(NASA-CR-170993) EXHIBIT D MODULAR DESIGN ATTITUDE CONTROL SYSTEM STUDY Progress Report, Jan. - Feb. 1984 (Bendix Corp.) 69 p HC A04/MF A01 CSCL 22B

N84-20625

Unclas
G3/18 18805

CONTRACT NAS8-33979

EXHIBIT D
MODULAR DESIGN ATTITUDE CONTROL SYSTEM STUDY
PROGRESS REPORT

JANUARY - FEBRUARY 1984

PREPARED FOR:

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1.0 BACKGROUND

Models of flexible spacecraft have traditionally been an assembly of rigid bodies one of which has been identified as a central body that contains all sensors and actuators. As the overall size of spacecraft space platforms increases, central body control will no longer be adequate for meeting specified performance levels. The overall goal of this project is to develop and implement new models with advanced attitude controls which can meet the performance specifications of the future.

Under Exhibit B of this contract, the following tasks were accomplished:

1. Development of a dynamically equivalent four body approximation of the NASTRAN finite element model supplied for the MSFC/hybrid deployable truss to support the digital computer simulation of the ten body model of the flexible space platform that incorporates the four body truss model.

2. Generation of coefficients for sensitivity of state variables of the linearized model of the three axes rotational dynamics of the prototype flexible spacecraft with respect to the model's parameters.

3. Evaluation of software changes required to accomodate addition of another rigid body to the five body model of the rotational dynamics of the prototype flexible spacecraft.

4. Comparison of effectiveness of attitude control for actuators on two bodies of the six body model of the prototype flexible space platform with that for actuators restricted to one body of the same model.

The models and procedures utilized and the results obtained were presented in the Exhibit B final report of March 18, 1983 (1).

Additional study effort concerning control with incomplete state feedback was not included in the above cited report because discussions between Marshall Space Flight Center and Bendix technical representatives prior to and following the oral presentation of the report resulted in the delineation of a new direction to be pursued in this aspect of the study contract. A written statement of the overall objective and tasks to be accomplished in support of this new direction was submitted to Dr. Henry Waites of Marshall Space Flight Center within a short time following the oral presentation of the report.

2.0 CURRENT EFFORT

The principal objective of the current work on this study contract is the generation of a series of linear observers to support the application of modular attitude control to a series of state variable models of flexible spacecraft for which some of the state variables are inaccessible. The specific tasks involved are the following:

1. Develop single axis state variable model of a prototype flexible spacecraft to be utilized in the comparison of different approaches to the development of modular attitude control systems. This model will consist of four rigid bodies serially connected by a flexible

suspension in such a way that motion is restricted to rotation about a common axis through the mass centers of the bodies.

2. Develop hierarchical modular attitude control of the single axis four body model for the following cases:

   a. State variable sensors and control actuators on all bodies,

   b. State variable sensors and control actuators on two adjacent bodies,

   c. State variable sensors and control actuators on the two end bodies,

   d. State variable sensors on three adjacent bodies with control actuators on two of these bodies.

3. Extend the work described above to the development of modular attitude control for the state variable three axis five body model of a prototype flexible spacecraft with some state variables inaccessable.

The following tasks were accomplished and described in the Exhibit C final report (2).

1. Development of two, three and four body single axis state variable models of a flexible spacecraft for which one or more state variables was inaccessible for direct measurement or observation.

2. Generation of a series of reduced state linear observers of the minimum orders required to reconstruct the inaccessible state variables of the single axis models that were developed in Task 1.

3. Comparison of the three and four body models to delineate the patterns that occur in the changes in the coefficient matrices of the model as a result of adding a rigid body.

3.0 PRESENT STATUS

During the month covered by this report work under Exhibit D of the contract in support of generating minimum order linear observers for the single axis models of a flexible spacecraft with damping was continued. In particular, reduced order linear observers were generated for the four body model with damping and up to seven of its eight scalar state variables inaccessible. The equations for synthesizing each observer were written with r_{11}, the ratio of the scalar damping constant, c_1, to the scalar spring constant, k_1; r_{22}, the ratio of the scalar damping constant, c_2, to the scalar spring constant, k_2, and r_{23}, the ratio of the scalar damping constant, c_3, to the scalar spring constant, k_3 as parameters. The observer synthesis equations also were written with $r_{11} = r_{22} = r_{33} = 0$ to represent the undamped case.

The observer synthesis equations for the four body single axis model with damping were then compared with those for the same model without damping. A general form was developed for these equations that encompassed the synthesis of the elements of the observer T matrix for the following conditions with respect to damping in the model.

1. No damping;

2. Damping only at the interface between bodies 1 and 2;

3. Damping only at the interface between bodies 2 and 3;

4. Damping only at the interface between bodies 3 and 4;

5. Damping at <u>both</u> the interface between body 1 and body 2 and the interface between body 2 and body 3;

6. Damping at <u>both</u> the interface between body 2 and body 3 and the interface between body 3 and body 4;

7. Damping at <u>both</u> the interface between body 1 and body 2 and the interface between body 3 and body 4;

8. Damping at all three interfaces.

A more detailed description of the development of the observer synthesis equations for the four body model with damping accompanies this report.

4.0 REFERENCES

1. Guidance Systems Division, The Bendix Corporation, "Modular Design Attitude Control System", Final Report, Contract NAS8-33979, Exhibit B, for George C. Marshall Space Flight Center, March 18, 1983.

2. Guidance Systems Division, The Bendix Corporation, "Modular Design Attitude Control System", Final Report, Contract NAS8-33979, Exhibit C for George C. Marshall Space Flight Center , January 24, 1984.

3. Luenberger, D. G., "Determining the state of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertation, Stanford University, 1963.

4. Luenberger, D. G., "Observers for Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, April 1966, pp. 190-197.

5. Luenberger, D. G., "An Introduction to Observers", IEEE Transactions on Automatic Control, Vol. AC-16, No. 6, December 1971, pp. 596-602.

6. Sage, A. P., Optimum Systems Control. Englewood Cliffs, N. J.: Prentice-Hall, Inc. 1968, pp. 306-312.

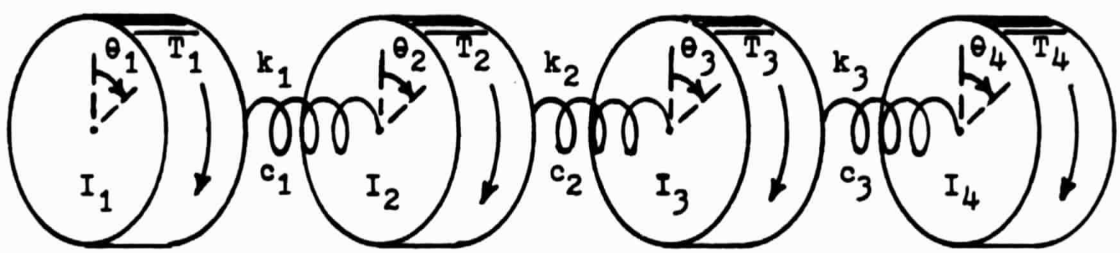
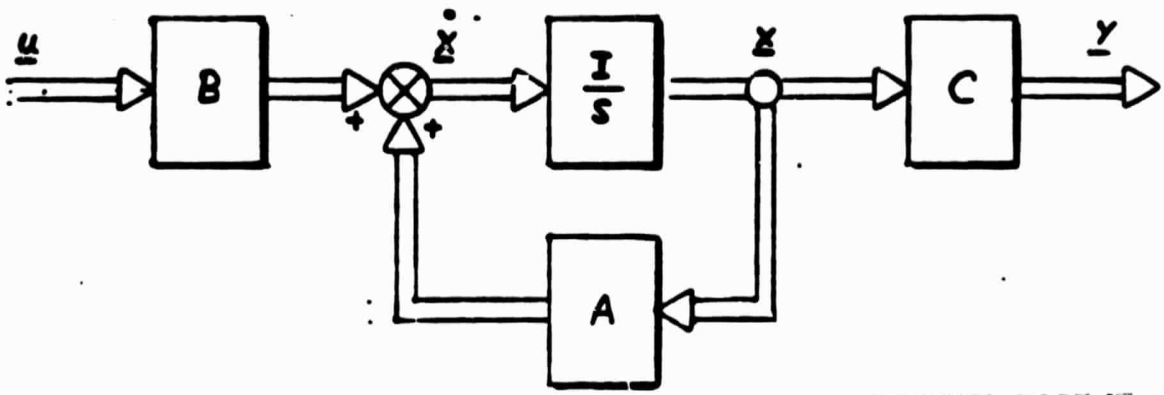


FIGURE 1

FOUR BODY SINGLE AXIS MODEL WITH DAMPING AT ALL THREE INTERFACES

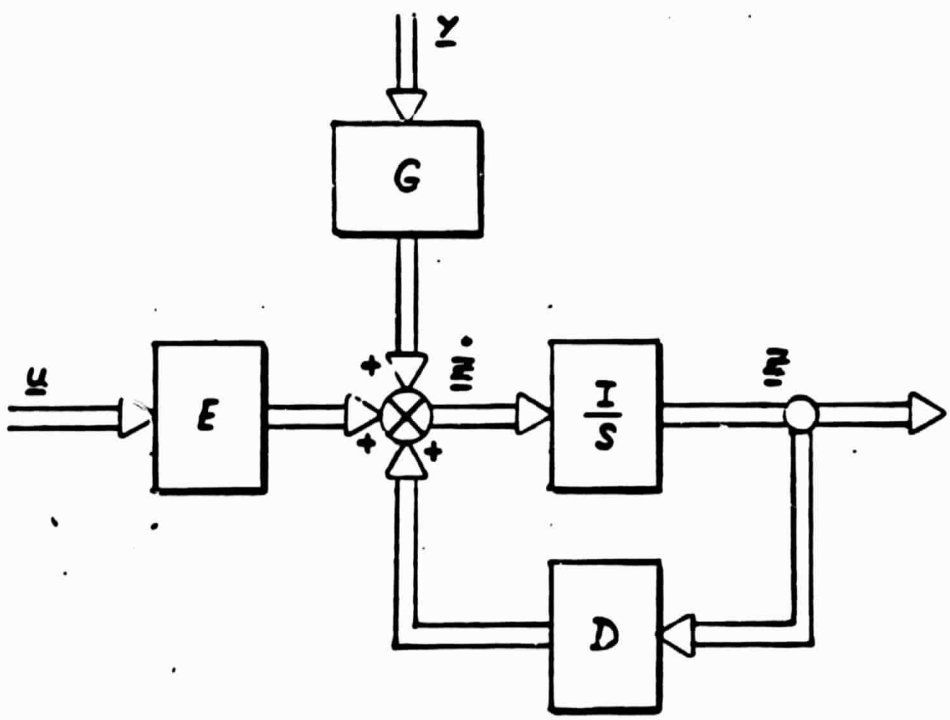
Linearized State Variable Model of the System to be Controlled



$\dot{\underline{x}} = A\underline{x} + B\underline{u}$

$\underline{y} = C\underline{x}$

Model of Linear Observer



$\dot{\underline{z}} = D\underline{z} + E\underline{u} + G\underline{y}$

$G\underline{y} = GC\underline{x} = F\underline{x}$

SUMMARY OF EQUATIONS FOR FOUR BODY SINGLE AXIS MODEL WITH DAMPING

MODEL EQUATIONS

The state variable form of the four body single axis model of a flexible spacecraft with damping may be expressed in the following form

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{1}$$

$$\underline{Y} = C\underline{x} \tag{2}$$

where:

$$\underline{x} = (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \theta_3, \dot{\theta}_3, \theta_4, \dot{\theta}_4)^T$$
$$= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T = \text{model state vector}$$

$$\underline{u} = (u_1, u_2, u_3, u_4)^T = \left(\frac{T_1}{I_1}, \frac{T_2}{I_2}, \frac{T_3}{I_3}, \frac{T_4}{I_4}\right)^T = \text{control vector}$$

$$\underline{Y} = (Y_1, \cdots, Y_m)^T = \text{vector of measured or observed states.}$$

$$A = \left[\begin{array}{cc:cc:cc:cc}
0 & 1 & 0 & 0 & & & & \\
-a_{23} & -r_{11}a_{23} & a_{23} & r_{11}a_{23} & \multicolumn{2}{c:}{[0]} & \multicolumn{2}{c}{[0]} \\
\hdashline
0 & 0 & 0 & 1 & 0 & 0 & & \\
a_{41} & r_{11}a_{41} & a_{43} & a_{44} & a_{45} & r_{22}a_{45} & \multicolumn{2}{c}{[0]} \\
\hdashline
 & & 0 & 0 & 0 & 1 & 0 & 0 \\
\multicolumn{2}{c:}{[0]} & a_{63} & r_{22}a_{63} & a_{65} & a_{66} & a_{67} & r_{33}a_{67} \\
\hdashline
 & & & & 0 & 0 & 0 & 1 \\
\multicolumn{2}{c:}{[0]} & \multicolumn{2}{c:}{[0]} & a_{85} & r_{33}a_{85} & -a_{85} & -r_{33}a_{85}
\end{array}\right] \tag{3}$$

$$a_{23} = \frac{k_1}{I_1}$$

$$a_{41} = \frac{k_1}{I_2}, \quad a_{45} = \frac{k_2}{I_2}, \quad a_{43} = -(a_{41} + a_{45}), \quad a_{44} = -(a_{41} r_{11} + a_{45} r_{22}) \tag{4}$$

$$a_{63} = \frac{k_2}{I_3}, \quad a_{67} = \frac{k_3}{I_3}, \quad a_{65} = -(a_{63} + a_{67}), \quad a_{66} = -(a_{63} r_{22} + a_{67} r_{33})$$

$$a_{85} = \frac{k_3}{I_4}$$

$$r_{ii} = \frac{c_i}{k_i}, \quad i = 1, 2, 3 \tag{5}$$

$$[0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{6}$$

$$B = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \tag{7}$$

C = measurement or observation matrix of dimension $m \times 8$, $m \leq 8$

CORRESPONDING REDUCED STATE LINEAR OBSERVER EQUATIONS

$$\dot{\underline{z}} = D\underline{z} + E\underline{u} + G\underline{y} \tag{8}$$

$$\underline{z} = T\underline{x} \tag{9}$$

where, for an observer of order $p = 8 - m$,

$D = p \times p$ observer state coefficient matrix (assumed diagonal)

$G = p \times m$ observer vector of observed states coefficient matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} & t_{17} & t_{18} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t_{p,1} & t_{p,2} & t_{p,3} & t_{p,4} & t_{p,5} & t_{p,6} & t_{p,7} & t_{p,8} \end{bmatrix} \tag{10}$$

$$E = TB = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} \\ \cdot & & & \\ \cdot & & & \\ t_{p,2} & t_{p,4} & t_{p,6} & t_{p,8} \end{bmatrix} \tag{11}$$

$\underline{z} = (z_1, \cdots, z_p)^T$ = observer state vector

OBSERVER SYNTHESIS EQUATIONS

The reduced state observer synthesis equation,

$$TA - DT = F = GC \tag{12}$$

reduces, under the assumption of a diagonal D matrix, to the following forms.

$$
\begin{bmatrix}
t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} & t_{17} & t_{18} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
t_{p,1} & t_{p,2} & t_{p,3} & t_{p,4} & t_{p,5} & t_{p,6} & t_{p,7} & t_{p,8}
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{23} & -r_{11}a_{23} & a_{23} & r_{12}a_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
a_{41} & r_{11}a_{41} & a_{43} & a_{44} & a_{45} & r_{22}a_{45} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & a_{63} & r_{22}a_{63} & a_{65} & a_{66} & a_{67} & r_{32}a_{67} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & a_{85} & r_{33}a_{85} & -a_{85} & -r_{33}a_{85}
\end{bmatrix}
$$

$$
-\begin{bmatrix}
d_{11} & & 0 \\
 & \ddots & \\
0 & & d_{pp}
\end{bmatrix}
\begin{bmatrix}
t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} & t_{17} & t_{18} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
t_{p,1} & t_{p,2} & t_{p,3} & t_{p,4} & t_{p,5} & t_{p,6} & t_{p,7} & t_{p,8}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{17} & f_{18} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
f_{p,1} & f_{p,2} & f_{p,3} & f_{p,4} & f_{p,5} & f_{p,6} & f_{p,7} & f_{p,8}
\end{bmatrix} \quad (13)
$$

$$
\begin{bmatrix}
-d_{ii} & -a_{23} & 0 & a_{41} & 0 & 0 & 0 & 0 \\
1 & -d'_{ii} & 0 & r_{11}a_{41} & 0 & 0 & 0 & 0 \\
0 & a_{23} & -d_{ii} & -(a_{41}+a_{45}) & 0 & a_{63} & 0 & 0 \\
0 & r_{11}a_{23} & 1 & -d''_{ii} & 0 & r_{22}a_{63} & 0 & 0 \\
0 & 0 & 0 & a_{45} & -d_{ii} & -(a_{63}+a_{67}) & 0 & a_{85} \\
0 & 0 & 0 & r_{22}a_{45} & 1 & -d'''_{ii} & 0 & r_{33}a_{85} \\
0 & 0 & 0 & 0 & 0 & a_{67} & -d_{ii} & -a_{85} \\
0 & 0 & 0 & 0 & 0 & r_{33}a_{67} & 1 & -d^{IV}_{ii}
\end{bmatrix}
\begin{bmatrix} t_{i1} \\ t_{i2} \\ t_{i3} \\ t_{i4} \\ t_{i5} \\ t_{i6} \\ t_{i7} \\ t_{i8} \end{bmatrix}
=
\begin{bmatrix} f_{i1} \\ f_{i2} \\ f_{i3} \\ f_{i4} \\ f_{i5} \\ f_{i6} \\ f_{i7} \\ f_{i8} \end{bmatrix}
$$

where: $i = 1, 2, \cdots, p$ (14)

$$d'_{ii} = d_{ii} + a_{23} r_{11} \qquad (15)$$

$$d''_{ii} = d_{ii} + a_{41} r_{11} + a_{45} r_{22} \qquad (16)$$

$$d'''_{ii} = d_{ii} + a_{63} r_{22} + a_{67} r_{33} \qquad (17)$$

$$d^{IV}_{ii} = d_{ii} + a_{85} r_{33} \qquad (18)$$

$$t_{i1} = d'_{ii} t_{i2} - r_{11} a_{41} t_{i4} + f_{i2} \qquad (19)$$

$$t_{i3} = -a_{23} r_{11} t_{i2} + d''_{ii} t_{i4} - a_{63} r_{22} t_{i6} + f_{i4} \qquad (20)$$

$$t_{i5} = -a_{45} r_{22} t_{i4} + d'''_{ii} t_{i6} - a_{85} r_{33} t_{i8} + f_{i6} \qquad (21)$$

$$t_{i7} = -a_{67} r_{33} t_{i6} + d^{IV}_{ii} t_{i8} + f_{i8} \qquad (22)$$

$$-d_{ii}\, t_{i1} - a_{23} r_{11}\, t_{i2} + a_{41}\, t_{i4} = f_{i1} \tag{23}$$

$$a_{23}\, t_{i2} - d_{ii}\, t_{i3} - (a_{41} + a_{45})\, t_{i4} + a_{63}\, t_{i6} = f_{i3} \tag{24}$$

$$a_{45}\, t_{i4} - d_{ii}\, t_{i5} - (a_{63} + a_{67})\, t_{i6} + a_{85}\, t_{i8} = f_{i5} \tag{25}$$

$$a_{67}\, t_{i6} - d_{ii}\, t_{i7} - a_{85}\, t_{i8} = f_{i7} \tag{26}$$

Substitution of equation (19) in (23), (20) in (24), (21) in (25) and (22) in (26) yields the following.

$$\begin{bmatrix} -(d_{ii}^2 + a'_{23}) & a'_{41} & 0 & 0 \\ a'_{23} & -(d_{ii}^2 + a'_{41} + a'_{45}) & a'_{63} & 0 \\ 0 & a'_{45} & -(d_{ii}^2 + a'_{63} + a'_{67}) & a'_{85} \\ 0 & 0 & a'_{67} & -(d_{ii}^2 + a'_{85}) \end{bmatrix} \begin{bmatrix} t_{i2} \\ t_{i4} \\ t_{i5} \\ t_{i8} \end{bmatrix}$$

$$= \begin{bmatrix} f_{i1} + d_{ii} f_{i2} \\ f_{i3} + d_{ii} f_{i4} \\ f_{i5} + d_{ii} f_{i6} \\ f_{i7} + d_{ii} f_{i8} \end{bmatrix} \tag{27}$$

where:

$$a'_{23} = a_{23}(1 + r_{11} d_{ii}) \tag{28}$$

$$a'_{41} = a_{41}(1 + r_{11} d_{ii}) \tag{29}$$

$$a'_{45} = a_{45}(1 + r_{22} d_{ii}) \tag{30}$$

$$a'_{63} = a_{63}(1 + r_{22} d_{ii}) \tag{31}$$

$$a'_{67} = a_{67}(1 + r_{33} d_{ii}) \tag{32}$$

$$a'_{85} = a_{85}(1 + r_{33} d_{ii}) \tag{33}$$

$$\Delta'_{i4} = \begin{vmatrix} -(d_{ii}^2 + a'_{23}) & a'_{41} & 0 & 0 \\ a'_{23} & -(d_{ii}^2 + a'_{41} + a'_{45}) & a'_{63} & 0 \\ 0 & a'_{45} & -(d_{ii}^2 + a'_{63} + a'_{67}) & a'_{85} \\ 0 & 0 & a'_{67} & -(d_{ii}^2 + a'_{85}) \end{vmatrix}$$

$$= -(d_{ii}^2 + a'_{23})(\Delta'_{i4})_{1,1} - a'_{23}(\Delta'_{i4})_{2,1} \qquad (34)$$

where $(\Delta'_{i4})_{i,j} = \Delta'_{i4}$ without the elements of the ith row and jth column

$$(\Delta'_{i4})_{1,1} = \begin{vmatrix} -(d_{ii}^2 + a'_{41} + a'_{45}) & a'_{63} & 0 \\ a'_{45} & -(d_{ii}^2 + a'_{63} + a'_{67}) & a'_{85} \\ 0 & a'_{67} & -(d_{ii}^2 + a'_{85}) \end{vmatrix}$$

$$= -(d_{ii}^2 + a'_{41} + a'_{45})[d_{ii}^4 + (a'_{63} + a'_{67} + a'_{85})d_{ii}^2 + a'_{63}a'_{85}]$$
$$+ a'_{45}a'_{63}(d_{ii}^2 + a'_{85})$$
$$= -[d_{ii}^6 + (a'_{41} + a'_{45} + a'_{63} + a'_{67} + a'_{85})d_{ii}^4 + (a'_{41}a'_{63} + a'_{41}a'_{67} + a'_{41}a'_{85}$$
$$+ a'_{45}a'_{67} + a'_{45}a'_{85} + a'_{63}a'_{85})d_{ii}^2 + a'_{41}a'_{63}a'_{85}]$$

$$-(d_{ii}^2 + a'_{23})(\Delta'_{i4})_{1,1} = d_{ii}^8 + (a'_{23} + a'_{41} + a'_{45} + a'_{63} + a'_{67} + a'_{85})d_{ii}^6 + (a'_{23}a'_{41} + a'_{23}a'_{45}$$
$$+ a'_{23}a'_{63} + a'_{23}a'_{67} + a'_{23}a'_{85} + a'_{41}a'_{63} + a'_{41}a'_{67} + a'_{41}a'_{85} + a'_{45}a'_{67}$$
$$+ a'_{45}a'_{85} + a'_{63}a'_{85})d_{ii}^4 + (a'_{23}a'_{41}a'_{63} + a'_{23}a'_{41}a'_{67} + a'_{23}a'_{41}a'_{85}$$
$$+ a'_{23}a'_{45}a'_{67} + a'_{23}a'_{45}a'_{85} + a'_{23}a'_{63}a'_{85} + a'_{41}a'_{63}a'_{85})d_{ii}^2 + a'_{23}a'_{41}a'_{63}a'_{85}]$$

$$(\Delta'_{i4})_{2,1} = \begin{vmatrix} a'_{41} & 0 & 0 \\ a'_{45} & -(d_{ii}^2+a'_{63}+a'_{67}) & a'_{85} \\ 0 & a'_{67} & -(d_{ii}^2+a'_{85}) \end{vmatrix}$$

$$= a'_{41}\left[d_{ii}^4+(a'_{63}+a'_{67}+a'_{85})d_{ii}^2+a'_{63}a'_{85}\right]$$

$$-a'_{23}(\Delta'_{i4})_{2,1} = -a'_{23}a'_{41}\left[d_{ii}^4+(a'_{63}+a'_{67}+a'_{85})d_{ii}^2+a'_{63}a'_{85}\right]$$

$$\begin{aligned}\Delta'_{i4} = \Big[d_{ii}^6 &+ (a'_{23}+a'_{41}+a'_{45}+a'_{63}+a'_{67}+a'_{85})d_{ii}^4+(a'_{23}a'_{45}-a'_{23}a'_{67}+a'_{23}a'_{63} \\ &+a'_{23}a'_{85}+a'_{41}a'_{63}+a'_{41}a'_{67}+a'_{41}a'_{85}+a'_{45}a'_{67}+a'_{45}a'_{85}+a'_{67}a'_{85})d_{ii}^2 \\ &+(a'_{23}a'_{45}a'_{67}+a'_{23}a'_{45}a'_{85}+a'_{23}a'_{63}a'_{85}+a'_{41}a'_{63}a'_{85})\Big]\,d_{ii}^2 \qquad (35)\end{aligned}$$

$$(\Delta'_{i4})_{3,1} = \begin{vmatrix} a'_{41} & 0 & 0 \\ -(d_{ii}^2+a'_{41}+a'_{45}) & a'_{63} & 0 \\ 0 & a'_{67} & -(d_{ii}^2+a'_{85}) \end{vmatrix} = -a'_{41}a'_{63}(d_{ii}^2+a'_{85})$$

$$(\Delta'_{i4})_{4,1} = \begin{vmatrix} a'_{41} & 0 & 0 \\ -(d_{ii}^2+a'_{41}+a'_{45}) & a'_{63} & 0 \\ a'_{45} & -(d_{ii}^2+a'_{63}+a'_{67}) & a'_{85} \end{vmatrix} = a'_{41}a'_{63}a'_{85}$$

$$(\Delta'_{i4})_{1,2} = \begin{vmatrix} a'_{23} & a'_{63} & 0 \\ 0 & -(d_{ii}^2+a'_{63}+a'_{67}) & a'_{85} \\ 0 & a'_{67} & -(d_{ii}^2+a'_{85}) \end{vmatrix}$$

$$= a'_{23}\left[d_{ii}^4+(a'_{63}+a'_{67}+a'_{85})d_{ii}^2+a'_{63}a'_{85}\right]$$

$$(\Delta'_{i4})_{2,2} = \begin{vmatrix} -(d_{ii}^2+a'_{23}) & 0 & 0 \\ 0 & -(d_{ii}^2+a'_{63}+a'_{67}) & a'_{85} \\ 0 & a'_{67} & -(d_{ii}^2+a'_{85}) \end{vmatrix}$$

$$= -(d_{ii}^2+a'_{23})[d_{ii}^4+(a'_{63}+a'_{67}+a'_{85})d_{ii}^2+a'_{63}a'_{85}]$$

$$(\Delta'_{i4})_{3,2} = \begin{vmatrix} -(d_{ii}^2+a'_{23}) & 0 & 0 \\ a'_{23} & a'_{63} & 0 \\ 0 & a'_{67} & -(d_{ii}^2+a'_{85}) \end{vmatrix} = a'_{63}(d_{ii}^2+a'_{23})(d_{ii}^2+a'_{85})$$

$$(\Delta'_{i4})_{4,2} = \begin{vmatrix} -(d_{ii}^2+a'_{23}) & 0 & 0 \\ a'_{23} & a'_{63} & 0 \\ 0 & -(d_{ii}^2+a'_{63}+a'_{67}) & a'_{85} \end{vmatrix} = -a'_{63}a'_{85}(d_{ii}^2+a'_{23})$$

$$(\Delta'_{i4})_{1,3} = \begin{vmatrix} a'_{23} & -(d_{ii}^2+a'_{41}+a'_{45}) & 0 \\ 0 & a'_{45} & a'_{85} \\ 0 & 0 & -(d_{ii}^2+a'_{85}) \end{vmatrix} = -a'_{23}a'_{45}(d_{ii}^2+a'_{85})$$

$$(\Delta'_{i4})_{2,3} = \begin{vmatrix} -(d_{ii}^2+a'_{23}) & a'_{41} & 0 \\ 0 & a'_{45} & a'_{85} \\ 0 & 0 & -(d_{ii}^2+a'_{85}) \end{vmatrix} = a'_{45}(d_{ii}^2+a'_{23})(d_{ii}^2+a'_{85})$$

$$(\Delta'_{i4})_{3,3} = \begin{vmatrix} -(d_{ii}^2+a'_{23}) & a'_{41} & 0 \\ a'_{23} & -(d_{ii}^2+a'_{41}+a'_{45}) & 0 \\ 0 & 0 & -(d_{ii}^2+a'_{85}) \end{vmatrix}$$

$$= -(d_{ii}^2+a'_{85})[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}]$$

$$(\Delta'_{ii})_{4,3} = \begin{vmatrix} -(d_{ii}^2+a'_{23}) & a'_{41} & 0 \\ a'_{23} & -(d_{ii}^2+a'_{41}+a_{45}) & 0 \\ 0 & a'_{45} & a'_{85} \end{vmatrix}$$

$$= a'_{85}\left[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}\right]$$

$$(\Delta'_{ii})_{1,4} = \begin{vmatrix} a'_{23} & -(d_{ii}^2+a'_{41}+a'_{45}) & a'_{63} \\ 0 & a'_{45} & -(d_{ii}^2+a'_{63}+a'_{67}) \\ 0 & 0 & a'_{67} \end{vmatrix} = a'_{23}a'_{45}a'_{67}$$

$$(\Delta'_{ii})_{2,4} = \begin{vmatrix} -(d_{ii}^2+a'_{23}) & a'_{41} & 0 \\ 0 & a'_{45} & -(d_{ii}^2+a'_{63}+a'_{67}) \\ 0 & 0 & a'_{67} \end{vmatrix} = -a'_{45}a_{67}(d_{ii}^2+a_{23})$$

$$(\Delta_{ii})_{3,4} = \begin{vmatrix} -(d_{ii}^2+a'_{23}) & a'_{41} & 0 \\ a'_{23} & -(d_{ii}^2+a'_{41}+a'_{45}) & a'_{63} \\ 0 & 0 & a'_{67} \end{vmatrix}$$

$$= a'_{67}\left[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}\right]$$

$$(\Delta'_{ii})_{4,4} = \begin{vmatrix} -(d_{ii}^2+a'_{23}) & a'_{41} & 0 \\ a'_{23} & -(d_{ii}^2+a'_{41}+a'_{45}) & a'_{63} \\ 0 & a'_{45} & -(d_{ii}^2+a'_{63}+a'_{67}) \end{vmatrix}$$

$$= -(d_{ii}^2+a'_{63}+a'_{67})\left[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}\right]$$

$$t_{i2} = \frac{(\Delta_{i4}')_{1,1}(f_{i1}+d_{ii}f_{i2}) - (\Delta_{i4}')_{2,1}(f_{i3}+d_{ii}f_{i4}) + (\Delta_{i4}')_{3,1}(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}'}$$

$$- \frac{(\Delta_{i4}')_{4,1}(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}'}$$

$$= - \frac{[d_{ii}^6 + (a_{41}'+a_{45}'+a_{63}'+a_{67}'+a_{85}')d_{ii}^4 + (a_{41}'a_{63}'+a_{41}'a_{67}'+a_{41}'a_{85}'+a_{45}'a_{67}'}{\Delta_{i4}'}$$

$$+ \frac{a_{45}'a_{85}'+a_{63}'a_{85}')d_{ii}^2 + a_{41}'a_{63}'a_{85}'](f_{i1}+d_{ii}f_{i2})}{\Delta_{i4}'}$$

$$- \frac{a_{41}'[d_{ii}^4+(a_{63}'+a_{67}'+a_{85}')d_{ii}^2+a_{63}'a_{85}'](f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}'}$$

$$- \frac{a_{41}'a_{63}'(d_{ii}^2+a_{85}')(f_{i5}+d_{ii}f_{i6}) + a_{41}'a_{63}'a_{85}'(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}'} \qquad (36)$$

$$t_{i4} = - \frac{(\Delta_{i4}')_{1,2}(f_{i1}+d_{ii}f_{i2}) - (\Delta_{i4}')_{2,2}(f_{i3}+d_{ii}f_{i4}) + (\Delta_{i4}')_{3,2}(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}'}$$

$$+ \frac{(\Delta_{i4}')_{4,2}(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}'}$$

$$= - \frac{a_{23}'[d_{ii}^4+(a_{63}'+a_{67}'+a_{85}')d_{ii}^2+a_{63}'a_{85}'](f_{i1}+d_{ii}f_{i2})}{\Delta_{i4}'}$$

$$- \frac{(d_{ii}^2+a_{23}')[d_{ii}^4+(a_{63}'+a_{67}'+a_{85}')d_{ii}^2+a_{63}'a_{85}'](f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}'}$$

$$- \frac{a_{63}'(d_{ii}^2+a_{23}')(d_{ii}^2+a_{85}')(f_{i5}+d_{ii}f_{i6}) + a_{63}'a_{85}'(d_{ii}^2+a_{23}')(f_{i7}-d_{ii}f_{i8})}{\Delta_{i4}'} \qquad (37)$$

$$t_{i6} = \frac{(\Delta'_{i4})_{1,3}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i4})_{2,3}(f_{i3}+d_{ii}f_{i4}) + (\Delta'_{i4})_{3,3}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}}$$

$$- \frac{(\Delta'_{i4})_{4,3}(f_{i7}+d_{ii}f_{i8})}{\Delta'_{i4}}$$

$$= - \frac{a'_{23}a'_{45}(d_{ii}^2+a'_{85})(f_{i1}+d_{ii}f_{i2}) + a'_{45}(d_{ii}^2+a'_{23})(d_{ii}^2+a'_{85})(f_{i3}+d_{ii}f_{i4})}{\Delta'_{i4}}$$

$$- \frac{(d_{ii}^2+a'_{85})\left[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}\right](f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}}$$

$$- \frac{a'_{85}\left[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}\right](f_{i7}+d_{ii}f_{i8})}{\Delta'_{i4}} \qquad (38)$$

$$t_{i8} = - \frac{(\Delta'_{i4})_{1,4}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i4})_{2,4}(f_{i3}+d_{ii}f_{i4}) + (\Delta'_{i4})_{3,4}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}}$$

$$+ \frac{(\Delta'_{i4})_{4,4}(f_{i7}+d_{ii}f_{i8})}{\Delta'_{i4}}$$

$$= - \frac{a'_{23}a'_{45}a'_{67}(f_{i1}+d_{ii}f_{i2}) + a'_{45}a'_{67}(d_{ii}^2+a'_{23})(f_{i3}-d_{ii}f_{i4})}{\Delta'_{i4}}$$

$$- \frac{a'_{67}\left[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}\right](f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}}$$

$$- \frac{(d_{ii}^2+a'_{63}+a'_{67})\left[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}\right](f_{i7}+d_{ii}f_{i8})}{\Delta'_{i4}} \qquad (39)$$

$$\epsilon_{i1} = \frac{[d'_{ii}(\Delta'_{i4})_{1,1} + a_{41} r_{11} (\Delta'_{i4})_{1,2}] f_{i1}}{\Delta'_{i4}}$$

$$+ \frac{[(\cancel{d_{ii} d'_{ii}} - d^2_{ii} - a_{23})(\Delta'_{i4})_{1,1} + (\cancel{a_{41} r_{11} d_{ii}} - a_{41})(\Delta'_{i4})_{1,2}] f_{i2}}{\Delta'_{i4}}$$

$$- \frac{[d^4_{ii} + (a'_{63} + a'_{67} + a'_{85}) d^2_{ii} + a'_{63} a'_{85}][\cancel{a_{41} r_{11}(d^2_{ii} + a'_{23})} - a_{41} d_{ii}](f_{i3} + d_{ii} f_{i4})}{\Delta'_{i4}}$$

$$+ \frac{a'_{63}(d^2_{ii} + a'_{85})[-a_{41} d_{ii} + \cancel{a_{41} r_{11}(d^2_{ii} + a'_{23})}](f_{i5} + d_{ii} f_{i6})}{\Delta'_{i4}}$$

$$+ \frac{[\cancel{a_{41} r_{11}(d^2_{ii} + a'_{23})} - a_{41} d_{ii}] a'_{63} a'_{85} (f_{i7} + d_{ii} f_{i8})}{\Delta'_{i4}}$$

$$= \frac{[d'_{ii}(\Delta'_{i4})_{1,1} + a_{41} r_{11} (\Delta'_{i4})_{1,2}] f_{i1}}{\Delta'_{i4}}$$

$$+ \frac{a_{23} d^2_{ii}[d^4_{ii} + (a'_{45} + a'_{63} + a'_{67} + a'_{85}) d^2_{ii} + a'_{45} a'_{67} + a'_{45} a'_{85} + a'_{63} a'_{85}] f_{i2}}{\Delta'_{i4}}$$

$$+ \frac{a_{41} d_{ii}[d^4_{ii} + (a'_{63} + a'_{67} + a'_{85}) d^2_{ii} + a'_{63} a'_{85}](f_{i3} + d_{ii} f_{i4})}{\Delta'_{i4}}$$

$$- \frac{a_{41} a'_{63} d_{ii}(d^2_{ii} + a'_{85})(f_{i5} + d_{ii} f_{i6}) + a_{41} a'_{63} a'_{85} d_{ii}(f_{i7} + d_{ii} f_{i8})}{\Delta'_{i4}} \qquad (40)$$

where
$$\Delta'_{i4} = -(d^2_{ii} + a'_{23})(\Delta'_{i4})_{1,1} - a'_{41}(\Delta'_{i4})_{1,2}$$

$$a_{23}(\Delta'_{i4})_{1,1} + a_{41}(\Delta'_{i4})_{1,2} =$$
$$-a_{23} d^2_{ii}[d^4_{ii} + (a'_{45} + a'_{63} + a'_{67} + a'_{85}) d^2_{ii} + a'_{45} a'_{67} + a'_{45} a'_{85} + a'_{63} a'_{85}]$$

$$d'_{ii}(\Delta'_{i4})_{1,1} + a_{41} r_{11}(\Delta'_{i4})_{1,2} = (d_{ii} + a_{23} r_{1})(\Delta'_{i4})_{1,1} + a_{41} r_{11}(\Delta'_{i4})_{1,2}$$
$$= -[d'_{ii} d^5_{ii} + (a'_{41} + a'_{45} + a'_{63} + a'_{67} + a'_{85}) d'_{ii} d^3_{ii} - a'_{23} a_{41} r_{11} d^3_{ii} + (a'_{45} a'_{63} + a'_{41} a'_{67}$$
$$+ a'_{41} a'_{85} + a'_{45} a'_{67} + a'_{45} a'_{85} + a'_{63} a'_{85}) d'_{ii} d_{ii} - a'_{23} a_{41}(a'_{63} + a'_{67} + a'_{85}) r_{11} d_{ii} + a'_{41} a'_{63} a'_{85}] d_{ii}$$

$$t_{i7} = -\frac{\left[a_{67} r_{33} (\Delta'_{i4})_{1,3} + d_{ii}^{IV} (\Delta'_{i4})_{1,4}\right](f_{i1} + d_{ii} f_{i2})}{\Delta'_{i4}}$$

$$+ \frac{\left[a_{67} r_{33} (\Delta'_{i4})_{2,3} + d_{ii}^{IV} (\Delta'_{i4})_{2,4}\right](f_{i3} + d_{ii} f_{i4})}{\Delta'_{i4}}$$

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$$- \frac{\left[a_{67} r_{33} (\Delta'_{i4})_{3,3} + d_{ii}^{IV} (\Delta'_{i4})_{3,4}\right](f_{i5} + d_{ii} f_{i6})}{\Delta'_{i4}}$$

$$+ \frac{\left[a_{67} r_{33} (\Delta'_{i4})_{4,3} + d_{ii}^{IV} (\Delta'_{i4})_{4,4}\right] f_{i7}}{\Delta'_{i4}}$$

$$+ \frac{\left[(\overset{0}{\cancel{a_{67} r_{33}}} d_{ii} - a_{67})(\Delta'_{i4})_{4,3} + (\overset{0}{\cancel{d_{ii} d_{ii}^{IV}}} - d_{ii}^{2} - a_{85})(\Delta'_{i4})_{4,4}\right] f_{i8}}{\Delta'_{i4}}$$

where $\Delta'_{i4} = -a'_{67} (\Delta'_{i4})_{4,3} - (d_{ii}^{2} + a'_{85})(\Delta'_{i4})_{4,4}$

$$t_{i7} = \frac{\left[\overset{0}{\cancel{a'_{23} a'_{45} a_{67} r_{33} (d_{ii}^{2} + a_{85})}} - a'_{23} a'_{45} a_{67} d_{ii}\right](f_{i1} + d_{ii} f_{i2})}{\Delta'_{i4}}$$

$$+ \frac{a'_{45}(d_{ii}^{2} + a'_{23})\left[\overset{0}{\cancel{a_{67} r_{33} (d_{ii}^{2} + a_{85})}} - a_{67} d_{ii}\right](f_{i3} + d_{ii} f_{i4})}{\Delta'_{i4}}$$

$$+ \frac{\left[d_{ii}^{4} + (a'_{23} + a'_{41} + a'_{45}) d_{ii}^{2} + a'_{23} a'_{45}\right]\left[\overset{0}{\cancel{a_{67} r_{33} (d_{ii}^{2} + a_{85})}} - a_{67} d_{ii}\right](f_{i5} + d_{ii} f_{i6})}{\Delta'_{i4}}$$

$$+ \frac{\left[a_{67} r_{33} (\Delta'_{i4})_{4,3} + d_{ii}^{IV} (\Delta'_{i4})_{4,4}\right] f_{i7}}{\Delta'_{i4}}$$

$$- \frac{\left[a_{67} (\Delta'_{i4})_{4,3} + a_{85} (\Delta'_{i4})_{4,4}\right] f_{i8}}{\Delta'_{i4}} \qquad (41)$$

$$a_{67} (\Delta'_{i4})_{4,3} + a_{85} (\Delta'_{i4})_{4,4} =$$

$$\left[\overset{0}{\cancel{a_{67} a_{85}}} - a_{85} d_{ii}^{2} - a'_{67} a_{85} - \overset{0}{\cancel{a'_{67} a_{85}}}\right]\left[d_{ii}^{4} + (a'_{23} + a'_{41} + a'_{45}) d_{ii}^{2} + a'_{23} a'_{45}\right]$$

Four Body Model (cont'd)

$$a_{23} r_{11} (\Delta'_{i4})_{1,1} =$$

$$-a_{23} r_{11} \left[d_{ii}^6 + (a'_{41} + a'_{45} + a'_{63} + a'_{67} + a'_{85}) d_{ii}^4 + (a'_{41}a'_{63} + a'_{41}a'_{67} + a'_{41}a'_{85} + a'_{45}a'_{67} + a'_{45}a'_{85} + a'_{63}a'_{85}) d_{ii}^2 + a'_{41}a'_{63}a'_{85} \right]$$

$$a_{23} r_{11} (\Delta'_{i4})_{1,1} + d''_{ii} (\Delta'_{i4})_{1,2} + a_{63} r_{22} (\Delta'_{i4})_{1,3} =$$

$$-a_{23} r_{11} d_{ii}^6 + a'_{23} d''_{ii} d_{ii}^4 - (a_{23}a'_{41} + a_{23}a'_{45} + a_{23}a'_{63} + a_{23}a'_{67} + a_{23}a'_{85}) r_{11} d_{ii}^4$$
$$+ (a'_{23}a'_{63} + a'_{23}a'_{67} + a'_{23}a'_{85}) d''_{ii} d_{ii}^2 - (a_{23}a'_{41}a'_{63} + a_{23}a'_{41}a'_{67} + a_{23}a'_{41}a'_{85}$$
$$+ a_{23}a'_{45}a'_{67} + a_{23}a'_{45}a'_{85} + a_{23}a'_{63}a'_{85}) r_{11} d_{ii}^2 - a'_{23}a_{45}a_{63} r_{22} d_{ii}^2 - a_{23}a'_{41}a'_{63}a'_{85} r_{11}$$

$$t_{i3} = -\frac{\left[a_{23} r_{11} (\Delta'_{i4})_{1,1} + d''_{ii} (\Delta'_{i4})_{1,2} + a_{63} r_{22} (\Delta'_{i4})_{1,3}\right](f_{i1} + d_{ii} f_{i2})}{\Delta'_{i4}}$$

$$+ \frac{\left[a_{23} r_{11} (\Delta'_{i4})_{2,1} + d''_{ii} (\Delta'_{i4})_{2,2} + a_{63} r_{22} (\Delta'_{i4})_{2,3}\right] f_{i3}}{\Delta'_{i4}}$$

$$- \frac{d_{ii}^2 \left[(a_{41} + a_{45}) d_{ii}^4 + (a'_{23}a_{45} + a_{41}a'_{63} + a_{41}a'_{67} + a_{41}a'_{85} + a_{45}a'_{67} + a_{45}a'_{85}) d_{ii}^2 \right.}{\Delta'_{i4}}$$

$$+ \frac{\left. a'_{23}a_{45}a'_{67} + a'_{23}a_{45}a'_{85} + a_{41}a'_{63}a'_{85} \right] f_{i4}}{\Delta'_{i4}}$$

$$- \frac{d_{ii}(d_{ii}^2 + a'_{85}) \left\{ a_{63} d_{ii}^2 + \left[a_{41}a_{63}(r_{11} - r_{22}) - a'_{23}a_{63}r_{22} \right] d_{ii} + a'_{23}a'_{63} \right\} (f_{i5} + d_{ii} f_{i6})}{\Delta'_{i4}}$$

$$- \frac{a'_{85} d_{ii} \left\{ a_{63} d_{ii}^2 + \left[a_{41}a_{63}(r_{11} - r_{22}) - a'_{23}a_{63}r_{22} \right] d_{ii} + a'_{23}a'_{63} \right\} (f_{i7} + d_{ii} f_{i8})}{\Delta'_{i4}} \qquad (65)$$

$$\epsilon_{i5} = \frac{[a_{45} r_{22}(\Delta'_{i4})_{1,2} + d'''_{ii}(\Delta'_{i4})_{1,3} + a_{85} r_{33}(\Delta'_{i4})_{1,4}](f_{i1} + d_{ii} f_{i2})}{\Delta'_{i4}}$$

$$- \frac{[a_{45} r_{22}(\Delta'_{i4})_{2,2} + d'''_{ii}(\Delta'_{i4})_{2,3} + a_{85} r_{33}(\Delta'_{i4})_{2,4}](f_{i3} + d_{ii} f_{i4})}{\Delta'_{i4}}$$

$$+ \frac{[a_{45} r_{22}(\Delta'_{i4})_{3,2} + d'''_{ii}(\Delta'_{i4})_{3,3} + a_{85} r_{33}(\Delta'_{i4})_{3,4}] f_{i5}}{\Delta'_{i4}}$$

$$+ \frac{[(\cancel{a_{45} r_{22}}^{\,0} d_{ii} - a_{45})(\Delta'_{i4})_{3,2} + (\cancel{d_{ii} d_{ii}}^{\,0} \cancel{- d_{ii}^2} - a_{63} - a_{67})(\Delta'_{i4})_{3,3}}{\Delta'_{i4}}$$

$$+ \frac{(\cancel{a_{85} r_{33}}^{\,0} d_{ii} - a_{85})(\Delta'_{i4})_{3,4}] f_{i6}}{\Delta'_{i4}}$$

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$$- \frac{[a_{45} r_{22}(\Delta'_{i4})_{4,2} + d'''_{ii}(\Delta'_{i4})_{4,3} + a_{85} r_{33}(\Delta'_{i4})_{4,4}](f_{i7} + d_{ii} f_{i8})}{\Delta'_{i4}}$$

where:

$$\Delta'_{i4} = -a'_{45}(\Delta'_{i4})_{3,2} - (d_{ii}^2 + a'_{63} + a'_{67})(\Delta'_{i4})_{3,3} - a'_{85}(\Delta'_{i4})_{3,4}$$

$$(a_{63} + a_{67})(\Delta'_{i4})_{3,3} + a_{85}(\Delta'_{i4})_{3,4} =$$

$$[d_{ii}^4 + (a'_{23} + a'_{41} + a'_{45}) d_{ii}^2 + a'_{23} a'_{45}][\cancel{a'_{67} a_{85}} - (a_{63} + \cancel{a_{67}}^{\,0}) a'_{85} - (a_{63} + a_{67}) d_{ii}^2]$$

$$= -[(a_{63} + a_{67}) d_{ii}^2 + a_{63} a'_{85}][d_{ii}^4 + (a'_{23} + a'_{41} + a'_{45}) d_{ii}^2 + a'_{23} a'_{45}]$$

$$= -[(a_{63} + a_{67}) d_{ii}^6 + (a_{63} a'_{85} + a'_{23} a_{63} + a'_{41} a_{63} + a'_{45} a_{63} + a'_{23} a_{67}$$

$$+ a'_{41} a_{67} + a'_{45} a_{67}) d_{ii}^4 + (a'_{23} a_{63} a'_{85} + a'_{41} a_{63} a'_{85} + a'_{45} a_{63} a'_{85}$$

$$+ a'_{23} a'_{45} a_{63} + a'_{23} a'_{45} a_{67}) d_{ii}^2 + a'_{23} a'_{45} a_{63} a'_{85}]$$

$$a_{45}(\Delta'_{i4})_{3,2} = a_{45} a'_{63}[d_{ii}^4 + (a'_{23} + a'_{85}) d_{ii}^2 + a'_{23} a'_{85}]$$

$$a_{45}(\Delta'_{i4})_{3,2} + (a_{63} + a_{67})(\Delta'_{i4})_{3,3} + a_{85}(\Delta'_{i4})_{3,4} =$$

$$-[(a_{63} + a_{67}) d_{ii}^4 + (a_{63} a'_{85} + a'_{23} a_{63} + a'_{41} a_{63} + a'_{23} a_{67} + a'_{41} a_{67} + a'_{45} a_{67}) d_{ii}^2$$

$$+ (a'_{23} a_{63} a'_{85} + a'_{41} a_{63} a'_{85} + a'_{23} a'_{45} a_{67})] d_{ii}^2$$

$$t_{i3} = -\frac{a'_{23} a_{45} d_{ii}[d_{ii}^2 + a_{67}(r_{33}-r_{22})d_{ii} + a'_{85}](f_{i1}+d_{ii} f_{i2})}{\Delta'_{i4}}$$

$$-\frac{a_{45} d_{ii}(d_{ii}^2 + a'_{23})[d_{ii}^2 + a_{67}(r_{33}-r_{22})d_{ii} + a'_{85}](f_{i3}+d_{ii} f_{i4})}{\Delta'_{i4}}$$

$$+\frac{[a_{45} r_{22}(\Delta'_{i4})_{3,2} + d'''_{ii}(\Delta'_{i4})_{3,3} + a_{85} r_{33}(\Delta'_{i4})_{3,4}] f_{i5}}{\Delta'_{i4}}$$

$$-\frac{d_{ii}^2[(a_{63}+a_{67}) d_{ii}^4 + (a_{63}a'_{85} + a'_{23}a_{67} + a'_{41}a_{63} + a'_{21}a_{67} + a'_{41}a_{67} + a'_{45}a_{67}) d_{ii}^2}{\Delta'_{i4}}$$

$$+\frac{a'_{23}a_{63}a'_{85} + a'_{41}a_{63}a'_{85} + a'_{23}a'_{45}a_{67}] f_{i6}}{\Delta'_{i4}}$$

$$-\frac{[a_{45} r_{22}(\Delta'_{i4})_{4,2} + d'''_{ii}(\Delta'_{i4})_{4,3} + a_{85} r_{33}(\Delta'_{i4})_{4,4}](f_{i7}+d_{ii} f_{i8})}{\Delta'_{i4}} \qquad (43)$$

$$a_{45} r_{22}(\Delta'_{i4})_{1,2} + d'''_{ii}(\Delta'_{i4})_{1,3} + a_{85} r_{33}(\Delta'_{i4})_{1,4} =$$

$$a'_{23} a'_{45} r_{22}[d_{ii}^4 + (a'_{63}+a'_{67}+a'_{85})d_{ii}^2 + a'_{63}a'_{85}] + a'_{23}a'_{45}[a'_{67}a_{85}r_{33} - d'''_{ii}d_{ii}^2 - a'_{85}d'''_{ii}]$$

$$= a'_{23} a_{45} r_{22}[d_{ii}^4 + (a'_{63}+a'_{67}+a'_{85})d_{ii}^2 + a'_{63}a'_{85}]$$

$$+ a'_{23}a'_{45}[\cancelto{0}{a'_{67}a_{85}r_{33}} - a'_{85}(d_{ii} + a_{63}r_{22} + \cancelto{0}{a_{67}r_{33}}) - d'''_{ii}d_{ii}^2]$$

$$= a'_{23}\{a_{45}r_{22}[d_{ii}^4 + (a'_{63}+a'_{67}+a'_{85})d_{ii}^2 + \cancelto{0}{a'_{63}a_{85}}] - a'_{45}[d'''_{ii}d_{ii}^2 + a'_{85}(d_{ii} + \cancelto{0}{a_{63}r_{22}})]\}$$

$$= a'_{23}\{a_{45}r_{22}[\cancelto{0}{d_{ii}^3} + (\cancelto{0}{a'_{63}} + a'_{67} + \cancelto{0}{a'_{85}})d_{ii}] - a_{45}d_{ii}^2 - a'_{45}(\cancel{a_{63}r_{22}} + a_{67}r_{33})d_{ii} - a_{45}a'_{85}\}d_{ii}$$

$$= -a'_{23}a_{45}d_{ii}[d_{ii}^2 + a_{67}(r_{33}-r_{22})d_{ii} + a'_{85}]$$

$$-a_{45}r_{22}(\Delta'_{i4})_{2,2} - d'''_{ii}(\Delta'_{i4})_{2,3} - a_{85}r_{33}(\Delta'_{i4})_{2,4}$$

$$= (d_{ii}^2 + a'_{23})\{a_{45}r_{22}[d_{ii}^4 + (\cancelto{0}{a'_{63}} + a'_{67} + \cancelto{0}{a'_{85}})d_{ii}^2 + \cancelto{0}{a'_{63}a_{85}}] - a'_{45}d_{ii}^2(d_{ii} + \cancelto{0}{a_{63}r_{22}} + a_{67}r_{33})$$

$$- a_{45}a'_{85}(d_{ii} + \cancelto{0}{a_{63}r_{22}} + \cancelto{0}{a_{67}r_{33}}) + \cancelto{0}{a'_{45}a'_{67}a_{85}r_{33}}\}$$

$$= -(d_{ii}^2 + a'_{23})\; a_{45}d_{ii}[d_{ii}^2 + a_{67}(r_{33}-r_{22})d_{ii} + a'_{85}]$$

$$d_{AA}'''(\Delta_{A4}')_{3,3}+a_{85}r_{33}(\Delta_{A4}')_{3,4}=$$

$$[d_{AA}^4+(a_{23}'+a_{41}'+a_{45}')d_{AA}^2+a_{23}'a_{45}'][\cancel{a_{67}'a_{85}r_{33}}-d_{AA}^2(d_{AA}+a_{63}r_{22}+a_{67}r_{33})$$

$$-a_{85}'(d_{AA}+a_{63}r_{22}+\cancel{a_{67}r_{33}})],$$

$$=-[d_{AA}'''d_{AA}^2+a_{85}'(d_{AA}+a_{63}r_{22})][d_{AA}^4+(a_{23}'+a_{41}'+a_{45}')d_{AA}^2+a_{23}'a_{45}']$$

$$=-[d_{AA}'''d_{AA}^6+(a_{23}'+a_{41}'+a_{45}')d_{AA}'''d_{AA}^4+a_{23}'a_{45}'d_{AA}'''d_{AA}^2+a_{85}'(d_{AA}+a_{63}r_{22})d_{AA}^4$$

$$+(a_{23}'+a_{41}'+a_{45}')a_{85}'(d_{AA}+a_{63}r_{22})d_{AA}^2+a_{23}'a_{45}'a_{85}'(d_{AA}+a_{63}r_{22})]$$

$$a_{45}r_{22}(\Delta_{A4}')_{3,2}=a_{45}a_{63}'r_{22}[d_{AA}^4+(a_{23}'+a_{85}')d_{AA}^2+a_{23}'a_{85}']$$

$$a_{45}r_{22}(\Delta_{A4}')_{3,2}+d_{AA}'''(\Delta_{A4}')_{3,3}+a_{85}r_{33}(\Delta_{A4}')_{3,4}=$$

$$-[d_{AA}'''d_{AA}^5+(a_{23}'+a_{41}'+a_{45}')d_{AA}'''d_{AA}^3+a_{85}'d_{AA}^3(d_{AA}+a_{63}r_{22})-a_{45}a_{63}'r_{22}d_{AA}^3$$

$$+a_{23}'a_{45}'d_{AA}(d_{AA}+\cancel{a_{63}r_{22}}+a_{67}r_{33})-a_{45}a_{63}'(\cancel{a_{23}'}+a_{85}')r_{22}d_{AA}$$

$$+a_{23}'a_{45}'a_{85}']d_{AA}$$

$$=-[d_{AA}'''d_{AA}^5+(a_{23}'+a_{41}'+a_{45}')d_{AA}'''d_{AA}^3+a_{85}'(d_{AA}+a_{63}r_{22})d_{AA}^3$$

$$-a_{45}a_{63}'r_{22}d_{AA}(d_{AA}^2+a_{85}')+a_{23}'a_{45}'(d_{AA}+a_{67}r_{33})+a_{23}'a_{45}'a_{85}']d_{AA}$$

$$d_{AA}'''(\Delta_{A4}')_{4,3}+a_{85}r_{33}(\Delta_{A4}')_{4,4}=$$

$$[d_{AA}^4+(a_{23}'+a_{41}'+a_{45}')d_{AA}^2+a_{23}'a_{45}'][a_{85}'(d_{AA}+a_{63}r_{22}+\cancel{a_{67}r_{33}})-a_{85}r_{33}(d_{AA}^2+a_{63}'-\cancel{a_{67}'})]$$

$$=[d_{AA}^4+(a_{23}'+a_{41}'+a_{45}')d_{AA}^2+a_{23}'a_{45}']\,a_{85}\{d_{AA}+a_{63}(r_{22}-r_{33})]$$

$$=a_{85}\{d_{AA}^4[d_{AA}+a_{63}(r_{22}-r_{33})]+(a_{23}'+a_{41}'+a_{45}')d_{AA}^2[d_{AA}+a_{63}(r_{22}-r_{33})]$$

$$+a_{23}'a_{45}'[d_{AA}+a_{63}(r_{22}-r_{33})]\}$$

$$a_{45}r_{22}(\Delta_{A4}')_{4,2}=-a_{45}a_{63}'a_{85}'r_{22}(d_{AA}^2+a_{23}')$$

$$a_{45}r_{22}(\Delta_{A4}')_{4,2}+d_{AA}'''(\Delta_{A4}')_{4,3}+a_{85}r_{33}(\Delta_{A4}')_{4,4}=$$

$$a_{85}\{[d_{AA}+a_{63}(r_{22}-r_{33})]d_{AA}^4+(a_{23}'+a_{41}'+a_{45}')d_{AA}^2(d_{AA}-a_{63}r_{33})$$

$$+(a_{23}'+a_{41}')a_{63}r_{22}d_{AA}^2$$

If damping is removed from all three interfaces of the model:
$r_{11}\to 0$, $r_{22}\to 0$, $r_{33}\to 0$, $a'_{23}\to a_{23}$, $a'_{41}\to a_{41}$, $a'_{45}\to a_{45}$, $a'_{63}\to a_{63}$, $a'_{67}\to a_{67}$, $a'_{85}\to a_{85}$, $d'_{ii}\to d_{ii}$, $d''_{ii}\to d_{ii}$, $d'''_{ii}\to d_{ii}$, $d^{IV}_{ii}\to d_{ii}$, $\Delta'_{i4}\to\Delta_{i4}$ and $(\Delta'_{i4})_{ij}\to(\Delta_{i4})_{ij}$ where $(\Delta_{i4})_{ij}=\Delta_{i4}$ without the elements of the ith row and jth column and

$$\Delta_{i4}=d_{ii}^2\Big[d_{ii}^6+(a_{23}+a_{41}+a_{45}+a_{63}+a_{67}+a_{85})d_{ii}^4+(a_{23}a_{45}+a_{23}a_{63}+a_{23}a_{67}+a_{23}a_{85}+a_{41}a_{63}+a_{41}a_{67}+a_{41}a_{85}+a_{45}a_{67}+a_{45}a_{85}+a_{63}a_{85})d_{ii}^2+a_{23}a_{45}a_{67}+a_{23}a_{45}a_{85}+a_{23}a_{63}a_{85}+a_{41}a_{63}a_{85}\Big]$$

Then

$$t_{i1}=\frac{d_{ii}(\Delta_{i4})_{1,1}f_{i1}}{\Delta_{i4}}+\frac{a_{23}d_{ii}^2\left[d_{ii}^4+(a_{45}+a_{63}+a_{67}-a_{85})d_{ii}^2+a_{45}a_{67}+a_{45}a_{85}+a_{63}a_{85}\right]f_{i2}}{\Delta_{i4}}+\frac{a_{41}d_{ii}\left[d_{ii}^4+(a_{63}+a_{67}+a_{85})d_{ii}^2+a_{63}a_{85}\right](f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}-\frac{a_{41}a_{63}d_{ii}(d_{ii}^2+a_{85})(f_{i5}+d_{ii}f_{i6})+a_{41}a_{63}a_{85}d_{ii}(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}} \qquad (44)$$

$$t_{i2}=\frac{(\Delta_{i4})_{1,1}(f_{i1}+d_{ii}f_{i2})-a_{41}\left[d_{ii}^4+(a_{63}+a_{67}+a_{85})d_{ii}^2+a_{63}a_{85}\right](f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}-\frac{a_{41}a_{63}(d_{ii}^2+a_{85})(f_{i5}+d_{ii}f_{i6})+a_{41}a_{63}a_{85}(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}} \qquad (45)$$

$$t_{i3} = -\frac{d_{ii}(\Delta_{i4})_{1,2}(f_{i1}+d_{ii}f_{i2}) - d_{ii}(\Delta_{i4})_{2,2}f_{i3}}{\Delta_{i4}}$$

$$-\frac{d_{ii}^2\left[(a_{41}+a_{45})d_{ii}^4 + (a_{23}a_{45}+a_{41}a_{63}+a_{41}a_{67}+a_{41}a_{85}+a_{45}a_{67}+a_{45}a_{85})d_{ii}^2\right.}{\Delta_{i4}}$$

$$+\frac{\left.a_{23}a_{45}a_{67}+a_{23}a_{45}a_{85}+a_{41}a_{63}a_{85}\right]f_{i4}}{\Delta_{i4}}$$

$$-\frac{a_{63}d_{ii}(d_{ii}^2+a_{23})(d_{ii}^2+a_{85})(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}}$$

$$-\frac{a_{63}a_{85}d_{ii}(d_{ii}^2+a_{23})(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}} \qquad (46)$$

$$t_{i4} = -\frac{(\Delta_{i4})_{1,2}(f_{i1}+d_{ii}f_{i2}) - (\Delta_{i4})_{2,2}(f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$-\frac{a_{63}(d_{ii}^2+a_{23})(d_{ii}^2+a_{85})(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}}$$

$$-\frac{a_{63}a_{85}(d_{ii}^2+a_{23})(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}} \qquad (47)$$

$$t_{i5} = -\frac{a_{23}a_{45}d_{ii}(d_{ii}^2+a_{85})(f_{i1}+d_{ii}f_{i2})}{\Delta_{i4}}$$

$$+\frac{a_{45}d_{ii}(d_{ii}^2+a_{23})(d_{ii}^2+a_{85})(f_{i3}+d_{ii}f_{i4}) + d_{ii}(\Delta_{i4})_{3,3}f_{i5}}{\Delta_{i4}}$$

$$-\frac{d_{ii}^2\left[(a_{63}+a_{67})d_{ii}^4 + (a_{63}a_{85}+a_{23}a_{63}+a_{41}a_{63}+a_{23}a_{67}+a_{41}a_{67}+a_{45}a_{67})d_{ii}^2\right.}{\Delta_{i4}}$$

$$+\frac{\left.a_{23}a_{63}a_{85}+a_{41}a_{63}a_{85}+a_{23}a_{45}a_{67}\right]f_{i6} - d_{ii}(\Delta_{i4})_{4,3}(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}} \qquad (48)$$

$$t_{i6} = -\frac{a_{23}a_{45}(d_{ii}^2+a_{85})(f_{i1}+d_{ii}f_{i2}) + a_{45}(d_{ii}^2+a_{23})(d_{ii}^2+a_{85})(f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$+\frac{(\Delta_{i4})_{3,3}(f_{i5}+d_{ii}f_{i6}) - (\Delta_{i4})_{4,3}(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}} \qquad (49)$$

$$t_{i7} = -\frac{a_{23}a_{45}a_{67}d_{ii}(f_{i1}+d_{ii}f_{i2}) + a_{45}a_{67}d_{ii}(d_{ii}^2+a_{23})(f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$-\frac{a_{67}d_{ii}[d_{ii}^4+(a_{23}+a_{41}+a_{45})d_{ii}^2+a_{23}a_{45}](f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}}$$

$$-\frac{d_{ii}(d_{ii}^2+a_{63}+a_{67})[d_{ii}^4+(a_{23}+a_{41}+a_{45})d_{ii}^2+a_{23}a_{45}]f_{i7}}{\Delta_{i4}}$$

$$+\frac{a_{85}(d_{ii}^2+a_{63})[d_{ii}^4+(a_{23}+a_{41}+a_{45})d_{ii}^2+a_{23}a_{45}]f_{i8}}{\Delta_{i4}} \qquad (50)$$

$$t_{i8} = -\frac{a_{23}a_{45}a_{67}(f_{i1}+d_{ii}f_{i2}) + a_{45}a_{67}(d_{ii}^2+a_{23})(f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$-\frac{a_{67}[d_{ii}^4+(a_{23}+a_{41}+a_{45})d_{ii}^2+a_{23}a_{45}](f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}}$$

$$-\frac{(d_{ii}^2+a_{63}+a_{67})[d_{ii}^4+(a_{23}+a_{41}+a_{45})d_{ii}^2+a_{23}a_{45}](f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}} \qquad (51)$$

Comparison of equation (45) with equation (36), equation (47) with equation (37), equation (49) with equation (38), equation (50) with equation (41) and equation (51) with equation (39) reveals that the equations for generating t_{i2}, t_{i4}, t_{i6}, t_{i7} and t_{i8} retain the same form when damping is added to all three interfaces of the four body model if a'_{23}, a'_{41}, a'_{45}, a'_{63}, a'_{67} and a'_{85} are substituted for a_{23}, a_{41}, a_{45}, a_{63}, a_{67} and a_{85}, respectively. Comparison of equation (44) with equation (40), equation (46) with equation (42) and equation (48) with equation (43) reveals that the denominators of the equations for generating t_{i1}, t_{i3} and t_{i5} retain the same form when damping is added at all three interfaces of the four body model if the substitutions listed above are applied to these equations.

The numerator of the equation for generating t_{i1} for the four body model with damping at all three interfaces may be written in the same form as the numerator of the corresponding equation for the undamped system by substituting for d_{ii} in the coefficient of f_{i1} the following.

$$d^{v}_{ii} = d_{ii} + \frac{a_{23}(\Delta'_{i4})_{1,1} + a_{41}(\Delta'_{i4})_{1,2}\, r_{11}}{(\Delta'_{i4})_{1,1}} \qquad (52)$$

Then

$$t_{i1} = \frac{d_{ii}^{V} (\Delta'_{i4})_{1,1} f_{i1}}{\Delta'_{i4}}$$

$$+ \frac{a_{23} d_{ii}^2 [d_{ii}^4 + (a'_{45}+a'_{63}+a'_{67}+a'_{85}) d_{ii}^2 + a'_{45}a'_{67} + a'_{45}a'_{85} + a'_{63}a'_{85}] f_{i2}}{\Delta'_{i4}}$$

$$+ \frac{a_{41} d_{ii} [d_{ii}^4 + (a'_{63}+a'_{67}+a'_{85}) d_{ii}^2 + a'_{63}a'_{85}] (f_{i3} + d_{ii} f_{i4})}{\Delta'_{i4}}$$

$$- \frac{a_{41}a'_{63} d_{ii} (d_{ii}^2 + a'_{85})(f_{i5} + d_{ii} f_{i6}) + a_{41}a'_{63}a'_{85} d_{ii} (f_{i7} + d_{ii} f_{i8})}{\Delta'_{i4}} \qquad (53)$$

The <u>numerator</u> of the equation for generating t_{i3} for the four body model with damping at all three interfaces may be written in the same form as the <u>numerator</u> of the corresponding equation for the same model without any damping by means of the following substitutions: In the coefficients of f_{i1} and f_{i2}, let

$$d_{ii}^{VI} = d_{ii} + \frac{a_{23}(\Delta'_{i4})_{1,1} + a_{41}(\Delta'_{i4})_{1,2}}{(\Delta'_{i4})_{1,2}} r_{11} + \frac{a_{45}(\Delta'_{i4})_{1,2} + a_{63}(\Delta'_{i4})_{1,3}}{(\Delta'_{i4})_{1,2}} r_{22} \qquad (54)$$

In the coefficients of f_{i3} and f_{i4}, let

$$d_{ii}^{VII} = d_{ii} + \frac{a_{23}(\Delta'_{i4})_{2,1} + a_{41}(\Delta'_{i4})_{2,2}}{(\Delta'_{i4})_{2,2}} r_{11} + \frac{a_{45}(\Delta'_{i4})_{2,2} + a_{63}(\Delta'_{i4})_{2,3}}{(\Delta'_{i4})_{2,2}} r_{22} \qquad (55)$$

In the coefficients of f_{i5}, f_{i6}, f_{i7} and f_{i8} let <u>one</u> d_{ii} in the term of highest order in d_{ii} within the common coefficient be changed to the following form:

$$d_{ii}^{VIII} = d_{ii} + a_{41}(r_{11} - r_{22}) - a'_{23} r_{22} \qquad (56)$$

Then

$$t_{i3} = -\frac{d_{ii}^{VI}(\Delta'_{i4})_{1,2}(f_{i1} + d_{ii} f_{i2}) - d_{ii}^{VII}(\Delta'_{i4})_{2,2} f_{i3}}{\Delta'_{i4}}$$

$$- \frac{d_{ii}^2[(a_{41} + a_{45})d_{ii}^4 + (a'_{23}a_{45} + a_{41}a'_{63} + a_{41}a'_{67} + a_{41}a'_{85} + a_{45}a'_{67} + a_{45}a'_{85})d_{ii}^2}{\Delta'_{i4}}$$

$$+ \frac{a'_{23}a_{45}a'_{67} + a'_{23}a_{45}a'_{85} + a_{41}a'_{63}a'_{85}] f_{i4}}{\Delta'_{i4}}$$

$$- \frac{d_{ii}(d_{ii}^2 + a'_{85})(a_{63} d_{ii} d_{ii}^{VIII} + a'_{63}a'_{23})(f_{i5} + d_{ii} f_{i6})}{\Delta'_{i4}}$$

$$- \frac{a'_{85} d_{ii}(a_{63} d_{ii} d_{ii}^{VIII} + a'_{63}a'_{23})(f_{i7} + d_{ii} f_{i8})}{\Delta'_{i4}} \qquad (57)$$

The <u>numerator</u> of the equation for generating t_{i5} for the four body model with damping at all three interfaces may be written in the same form as the <u>numerator</u> of the corresponding equation for the four body model without any damping with the following substitutions.

In the coefficients of f_{i1}, f_{i2}, f_{i3} and f_{i4} let one of the d_{ii}'s in the term of highest order in d_{ii} within the common coefficient be changed to the following form.

$$d_{ii}^{IX} = d_{ii} + a_{67}(r_{33} - r_{22}) \tag{58}$$

In the coefficient of f_{i5} let d_{ii}''' be changed to:

$$d_{ii}^{X} = d_{ii} + \frac{a_{45}(\Delta_{i4}')_{3,2} + a_{63}(\Delta_{i4}')_{3,3}}{(\Delta_{i4}')_{3,3}} r_{22} + \frac{a_{67}(\Delta_{i4}')_{3,3} + a_{85}(\Delta_{i4}')_{3,4}}{(\Delta_{i4}')_{3,3}} r_{33} \tag{59}$$

In the coefficient of f_{i7} let d_{ii}''' be changed to:

$$d_{ii}^{XI} = d_{ii} + \frac{a_{45}(\Delta_{i4}')_{4,2} + a_{63}(\Delta_{i4}')_{4,3}}{(\Delta_{i4}')_{4,3}} r_{22} + \frac{a_{67}(\Delta_{i4}')_{4,3} + a_{85}(\Delta_{i4}')_{4,4}}{(\Delta_{i4}')_{4,3}} r_{33} \tag{60}$$

Then

$$
\begin{aligned}
t_{i5} = &- \frac{a_{23}' a_{45} d_{ii}(d_{ii} d_{ii}^{IX} + a_{85}')(f_{i1} + d_{ii} f_{i2})}{\Delta_{i4}'} \\
&- \frac{a_{45} d_{ii}(d_{ii}^2 + a_{23}')(d_{ii} d_{ii}^{IX} + a_{85}')(f_{i3} + d_{ii} f_{i4})}{\Delta_{i4}'} \\
&+ \frac{d_{ii}^{X}(\Delta_{i4}')_{3,3} f_{i5} - d_{ii}^2 [(a_{63} + a_{67}) d_{ii}^4}{\Delta_{i4}'} \\
&+ \frac{(a_{63} a_{85}' + a_{23}' a_{63} + a_{41}' a_{63} + a_{23}' a_{67} + a_{41}' a_{67} + a_{45}' a_{67}) d_{ii}^2}{\Delta_{i4}'} \\
&+ \frac{a_{23}' a_{63} a_{85}' + a_{41}' a_{63} a_{85}' + a_{23}' a_{45}' a_{67}] f_{i6} - d_{ii}^{XI}(\Delta_{i4}')_{4,3}(f_{i7} + d_{ii} f_{i8})}{\Delta_{i4}'}
\end{aligned}
\tag{61}
$$

The elements, t_{i1} through t_{i8} $(i = 1, \cdots, p)$, of the T matrix of the linear observer of order p corresponding to the four body single axis model with one or more inaccessible states were found to be affected by by the addition of damping only at the interface between bodies 1 and 2 as follows.

1. The scalars, a_{23} and a_{41}, were modified to a'_{23} and a'_{41}, respectively, in the equations for generating t_{i2}, t_{i4}, t_{i5}, t_{i6}, t_{i7} and t_{i8}, the common denominators, Δ'_{i4}, of the equations for generating t_{i1} and t_{i3} and $(\Delta'_{i4})_{i,j}$ which is Δ'_{i4} without the elements of the ith row and jth column [a'_{23} and a'_{41} are defined in equations (28) and (29).]

2. In the <u>numerator</u> of the equation for generating t_{i1}, (53), the scalar, d_{ii}, common to all terms in the coefficient of f_{i1} was modified to d_{ii}^{V} which is defined in equation (52).

3. In the <u>numerator</u> of the equation for generating t_{i3}, (57), the following changes occurred.

   a. The scalar, d_{ii}, common to all terms in the coefficients of f_{i1} and f_{i2} was modified to:

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$$d_{ii}^{VI} = d_{ii} + \frac{a_{23}(\Delta'_{i4})_{1,1} + a_{41}(\Delta'_{i4})_{1,2}}{(\Delta'_{i4})_{1,2}} r_{11} \qquad (62)$$

b. The scalar, d_{ii}, common to all of the terms in the coefficient of f_{i3} was modified to:

$$d_{ii}^{VII} = d_{ii} + \frac{a_{23}(\Delta'_{i4})_{2,1} + a_{41}(\Delta'_{i4})_{2,2}}{(\Delta'_{i4})_{2,2}} r_{11} \qquad (63)$$

c. One d_{ii} in the term of highest order in d_{ii} in the coefficients of f_{i5}, f_{i6}, f_{i7} and f_{i8} was modified to:

$$d_{ii}^{VIII} = d_{ii} + a_{41} r_{11} \qquad (64)$$

d. The scalar, a_{23}, was changed to a'_{23} which is defined in equation (28)

Addition of damping only to the interface between body 2 and body 3 had the following effects.

1. The scalars, a_{45} and a_{63}, were modified to a'_{45} and a'_{63}, respectively, in the equations for generating t_{i1}, t_{i2}, t_{i4}, t_{i6}, t_{i7} and t_{i8}, the common denominator, Δ'_{i4}, of the equations for generating t_{i3} and t_{i5} and $(\Delta'_{i4})_{i,j}$ which is Δ'_{i4} without the elements of the ith row and jth column. [a'_{45} and a'_{63} are defined in equations (30) and (31).]

2. In the numerator of the equation for generating t_{i3}, (57), the following changes occurred.

a. The scalar, d_{ii}, common to all terms in the coefficients of f_{i1} and f_{i2} was modified to:

$$d_{ii}^{VI} = d_{ii} + \frac{a_{45}(\Delta_{i4}^{!})_{1,2} + a_{63}(\Delta_{i4}^{!})_{1,3}}{(\Delta_{i4}^{!})_{1,2}} r_{22} \tag{65}$$

b. The scalar, d_{ii}, common to all terms in the coefficient of f_{i3} was modified to:

$$d_{ii}^{VII} = d_{ii} + \frac{a_{45}(\Delta_{i4}^{!})_{2,2} + a_{63}(\Delta_{i4}^{!})_{2,3}}{(\Delta_{i4}^{!})_{2,2}} r_{22} \tag{66}$$

c. One d_{ii} in the term of highest order in d_{ii} in the coefficients of f_{i5}, f_{i6}, f_{i7} and f_{i8} was modified to:

$$d_{ii}^{VIII} = d_{ii} - (a_{23}' + a_{41}) r_{22} \tag{67}$$

d. The scalar, a_{63}, was changed to a_{63}' except where it is associated with the term of highest order in d_{ii} in the coefficients of f_{i5}, f_{i6}, f_{i7} and f_{i8}

3. In the numerator of the equation for generating t_{i5}, (61), the following changes occurred.

a. One d_{ii} in the term of highest order in d_{ii} in the coefficients of f_{i1}, f_{i2}, f_{i3} and f_{i4} was modified to:

$$d_{ii}^{IX} = d_{ii} - a_{67} r_{22} \tag{68}$$

b. The scalar, d_{ii}, common to all terms in the coefficient of f_{i5} was modified to:

$$d_{ii}^{X} = d_{ii} + \frac{a_{67}(\Delta_{i4}')_{3,3} + a_{85}(\Delta_{i4}')_{3,4}}{(\Delta_{i4}')_{3,3}} r_{22} \tag{69}$$

c. The scalar, d_{ii}, common to all terms in the coefficients of f_{i7} and f_{i8} was modified to:

$$d_{ii}^{XI} = d_{ii} + \frac{a_{67}(\Delta_{i4}')_{4,3} + a_{85}(\Delta_{i4}')_{4,4}}{(\Delta_{i4}')_{4,3}} r_{22} \tag{70}$$

d The scalar, a_{45}, was modified to a_{45}' only in the coefficient of f_{i6}.

Addition of damping only to the interface between body 3 and body 4 had the following effects.

1. The scalars, a_{67} and a_{85}, were modified to a'_{67} and a'_{85}, respectively, in the equations for generating $t_{i1}, t_{i2}, t_{i3}, t_{i4}, t_{i6}$ and t_{i8}, the common denominator, Δ'_{i4}, of the equations for generating t_{i5} and t_{i7} and $(\Delta'_{i4})_{i,j}$, which is Δ'_{i4} without the elements of the ith row and jth column. [a'_{67} and a'_{85} are defined in equations (32) and (33).]

2. In the <u>numerator</u> of the equation for generating t_{i5}, (61), the following changes occurred.

   a. One scalar d_{ii} in the term of highest order in d_{ii} in the coefficients of f_{i1}, f_{i2}, f_{i3} and f_{i4} was modified to:

$$d_{ii}^{\prime x} = d_{ii} + a_{67} r_{33} \tag{71}$$

   b. The scalar d_{ii} common to all terms in the coefficient of f_{i5} was modified to:

$$d_{ii}^{x} = d_{ii} + \frac{a_{67}(\Delta'_{i4})_{3,3} + a_{85}(\Delta'_{i4})_{3,4}}{(\Delta'_{i4})_{3,3}} r_{33} \tag{72}$$

c. The scalar d_{ii} common to all terms in the coefficients of f_{i7} and f_{i8} was modified to:

$$d_{ii}^{XI} = d_{ii} + \frac{a_{67}(\Delta'_{i4})_{4,3} + a_{85}(\Delta'_{i4})_{4,4}}{(\Delta'_{i4})_{4,3}} r_{23} \qquad (73)$$

d. The scalar, a_{85}, was modified to a'_{85}.

3. In the numerator of the equation for generating t_{i7}, (41), the following changes occurred.

a. One scalar d_{ii} in the term of highest order in d_{ii} in the coefficient of f_{i7} was modified to d_{ii}^{IV} which is defined in equation (18).

b. The scalar, a_{67}, was modified to a'_{67} only in the coefficient of f_{i7} with a'_{67} defined in equation (32).

Addition of damping to both the interface between body 1 and body 2 and the interface between body 2 and body 3 had the following effects.

1. The scalars, a_{23}, a_{41}, a_{45} and a_{63} were modified to a'_{23}, a'_{41}, a'_{45} and a'_{63}, respectively, in the equations for generating t_{i2}, t_{i4}, t_{i6}, t_{i7} and t_{i8}, the common denominator, Δ'_{i4}, of the equations for generating t_{i1}, t_{i3} and t_{i5} and $(\Delta'_{i4})_{ij}$ which is Δ'_{i4} without the elements of the ith row and jth column.

[a'_{23}, a'_{41}, a'_{45} and a'_{63} are defined in equations (28) through (31).]

2. In the <u>numerator</u> of the equation for generating t_{i1}, (53), the scalar, d_{ii}, common to f_{i1} was modified to d_{ii}^{V} which is defined in equation (52).

3. In the <u>numerator</u> of the equation for generating t_{i3}, (57), the following changes occurred.

   a. The scalar, d_{ii}, common to all terms in the coefficients of f_{i1} and f_{i2} was modified to d_{ii}^{VI} which is defined in equation (54).

   b. The scalar, d_{ii}, common to all terms in the coefficient of f_{i3} was modified to d_{ii}^{VII} which is defined in equation (55).

   c. One d_{ii} in the term of highest order in d_{ii} in the coefficients of f_{i5}, f_{i6}, f_{i7} and f_{i8} was modified to d_{ii}^{VIII} which is defined in equation (56).

   d. The scalar, a_{23}, was changed to a'_{23}.

   e. The scalar, a_{63}, was changed to a'_{63} except where it multiplies the term of highest order in d_{ii} in the coefficients of f_{i5}, f_{i6}, f_{i7} and f_{i8}

4. In the numerator of the equation for generating t_{i5}, (61), the following changes occurred.

   a. One scalar d_{ii} in the term of highest order in d_{ii} in the coefficients of f_{i1}, f_{i2}, f_{i3} and f_{i4} was modified to d_{ii}^{IX} as defined in equation (68)

   b. The scalar, d_{ii}, common to all terms in the coefficient of f_{i5} was modified to d_{ii}^{X} as defined in equation (69)

   c. The scalar, d_{ii}, common to all terms in the coefficients of f_{i7} and f_{i8} was modified to d_{ii}^{XI} as defined in equation (70)

   d The scalar, a_{45}, was changed to a_{45}' only in the coefficient of f_{i6}.

Addition of damping to both the interface between body 2 and body 3 and the interface between body 2 and body 3 had the following effects.

1. The scalars, a_{45}, a_{63}, a_{67} and a_{85}, were modified to a_{45}', a_{63}', a_{67}' and a_{85}', respectively in the equations for generating t_{i1}, t_{i2}, t_{i4}, t_{i6} and t_{i8}, the common denominator, Δ_{i4}', of the equations for generating t_{i3}, t_{i5} and t_{i7} and $(\Delta_{i4}')_{ij}$ which is Δ_{i4}' without the elements of the ith row and jth column

[a'_{45}, a'_{62}, a'_{63} and a'_{85} are defined in equations (30) through (33).]

2. In the numerator of the equation for generating t_{i3}, (57), the following changes occurred.

   a. The scalar, d_{ii}, common to all terms in the coefficients of f_{i1} and f_{i2} was modified to d_{ii}^{VI} of the form defined in equation (65)

   b. The scalar, d_{ii}, common to all terms in the coefficient of f_{i3} was modified to d_{ii}^{VII} of the form given in equation (66)

   c. One d_{ii} in the term of highest order in d_{ii} in the coefficients of f_{i5}, f_{i6}, f_{i7} and f_{i8} was modified to d_{ii}^{VIII} of the form given by equation (67)

   d. The scalar, a_{63}, was changed to a'_{63} except where it multiplies the term of highest order in d_{ii} in the coefficients of f_{i5}, f_{i6}, f_{i7} and f_{i8}.

3. In the numerator of the equation for generating t_{i5}, (61), the following changes occurred.

   a. One scalar d_{ii} in the term of highest order in d_{ii} in coefficients of f_{i1}, f_{i2}, f_{i3} and f_{i4} was modified to d_{ii}^{IX} of the form defined in equation (58)

b. The scalar, d_{ii}, common to all terms in the coefficient of f_{i5} was modified to d_{ii}^{X} in the form given in equation (59).

c. The scalar, d_{ii}, common to all terms in the coefficients of f_{i7} and f_{i8} was modified to d_{ii}^{XI} of the form given in equation (60).

d. The scalar, a_{45}, was modified to a_{45}' only in the coefficient of f_{i6}.

e. The scalar, a_{85}, was modified to a_{85}' which is defined in equation (33).

4. In the <u>numerator</u> of the equation for generating t_{i7}, (41), the following changes occurred

a. One scalar d_{ii} in the term of highest order in d_{ii} in the coefficient of f_{i7} was modified to d_{ii}^{IV} which is defined in equation (18).

b. The scalars, a_{75} and a_{63} were modified to a_{75}' and a_{63}', respectively, which are defined in equations (30) and (31).

c. The scalar, a_{67}, was modified to a_{67}' only in the coefficient of f_{i7}.

Addition of damping to <u>both</u> the interface between body 1 and body 2 and the interface between body 3 and body 4 had the following effects.

1. The scalars, a_{23}, a_{41}, a_{67} and a_{85}, were modified to a'_{23}, a'_{41}, a'_{67} and a'_{85}, respectively, in the equations for generating t_{i2}, t_{i4}, t_{i6} and t_{i8}, the common denominator, Δ'_{i4}, of t_{i1}, t_{i3}, t_{i5} and t_{i7} and $(\Delta'_{i4})_{i,j}$ which is Δ'_{i4} without the elements of the ith row and jth column. [a'_{23}, a'_{41}, a'_{67} and a'_{85} are defined in equations (28), (29), (32) and (33).]

2. In the <u>numerator</u> of the equation for generating t_{i1}, (53), the following changes occurred

   a. The scalar d_{ii} common to all terms in the coefficient of f_{i1} was modified to d_{ii}^{v} which is defined in equation (52).

   b. The scalars, a_{67} and a_{85}, were modified to a'_{67} and a'_{85}, respectively, where a'_{67} and a'_{85} are defined in equations (32) and (33).

3. In the numerator of the equation for generating t_{i3}, (57), the following changes occurred.

a. The scalar $d_{\hat{i}\hat{a}}$ common to all terms in the coefficients of $f_{\hat{i}1}$ and $f_{\hat{i}2}$ was modified to $d_{\hat{i}\hat{a}}^{VI}$ as defined in equation (62).

b. The scalar $d_{\hat{a}\hat{a}}$ common to all terms in the coefficient of $f_{\hat{i}3}$ was modified to the form of $d_{\hat{a}\hat{a}}^{VII}$ given in equation (63).

c. One $d_{\hat{i}\hat{a}}$ in the term of highest order in $d_{\hat{i}\hat{a}}$ in the coefficients of $f_{\hat{i}5}$, $f_{\hat{i}6}$, $f_{\hat{i}7}$ and $f_{\hat{i}8}$ was modified to the form of $f_{\hat{a}\hat{a}}^{VIII}$ given by equation (64).

d. The scalars, a_{23}, a_{67} and a_{85}, were modified to a_{23}', a_{67}' and a_{85}', respectively.

4. In the <u>numerator</u> of the equation for generating $t_{\hat{i}5}$, (61), the following changes occurred

a. One scalar $d_{\hat{i}\hat{a}}$ in the term of highest order in $d_{\hat{i}\hat{a}}$ in the coefficients of $f_{\hat{i}1}$, $f_{\hat{i}2}$, $f_{\hat{i}3}$ and $f_{\hat{i}4}$ was modified to the form of $d_{\hat{a}\hat{a}}^{IX}$ given in equation (71).

b. The scalar $d_{\hat{i}\hat{a}}$ common to all terms in the coefficient of $f_{\hat{i}5}$ was modified to $d_{\hat{a}\hat{a}}^{X}$ as given by equation (72).

c. The scalar d_{ii} common to all terms in the coefficients of f_{i7} and f_{i8} was modified to the form of d_{ii}^{XI} given by equation (73).

d. The scalars, a_{23}, a_{41}, and a_{85}, were modified to a'_{23}, a'_{41}, and a'_{85}, respectively.

5. In the <u>numerator</u> of the equation for generating ϵ_{i7}, (41), the following changes occurred.

a. One scalar d_{ii} in the term of highest order in d_{ii} in the coefficient of f_{i7} was modified to d_{ii}^{VI} which is defined in equation (18).

b. The scalars, a_{23}, a_{41}, and a_{85}, were modified to the scalars, a'_{23}, a'_{41} and a'_{85}, respectively.

c. The scalar, a_{67}, was modified to a'_{67} only in the coefficient of f_{i7}.

Addition of damping to all three interfaces of the four body single axis model had the following effects.

1. The scalars, a_{23}, a_{41}, a_{45}, a_{63}, a_{67} and a_{85}, were modified to a'_{23}, a'_{41}, a'_{45}, a'_{63}, a'_{67} and a'_{85}, respectively, in the equations for generating t_{i2}, t_{i4}, t_{i6} and t_{i8}, the common denominator, Δ'_{i4}, of the equations for generating t_{i1}, t_{i3}, t_{i5} and t_{i7} and $(\Delta'_{i4})_{ij}$ which is Δ'_{i4} without the elements of the ith row and jth column. The modified scalars, a'_{23}, a'_{41}, a'_{45}, a'_{63}, a'_{67} and a'_{85} are defined in equations (28) through (33).

2. In the numerator of the equation for generating t_{i1}, (53), the following changes occurred.

   a. The scalar d_{ii} common to all terms in the coefficient of f_{i1} was modified to d_{ii}^{V} which is defined in equation (52)

   b. The scalars, a_{45}, a_{63}, a_{67} and a_{85} were modified to the scalars a'_{45}, a'_{63}, a'_{67} and a'_{85}, respectively.

3. In the numerator of the equation for generating t_{i3}, (57), the following changes occurred.

   a. The scalar d_{ii} common to all terms in the coefficients of f_{i1} and f_{i2} was modifed to the form of d_{ii}^{VI} given by equation (54).

b. The scalar d_{ii} common to all terms in the coefficient of f_{i3} was modified to the form of d_{ii}^{VII} given in equation (55).

c. One d_{ii} in the term of highest order in d_{ii} in the coefficients of f_{i5}, f_{i6}, f_{i7} and f_{i8} was modified to the form of d_{ii}^{VIII} given in equation (56)

d. The scalars, a_{23}, a_{67} and a_{85}, were modified to a_{23}', a_{67}' and a_{85}' respectively.

e. The scalar, a_{63}, was modified to a_{63}' except where it is mulitplying the terms of highest order in d_{ii} in the coefficeonts of f_{i5}, f_{i6}, f_{i7} and f_{i8}

4. In the numerator of the equation for generating t_{i5}, (61), the following changes occurred.

a. One d_{ii} in the term of highest order in d_{ii} in the coefficients of f_{i1}, f_{i2}, f_{i3} and f_{i4} was modified to the form of d_{ii}^{IX} given in equation (58).

b. The scalar d_{ii} common to all terms in the coefficient of f_{i5} was modified to the form of d_{ii}^{X} given by equation (59).

c. The scaler d_{ii} common to all terms in the coefficients of f_{i7} and f_{i8} was modified to the form of d_{ii}^{XI} defined in equation (60).

d. The scalars, a_{23}, a_{41} and a_{85}, were modified to a'_{23}, a'_{41} and a'_{85}, respectively.

e. The scalar, a_{45}, was modified to a'_{45} only in the coefficient of f_{i6}.

5. In the numerator of the equation for generating t_{i7}, (41), the following changes occurred.

a. One scalar d_{ii} in the term of highest order in d_{ii} in the coefficient of f_{i7} was modified to d_{ii}^{IV} which is defined in equation (18).

b. The scalars, a_{23}, a_{41}, a_{45} and a_{63} were modified to a'_{23}, a'_{41}, a'_{45} and a'_{63}, respectively.

c. The scalar, a_{67}, was modified to a'_{67} only in the coefficient of f_{i7}

FIRST ORDER OBSERVERS ($p=1$)

An observer of order at least one is required when only one of the eight scaler variables of the four body model is inaccessible. Therefore, the total number of first order observers that can be generated for the four body model is given by:

$$C_1^8 = 8$$

The first order form of the linear observer equation is as follows.

$$\dot{z} = d z + E\underline{u} + G\underline{y} \qquad (74)$$

The F and T matrices associated with a first order observer for the four body model then reduce to the following row forms.

$$F = [\, f_1 \;\; f_2 \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; f_8 \,] \qquad (75)$$

$$T = [\, t_1 \;\; t_2 \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; t_8 \,] \qquad (76)$$

The first order observer synthesis equations for the four body single axis model with damping at all three interfaces are then of the same form as equations (35) through (39), (41) and (53) through (61) with $i=1$ and one $f_i = 0$ $(i = 1, 2, \cdots, 8)$. The corresponding equations for the four body model are obtained from equations (44) through (51) with $i=1$ and one $f_i = 0$ $(i = 1, 2, \cdots, 8)$.

<u>Example:</u> Generation of First Order Observer For the Four Body Model with Damping at all Interfaces

Suppose the scalar state representing the angular rate of body 4, x_8, is inaccessible. Then $f_8 = 0$ and the observer synthesis equations reduce to the following forms

$$\epsilon_1 = \frac{d^v(\Delta_4')_{1,1} f_1}{\Delta_4'}$$

$$+ \frac{a_{23} d^2[d^4 + (a_{45}' + a_{63}' + a_{67}' + a_{85}')d^2 + a_{45}'a_{67}' + a_{45}'a_{85}' + a_{63}'a_{85}']f_2}{\Delta_4'}$$

$$+ \frac{a_{41} d[d^4 + (a_{63}' + a_{67}' + a_{85}')d^2 + a_{63}'a_{85}'](f_3 + df_4)}{\Delta_4'}$$

$$- \frac{a_{41} a_{63}' d(d^2 + a_{85}')(f_5 + df_6) + a_{41} a_{63}' a_{85}' df_7}{\Delta_4'} \qquad (77)$$

$$\epsilon_2 = \frac{(\Delta_4')_{1,1}(f_1 + df_2) - a_{41}'[d^4 + (a_{63}' + a_{67}' + a_{85}')d^2 + a_{63}'a_{85}'](f_3 + df_4)}{\Delta_4'}$$

$$- \frac{a_{41}'a_{63}'(d^2 + a_{85}')(f_5 + df_6) + a_{41}'a_{63}'a_{85}' f_7}{\Delta_4'} \qquad (78)$$

where:

$$\Delta_4' = d^2[d^6 + (a_{23}' + a_{41}' + a_{45}' + a_{63}' + a_{67}' + a_{85}')d^4 + (a_{23}'a_{45}' + a_{23}'a_{63}' + a_{23}'a_{67}'$$
$$+ a_{23}'a_{85}' + a_{41}'a_{63}' + a_{41}'a_{67}' + a_{41}'a_{85}' + a_{45}'a_{67}' + a_{45}'a_{85}' + a_{63}'a_{85}')d^2$$
$$+ a_{23}'a_{45}'a_{67}' + a_{23}'a_{45}'a_{85}' + a_{23}'a_{63}'a_{85}' + a_{41}'a_{63}'a_{85}'] \qquad (79)$$

$$d^v = d + \frac{a_{23}(\Delta_4')_{1,1} + a_{41}(\Delta_4')_{1,2}}{(\Delta_4')_{1,1}} r_{11} \qquad (80)$$

$$t_3 = -\frac{d^{VI}(\Delta_4')_{1,2}(f_1 + d f_2) - d^{VII}(\Delta_4')_{22} f_3}{\Delta_4'}$$

$$-\frac{d^2[(a_{41}+a_{45})d^4 + (a_{23}'a_{45} + a_{41}a_{63}' + a_{41}a_{67}' + a_{41}a_{85}' + a_{45}a_{67}' + a_{45}a_{85}')d^2}{\Delta_4'}$$

$$+\frac{a_{23}'a_{45}a_{67}' + a_{23}'a_{45}a_{85}' + a_{41}a_{63}'a_{85}']f_4}{\Delta_4'}$$

$$-\frac{d(d^2 + a_{85}')(a_{63} d d^{VIII} + a_{63}'a_{23}')(f_5 + d f_6)}{\Delta_4'}$$

$$-\frac{a_{85}' d(a_{63} d d^{VIII} + a_{63}'a_{23}')f_7}{\Delta_4'} \qquad (81)$$

$$t_4 = -\frac{(\Delta_4')_{1,2}(f_1 + d f_2) - (\Delta_4')_{2,2}(f_3 + d f_-)}{\Delta_4'}$$

$$-\frac{a_{63}'(d^2 + a_{23}')(d^2 + a_{85}')(f_5 + d f_6) - a_{63}a_{85}(d^2 + a_{23}')f_7}{\Delta_4'} \qquad (82)$$

$$d^{VI} = d + \frac{a_{23}(\Delta_4')_{1,1} + a_{41}(\Delta_4')_{1,2}}{(\Delta_4')_{1,2}} r_{11} + \frac{a_{45}(\Delta_4')_{1,2} + a_{63}(\Delta_4')_{1,3}}{(\Delta_4')_{1,2}} r_{22} \qquad (83)$$

$$d^{VII} = d + \frac{a_{23}(\Delta_4')_{2,1} + a_{41}(\Delta_4')_{2,2}}{(\Delta_4')_{2,2}} r_{11} + \frac{a_{45}(\Delta_4')_{2,2} + a_{63}(\Delta_4')_{2,3}}{(\Delta_4')_{2,2}} r_{22} \qquad (84)$$

$$d^{VIII} = d + a_{41}(r_{11} - r_{22}) - a_{23}' r_{22} \qquad (85)$$

$$t_5 = -\frac{a'_{23} a_{45} d (d d^{1x} + a'_{85})(f_1 + d f_2)}{\Delta'_4}$$

$$- \frac{a_{45} d (d^2 + a'_{23})(d d^{1x} + a'_{85})(f_3 + d f_4) - d^{x} (\Delta'_4)_{3,3} f_5}{\Delta'_4}$$

$$- \frac{d^2 [(a_{63} + a_{67}) d^4 + (a_{63} a'_{85} + a'_{23} a_{67} + a'_{41} a_{63} + a'_{23} a_{67} + a'_{41} a_{67} + a'_{45} a_{67}) d^2}{\Delta'_4}$$

$$+ \frac{a'_{23} a_{63} a'_{85} + a'_{41} a'_{63} a'_{85} + a'_{23} a'_{45} a_{67}] f_6 - d^{x1} (\Delta'_4)_{4,3} f_7}{\Delta'_4} \qquad (86)$$

$$t_6 = -\frac{a'_{23} a'_{45} (d^2 + a'_{85})(f_1 + d f_2) + a'_{45} (d^2 + a'_{23})(d^2 + a'_{85})(f_3 + d f_4)}{\Delta'_4}$$

$$+ \frac{(\Delta'_4)_{3,3} (f_5 + d f_6) - (\Delta'_4)_{4,3} f_7}{\Delta'_4} \qquad (87)$$

$$d^{1x} = d + a_{67} (r_{33} - r_{22}) \qquad (88)$$

$$d^{x} = d + \frac{a_{45} (\Delta'_4)_{3,2} + a_{63} (\Delta'_4)_{3,3}}{(\Delta'_4)_{3,3}} r_{22} + \frac{a_{67} (\Delta'_4)_{3,3} + a_{85} (\Delta'_4)_{3,4}}{(\Delta'_4)_{3,3}} r_{33} \qquad (89)$$

$$d^{x1} = d + \frac{a_{45} (\Delta'_4)_{4,2} + a_{63} (\Delta'_4)_{4,3}}{(\Delta'_4)_{4,3}} r_{22} + \frac{a_{67} (\Delta'_4)_{4,3} + a_{85} (\Delta'_4)_{4,4}}{(\Delta'_4)_{4,3}} r_{33} \qquad (90)$$

$$\epsilon_7 = -\frac{a'_{23}a'_{45}a_{67}\,d(f_1+df_2) + a'_{45}a_{67}(d^2+a'_{23})d(f_3+df_4)}{\Delta'_4}$$

$$-\frac{[d^4+(a'_{23}+a'_{41}+a'_{45})d^2+a'_{23}a'_{45}]a_{67}d(f_5+df_6)}{\Delta'_4}$$

$$-\frac{[d^{IV}(d^2+a'_{63})+a'_{67}d][d^4+(a'_{23}+a'_{41}+a'_{45})d^2+a'_{23}a'_{45}]f_7}{\Delta'_4} \qquad (91)$$

$$\epsilon_8 = -\frac{a'_{23}a'_{45}a'_{67}(f_1+df_2) + a'_{45}a'_{67}(d^2+a'_{23})(f_3+df_4)}{\Delta'_4}$$

$$-\frac{a_{67}[d^4+(a'_{23}+a'_{41}+a'_{45})d^2+a'_{23}a'_{45}](f_5+df_6)}{\Delta'_4}$$

$$-\frac{(d^2+a_{63}+a_{67})[d^4+(a'_{23}+a'_{41}+a'_{45})d^2+a'_{23}a'_{45}]f_7}{\Delta'_4} \qquad (92)$$

$$d^{IV} = d + a_{85}r_{33} \qquad (93)$$

$(\Delta'_4)_{i,j}$ = Δ'_4 without the elements of the ith row and jth column

$$\begin{aligned} a'_{23} &= a_{23}(1+r_{11}d) \\ a'_{41} &= a_{41}(1+r_{11}d) \\ a'_{45} &= a_{45}(1+r_{22}d) \\ a'_{63} &= a_{63}(1+r_{22}d) \\ a'_{67} &= a_{67}(1+r_{33}d) \\ a'_{85} &= a_{85}(1+r_{33}d) \end{aligned} \qquad (94)$$

If all damping is removed from the model, $r_{11}\to 0$, $r_{22}\to 0$, $r_{33}\to 0$, $a'_{23}\to a_{23}$, $a'_{41}\to a_{41}$; $a'_{45}\to a_{45}$, $a'_{63}\to a_{63}$, $a'_{67}\to a_{67}$, $a'_{85}\to a_{85}$, $d'\to d$, $d''\to d$, $d'''\to d$, $d^{IV}\to d$, $d^{V}\to d$, $d^{VI}\to d$, $d^{VII}\to d$, $d^{IX}\to d$, $d^{X}\to d$, $d^{XI}\to d$, $\Delta'_4\to\Delta_4$ and $(\Delta'_4)_{i,j}\to(\Delta_4)_{i,j}$ where $(\Delta_4)_{i,j}$ = Δ_4 without the elements of the ith row and jth column and Δ_4 =

$$
\begin{aligned}
&d^2[d^6+(a_{23}+a_{41}+a_{45}+a_{63}+a_{67}+a_{85})d^4+(a_{23}a_{45}+a_{23}a_{63}+a_{23}a_{67}\\
&\quad+a_{23}a_{85}+a_{41}a_{63}+a_{41}a_{67}+a_{41}a_{85}+a_{45}a_{67}+a_{45}a_{85}+a_{63}a_{85})d^2\\
&\quad+a_{23}a_{45}a_{67}+a_{23}a_{45}a_{85}+a_{23}a_{63}a_{85}+a_{41}a_{63}a_{85}]
\end{aligned}
\tag{95}
$$

Then

$$
\begin{aligned}
t_1 &= \frac{d(\Delta_4)_{1,1}f_1}{\Delta_4}\\
&+\frac{a_{23}d^2[d^4+(a_{45}+a_{63}+a_{67}+a_{85})d^2+a_{45}a_{67}+a_{45}a_{85}+a_{63}a_{85}]f_2}{\Delta_4}\\
&+\frac{a_{41}d[d^4+(a_{63}+a_{67}+a_{85})d^2+a_{63}a_{85}](f_3+df_4)}{\Delta_4}\\
&-\frac{a_{41}a_{63}d(d^2+a_{85})(f_5+df_6)+a_{41}a_{63}a_{85}df_7}{\Delta_4}
\end{aligned}
\tag{96}
$$

$$
\begin{aligned}
t_2 &= \frac{(\Delta_4)_{1,1}(f_1+df_2)-a_{41}[d^4+(a_{63}+a_{67}+a_{85})d^2+a_{63}a_{85}](f_3+df_4)}{\Delta_4}\\
&-\frac{a_{41}a_{63}(d^2+a_{85})(f_5+df_6)+a_{41}a_{63}a_{85}f_7}{\Delta_4}
\end{aligned}
\tag{97}
$$

$$t_3 = -\frac{d(\Delta_4)_{1,2}(f_1+df_2) - d(\Delta_4)_{2,2} f_3}{\Delta_4}$$

$$-\frac{d^2\left[(a_{41}+a_{45})d^4 + (a_{23}a_{45} + a_{41}a_{63} + a_{41}a_{67} + a_{41}a_{85} + a_{45}a_{67} + a_{45}a_{85})d^2\right.}{\Delta_4}$$

$$+\frac{\left. a_{23}a_{45}a_{67} + a_{23}a_{45}a_{85} + a_{41}a_{63}a_{85}\right] f_4}{\Delta_4}$$

$$-\frac{a_{63}d(d^2+a_{23})(d^2+a_{85})(f_5+df_6) + a_{63}a_{85}d(d^2+a_{23})f_7}{\Delta_4} \qquad (98)$$

$$t_4 = -\frac{(\Delta_4)_{1,2}(f_1+df_2) - (\Delta_4)_{2,2}(f_3+df_4) + a_{63}(d^2+a_{23})(d^2+a_{85})(f_5+df_6)}{\Delta_4}$$

$$-\frac{a_{63}a_{85}(d^2+a_{23})f_7}{\Delta_4} \qquad (99)$$

$$t_5 = -\frac{a_{23}a_{45}d(d^2+a_{85})(f_1+df_2) - a_{45}d(d^2+a_{23})(d^2+a_{85})(f_3+df_4)}{\Delta_4}$$

$$+\frac{d(\Delta_4)_{3,3}f_5 - d^2\left[(a_{63}+a_{67})d^4 + (a_{63}a_{83} + a_{23}a_{63} + a_{41}a_{63} + a_{23}a_{67}\right.}{\Delta_4}$$

$$+\frac{\left. a_{41}a_{67} + a_{45}a_{67})d^2 + a_{23}a_{63}a_{85} + a_{41}a_{63}a_{85} + a_{23}a_{45}a_{67}\right] f_6}{\Delta_4}$$

$$-\frac{d(\Delta_4)_{4,3}f_7}{\Delta_4} \qquad (100)$$

$$t_6 = -\frac{a_{23}a_{45}(d^2+a_{85})(f_1+df_2) + a_{45}(d^2+a_{23})(d^2+a_{85})(f_3+df_4)}{\Delta_4}$$

$$+\frac{(\Delta_4)_{3,3}(f_5+df_6) - (\Delta_4)_{4,3}f_7}{\Delta_4} \qquad (101)$$

$$t_7 = d\,t_8 \tag{102}$$

$$t_8 = -\frac{a_{23}a_{45}a_{67}(f_1 + d f_2) + a_{45}a_{67}(d^2 + a_{23})(f_3 + d f_4)}{\Delta_4}$$

$$- \frac{a_{67}\left[d^4 + (a_{23} + a_{41} + a_{45})d^2 + a_{23}a_{45}\right](f_5 + d f_6)}{\Delta_4}$$

$$- \frac{(d^2 + a_{63} + a_{67})\left[d^4 + (a_{23} + a_{41} + a_{45})d^2 + a_{23}a_{45}\right]f_7}{\Delta_4} \tag{103}$$

OBSERVERS OF INTERMEDIATE ORDER

When an intermediate number of the eight scalar states of the four body model is inaccessible ($p = 2, 3, \cdots$ or 6), the minimum order of the reduced state linear observer required to reconstruct the inaccessible scalar states equals p which also is the number of null columns in the measurement or observation matrix, C, and the F matrix. The general forms of the D, F and T matrices are given in equations (10), (11) and (13). If any p of the eight scalar state-variables of the four body model are inaccessible, then the total number of observers of order p that can be generated for the four body model is:

$$n_p = C_p^8 = \frac{8!}{p!\,(8-p)!} \tag{104}$$

<u>Example:</u> Generation of Second Order Observer for Four Body Model with Damping at all Interfaces

Suppose that the scalar states, x_7 and x_8, which represent the angular position and rate, respectively, of body 4, are inaccessible. The minimum order of the linear observer required to reconstruct these inaccessible state variables is $p = 2$. The corresponding observer synthesis equations are then given by equations (35) through (39), (41) and (53) through (61) with $i = 1, 2$ and $f_{i7} = f_{i8} = 0$. The observer synthesis equations for the same model without any damping are given by equations (44) through (51) with $i = 1, 2$ and $f_{i7} = f_{i8} = 0$.

$$\epsilon_{i1} = \frac{d_{ii}^{v} (\Delta_{i4}) f_{i1}}{\Delta'_{i4}}$$

$$+ \frac{a_{23} d_{ii}^{2} [d_{ii}^{4} + (a'_{45} + a'_{63} + a'_{67} + a'_{85}) d_{ii}^{2} + a'_{45} a'_{67} + a'_{45} a'_{85} + a'_{63} a'_{85}] f_{i2}}{\Delta'_{i4}}$$

$$+ \frac{a_{41} d_{ii} [d_{ii}^{4} + (a'_{45} + a'_{63} + a'_{67} + a'_{85}) d_{ii}^{2} + a'_{63} a'_{85}] (f_{i3} + d_{ii} f_{i4})}{\Delta'_{i4}}$$

$$- \frac{a_{41} a'_{63} d_{ii} (d_{ii}^{2} + a'_{85}) (f_{i5} + d_{ii} f_{i6})}{\Delta'_{i4}} \qquad (105)$$

$i = 1, 2$

$$d_{ii}^{v} = d_{ii} + \frac{a_{23} (\Delta'_{i4})_{1,1} + a_{41} (\Delta'_{i4})_{1,2}}{(\Delta'_{i4})_{1,1}} r_{11} \qquad (52)$$

$$\epsilon_{i2} = \frac{(\Delta'_{i4})_{1,1}(f_{i1}+d_{ii}f_{i2}) - a'_{41}\left[d_{ii}^4 + (a'_{63}+a'_{67}+a'_{85})d_{ii}^2 + a'_{63}a'_{85}\right](f_{i3}+d_{ii}f_{i4})}{\Delta'_{i4}}$$

$$- \frac{a'_{41}a'_{63}(d_{ii}^2+a'_{85})(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \qquad (167)$$

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$$\epsilon_{i3} = -\frac{d_{ii}^{VI}(\Delta'_{i4})_{1,2}(f_{i1}+d_{ii}f_{i2}) - d_{ii}^{VII}(\Delta'_{i4})_{2,2}f_{i3}}{\Delta'_{i4}}$$

$$-\frac{d_{ii}^2\left[(a_{41}+a_{45})d_{ii}^4 + (a'_{23}a_{45}+a_{41}a'_{63}+a_{41}a'_{67}+a_{41}a'_{85}+a_{45}a'_{67}+a_{45}a'_{85})d_{ii}^2\right.}{\Delta'_{i4}}$$

$$+\frac{\left.a'_{23}a_{45}a'_{67}+a'_{23}a_{45}a'_{85}+a_{41}a'_{63}a'_{85}\right]f_{i4}}{\Delta'_{i4}}$$

$$-\frac{d_{ii}(d_{ii}^2+a'_{85})(a_{63}d_{ii}d_{ii}^{VIII}+a'_{63}a'_{23})(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \qquad (168)$$

$$d_{ii}^{VI} = d_{ii} + \frac{a_{23}(\Delta'_{i4})_{1,1} + a_{41}(\Delta'_{i4})_{1,2}}{(\Delta'_{i4})_{1,2}}r_{11} + \frac{a_{45}(\Delta'_{i4})_{1,2} + a_{63}(\Delta'_{i4})_{1,3}}{(\Delta'_{i4})_{1,2}}r_{22} \qquad (54)$$

$$d_{ii}^{VII} = d_{ii} + \frac{a_{23}(\Delta'_{i4})_{2,1} + a_{41}(\Delta'_{i4})_{2,2}}{(\Delta'_{i4})_{2,2}}r_{11} + \frac{a_{45}(\Delta'_{i4})_{2,2} + a_{63}(\Delta'_{i4})_{2,3}}{(\Delta'_{i4})_{2,2}}r_{22} \qquad (55)$$

$$d_{ii}^{VIII} = d_{ii} + a_{41}(r_{11}-r_{22}) - a'_{23}r_{22} \qquad (56)$$

$$\Delta'_{i4} = d_{ii}^2\left[d_{ii}^6 + (a'_{23}+a'_{41}+a'_{45}+a'_{63}+a'_{67}+a'_{85})d_{ii}^4 + (a'_{23}a'_{45}+a'_{23}a'_{63}+a'_{23}a'_{67}\right.$$
$$+a'_{23}a'_{85}+a'_{41}a'_{63}+a'_{41}a'_{67}+a'_{41}a'_{85}+a'_{45}a'_{67}+a'_{45}a'_{85}+a'_{67}a'_{85})d_{ii}^2$$
$$\left.+a'_{23}a'_{45}a'_{67}+a'_{23}a'_{45}a'_{85}+a'_{23}a'_{63}a'_{85}+a'_{41}a'_{63}a'_{85}\right]$$

$(\Delta'_{i4})_{i,j} = \Delta'_{i4}$ without the elements of the ith row and jth column.

$$t_{i4} = -\frac{(\Delta'_{i4})_{1,2}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i4})_{2,2}(f_{i3}+d_{ii}f_{i4})}{\Delta'_{i4}}$$

$$-\frac{a'_{63}(d_{ii}^2+a'_{23})(d_{ii}^2+a'_{85})(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \qquad (109)$$

$$t_{i5} = -\frac{a'_{23}a_{45}d_{ii}(d_{ii}d_{ii}^{1x}+a'_{85})(f_{i1}+d_{ii}f_{i2})}{\Delta'_{i4}}$$

$$-\frac{a_{45}d_{ii}(d_{ii}^2+a'_{23})(d_{ii}d_{ii}^{1x}+a'_{85})(f_{i3}+d_{ii}f_{i4}) - d_{ii}^{x}(\Delta'_{i4})_{3,3}f_{i5}}{\Delta'_{i4}}$$

$$-\frac{d_{ii}^2[(a_{63}+a_{67})d_{ii}^4+(a_{63}a'_{85}+a'_{23}a_{63}+a'_{41}a_{63}+a'_{23}a_{67}+a'_{41}a_{67}+a'_{45}a_{67})d_{ii}^2}{\Delta'_{i4}}$$

$$+\frac{a'_{23}a_{63}a'_{85}+a'_{41}a_{63}a'_{85}+a'_{23}a'_{45}a_{67}]f_{i6}}{\Delta'_{i4}} \qquad (110)$$

$$d_{ii}^{1x} = d_{ii} + a_{67}(r_{33}-r_{22}) \qquad (58)$$

$$d_{ii}^{x} = d_{ii} + \frac{a_{45}(\Delta'_{i4})_{3,2}+a_{63}(\Delta'_{i4})_{3,3}}{(\Delta'_{i4})_{3,3}}r_{22} + \frac{a_{67}(\Delta'_{i4})_{3,3}+a_{85}(\Delta'_{i4})_{3,4}}{(\Delta'_{i4})_{3,3}}r_{33} \qquad (59)$$

$$t_{i6} = -\frac{a'_{23}a'_{45}(d_{ii}^2+a'_{85})(f_{i1}+d_{ii}f_{i2}) + a'_{45}(d_{ii}^2+a'_{23})(d_{ii}^2+a'_{85})(f_{i3}+d_{ii}f_{i4})}{\Delta'_{i4}}$$

$$+\frac{(\Delta'_{i4})_{3,3}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \qquad (111)$$

$$t_{i7} = -\frac{a'_{23}a'_{45}a_{67}d_{ii}(f_{i1}+d_{ii}f_{i2}) + a'_{45}a_{67}d_{ii}(d_{ii}^2+a'_{23})(f_{i3}+d_{ii}f_{i4})}{\Delta'_{i4}}$$

$$-\frac{[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}]a_{67}d_{ii}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \qquad (112)$$

$$t_{i8} = -\frac{a'_{23}a'_{45}a'_{63}(f_{i1}+d_{ii}f_{i2}) + a'^{\,2}_{45}a'_{67}(d_{ii}^2+a'_{23})(f_{i3}+d_{ii}f_{i4})}{\Delta'_{i4}}$$

$$-\frac{a'_{67}\left[d_{ii}^4+(a'_{23}+a'_{41}+a'_{45})d_{ii}^2+a'_{23}a'_{45}\right](f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \qquad (113)$$

The scalers, a'_{23}, a'_{41}, a'_{45}, a'_{63}, a'_{67} and a'_{85} are defined in terms of a_{23}, a_{41}, a_{45}, a_{63}, a_{67} and a_{85}, respectively, in equations (28) through (33).

If all damping is removed from the four body model, $r_{11}\to 0$, $r_{22}\to 0$, $r_{33}\to 0$, $a'_{23}\to a_{23}$, $a'_{41}\to a_{41}$, $a'_{45}\to a_{45}$, $a'_{63}\to a_{63}$, $a'_{67}\to a_{67}$, $a'_{85}\to a_{85}$, $d'_{ii}\to d_{ii}$, $d''_{ii}\to d_{ii}$, $d'''_{ii}\to d_{ii}$, $d^{IV}_{ii}\to d_{ii}$, $d^{V}_{ii}\to d_{ii}$, $d^{VI}_{ii}\to d_{ii}$, $d^{VII}_{ii}\to d_{ii}$, $d^{VIII}_{ii}\to d_{ii}$, $d^{IX}_{ii}\to d_{ii}$, $d^{X}_{ii}\to d_{ii}$, $\Delta'_{i4}\to\Delta_{i4}$ and $(\Delta'_{i4})_{ij}\to(\Delta_{i4})_{ij}$ where $\Delta_{i4}=$

$$d_{ii}^2\left[d_{ii}^6+(a_{23}+a_{41}+a_{45}+a_{63}+a_{67}+a_{85})d_{ii}^4+(a_{23}a_{45}+a_{23}a_{63}+a_{23}a_{67}+a_{23}a_{85}+a_{41}a_{63}+a_{41}a_{67}+a_{41}a_{85}+a_{45}a_{67}+a_{45}a_{85}+a_{63}a_{85})d_{ii}^2+a_{23}a_{45}a_{67}+a_{23}a_{45}a_{85}+a_{23}a_{63}a_{85}+a_{41}a_{63}a_{85}\right]$$

Then

$$t_{i1} = \frac{d_{ii}(\Delta_{i4})_{11}f_{i1}}{\Delta_{i4}}$$

$$+\frac{a_{23}d_{ii}^2\left[d_{ii}^4+(a_{45}+a_{63}+a_{67}+a_{85})d_{ii}^2+a_{45}a_{67}+a_{45}a_{85}+a_{63}a_{85}\right]f_{i2}}{\Delta_{i4}}$$

$$+\frac{a_{41}d_{ii}\left[d_{ii}^4+(a_{63}+a_{67}+a_{85})d_{ii}^2+a_{63}a_{85}\right](f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$-\frac{a_{41}a_{63}d_{ii}(d_{ii}^2+a_{85})(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}} \qquad (114)$$

$i=1,2$

$$t_{i2} = \frac{(\Delta_{i4})_{1,1}(f_{i1}+d_{ii}f_{i2}) - a_{41}\left[d_{ii}^4+(a_{63}+a_{67}+a_{85})d_{ii}^2+a_{63}a_{85}\right](f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$- \frac{a_{41}a_{63}(d_{ii}^2+a_{85})(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}} \qquad (115)$$

$$t_{i3} = - \frac{d_{ii}(\Delta_{i4})_{1,2}(f_{i1}+d_{ii}f_{i2}) - d_{ii}(\Delta_{i4})_{2,2}f_{i3}}{\Delta_{i4}}$$

$$- \frac{d_{ii}^2\left[(a_{41}+a_{45})d_{ii}^4+(a_{23}a_{45}+a_{41}a_{63}+a_{41}a_{67}+a_{41}a_{85}+a_{45}a_{67}+a_{45}a_{85})d_{ii}^2\right.}{\Delta_{i4}}$$

$$+ \frac{\left.a_{23}a_{45}a_{67}+a_{23}a_{45}a_{85}+a_{41}a_{63}a_{85}\right]f_{i4}}{\Delta_{i4}}$$

$$- \frac{a_{63}d_{ii}(d_{ii}^2+a_{23})(d_{ii}^2+a_{85})(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}} \qquad (116)$$

$$t_{i4} = - \frac{(\Delta_{i4})_{1,2}(f_{i1}+d_{ii}f_{i2}) - (\Delta_{i4})_{2,2}(f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$- \frac{a_{63}a_{85}(d_{ii}^2+a_{23})(d_{ii}^2+a_{85})(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}} \qquad (117)$$

$$t_{i5} = - \frac{a_{23}a_{45}d_{ii}(d_{ii}^2+a_{85})(f_{i1}+d_{ii}f_{i2})}{\Delta_{i4}}$$

$$+ \frac{a_{45}d_{ii}(d_{ii}^2+a_{23})(d_{ii}^2+a_{85})(f_{i3}+d_{ii}f_{i4}) + d_{ii}(\Delta_{i4})_{3,3}f_{i5}}{\Delta_{i4}}$$

$$- \frac{d_{ii}^2\left[(a_{63}+a_{67})d_{ii}^4+(a_{63}a_{85}+a_{23}a_{63}+a_{41}a_{63}+a_{23}a_{67}+a_{41}a_{67}+a_{45}a_{67})d_{ii}^2\right.}{\Delta_{i4}}$$

$$+ \frac{\left.a_{23}a_{63}a_{85}+a_{41}a_{63}a_{85}+a_{23}a_{45}a_{67}\right]f_{i6}}{\Delta_{i4}} \qquad (118)$$

$$t_{i6} = -\frac{a_{23}a_{45}(d_{ii}^2+a_{35})(f_{i1}+d_{ii}f_{i2}) + a_{45}(d_{ii}^2+a_{23})(d_{ii}^2+a_{35})(f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$+\frac{(\Delta_{i4})_{3,3}(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}} \qquad (119)$$

$$t_{i7} = d_{ii}\, t_{i8} \qquad (120)$$

$$t_{i8} = -\frac{a_{23}a_{45}a_{67}(f_{i1}+d_{ii}f_{i2}) + a_{45}a_{67}(d_{ii}^2+a_{23})(f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$-\frac{a_{67}\left[d_{ii}^4+(a_{23}+a_{41}+a_{45})d_{ii}^2+a_{23}a_{45}\right](f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}} \qquad (121)$$

SEVENTH ORDER OBSERVERS (p=7)

When any seven of the eight scalar state variables of the four body model are inaccessible, a linear observer of order of at least seven is required to reconstruct the inaccessible states. The total number of seventh order observers that can be generated for the four body model may be expressed as follows

$$n_7 = C_7^8 = 8 \qquad (122)$$

The general forms of the corresponding D, F and T matrices are given by equations (10), (11) and (13) with p=7. The synthesis equations for the seventh order observer are given by equations (35) through (39), (41) and (53) through (61) with $f_{1j} = f_{2j} = \cdots = f_{7j} = 0$ for seven of the eight values of the subscript, j.

<u>Example:</u> Generation of Seventh Order Observer for Four Body Model with Damping at all Interfaces.

Suppose only the scalar state variable representing the angular position of body 1, x_1, is accessible. Then the remaining scalar states, $x_2, x_3, \cdots x_8$, are inaccessible, $f_{i2} = f_{i3} = \cdots = f_{i8} = 0$ for $i = 1, 2, \cdots, 7$ and the observer synthesis equations reduce to the following forms.

$$t_{i1} = \frac{d_{ii}^{V}(\Delta'_{i4})_{1,1} f_{i1}}{\Delta'_{i4}} \qquad i = 1, 2, \cdots, 7 \tag{123}$$

$$d_{ii}^{V} = d_{ii} + \frac{a_{23}(\Delta'_{i4})_{1,1} + a_{41}(\Delta'_{i4})_{1,2}}{(\Delta'_{i4})_{1,1}} r_{11} \tag{52}$$

$$t_{i2} = \frac{(\Delta'_{i4})_{1,1} f_{i1}}{\Delta'_{i4}} \tag{124}$$

$$t_{i3} = -\frac{d_{ii}^{VI}(\Delta'_{i4})_{1,2} f_{i1}}{\Delta'_{i4}} \tag{125}$$

$$d_{ii}^{VI} = d_{ii} + \frac{a_{23}(\Delta'_{i4})_{1,1} + a_{41}(\Delta'_{i4})_{1,2}}{(\Delta'_{i4})_{1,2}} r_{11} + \frac{a_{45}(\Delta'_{i4})_{1,2} + a_{63}(\Delta'_{i4})}{(\Delta'_{i4})_{1,2}} r_{22} \tag{54}$$

$$t_{i4} = -\frac{(\Delta'_{i4})_{1,2} f_{i1}}{\Delta'_{i4}} \tag{126}$$

$$t_{i5} = -\frac{a'_{23} a_{45} d_{ii}(d_{ii} d_{ii}^{IX} + a'_{85}) f_{i1}}{\Delta'_{i4}} \tag{127}$$

$$d_{ii}^{IX} = d_{ii} + a_{67}(r_{33} - r_{22}) \tag{58}$$

$$t_{i6} = -\frac{a'_{23} a'_{45}(d_{ii}^{2} + a'_{85}) f_{i1}}{\Delta'_{i4}} \tag{128}$$

$$t_{i7} = -\frac{a'_{23} a'_{45} a_{67} d_{ii} f_{i1}}{\Delta'_{i4}} \tag{129}$$

$$t_{i8} = -\frac{a'_{23}a'_{45}a'_{67}f_{i1}}{\Delta'_{i4}} \qquad (130)$$

$$\Delta'_{i4} = d_{ii}^2\,[d_{ii}^6 + (a'_{23}+a'_{41}+a'_{45}+a'_{63}+a'_{67}+a'_{85})d_{ii}^4 + (a'_{23}a'_{45}+a'_{23}a'_{63}+a'_{23}a'_{67} + a'_{23}a'_{85}+a'_{41}a'_{63}+a'_{41}a'_{67}+a'_{41}a'_{85}+a'_{45}a'_{67}+a'_{45}a'_{85}+a'_{63}a'_{85})d_{ii}^2 + a'_{23}a'_{45}a'_{67}+a'_{23}a'_{45}a'_{85}+a'_{23}a'_{63}a'_{85}+a'_{41}a'_{63}a'_{85}]$$

$(\Delta'_{i4})_{i,j} = \Delta'_{i4}$ without the elements of the ith row and jth column
The scalars, a'_{23}, a'_{41}, a'_{45}, a'_{63}, a'_{67} and a'_{85} are defined in terms of a_{23}, a_{41}, a_{45}, a_{63}, a_{67} and a_{85}, respectively, in equations (28) through (33)

The observer synthesis equations for the same model without any damping are given by equations (44) through (51) with $i = 1, 2, \cdots, 7$ and $f_{i2} = f_{i3} = \cdots = f_{i8} = 0$. They reduce to the following forms.

$$t_{i1} = d_{ii}\,t_{i2} \qquad i = 1, 2, \cdots, 7 \qquad (131)$$

$$t_{i2} = \frac{(\Delta_{i4})_{1,1}f_{i1}}{\Delta_{i4}} \qquad (132)$$

$$t_{i3} = d_{ii}\,t_{i4} \qquad (133)$$

$$t_{i4} = -\frac{(\Delta_{i4})_{1,2}f_{i1}}{\Delta_{i4}} \qquad (134)$$

$$t_{i5} = d_{ii}\,t_{i6} \qquad (135)$$

$$t_{i6} = -\frac{a_{23}a_{45}(d_{ii}^2 + a_{85})f_{i1}}{\Delta_{i4}} \qquad (136)$$

$$t_{i7} = d_{ii}\, t_{i8} \qquad (137)$$

$$t_{i8} = -\frac{a_{23}a_{45}a_{67}f_{i1}}{\Delta_{i4}} \qquad (138)$$

where:

$$\Delta_{i4} = d_{ii}^2[d_{ii}^6 + (a_{23}+a_{41}+a_{45}+a_{63}+a_{67}+a_{85})d_{ii}^4 + a_{23}a_{45} + a_{23}a_{63} + a_{23}a_{67} + a_{23}a_{85} + a_{41}a_{63} + a_{41}a_{67} + a_{41}a_{85} + a_{45}a_{67} + a_{45}a_{85} + a_{63}a_{85})d_{ii}^2 + a_{23}a_{45}a_{67} + a_{23}a_{45}a_{85} + a_{23}a_{63}a_{85} + a_{41}a_{67}a_{85}]$$

$(\Delta_{i4})_{i,j} = \Delta_{i4}$ without the elements of the ith row and jth column