NASA Contractor Report 172293

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A FINITE DIFFERENCE SHCEME FOR THE EQUILIBRIUM
EQUATIONS OF ELASTIC BODIES

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Operated by the Universities Space Research Association

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A FINITE DIFFERENCE SCHEME FOR

THE EQUILIBRIUM EQUATIONS OF ELASTIC BODIES

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Abstract

A compact difference scheme is described for treating the first-order system of PDE's which describe the equilibrium equations of an elastic body. An algebraic simplification enables the solution to be obtained by standard direct or iterative techniques.

N84-17621#

INTRODUCTION

The conditions for the static equilibrium of an elastic body are described by an elliptic system of nine partial differential equations for the displacements and stresses. This paper describes a finite difference scheme which can be solved by standard direct or iterative methods and yields a solution which approximates smooth displacements with second-order accuracy. Iterative techniques can be attractive as a means of solving three-dimensional problems because they minimize storage requirements and present an algorithmic structure well-suited to advanced computer architectures. For material problems, these features can be useful for solving layered composite materials as well as materials with nonlinear properties.

As described here, a serious limitation of this method vis-a-vis finite element methods is that it is applicable only to bodies which can be subdivided into cube-like volume cells. However, a means of removing this restriction will be described in another paper.

Part I describes an algebraic approximation to the equilibrium conditions in a cell as expressed by tractions and displacements on the surface of the cell. The condition that traction forces balance across cell faces leads to an algebraic condition for equilibrium between any neighboring cells expressed solely in terms of displacements. A finite-sum approximation to the work due to tractions leads to an energy estimate and to a variational description of the algebraic equilibrium equations.

Part II illustrates this development for an isotropic material using a plane stress assumption to reduce the problem to two dimensions. Several simple iteration schemes are used to investigate the numerical convergence of the method when a singularity is present.

The methods described in this paper are closely related to those described by the authors in the context of a simpler problem [1].

PART I
General Development

I.1 The Equilibrium Problem

In this section we describe the equilibrium equations for an elastic body and attempt to motivate the origin of the finite difference scheme which will be described in the following section.

We consider a material body occupying a domain Ω on whose boundary, Γ, $\underline{n}$ is an outward unit normal; $\underline{u} = \left(u_1, u_2, u_3\right)^T$ denotes the displacement vector, $\tau = \left(\underline{\tau}_1, \underline{\tau}_2, \underline{\tau}_3\right)$ the symmetric stress tensor, and $\varepsilon = \left(\underline{\varepsilon}_1, \underline{\varepsilon}_2, \underline{\varepsilon}_3\right)$ the symmetric strain tensor.

In the absence of body forces, the equations of equilibrium are described by the following system of first-order partial differential equations:

$$a) \quad \partial_{x_1} \underline{\tau}_1 + \partial_{x_2} \underline{\tau}_2 + \partial_{x_3} \underline{\tau}_3 = 0,$$

(1)
$$b) \quad \varepsilon = \left(\text{grad } \underline{u} + \text{grad}^T \underline{u}\right)/2 \qquad \text{in } \Omega,$$

$$c) \quad \underline{\tau}_i = \sum_{j=1}^{3} c_{ij} \underline{\varepsilon}_j \qquad i = 1,2,3.$$

Here (a) states the conditions for the equilibrium of forces, (b) defines the strain tensor in terms of the displacement, and (c) is the constitutive relationship between stress and strain (Hooke's Law). In (c), (c_{ij}) involves 21 parameters. By assumption

(2)
$$\varepsilon^T \tau = \sum \varepsilon_{ij} \tau_{ij} > 0$$

with equality holding iff $\varepsilon = 0$.

On the boundary surface Γ, we let $\underline{p} \equiv \tau \cdot \underline{n}$ denote the traction due to the stress. Then boundary conditions associated with (1) are

$$\underline{u} = \underline{u}^0 \qquad \text{on} \quad \Gamma_1$$

(3)

$$\underline{p} = \underline{p}^0 \qquad \text{on} \quad \Gamma_2 \ ,$$

where Γ_1 and Γ_2 form a disjoint partition of Γ; if $\Gamma_1 = \phi$ then the solution of (1) will be determined to within a rigid body displacement.

As a result of solving this boundary value problem, the tractions $\underline{p}$ will be determined on Γ_1 and the displacements $\underline{u}$ on Γ_2. Thus the tractions on Γ, say $\underline{p}(\Gamma)$, will be related to the displacements on $\underline{u}(\Gamma)$ on Γ, a fact we indicate symbolically by

(4) $$\underline{p}(\Gamma) = R_\Omega \, \underline{u}(\Gamma).$$

We call the boundary operator R_Ω a transmission operator.

Let $\Pi(\Omega) \equiv \{\omega\}$ denote a partition of Ω into volume cells. Standard integration arguments show that $\underline{u}$ must be continuous and the surface tractions must balance across cell faces. Clearly, a necessary and sufficient condition for equilibrium in Ω is that any individual cell be in equilibrium with any neighboring cell.

Consider a cell ω whose volume is $0(h^3)$, where h is a representative length scale, and whose boundary surface γ consists of m faces. In equilibrium, the tractions $\underline{p}(\gamma)$ on γ will be related to the displacements

$\underline{u}(\gamma)$ on γ by means of a transmission operator R_ω, i.e., $\underline{p}(\gamma) = R_\omega \, \underline{u}(\gamma)$. Let $\langle \phi \rangle_\gamma$ indicate a vector whose m components represent the average values of ϕ on the faces of ω. We may then relate the average tractions $\langle \underline{p} \rangle_\gamma$ on the faces of ω to the average displacements on the faces by

$$(5) \qquad \langle \underline{p} \rangle_\gamma = R_\omega^h \, \langle \underline{u} \rangle_\gamma,$$

where R_ω^h is an $m \times m$ transmission matrix which is related to R_ω. When conditions for the balance of average surface tractions across cell faces are adjoined to (5), as well as boundary conditions for average tractions and displacements on Γ, we may expect that the resulting system of algebraic equations for average displacement values will yield an approximation to the equilibrium problem for (1) - (3) as $h \to 0$.

This approach can be made practical only if the transmission matrices R_ω^h in (5) can be approximated without an a priori knowledge of the transmission operators R_ω related to the continuous problem. A general method for constructing R_ω^h on arbitrary cells will be described in a separate paper by one of the authors.

In this paper we describe a simple construction for R_ω^h which results when cubical cells are employed. In this case, the equilibrium conditions (5) and the balance of traction conditions can be given a particularly simple form using finite difference notations and which we then call a compact finite difference scheme. Finite difference methods can then be used to obtain a finite sum energy estimate and also to characterize the average displacement values as the solution of a quadratic variational problem. As a result, the algebraic solution of the compact scheme can be obtained by either direct or

iterative methods using a priori facts about the structure of the algebraic system. This development is explained below.

I.2 A Compact Difference Scheme

In this section we describe a compact difference scheme (eq. (10)) which approximately describes the equilibrium of a cell due to displacements and tractions on the cell faces. We then develop an energy estimate and present a variational principle for the difference scheme. An appropriate method for solving the difference scheme is given in the following section.

We suppose that Ω can be partitioned into regular cubical cells whose faces are parallel to the coordinate axes; $\omega(\underline{x})$ indicates a cell with centerpoint at $\underline{x} = (x_1, x_2, x_3)$. We denote the average value of a function ϕ on a face whose centerpoint is $\underline{\xi}$ by $\phi(\underline{\xi})$. If $h_i \equiv \Delta x_i/2$, $i = 1,2,3$, the volume of ω is $\Delta\omega = 8h_1 h_2 h_3$.

Next, define central average and central divided difference operators μ_1 and δ_1 by

$$\mu_1 \phi(\underline{x}) \equiv \left(\phi(x_1 + h_1, x_2, x_3) + \phi(x_1 - h_1, x_2, x_3)\right)/2$$

(6)

$$\delta_1 \phi(\underline{x}) \equiv \left(\phi(x_1 + h_1, x_2, x_3) - \phi(x_1 - h_1, x_2, x_3)\right)/2h_1 .$$

The operators μ_2, δ_2, μ_3 and δ_3 are defined similarly.
Also define

$$\text{grad}_h \phi \equiv (\delta_1 \phi, \delta_2 \phi, \delta_3 \phi)^T$$

(7)

$$\text{div}_h \underline{u} \equiv \delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3 .$$

Finally, we write

$$(\mu\tau) \equiv (\mu_1 \, \underline{\tau}_1, \; \mu_2 \, \underline{\tau}_2, \; \mu_3 \, \underline{\tau}_3)$$

(8)

$$\varepsilon \equiv (\underline{\varepsilon}_1, \; \underline{\varepsilon}_2, \; \underline{\varepsilon}_3).$$

The restriction to cube-like cells simplifies the evaluation of surface tractions. Let $\xi_i^{\pm}$, $i = 1,2,3$, denote the centerpoints of the opposite faces of a cell ω. The outward normals $\underline{n}(\xi_i^{\pm})$ satisfy $\underline{n}(\xi_i^+) = -\underline{n}(\xi_i^-)$ so that the average surface tractions $\underline{p}(\xi_i^{\pm})$ are given by

(9)
$$\underline{p}(\xi_i^{\pm}) = \pm \, \underline{\tau}_i(\xi_i^{\pm}), \qquad\qquad i = 1,2,3.$$

In the following discussion the stress components arise as traction forces on the faces of cells. An important consequence is that then the conditions for the balance of average traction forces across cell faces simply reduce to the conditions that the jump in value of $\underline{\tau}_i$ vanish across a face $x_i = $ const., $i = 1,2,3$.

Corresponding to the equilibrium equations (1), we propose to consider the following compact scheme: in each cell ω

a) $\quad \delta_1 \, \underline{\tau}_1 + \delta_2 \, \underline{\tau}_2 + \delta_3 \, \underline{\tau}_3 = 0$

b) $\quad \mu_i \, \underline{\tau}_i = \sum\limits_{j=1}^{3} c_{ij} \, \underline{\varepsilon}_j, \qquad\qquad i = 1,2,3$

(10)

c) $\quad \varepsilon = (\text{grad}_h \, \underline{u} + \text{grad}_h^T \, \underline{u})/2$

d) $\quad \mu_i \, \underline{u} - \kappa_i^2 \, h_i^2 \, \delta_i \, \underline{\tau}_i = \mu_j \, \underline{u} - \kappa_j^2 \, h_j^2 \, \delta_j \, \underline{\tau}_j, \qquad \begin{array}{l} i \neq j \\[4pt] i,j = 1,2,3. \end{array}$

Here κ_i^2 indicates a positive diagonal matrix. Clearly, (a), (b), and (c) are consistent with (1); the significance of (d) will become clear from later developments.

Figure 1 indicates, with reference to a rectangular cell, the variables associated with the sides of the cell by the scheme (10).

If the strain tensor (ε) is eliminated from these equations there thus results a system of 18 algebraic equations for the 18 components of $\underline{u}$ and the 18 components of the tractions on the faces of a cell.

We now indicate how (10) leads to a finite-sum energy estimate. Recall that the work W done on the body can be evaluated from (1) by the use of integration-by-parts and the use of Gauss' Theorem with the result

$$(11) \qquad 2W = \sum_{i,j} \int_{\Omega} \varepsilon_{ij} \, \tau_{ij} \, d\omega = \int_{\Gamma_1} \underline{p}^T \, \underline{u}^0 \, \alpha\gamma + \int_{\Gamma_2} \underline{u}^T \, \underline{p}^0 \, d\gamma - \int_{\Omega} \underline{u}^T \, \text{div} \, \tau \, d\omega.$$

For finite differences summation-by-parts results from the identity

$$(12) \qquad \delta_i(\phi\psi) = (\mu_i \, \phi)(\delta_i \, \psi) + (\mu_i \, \psi)(\delta_i \, \phi), \qquad i = 1,2,3.$$

using (6). Also, using (7), Gauss' Theorem holds in the form

$$(13) \qquad \sum_{\omega\varepsilon\Omega} \text{div}_h \, \underline{v} \, \Delta\omega = \sum_{\Gamma} \underline{v}^T \, \underline{n} \, \Delta\gamma,$$

in which $\Delta\gamma$ is the area of a face of a cell on which $\underline{n}$ is the outward normal.

Using (12),

$$(14) \qquad \text{div}_h \, \underline{u}^T \, \tau = \sum_{i=1}^{3} \mu_i \, \underline{u}^T \cdot \delta_i \, \underline{\tau}_i + (\varepsilon, \tau)_h,$$

where, from (10c),

$$(15) \qquad (\varepsilon, \tau)_h \equiv \sum_{i=1}^{3} \underline{\varepsilon}_i^T \, (\mu_i \, \underline{\tau}_i).$$

Next, using (10d),

$$\sum_i (\mu_i \, \underline{u}^T)(\delta_i \, \underline{\tau}_i) = \underline{\lambda}^T \sum_i \delta_i \, \underline{\tau}_i + \sum_i \kappa_i^2 \, h_i^2 (\delta_i \, \underline{\tau}_i)^2$$

where $\underline{\lambda}$ is some constant. Thus, if we define

$$(16) \qquad [\varepsilon, \tau]_h \equiv (\varepsilon, \tau)_h + \sum_i \kappa_i^2 \, h_i^2 (\delta_i \, \underline{\tau}_i)^2,$$

then (14) assumes the form

$$(17) \qquad \text{div}_h \, \underline{u}^T \, \tau = [\varepsilon, \tau]_h + \underline{\lambda}^T \, \text{div}_h \, \tau.$$

Recalling (2) and (10b), we see that $(\varepsilon, \tau)_h$ is a positive definite function of the strain tensor; hence, from (16), so also is $[\varepsilon, \tau]_h$. Also, recalling (9), $\text{div}_h \, \underline{u}^T \, \tau$ is seen to represent the work per unit volume done by traction on the faces of a cell ω.

Summing (17) over cells in Ω

$$(18) \qquad \sum_{\omega \varepsilon \Omega} \mathrm{div}_h \, \underline{u}^T \tau \Delta \omega = \sum_{\omega \varepsilon \Omega} [\varepsilon, \tau]_h \, \Delta \omega + \sum_{\omega \varepsilon \Omega} \underline{\lambda}^T \, \mathrm{div}_h \tau \Delta \omega,$$

so that for a solution of the compact scheme (10)

$$(19) \qquad \sum_{\omega \varepsilon \Omega} \mathrm{div}_h \, \underline{u}^T \tau \Delta \omega = \sum_{\omega \varepsilon \Omega} [\varepsilon, \tau]_h \, \Delta \omega \geqslant 0.$$

Also, using Gauss' Theorem in the form (13), (18) can be written (compare (11))

$$(20) \qquad \sum_{\omega \varepsilon \Omega} [\varepsilon, \tau]_h \, \Delta \omega = \oint_{\Gamma_1} \underline{p}^T \, \underline{u}^0 \, \Delta \gamma + \oint_{\Gamma_2} \underline{u}^T \, \underline{p}^0 \, \Delta \gamma - \sum_{\omega \varepsilon \Omega} \underline{\lambda}^T \, \mathrm{div}_h \tau \Delta \omega,$$

where use has been made of (9). In this equation, $\underline{u}^0$, $\underline{p}^0$ are to be interpreted as the average values on cell faces on Γ of the data given by (3).

Consider a problem in which $\underline{u}$ is prescribed everywhere on Γ. The preceeding discussion can be used to verify that the average values of displacements on cell faces which solve the compact scheme (10) also solve the variational problem

$$(21) \qquad \min_{\underline{u}} \sum_{\omega \varepsilon \Omega} [\varepsilon, \tau]_h \, \Delta \omega = \oint_{\Gamma} \underline{p}^T \, \underline{u}^0 \, \Delta \gamma$$

for $\underline{u}$ satisfying the boundary conditions on Γ. The Euler conditions for this problem simply express the balance of traction forces, expressed in terms of $\underline{u}$ by the use of (10), across cell faces.

As mentioned earlier we will describe elsewhere how these ideas may be adopted to treat cells having general shapes. For this reason we shall not present here the details of the convergence argument as it applies to (10) except to cite the result: the solution $\underline{u}^h$ of (10) converges to the solution $\underline{u}$ of (1) with accuracy $O(h^2)$ in an ℓ_2-norm while $\underline{p}^h(u^h)$ converges to $\underline{p}(u)$ with an accuracy $O(h)$. These remarks apply, of course, only to sufficiently smooth solutions of (1).

An important feature of the compact scheme (10) is that it only employs values of the displacements and tractions which arise as average values on cell sides. This is in contrast to many finite element methods which employ edge and vertex values of $\underline{u}$.

I.3 Solution Method

The compact scheme (10) may be solved by direct algebraic techniques such as Gauss elimination, considering the displacements and tractions as unknown variables. A preferable approach, which we shall now describe, is to eliminate the traction variables so as to obtain an algebraic system involving only the displacement variables. We first indicate, in general form, the steps which lead to this elimination. The specific result upon which numerical calculations can be performed is given by equation (32).

Let $\gamma_i(\omega)$, $i = 1,2,\cdots,6$ indicate a face of a cell ω and write

$$[\underline{u}]_\gamma \equiv [\underline{u}(\gamma_1), \underline{u}(\gamma_2), \cdots, \underline{u}(\gamma_6)]^T,$$

(22)

$$[\underline{p}]_\gamma \equiv [\underline{p}(\gamma_1), \underline{p}(\gamma_2), \cdots, \underline{p}(\gamma_6)]^T,$$

where, for brevity, reference to ω has been omitted.

Eliminating the strains from (10) and recalling (9) we may solve (10) for the tractions $[\underline{p}]_\gamma$ in terms of the displacements $[\underline{u}]_\gamma$ to obtain

$$(23) \qquad [\underline{p}]_\gamma = R^h_\omega [\underline{u}]_\gamma ,$$

where R^h_ω is a block 6×6 transmission matrix associated with ω which we write in terms of its rows as

$$(24) \qquad R^h_\omega \equiv \begin{bmatrix} \underline{r}^h_\omega(\gamma_1) \\ \underline{r}^h_\omega(\gamma_2) \\ \vdots \\ \underline{r}^h_\omega(\gamma_6) \end{bmatrix} .$$

(A direct evaluation using (10) shows that this matrix is symmetric.) Thus (23) states

$$(25) \qquad \underline{p}(\gamma_i(\omega)) = \underline{r}^h_\omega(\gamma_i(\omega))[\underline{u}]_{\gamma(\omega)} , \qquad i = 1,2,\cdots,6.$$

As noted earlier, $\mathrm{div}_h \underline{u}^T \tau$ represents the work per unit volume due to the tractions arising from the displacements on the faces of ω. Let

$$\Delta\gamma = \mathrm{diag}(\Delta\gamma_1,\Delta\gamma_2,\cdots,\Delta\gamma_6)^T .$$

In terms of the notations just described and recalling (19) we then have

$$(26) \qquad 0 \leqslant \left(\mathrm{div}_h \, \underline{u}^T \, \tau\right)\Delta\omega = [\underline{u}]_\gamma^T \, \Delta\gamma \, [\underline{p}]_\gamma = [\underline{u}]_\gamma^T \, \Delta\gamma \, R_\omega^h [u]_\gamma \, .$$

This shows that R_ω^h is positive semidefinite since the equality in (26) holds, according to (19), iff $(\mu\epsilon) = 0$ and this is seen to hold iff $\underline{u}(\gamma_i) = \mathrm{constant}$, $i = 1,2,\cdots,6$.

Since $[\epsilon,\tau]_h = \mathrm{div}_h \, \underline{u}^T \, \tau$, the variational principle (21) also applies in the form

$$(27) \qquad \min_{\underline{u}} \, \sum_{\omega\epsilon\Omega} \, [\underline{u}]_{\gamma(\omega)}^T \, \Delta\gamma \, R_\omega^h \, [\underline{u}]_{\gamma(\omega)} \, .$$

The conditions for a minimum resulting from this problem are, as remarked earlier, simply the balance of average traction forces across any face $\hat{\gamma}$ common to any neighboring cells which, using (25), may be written as

$$(28) \qquad \underline{r}_\omega^h(\hat{\gamma})[\underline{u}]_{\gamma(\omega)} + \underline{r}_{\omega'}^h(\hat{\gamma})[\underline{u}]_{\gamma(\omega')} = 0 \, .$$

If $\hat{\gamma}$ lies on Γ_1, then also

$$(29a) \qquad \underline{u}(\hat{\gamma}) = \underline{u}^0(\hat{\gamma})$$

while if $\hat{\gamma}$ lies on Γ_2, then

$$(29b) \qquad \underline{r}_\omega^h(\hat{\gamma})[\underline{u}]_{\gamma(\omega)} = \underline{p}^0(\hat{\gamma}) \, .$$

Earlier remarks show that this system has a unique solution for $\underline{u}(\hat{\gamma})$ unless $\Gamma_1 = \phi$.

We now give the explicit result of applying (28) on cell faces. Using (10c)

(30)

$$\epsilon \equiv (\underline{\epsilon}_1, \underline{\epsilon}_2, \underline{\epsilon}_3) = \begin{bmatrix} \delta_1 u_1 & (\delta_2 u_1 + \delta_1 u_2)/2 & (\delta_3 u_1 + \delta_1 u_3)/2 \\ (\delta_2 u_1 + \delta_1 u_2)/2 & \delta_2 u_2 & (\delta_3 u_2 + \delta_2 u_3)/2 \\ (\delta_3 u_1 + \delta_1 u_3)/2 & (\delta_3 u_2 + \delta_2 u_3)/2 & \delta_3 u_3 \end{bmatrix}$$

Let

$$\rho_i = \kappa_i^2 \, h_i^2, \qquad\qquad i = 1,2,3,$$

(31)

$$\Delta = \rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1 \, ,$$

and let $\omega_{i,j,k}$ indicate a cell whose centerpoint is $\underline{x} = (i\Delta x_1, \, j\Delta x_2, \, k\Delta x_3)$. Then across the faces of $\omega_{i,j,k}$ incident with $\omega_{i+1,j,k}, \, \omega_{i,j+1,k}, \,$ and $\omega_{i,j,k+1}$ we find

(32a) $$\left[\Delta \sum_\ell c_{1\ell} \, \underline{\epsilon}_\ell + h_1 \, \rho_2 (\mu_1 - \mu_3) \underline{u} + h_1 \, \rho_3 (\mu_1 - \mu_2) \underline{u} \right]_{\omega_{i,j,k}}$$

$$= \left[\Delta \sum_\ell c_{1\ell} \, \underline{\epsilon}_\ell - h_1 \, \rho_2 (\mu_1 - \mu_3) \underline{u} - h_1 \, \rho_3 (\mu_1 - \mu_2) \underline{u} \right]_{\omega_{i+1,j,k}}$$

(32b)
$$\left[\Delta \sum_{\ell} c_{2\ell}\,\underline{\varepsilon}_{\ell} + h_2\,\rho_3\left(\mu_2 - \mu_1\right)\underline{u} + h_2\,\rho_1\left(\mu_2 - \mu_3\right)\underline{u}\right]_{\omega_{i,j,k}}$$

$$= \left[\Delta \sum_{\ell} c_{2\ell}\,\underline{\varepsilon}_{\ell} - h_2\,\rho_3\left(\mu_2 - \mu_1\right)\underline{u} - h_2\,\rho_1\left(\mu_2 - \mu_3\right)\underline{u}\right]_{\omega_{i,j+1,k}}$$

(32c)
$$\left[\Delta \sum_{\ell} c_{3\ell}\,\underline{\varepsilon}_{\ell} + h_3\,\rho_2\left(\mu_3 - \mu_1\right)\underline{u} + h_3\,\rho_1\left(\mu_3 - \mu_2\right)\underline{u}\right]_{\omega_{i,j,k}}$$

$$= \left[\Delta \sum_{\ell} c_{3\ell}\,\underline{\varepsilon}_{\ell} - h_3\,\rho_2\left(\mu_3 - \mu_1\right)\underline{u} - h_3\,\rho_1\left(\mu_3 - \mu_2\right)\underline{u}\right]_{\omega_{i,j,k+1}} .$$

In these equations we have left unspecified nine parameters which arise in the matrices $\kappa_i^2 = \rho_i\,h_i^{-2}$. These may, as indicated in the next section, be chosen for convenience (cf. [1]).

We call (32) the stress-eliminated form of the compact scheme (10).

In summary, then, the system of equilibrium equations (28), (29), in which R_ω^h is the transmission matrix for the compact scheme (10), arises as the Euler equations for a related positive definite quadratic variational problem. The system (32) is thus seen to be solvable by direct elimination methods without pivoting; if iterative methods are considered Gauss-Seidel and SOR methods are applicable.

Part II
Example of an Isotropic Material

II.1   Equilibrium Equations

We consider an isotropic material characterized by Young's modulus E and Poisson's ratio ν.   We use a plane stress assumption to formulate the problem in two dimensions, the $x_1 - x_2$ plane say. This involves setting $\underline{\tau}_3 = 0$ and assuming that the stress components $\tau_{11}, \tau_{22}, \tau_{21}$ are independent of x_3 (Timoshenko and Goodier [2]). In two dimensions, the stress-strain relationship (1c) may be written in component form as follows:

$$\tau_{11} = \zeta \, \varepsilon_{11} + \eta \, \varepsilon_{22}$$

$$\tau_{22} = \eta \, \varepsilon_{11} + \zeta \, \varepsilon_{22}$$

$$\tau_{12} = \tau_{21} = \sigma \, \varepsilon_{12}$$

or, in terms of the displacement $\underline{u}$,

$$\tau_{11} = \zeta \, \partial_{x_1} u_1 + \eta \, \partial_{x_2} u_2$$

(33)

$$\tau_{22} = \eta \, \partial_{x_1} u_1 + \zeta \, \partial_{x_2} u_2$$

$$\tau_{12} = \tau_{21} = \frac{1}{2} \, \sigma \left(\partial_{x_2} u_1 + \partial_{x_1} u_2 \right).$$

The parameters ζ, η and σ are given in terms of Young's modulus and Poisson's ratio as follows:

$$\zeta = \frac{E}{(1 - \nu^2)} \,, \qquad \eta = \frac{E\nu}{(1 - \nu^2)} \,, \qquad \sigma = \frac{E}{(1 + \nu)} \,.$$

The quantity $\frac{1}{2}\sigma$ is known as the shear modulus.

II.2 Method of Solution

In this section we write down the compact scheme for the two-dimensional case when square cells are employed and obtain the transmission matrix which relates the tractions and displacements in each cell. The properties of the resulting system are then discussed in the context of its iterative solution.

Analogous with (10), and upon elimination of the strains, we have the following compact scheme for the components of displacement and stress:

$$(34a) \qquad \begin{cases} \delta_{x_1} \tau_{11} + \delta_{x_2} \tau_{12} = 0 \\[2ex] \delta_{x_1} \tau_{21} + \delta_{x_2} \tau_{22} = 0 \end{cases} ,$$

$$(34b) \qquad \begin{cases} \mu_{x_1} \tau_{11} = \zeta \, \delta_{x_1} u_1 + \eta \, \delta_{x_2} u_2 \\[2ex] \mu_{x_2} \tau_{22} = \eta \, \delta_{x_1} u_1 + \zeta \, \delta_{x_2} u_2 , \\[2ex] \mu_{x_1} \tau_{21} = \mu_{x_2} \tau_{12} = \frac{1}{2} \sigma \left(\delta_{x_2} u_1 + \delta_{x_1} u_2 \right) \end{cases}$$

$$(34c) \quad \begin{cases} \left(\mu_{x_2} - \mu_{x_1}\right)u_1 = a^2 \, h^2 \, \delta_{x_2} \, \tau_{12} - c^2 \, h^2 \, \delta_{x_1} \, \tau_{11} \\ \\ \left(\mu_{x_2} - \mu_{x_1}\right)u_2 = b^2 \, h^2 \, \delta_{x_2} \, \tau_{22} - d^2 \, h^2 \, \delta_{x_1} \, \tau_{21} \end{cases}, $$

where $\kappa_1^2 = \mathrm{diag}(a^2, b^2)$, $\kappa_2^2 = \mathrm{diag}(c^2, d^2)$ in the notation of (10). Using (34a) and putting $\theta_1 = a^2 + c^2$, $\theta_2 = b^2 + d^2$, (34c) become

$$\left(\mu_{x_2} - \mu_{x_1}\right)u_1 = \theta_1 \, h^2 \, \delta_{x_2} \, \tau_{12}$$

$$\left(\mu_{x_2} - \mu_{x_1}\right)u_2 = \theta_2 \, h^2 \, \delta_{x_2} \, \tau_{22}.$$

Using (9) to eliminate the strains from (31) we may solve for the tractions $[\underline{p}]_\gamma$ in terms of the displacements $[\underline{u}]_\gamma$ in each cell ω to obtain

$$(35) \qquad\qquad [\underline{p}]_\gamma = R_\omega^h [\underline{u}]_\gamma \;,$$

where

$$[p]_\gamma = \left(\underline{p}(\xi_1^+), \; \underline{p}(\xi_2^+), \; \underline{p}(\xi_1^-), \; \underline{p}(\xi_2^-)\right)^T,$$

$$[\underline{u}]_\gamma = \left(\underline{u}(\xi_1^+), \; \underline{u}(\xi_2^+), \; \underline{u}(\xi_1^-), \; \underline{u}(\xi_2^-)\right)^T,$$

and in which the transmission matrix R_ω^h is given by

(36)
$$R_\omega^h = \frac{\Theta^{-1}}{2h} \begin{bmatrix} I + \Theta c_{11} & -(I - \Theta c_{12}) & I - \Theta c_{11} & -(I + \Theta c_{12}) \\ -(I - \Theta c_{21}) & I + \Theta c_{22} & -(I + \Theta c_{21}) & I - \Theta c_{22} \\ I - \Theta c_{11} & -(I + \Theta c_{12}) & I + \Theta c_{11} & -(I - \Theta c_{12}) \\ -(I + \Theta c_{21}) & I - \Theta c_{22} & -(I - \Theta c_{21}) & I + \Theta c_{22} \end{bmatrix}.$$

Here

$$c_{11} = \begin{bmatrix} \zeta & 0 \\ 0 & \tfrac{1}{2}\sigma \end{bmatrix}, \qquad c_{22} = \begin{bmatrix} \tfrac{1}{2}\sigma & 0 \\ 0 & \zeta \end{bmatrix},$$

$$c_{21}^T = c_{12} = \begin{bmatrix} 0 & \eta \\ \tfrac{1}{2}\sigma & 0 \end{bmatrix}, \qquad \Theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}.$$

The matrix R_ω^h is obviously symmetric and is easily shown to be positive semi-definite for positive θ_1 and θ_2. Balancing the tractions across vertical and horizontal faces common to neigboring cells, we obtain, with reference to Figure 2,

(37a) $2(I + \Theta c_{11})\underline{u}(P_0) + (I - \Theta c_{11})\big(\underline{u}(P_1) + \underline{u}(P_2)\big)$

$$= (I - \Theta c_{12})\big(\underline{u}(Q_1) + \underline{u}(Q_4)\big) + (I + \Theta c_{12})\big(\underline{u}(Q_0) + \underline{u}(Q_3)\big),$$

(37b) $2(I + \Theta c_{22})\underline{u}(Q_0) + (I - \Theta c_{22})\big(\underline{u}(Q_1) + \underline{u}(Q_2)\big)$

$$= (I - \Theta c_{21})\big(\underline{u}(P_1) + \underline{u}(P_4)\big) + (I + \Theta c_{21})\big(\underline{u}(P_0) + \underline{u}(P_3)\big).$$

These equations correspond to the stress-eliminated form (32). If $\Gamma_2 = \phi$, i.e., the values of the displacements are prescribed on the boundary, then the coefficient matrix of the above problem is symmetric and positive definite. These properties have an important consequence so far as the SOR method is concerned, since for such a system the convergence of SOR is guaranteed for any value of the relaxation factor α in the range, $0 < \alpha < 2$, (see Varga [3], Young [4]). Similar conclusions can be expected when mixed boundary conditions are considered.

A natural iterative method for solving (37) is line SOR. The method involves solving block tridgiagonal systems firstly along all horizontal lines and then along vertical lines. In the next section we will compare this and point relaxation methods to a simple problem.

II.3   Numerical Example

Consider a square domain on whose vertical edges the displacements are fixed and which experiences a uniform load along its top horizontal surface. This situation is illustrated in Figure 3.

The boundary conditions are

$$\underline{u} = (0,0)^T \qquad \text{on} \ x = 0,1$$

$$\underline{\tau}_2 = (0,0)^T \qquad \text{on} \ y = 0$$

$$\underline{\tau}_2 = (0,-1)^T \qquad \text{on} \ y = 1.$$

Our experiments were performed using the plane stress approximation with values of Young's modulus and Poisson's ratio given by $E = 10^7$, $\nu = 0.3$.

The equilibrium displacement u_1 is anti-symmetric and u_2 is symmetric about the line $x = 1/2$. However, in the computations that were performed, no advantage was taken of this symmetry.

For the choice of parameters $\theta_1 = \theta_2 = 2/\sigma$, an important simplification occurs in (34), viz.

$$I - \Theta A = \text{diag}(1 - 2\zeta/\sigma,\ 0),$$

$$I - \Theta \hat{A} = \text{diag}(0,\ 1 - 2\zeta/\sigma).$$

It is with the above values of θ_1 and θ_2 that the results in this section were obtained.

In our experiments we used the point Gauss-Seidel, point SOR. and line SOR methods. The parameters for the SOR methods were chosen to be the optimum ones for Laplace's equation. Initially, the value of $\underline{u}$ was taken to be zero at all the grid points. Keeping in mind that the method can be expected to yield only second-order accuracy, the iterations were terminated when the ℓ_2- norm of the residuals was less than 10^{-3}.

In Table I we show the dependence of the number of iterations required to attain the convergence criterion on the mesh size for various iterative schemes.

Table I. Dependence of Number of Iterations on Mesh Size

h	pt. Gauss-Seidel	pt. SOR	line SOR
1/2	42	34	--
1/4	100	54	30
1/8	325	97	51
1/16	1240	191	99

These results indicate that the rates of convergence of the Gauss-Seidel and SOR methods on this problem are $O(h^2)$ and $O(h)$ respectively. Table II contains the values of the ℓ_2-norms of the solutions to the stress-eliminated equations (37) for different grid sizes.

Table II. ℓ_2-Norms of Numerical Solution

h	$\|u_1\|_2$	$\|u_2\|_2$
1/2	0.0221	0.0979
1/4	0.0086	0.0710
1/8	0.0055	0.0603
1/16	0.0048	0.0568
1/32	0.0046	0.0556

We note that the convergence of $\|u_1\|_2$ is $O(h^2)$ while that of $\|u_2\|_2$ is $O(h^{3/2})$ as $h \to 0$. This degradation in behavior is due to the singularities which are located at the top corners.

The values of the tractions are calculated by substituting the values of $\underline{u}$ in (35). The components of displacement and traction display the relevant symmetric or anti-symmetric properties to four decimal places. There are two integral checks that can be carried out to verify the computations, namely

(35)
$$-\int_0^1 \tau_{11}(0,y)\,dy + \int_0^1 \tau_{11}(1,y)\,dy = 0,$$

$$\int_0^1 \tau_{21}(1,y)\,dy = -\frac{1}{2}.$$

These conditions express the integral condition

$$\int_\Gamma \tau \cdot \underline{n}\,d\gamma = 0$$

using the assumed symmetry of τ_{12} about $x = \frac{1}{2}$. These integrals were computed numerically using the midpoint rule. The first integral check held exactly while the second one was found to be correct to the number of decimal places specified in the displacement calculation.

Various plots are given of the solution. Figures 4, 5, 6 are contour plots of the stress componets $\tau_{11}, \tau_{22}, \tau_{12}$ respectively. These were obtained using 64 cells in each direction. Figures 7a and 7b show the principal stress vectors within each computational cell. The principal stress directions are defined to be those vectors $\underline{x} \neq 0$ which satisfy the eigenproblem

$$(\tau - \lambda I)\underline{x} = \underline{0}.$$
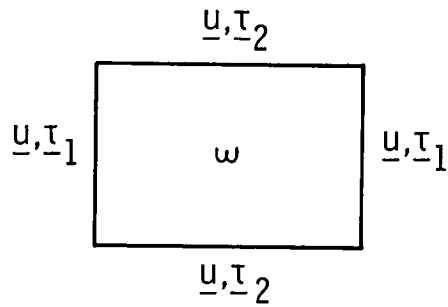
Concluding Remarks

The treatment of equilibrium with a volume force $\underline{f}$ requires no essential modifications to the method. In this case (1a) has the form $\sum_i \partial_{x_i} \underline{\tau}_i = \underline{f}$; correspondingly (10a) is modified to
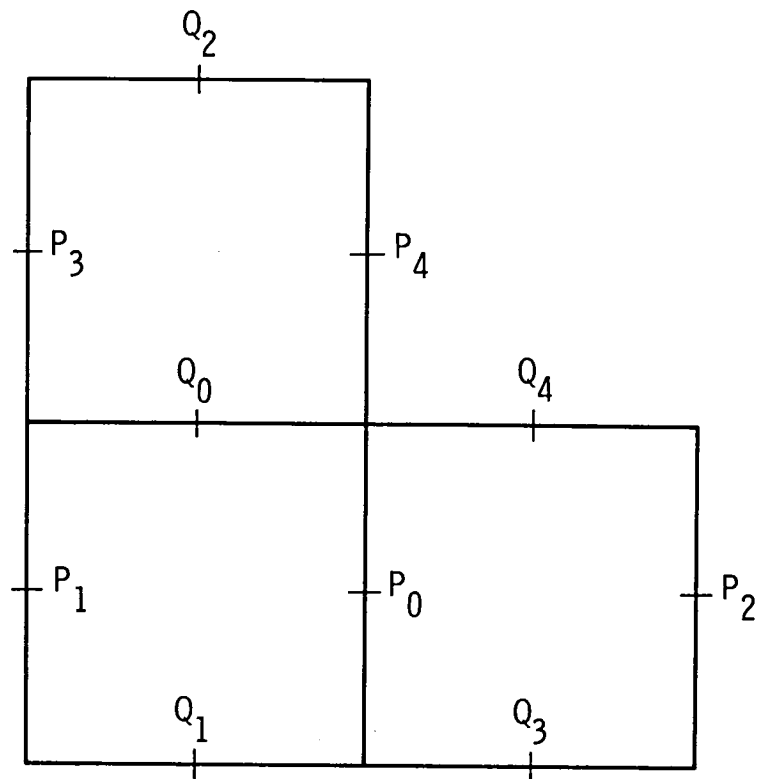
$$(10a') \qquad \sum_i \delta_{x_i} \underline{\tau}_i = \underline{f} \ .$$

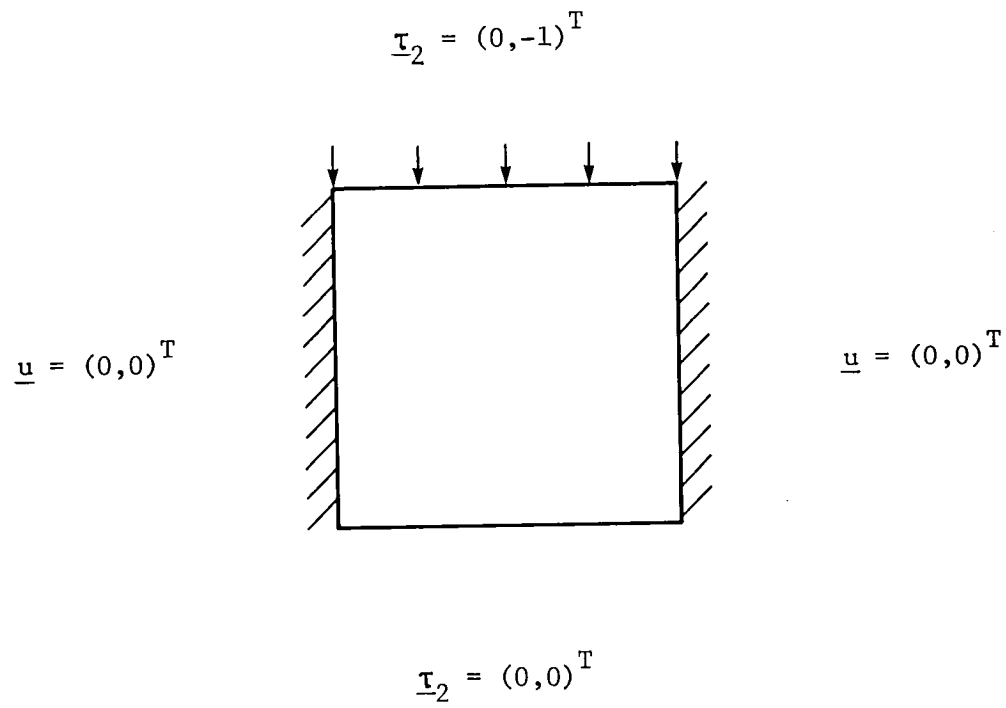We leave the details of the consequent developments to the reader.

Figure 1. Variables associated with cell sides by the difference equations in two dimensions.
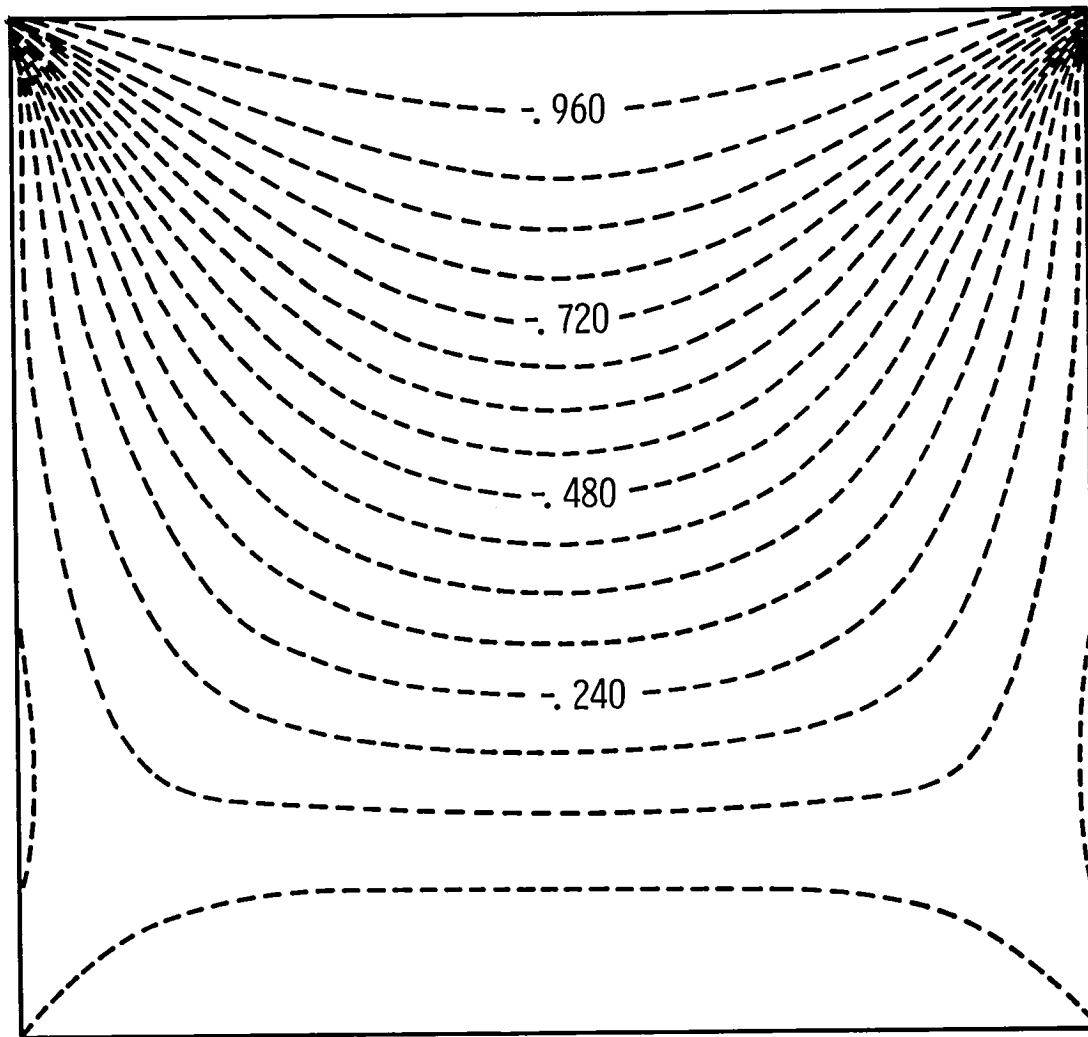

Figure 2. Points associated with the stress-eliminated equations (37).

Figure 3. Description of the plane stress test problem.
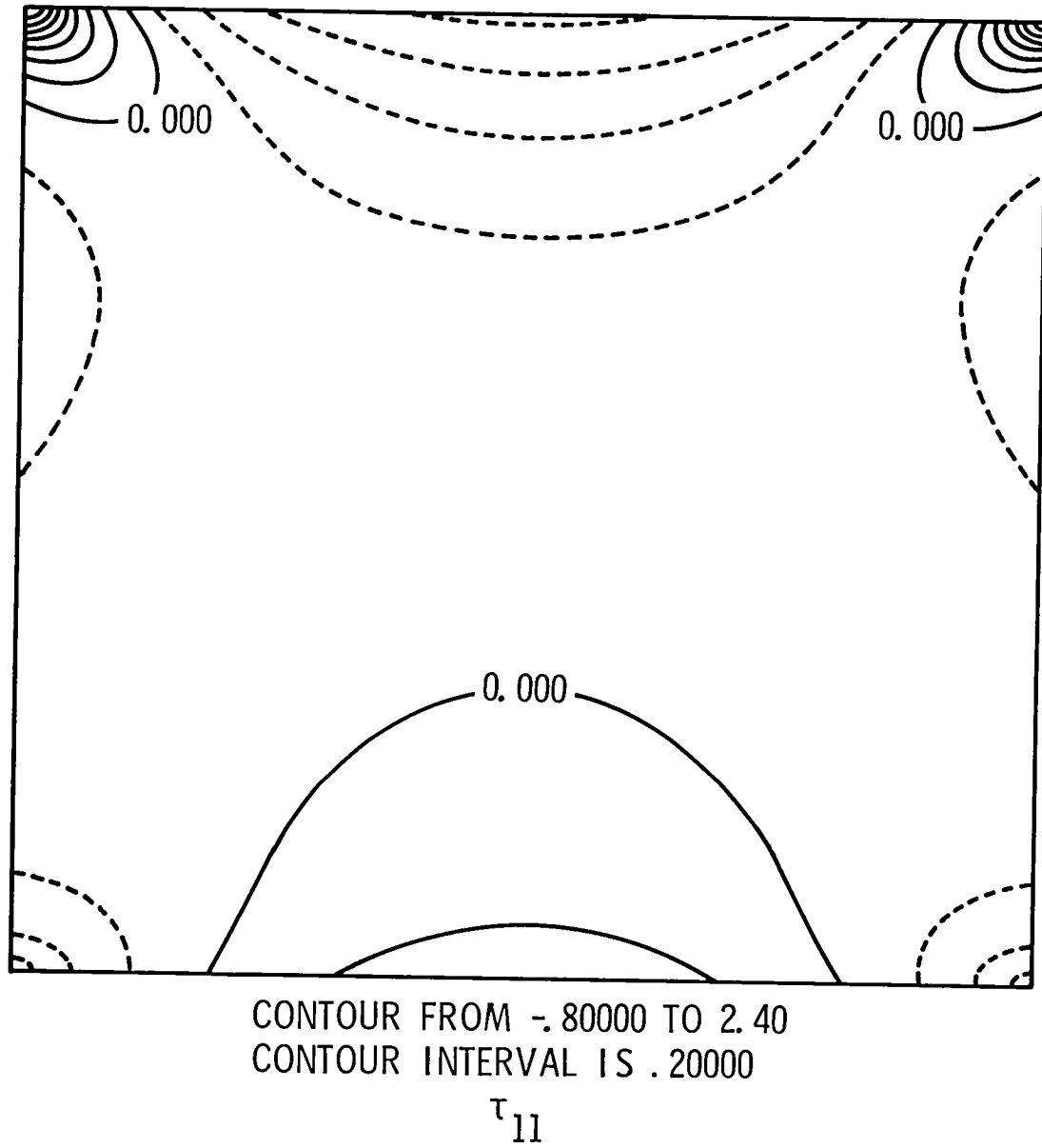
Figure 4. Plot of the stress component τ_{11}.

CONTOUR FROM -.96000 TO 0
CONTOUR INTERVAL IS .60000E-01

τ_{22}
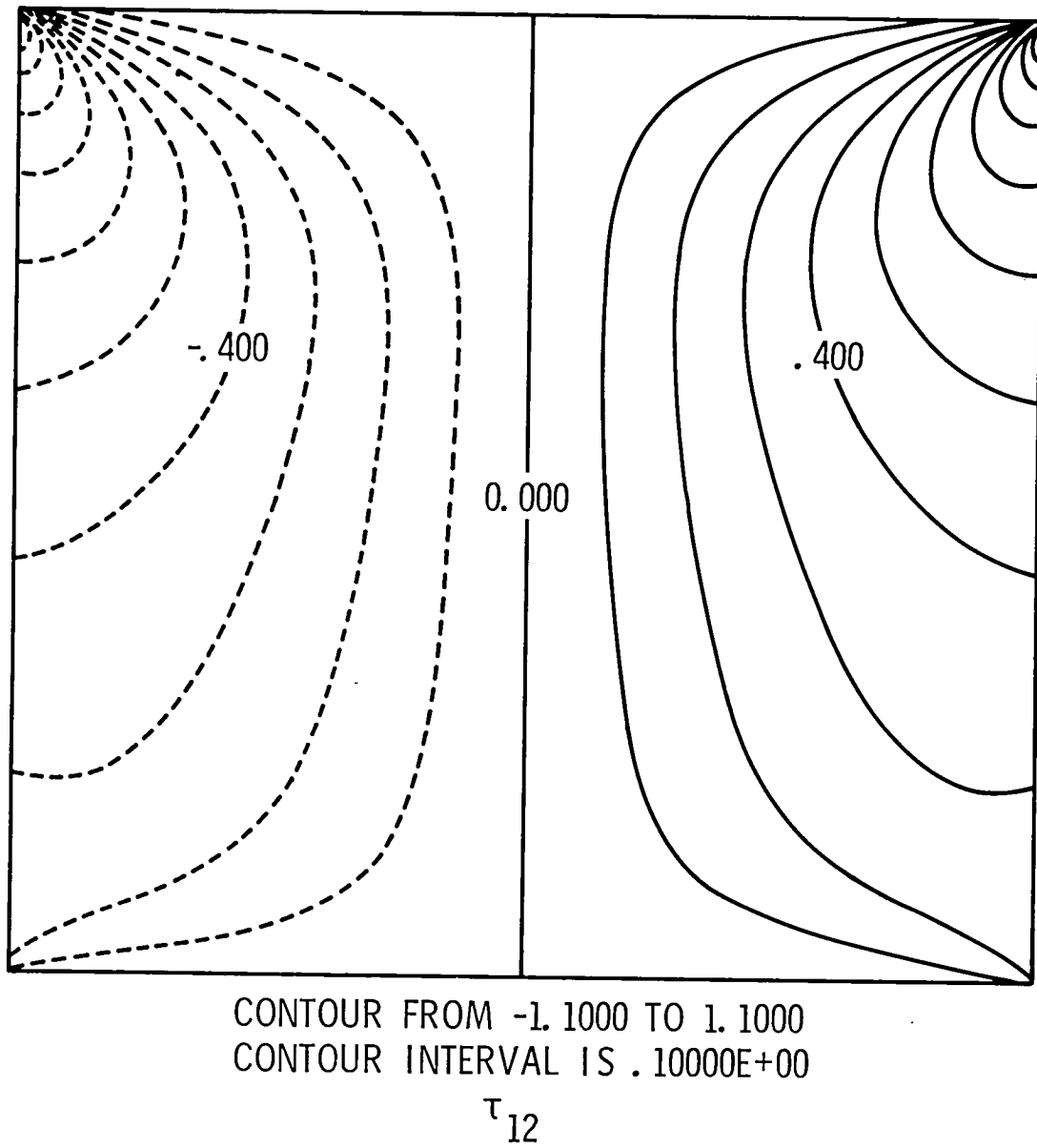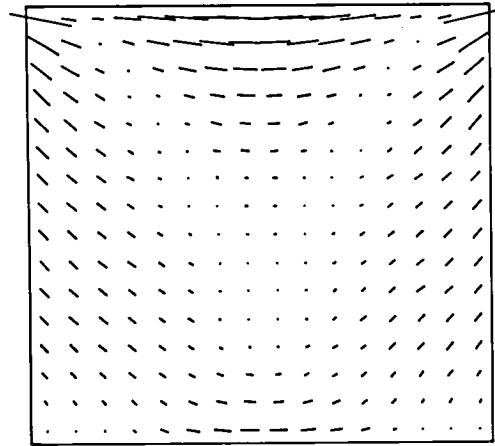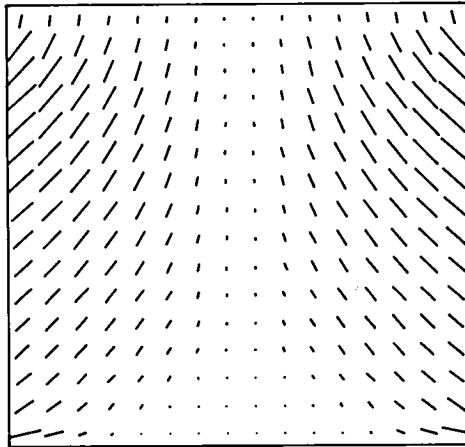
Figure 5. Plot of the Stress component τ_{22}.

CONTOUR FROM -1. 1000 TO 1. 1000
CONTOUR INTERVAL IS . 10000E+00

τ_{12}

Figure 6. Plot of the stress component τ_{12}.

Figure 7a

Figure 7b

Principal stress vectors

References

[1] T. N. PHILLIPS and M. E. ROSE, "A compact finite difference scheme for div(ρ grad u) $-$ q^2 u = 0," NASA CR-172165, June 1983.

[2] S. P. TIMOSHENKO and J. N. GOODIER, Theory of Elasticity, McGraw-Hill, New York, 1970.

[3] R. S. VARGA, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[4] D. M. YOUNG, Iterative Solution of Large Linear Systems, Academic Press, London and New York, 1971.

1. Report No. NASA CR-172293	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A Finite Difference Scheme for the Equilibrium Equations of Elastic Bodies		5. Report Date January 1984
		6. Performing Organization Code
7. Author(s) T. N. Phillips and M. E. Rose		8. Performing Organization Report No. 84-2
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665		10. Work Unit No.
		11. Contract or Grant No. NAS1-17070
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C 20546		13. Type of Report and Period Covered Contractor report
		14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Robert H. Tolson
Final Report

16. Abstract

A compact difference scheme is described for treating the first-order system of PDE's which describe the equilibrium equations of an elastic body. An algebraic simplification enables the solution to be obtained by standard direct or iterative techniques.

17. Key Words (Suggested by Author(s))
elasticity
finite difference
iterative methods
		18. Distribution Statement
		   39 Structural Mechanics
		   64 Numerical Analysis
		
		   Unclassified-Unlimited	

19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 32	22. Price A03