0.016 and 0.008). Referring to (15) note that for sudden cooling from inside $\theta_{ao}$ is negative resulting in tensile stresses at and near the inner boundary (Figures 3 and 5). For an embedded crack ($a<e<g<b$) the corresponding stress intensity factors are shown in Tables 1 and 2. The tables show the normalized stress intensity factors which are defined by

$$k_1(e,t) = \lim_{r\to e} \sqrt{2(e-r)}\, \sigma_{\theta\theta}(r,0,t) \;,\; \bar{k}(e) = \frac{k_1(e,t)}{k_o} \;, \tag{46}$$

$$k_1(g,t) = \lim_{r\to g} \sqrt{2(r-g)}\, \sigma_{\theta\theta}(r,0,t) \;,\; \bar{k}(g) = \frac{k_1(g,t)}{k_o} \;, \tag{47}$$

$$k_o = -\frac{E_2\alpha_1\theta_{ao}}{1-\nu_2} \sqrt{(g-e)/2} \quad . \tag{48}$$

Note that for longer crack lengths the outer tip $r=g$ of the crack would be in the compression zone and consequently the related stress intensity factor becomes negative.

In the case of the inner crack tip terminating at cylinder-reinforcement interface the stress intensity factor at the outer tip $r=g$ is shown in Figure 6. It may again be observed that as g increases the stress intensity factor $k_1(g,t)$ decreases and eventually becomes negative. In these calculations, too, it is assumed that $E_1=E_2=E$, $\nu_1=\nu_2=\nu$.

The effect of modulus ratio $E_1/E_2=E^*$ on the transient thermal stress $\sigma^T_{\theta 2}$ in the main cylinder is shown in Figure 7 for the case of plane strain and in Figure 8 for plane stress. The figures show the results at a fixed time $t^*=D_1t/b^2=0.0005$ and for $\alpha_2/\alpha_1 = 0.8$, $\nu_1=\nu_2=0.3$, $D_2/D_1 = 3$ and $k_2/k_1 = 3$. Tables 3 and 4 show the corresponding normalized stress intensity factors for an embedded crack. The results for the crack touching the membrane and for the broken membrane (i.e, for the edge crack) are given in Table 5. Note that for the broken membrane case generally the stress intensity factors are considerably higher.

Tables 1-4 and Figure 6 clearly show that an internal crack in a cylinder undergoing thermal shock at the inner boundary would always tend to propagate towards the inside wall, it may be arrested at the cylinder-reinforcement boundary if the reinforcement material is sufficiently tough, and under thermal shock alone it is not possible for the crack to propagate through the entire wall thickness of the cylinder. The tables also show that, generally for a given crack geometry the stress intensity factors first increase and, after going through a maximum, then start decreasing with increasing time. This behavior as well as the fact that for relatively long cracks the stress intensity factors at the two crack tips may have opposite signs is suggested by the thermal stress profiles shown in Figures 3, 5, 7 and 8.

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## 5. References

1. Stahn, D. and Blauel, J.G., "Experimentelle Untersuchungen zur Rissausbreitung in Glashohlzylindern unter Thermoschockbelastung", Vortrag XI, Int. Glaskongress Prag, Sammelband II, pp. 435-445, 1977.

2. Nied, H.F. and Erdogan, F., "Transient Thermal Stress Problem for a Circumferentially Cracked Hollow Cylinder", Journal of Thermal Stresses, Vol. 6, pp. 1-14, 1983.

3. Sternberg, E. and Chakravorty, J.G., "On Inertia Effects in a Transient Thermoelastic Problem", J. Appl. Mech., Vol. 26, Trans. ASME, pp. 503-508, 1959.

4. Sternberg, E. and Chakravorty, J.G., "Thermal Shock in an Elastic Body with a Spherical Cavity", Q. Appl. Math., Vol. 17, pp. 205-220, 1959.

5. Timoshenko, S. and Goodier, J.N., Theory of Elasticity, McGraw-Hill, 1951.

6. Tang, Renji and Erdogan, F. "Stress Intensity Factors in a Reinforced Thick-Walled Cylinder", Int. J. Engng. Science, Vol. 22, 1984 (to appear).

7. Erdogan, F., "Mixed Boundary Value Problems in Mechanics", Mechanics Today, S. Nemat-Nasser, ed., Vol. 4, pp. 1-86, 1978.

8. Bakioglu, M. and Erdogan, F., "Crack-Contact and the Free-End Problem for a Strip under Residual Stress", J. Appl. Mech., Vol. 44, Trans. ASME, pp. 41-44, 1977.

Table 1. Normalized stress intensity factors for an embedded crack in a reinforced cylinder subjected to transient thermal stresses ($\frac{a_o}{b}$ = 0.7968, $\frac{a}{b}$ = 0.8, $k_2/k_1$ = 3, $\frac{D_2}{D_1}$ = 3, $\frac{\alpha_2}{\alpha_1}$ = 0.8, $\frac{e}{b}$ = 0.84, $E_1=E_2$, $\nu_1=\nu_2$).

| $\frac{g-e}{b-a}$ | $t^*$=0.0005 | | $t^*$=0.001 | | $t^*$=0.005 | |
|---|---|---|---|---|---|---|
| | $\bar{k}(e)$ | $\bar{k}(g)$ | $\bar{k}(e)$ | $\bar{k}(g)$ | $\bar{k}(e)$ | $\bar{k}(g)$ |
| 0.05 | 0.11258 | 0.07698 | 0.16846 | 0.13549 | 0.11857 | 0.10180 |
| 0.10 | 0.09617 | 0.03173 | 0.15356 | 0.09109 | 0.11120 | 0.07845 |
| 0.20 | 0.06695 | -0.03653 | 0.12573 | 0.01508 | 0.09707 | 0.03509 |
| 0.30 | 0.04130 | -0.08115 | 0.09912 | -0.04545 | 0.08270 | -0.00419 |
| 0.40 | 0.01857 | -0.10986 | 0.07337 | -0.09284 | 0.06770 | -0.03938 |
| 0.50 | -0.00189 | -0.13062 | 0.04836 | -0.13134 | 0.05204 | -0.07128 |

Table 2. Normalized stress intensity factors for an embedded crack in a reinforced cylinder subjected to transient thermal stresses ($\frac{a_o}{b} = 0.7984$, $\frac{a}{b} = 0.8$, $\frac{k_2}{k_1} = 3$, $\frac{D_2}{D_1} = 3$, $\frac{\alpha_2}{\alpha_1} = 0.8$, $\frac{e}{b} = 0.84$, $E_1 = E_2$, $\nu_1 = \nu_2$).

| $\frac{g-e}{b-a}$ | $t^*=0.0005$ | | $t^*=0.001$ | | $t^*=0.005$ | |
|---|---|---|---|---|---|---|
| | $\bar{k}(e)$ | $\bar{k}(g)$ | $\bar{k}(e)$ | $\bar{k}(g)$ | $\bar{k}(e)$ | $\bar{k}(g)$ |
| 0.05 | 0.13677 | 0.09793 | 0.18877 | 0.15414 | 0.12371 | 0.10688 |
| 0.10 | 0.11899 | 0.04822 | 0.17328 | 0.10742 | 0.11642 | 0.08352 |
| 0.20 | 0.08715 | -0.02784 | 0.14430 | 0.02677 | 0.10247 | 0.04005 |
| 0.30 | 0.58931 | -0.07845 | 0.11639 | -0.03816 | 0.08822 | 0.00057 |
| 0.40 | 0.3376 | -0.11135 | 0.08916 | -0.08943 | 0.07328 | -0.03486 |
| 0.50 | 0.01106 | -0.13495 | 0.06259 | -0.13114 | 0.05764 | -0.06692 |

Table 3. The effect of the modulus ratio $E_1/E_2$ on the stress intensity factors in a reinforced cylinder, $k_n = -k_1/\alpha_1 E_2 \theta_{ao} \sqrt{(g-e)/2}$, $a_o/b = 0.7968$, $a/b = 0.8$, $e/b = 0.81$, $t^* = Dt/b^2 = 0.0005$, $\alpha_2/\alpha_1 = 0.8$, $k_2/k_1 = 3$, $D_2/D_1 = 3$, $\nu_1 = \nu_2 = 0.3$, the case of plane strain.

| $\frac{g-e}{b-a}$ | $E_1/E_2 = 1$ | | $E_1/E_2 = 2$ | | $E_1/E_2 = 6$ | | $E_1/E_2 = 10$ | |
|---|---|---|---|---|---|---|---|---|
| | $k_n(e)$ | $k_n(g)$ | $k_n(e)$ | $k_n(g)$ | $k_n(e)$ | $k_n(g)$ | $k_n(e)$ | $k_n(g)$ |
| 0.05 | 0.57556 | 0.49535 | 0.55469 | 0.47486 | 0.48101 | 0.40217 | 0.41844 | 0.34024 |
| 0.10 | 0.55370 | 0.39808 | 0.53000 | 0.37589 | 0.45059 | 0.29988 | 0.38554 | 0.23669 |
| 0.15 | 0.53438 | 0.31039 | 0.50762 | 0.28680 | 0.42198 | 0.20803 | 0.35420 | 0.14377 |
| 0.20 | 0.51465 | 0.23159 | 0.48504 | 0.20706 | 0.39366 | 0.12643 | 0.32335 | 0.06143 |
| 0.25 | 0.49342 | 0.16180 | 0.46139 | 0.13673 | 0.36513 | 0.05505 | 0.29263 | -0.01036 |
| 0.30 | 0.47052 | 0.10099 | 0.43654 | 0.07569 | 0.33636 | -0.00639 | 0.26207 | -0.07195 |

Table 4. The effect of the modulus ratio $E_1/E_2$ on the stress intensity factors in a reinforced cylinder, $k_n = -k_1/\alpha_1 E_2 \theta_{ao} \sqrt{(g-e)/2}$, $a_o/b = 0.7968$, $a/b = 0.8$, $e/b = 0.81$, $t^* = Dt/b^2 = 0.0005$, $\alpha_2/\alpha_1 = 0.8$, $k_2/k_1 = 3$, $D_2/D_1 = 3$, $\nu_1 = \nu_2 = 0.3$, the case of plane stress.

| $\frac{g-e}{b-a}$ | $E_1/E_2 = 1$ | | $E_1/E_2 = 2$ | | $E_1/E_2 = 6$ | | $E_1/E_2 = 10$ | |
|---|---|---|---|---|---|---|---|---|
| | $k_n(e)$ | $k_n(g)$ | $k_n(e)$ | $k_n(g)$ | $k_n(e)$ | $k_n(g)$ | $k_n(e)$ | $k_n(g)$ |
| 0.05 | 0.40280 | 0.34667 | 0.38800 | 0.33217 | 0.33564 | 0.28058 | 0.29124 | 0.23668 |
| 0.10 | 0.38732 | 0.27847 | 0.37022 | 0.26256 | 0.31276 | 0.20793 | 0.26620 | 0.16285 |
| 0.15 | 0.37362 | 0.21700 | 0.35405 | 0.19996 | 0.29130 | 0.14295 | 0.24263 | 0.09703 |
| 0.20 | 0.35965 | 0.16179 | 0.33784 | 0.14402 | 0.27048 | 0.08554 | 0.22017 | 0.03924 |
| 0.25 | 0.34468 | 0.11292 | 0.32100 | 0.09474 | 0.25002 | 0.03562 | 0.19862 | -0.01068 |
| 0.30 | 0.32858 | 0.07035 | 0.30344 | 0.05204 | 0.22987 | -0.00710 | 0.17791 | -0.05312 |

Table 5. The effect of $E_1/E_2$ on the normalized stress intensity factor $k_n(g)$ for a crack touching the membrane cylinder interface and for a fully broken membrane, $k_n(g) = -k_1(g,t)/E_2\alpha_1\theta_{ao}\sqrt{(g-a)}$, $a_o/b = 0.7968$, $a/b=0.8$, $e/a=1$, $t^*=D_1t/b^2=0.005$, $k_2/k_1=3$, $D_2/D_1 = 3$, $\nu_1 = \nu_2 = 0.3$, $\alpha_2/\alpha_1 = 0.8$, the case of plane strain.

| $\frac{g-e}{b-a}$ | Crack touching the membrane | | | broken membrane-edge crack | |
|---|---|---|---|---|---|
| | $E_1/E_2=1$ | $E_1/E_2 = 2$ | $E_1/E_2 = 10$ | $E_1/E_2 = 1$ | $E_1/E_2 = 2$ |
| 0.10 | 0.23866 | 0.22678 | 0.15943 | 0.39231 | 0.39201 |
| 0.20 | 0.20237 | 0.18697 | 0.11076 | 0.35107 | 0.34859 |
| 0.30 | 0.16164 | 0.14404 | 0.06290 | 0.32931 | 0.32314 |
| 0.40 | 0.11804 | 0.09940 | 0.01651 | 0.31389 | 0.30323 |
| 0.50 | 0.07348 | 0.05484 | -0.02742 | 0.29972 | 0.28361 |
| 0.60 | 0.02973 | 0.01190 | -0.06839 | 0.28100 | 0.25851 |
| 0.70 | -0.01222 | -0.02881 | -0.10741 | 0.24839 | 0.21943 |

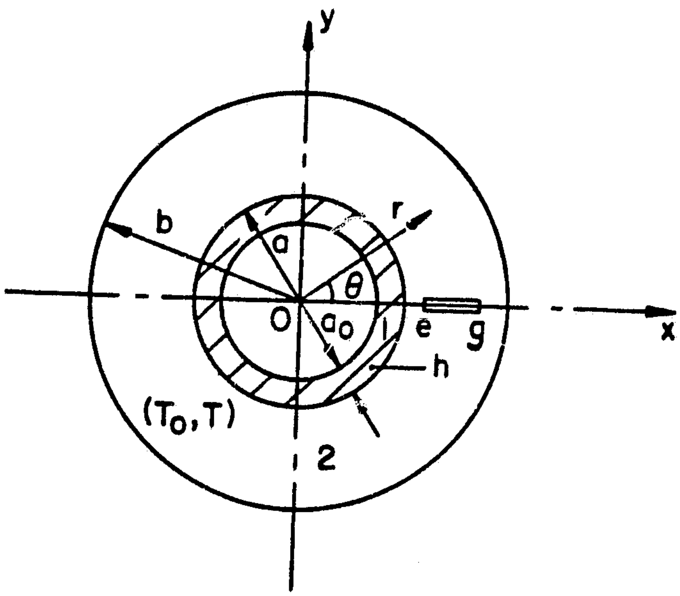


Figure 1. The geometry of the reinforced cylinder containing a radial crack.

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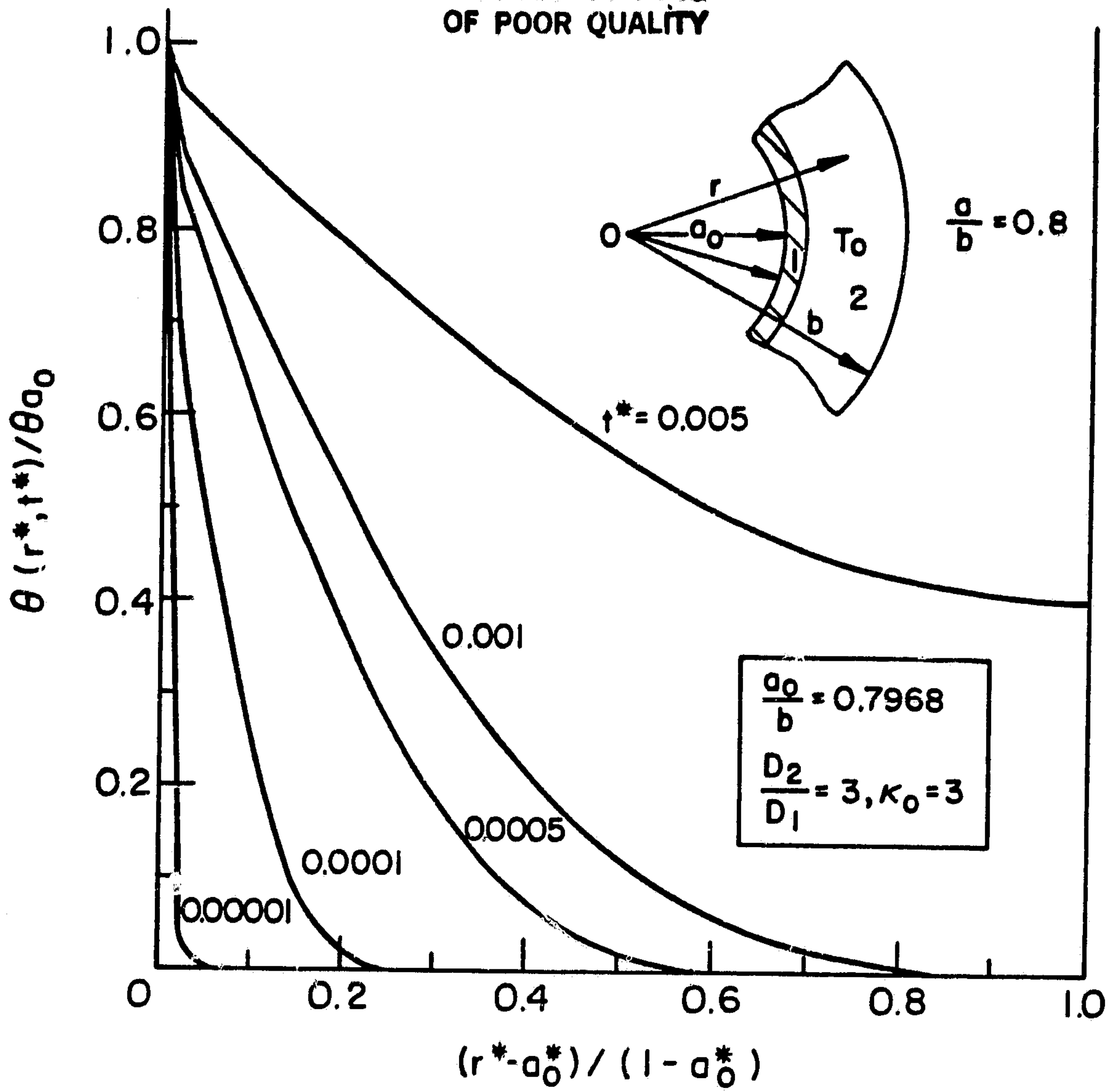


Figure 2. Transient temperature distribution in a hollow composite cylinder due to a sudden change $\theta_{ao}$ in the inner surface temperature $((a-a_0)/(b-a) = 0.016)$.

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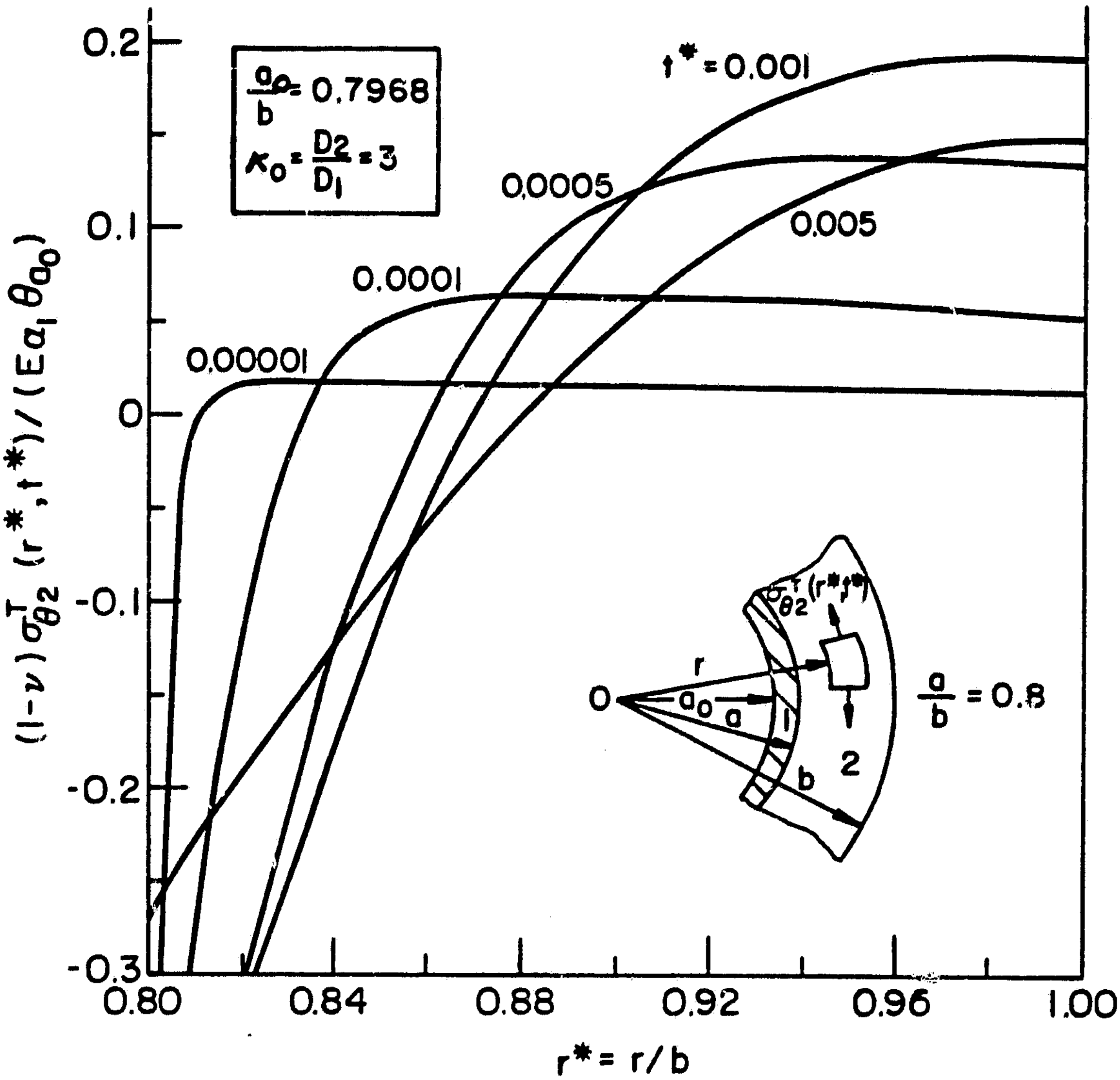


Figure 3. Hoop stress in a hollow composite cylinder caused by a sudden change $\theta_{ao}$ in the inner surface temperature $((a-a_o)/(b-a) = 0.016$ , $\alpha_2/\alpha_1 = 0.8)$.

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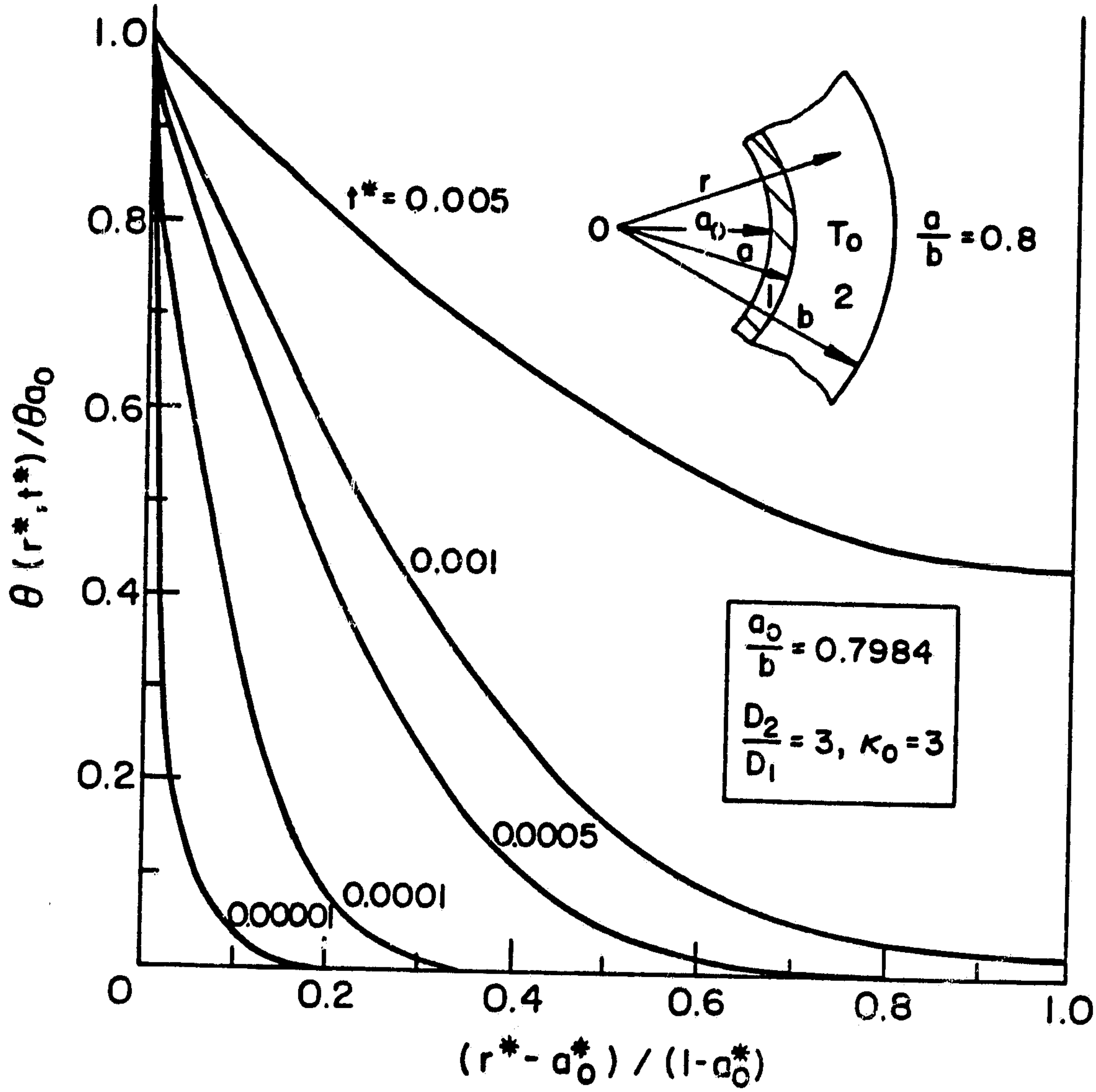


Figure 4. Transient temperature distribution in a hollow composite cylinder due to a sudden change in the inner surface temperature ($(a-a_0)/(b-a) = 0.008$).

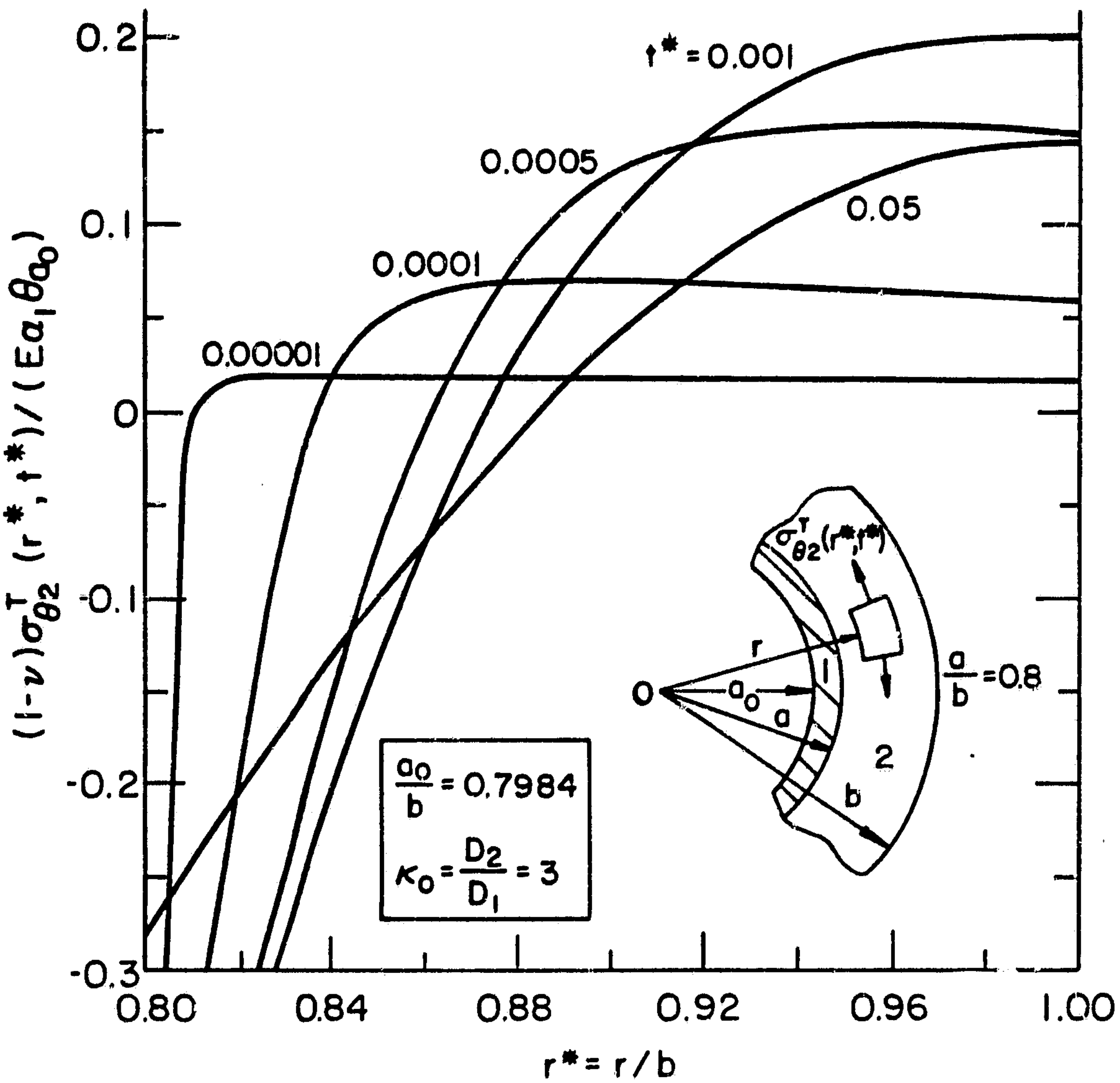


Figure 5. Hoop stress in a hollow composite cylinder caused by a sudden change in the inner surface temperature ($(a-a_0)/(b-a) = 0.008$, $\alpha_2/\alpha_1 = 0.8$).

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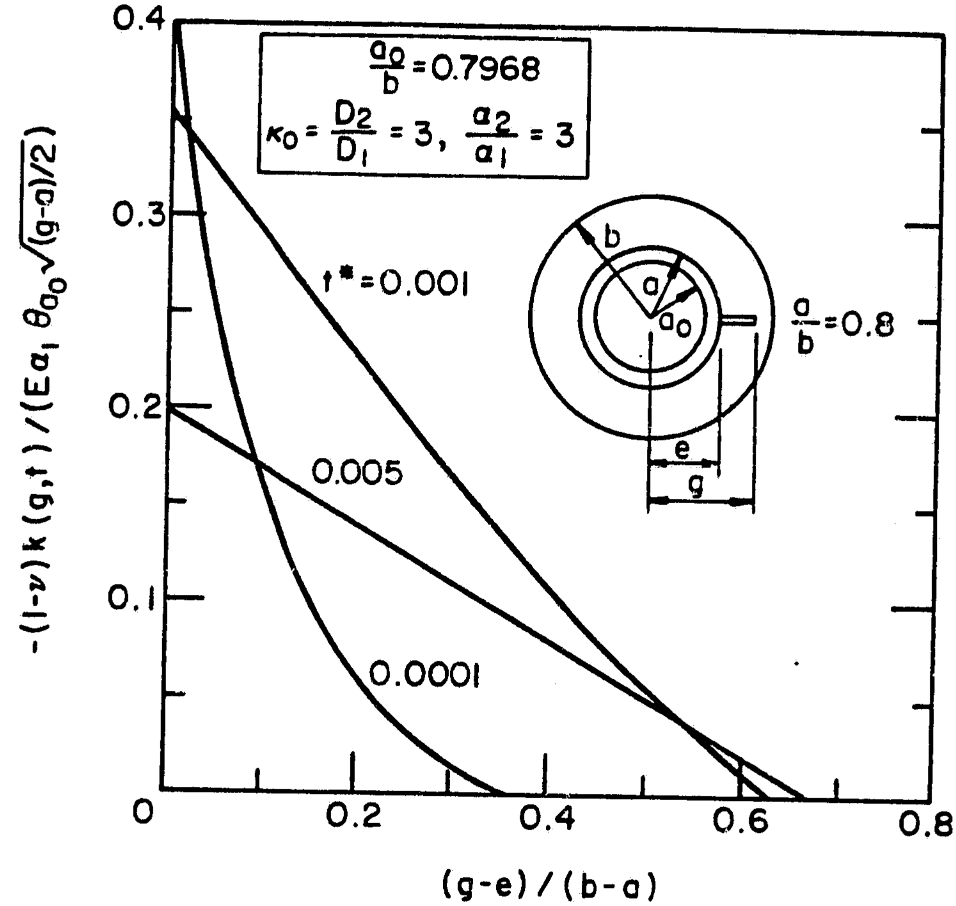


Figure 6. The normalized stress intensity factor $\bar{k}(g) = -k(g,t)/E\alpha_1\theta_{ao}\sqrt{(g-e)/2}/(1-\nu)$ in a reinforced cylinder which contains a crack terminating at the reinforcement-cylinder interface and which is subjected to a sudden change in the inner surface temperature ($E_1=E_2=E$, $\nu_1=\nu_2=\nu$, $(a-a_0)/(b-a) = 0.016$, the case of plane strain).

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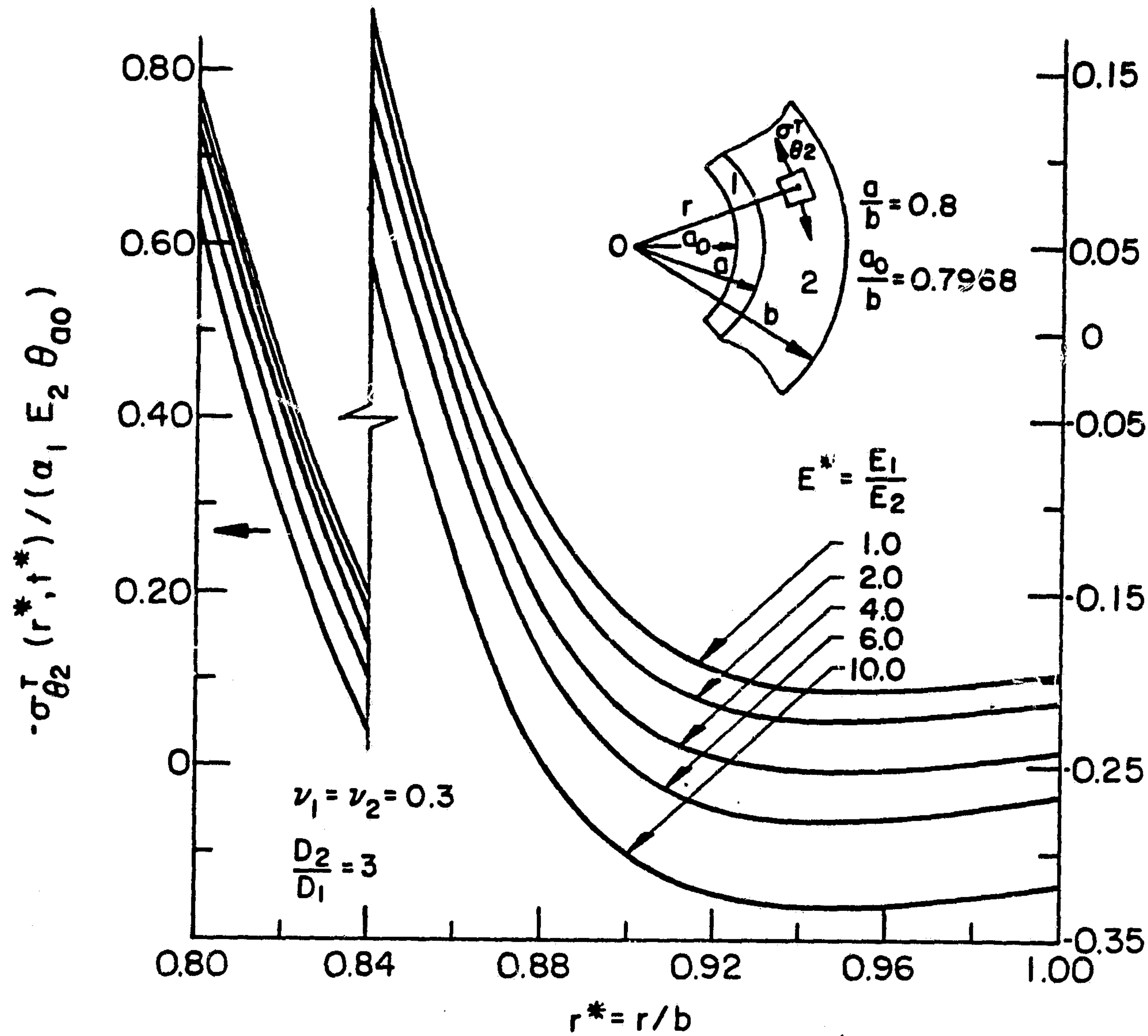


Figure 7. The influence of the modulus ratio $E_1/E_2$ on the distribution of hoop stress in a composite cylinder undergoing a sudden change in the inner surface temperature ($t^* = D_1 t/b^2 = 0.0005$, $(a-a_0)/(b-a) = 0.016$, $e/b = 0.81$, the case of plane strain).

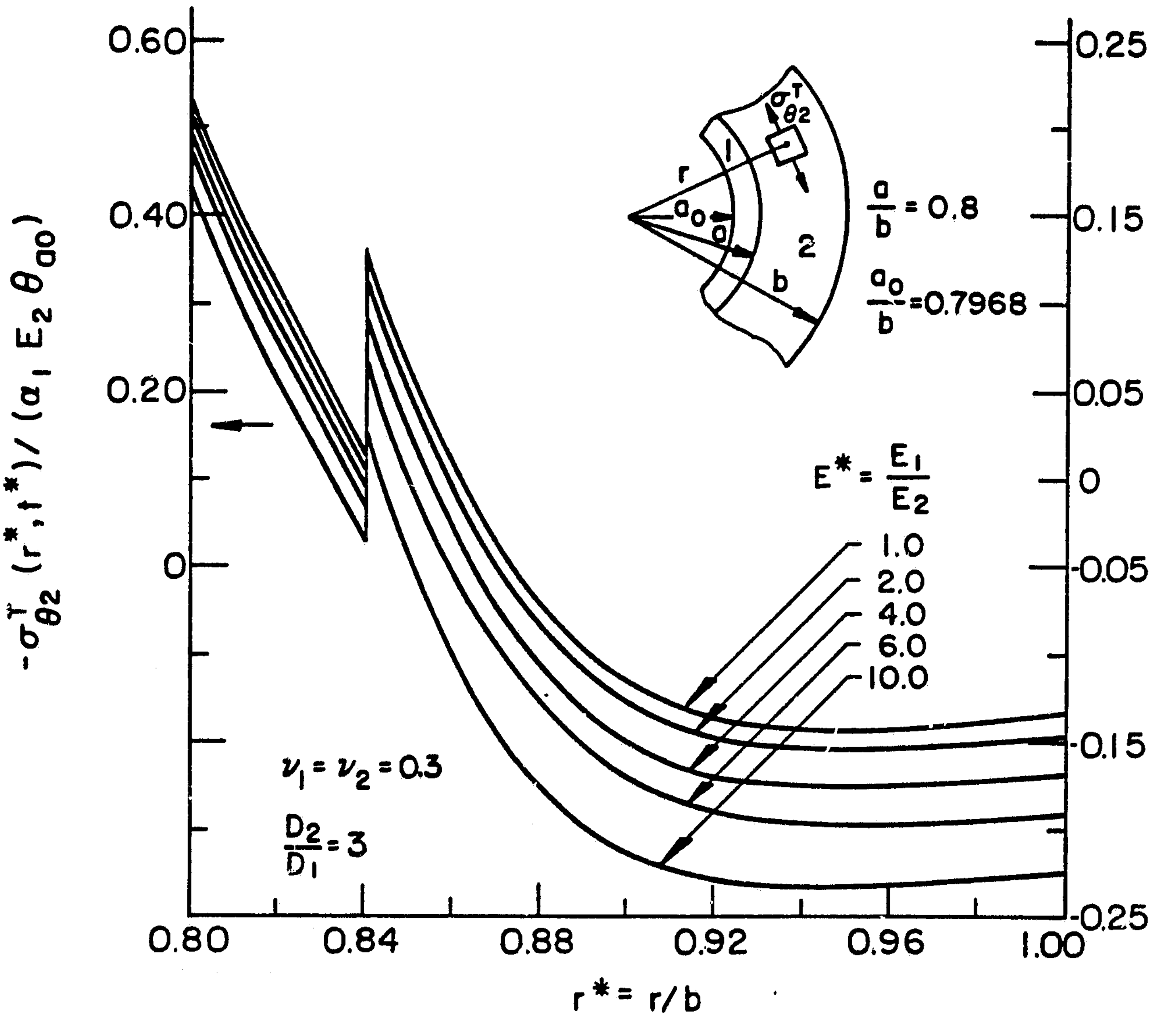


Figure 8. The influence of the modulus ratio $E_1/E_2$ on the distribution of hoop stress in a composite cylinder undergoing a sudden change in the inner surface temperature ($t^* = D_1 t/b^2 = 0.0005$, $(a-a_0)/(b-a) = 0.016$, $e/b = 0.81$, the case of plane stress.