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Dynamics Analysis of Electrodynamic Satellite Tethers

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"Equations of Motion and Numerical Solution Algorithms for the Tether"

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1. Introduction

In this report the equations of motion are developed for a perfectly flexible, inelastic tether with a satellite at its extremity. The tether is attached to a space vehicle in orbit. The tether is allowed to possess electrical conductivity. Also in this report a numerical solution algorithm to provide the motion of the tether and satellite system is presented.

This report has several purposes. 1) To provide a solution algorithm for the motion; 2) To allow an analysis into the physical and dynamical properties of the motion for various electrical currents and other parameters; 3) To provide a check on the developments and algorithms of other researchers; 4) To allow a determination of the approximations, if any, introduced in the developments of others; and, 5) To provide a set of exact, analytical differential equations that describe the motion.

The purpose of having exact, analytical differential equations has several benefits. The resulting differential equations can be solved by various existing standard numerical integration computer programs. The resulting differential equations allow the introduction of approximations that can lead to analytical, approximate general solutions. The differential equations allow more dynamical insight of the motion. Modifications, extensions, generalizations of the dynamical system are much easier to accomplish for the system of exact differential equations. And many numerical integration errors are more conveniently monitored and controlled. And finally, overall comprehension of the equations and solution algorithms is somewhat more straightforward.

2. Equations of Motion for the Tether-Satellite System

Consider an element or a segment of the tether with length ΔS and mass per unit length along the tether, ρ. Consider a tether having a negligible diameter relative to the length of the tether.

Throughout this report, the tether will be considered as inextensible or inelastic.

Figure 1 shows a portion of the tether, separated into several, adjacent, elements.

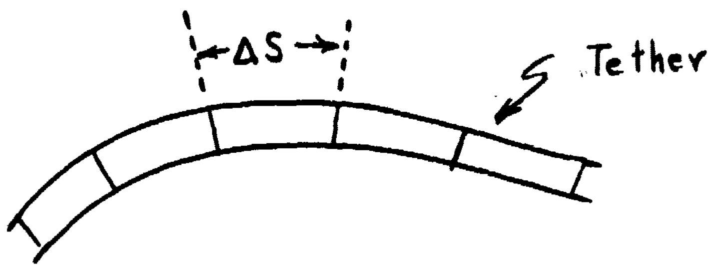

Figure 1

The force of one element on an adjacent element will be denoted by the vector $\overline{r}$. Since the forces between two adjacent elements (at the junction point) are equal and opposite, $\overline{R}$ on one element (1) due to an adjacent element (2), gives $-\overline{R}$ for the force on (2) due to (1). Let s denote the distance along the tether, where $s = 0$ at the point of attachment at the space vehicle and $s = L$ at the satellite, where L denotes the length of the tether. Then $\overline{R}$ will be a function of s, $\overline{R} = \overline{R}(s)$. The function $\overline{R}(s)$ will increase or decrease continuously as a function of s from one element to the next, as illustrated in Figure 2.

Consider the forces acting on an element as shown in Figure 2.

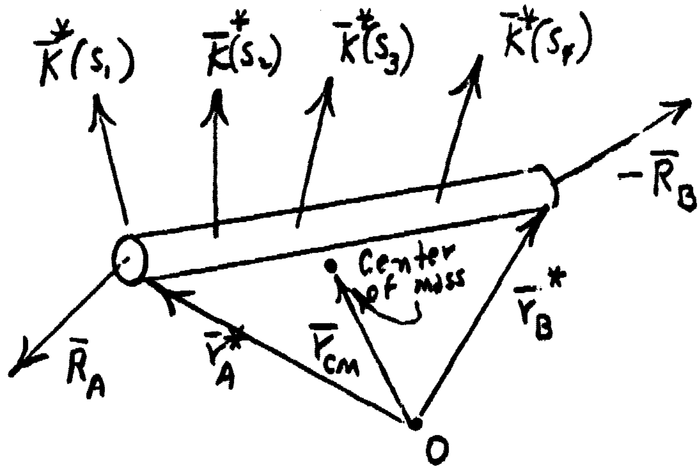

Figure 2

The forces $\bar{R}_A$ and $-\bar{R}_B$ are the forces due to each adjacent element. The forces $\bar{k}^*(s_1)$, $\bar{k}^*(s_2)$, are other external forces such as gravitational forces, electrodynamical forces, etc., acting at various locations $s_1, s_2, s_3, \ldots,$ on the element where s is the distance along the tether.

Define vectors $\bar{r}_A^*$, $\bar{r}_B^*$, and $\bar{r}_{cm}^*$, from a reference origin 0 to each end of the element and to the center of mass of the element, cm, as is depicted in Figure 2. Note that the point cm is not necessarily situated on the element.

The first equations of motion for the element ΔS is the sum of the forces acting on ΔS equal to the time rate of change of the linear angular momentum of the center of mass of the element, cm, which, in this case, is equivalent to the mass of the element times the acceleration of cm, or

$$\Delta S \cdot \rho \cdot \ddot{\bar{r}}^{*}_{cm} = R_A - R_B + \sum_{i=1}^{n} k_i^{*} \tag{1}$$

where dots denote differentiation with respect to time, and ρ is the mass per unit length of the tether.

The restriction placed on the origin O is that it be an inertial point.

The second equation of motion is the sum of the moments about cm equal to the time rate of change of the angular momentum about cm, or

$$\dot{\bar{H}}_{cm} = \sum_{i=1}^{N} \bar{M}_i^{cm} \tag{2}$$

The forces $\bar{k}^{*}(s)$ in equation (1) are not discrete forces but are distributed uniformly and continuously over the element, hence the summation given in equation (1) over $\bar{k}^{*}(s)$ can be replaced by $\bar{k}(s)$, and the total force is determined by:

$$\int_A^B \bar{k}(s)ds$$

where the limits indicate that the integration is taken from $\bar{r}_A$ to $\bar{r}_B$ and similarly the moments due to $\bar{k}$ (s) about the point cm, M_1^{cm} in equation (2), are:

$$\int_A^B (\bar{r}_s^* - \bar{r}_{cm}^*) \times \bar{k}(s)ds$$

where $\bar{r}_s^*$ is the vector to the point s on the tether.

Equation (1) becomes

$$\Delta S \cdot \rho \cdot \ddot{\bar{r}}_{cm}^* = \bar{R}_A - \bar{R}_B + \int_A^B \bar{k}(s)ds \tag{3}$$

and equation (2) becomes

$$\frac{d}{dt}[\Delta \bar{H}_{cm}] = [\bar{r}_A^* - \bar{r}_{cm}^*] \times \bar{R}_B + \int_A^B \bar{M}_T \, ds + \int_A^B \bar{M}_B \, ds \tag{4}$$

where $\Delta \bar{H}_{cm}$ is the angular momentum of the element at s, with length ΔS, about the center of mass of the element. The quantities $\bar{M}_T$ and $\bar{M}_B$ are the torsional moment and the bending moment, respectively, per unit length.

The vectors $\bar{R}_A$, and $-\bar{R}_B$ can each be decomposed into two component vectors at right angles to each other, one tangent to the element and one perpendicular or normal to the element, as shown in Figure 3.

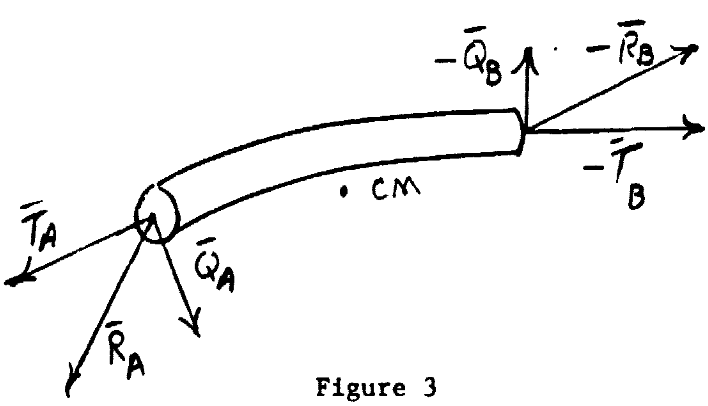

Figure 3

The vectors $\overline{T}_A$ and $-\overline{T}_B$, components of $\overline{R}_A$, and $-\overline{R}_B$ along the element are referred to as tensile stress. The vectors $\overline{Q}_A$ and $-\overline{Q}_B$ are the shear stress, since they are normal to the element.

Dividing equation (3) by Δ S and using the component vectors of $\overline{R}_A$ and $\overline{R}_B$ gives

$$\rho \cdot \ddot{\overline{r}}^{*}_{cm} = \frac{\overline{T}_A - \overline{T}_B}{\Delta S} + \frac{\overline{Q}_A - \overline{Q}_B}{\Delta S} \frac{1}{\Delta S} \int_A^B \overline{k}(s)ds \qquad (5)$$

Letting $\Delta \overline{T} = \overline{T}_A - \overline{T}_B$ and $\Delta \overline{Q} = \overline{Q}_A - \overline{Q}_B$, and taking the limit gives

$$\lim_{\Delta S \to 0} - \frac{\Delta\overline{T}}{\Delta S} - \frac{\Delta\overline{Q}}{\Delta S} = \frac{\partial\overline{T}}{\partial s} - \frac{\partial\overline{Q}}{\partial s}$$

Also

$$\lim_{\Delta S \to 0} \frac{1}{\Delta S} \int_A^B \overline{k}(s)ds = \overline{K}(s) \ .$$

where $\overline{K}(s)$ is a force per unit length at s. This result follows since as $\Delta S \to 0$, this implies that point A (or point B) approaches, in the limit, point B (or point A).

Equation (5) becomes

$$\rho \cdot \ddot{\overline{r}}^{*}_{cm} = - \frac{\partial\overline{T}}{\partial S} - \frac{\partial\overline{Q}}{\partial S} + \overline{K}(s) \qquad (6)$$

Equation (6) is the translational equation of motion (for the center of mass) of the infinitessimal element ds, including torsional and bending moments, for a non-flexible, inelastic tether.

In equation (4), dividing through by Δ s and using the component vectors for $\overline{R}_A$ and $\overline{R}_B$:

$$\frac{d}{dt}\frac{(\Delta H_{cm})}{\Delta s} = \frac{(\overline{r}^*_A - r^*_{cm})}{\Delta s} x\overline{T}_A + \frac{(\overline{r}^*_A - \overline{r}^*_{cm})}{\Delta s} x\overline{Q}_A \quad \frac{\overline{r}^*_A - \overline{r}^*_{cm}}{\Delta s} xT_B$$

$$- \frac{\overline{r}^*_B - \overline{r}^*_{cm}}{\Delta s} x\overline{Q}_B + \frac{1}{\Delta s}\int_A^B (\overline{r}^*_{(s)} - \overline{r}^*_{cm})\, x\overline{K}(s)ds$$

$$+ \frac{1}{\Delta s}\int_A^B M_T ds + \frac{1}{\Delta s}\int_A^B M_B ds \qquad (7)$$

Taking the limit of equation (7), as Δ s$\rightarrow$0 gives:

$$\lim_{\Delta S\rightarrow 0} \frac{d}{dt}\left[\frac{\Delta H_{cm}}{\Delta S}\right] = \frac{d}{dt}\frac{\partial H_{cm}}{\partial S}$$

$$\lim_{\Delta S\rightarrow 0} \frac{\overline{r}^*_A - \overline{r}^*_{cm}}{\Delta S} xT_A = \frac{d\overline{r}^*}{ds} xT_A = 0$$

$$\lim_{\Delta S\rightarrow 0} \frac{r^*_B - r^*_{cm}}{\Delta S} xT_B = - \frac{dr^*}{ds} xT_B = 0$$

and letting Q_A and Q_B go to Q as Δ s 0:

$$\lim_{\Delta S \to 0} \left[\frac{r_A^* - r_{cm}^*}{\Delta S} x Q_A + \frac{r_A^* - r_{cm}^*}{\Delta S} x(-Q_B) \right] =$$

$$\lim_{\Delta S \to 0} \left[\frac{r_A^* - r_{cm}^*}{\Delta S} x Q_A + \frac{r_{cm}^* - r_B^*}{\Delta S} x \overline{Q}_B \right] =$$

$$\lim_{\Delta S \to 0} = - \frac{\overline{r}_B^* - \overline{r}_{cm}^* \quad \overline{r}_{cm}^* - \overline{r}_A^*}{\Delta S} x \overline{Q} =$$

$$\lim_{\Delta S \to 0} = - \frac{r_B^* - r_A^*}{\Delta S} x \overline{Q} = - \frac{d\overline{r}^*}{ds} x \overline{Q}$$

where the direction $\overline{r}_A^*$ to $\overline{r}_B^*$ is chosen as the direction of $d\overline{r}^*$. Also,

$$\lim_{\Delta S \to 0} \frac{1}{\Delta S} \int_A^B \left(\overline{r}(s)^* - \overline{r}_{cm}^* \right) x \overline{K}(s) ds = 0$$

since $r(s)^* - r_{cm}^* = 0$ and $ds/\Delta S = 1$ in the limit. Also,

$$\lim_{\Delta S \to 0} \frac{1}{\Delta S} \int_A^B M_T \, ds = M_T$$

and

$$\lim_{\Delta S \to 0} \frac{1}{\Delta S} \int_A^B M_B \, ds = M_B$$

So equation (9) becomes

$$\frac{d}{dt}\frac{\partial \overline{H}_{cm}}{\partial s} = -\frac{d\overline{r}^*}{ds} \times \overline{Q} + M_T + \overline{M}_B \tag{8}$$

Equation (8) is the rotational equation of motion for the infinitessimal element ds about its center of mass, including torsional and bending moments, for a non-flexible, inelastic tether.

3. A Perfectly Flexible Tether

The torsional moment M_T, is the resistance of the element to twisting about its axial direction. In this report, inclusion of the torsional moment is not necessary for the tether. Hence $M_T = 0$ in equation (8).

The bending moment is the resistance of the element to bending about an axis perpendicular or normal to the physical element. The bending moment is determined by the properties of the tether. For this report, it will be assumed that the tether is perfectly flexible, or has no stiffness, and that the bending moment is negligible. This is probably a good assumption for many types of long, thin tethers. However, if the diameter of the tether were appreciable, this assumption might need evaluation, depending on the physical properties of the tether. Hence, in this report,

$$\overline{M}_B = 0$$

in equation (8).

For a perfectly flexible tether, with no bending moment, the vector $\overline{Q}$ in equations (6) and (8) will be set to zero. (See, for example, Shames, <u>Engineering Mechanics, Statics and Dynamics</u>, Prentice Hall, 2nd edition, p. 148, for the proof that $\overline{Q} = 0$ for a perfectly flexible cable in equilibrium (zero acceleration). Extension of the proof to the case at hand, the non-equilibrium, non-zero acceleration case, has not been given. However, it will be assumed here that for small to moderate accelerations of the tether, setting $Q = 0$ will provide sufficient accuracy.)

Equations (6) and (8) become

$$\rho\, \overline{r}^{*}_{cm} = -\frac{\partial \overline{T}}{\partial s} + \overline{K}(s) \tag{9}$$

$$\frac{d}{dt}\frac{\partial \overline{H}_{cm}}{\partial s} = 0 \tag{10}$$

In equation (9), the vector $\overline{T}$ lies along the tangent to the tether at the point s, as discussed above. Equation (10) can be integrated immediately giving

$$\frac{\partial \overline{H}_{cm}}{\partial s} = \text{constant} = c(s) \tag{11}$$

providing the result that the angular momentum of <u>each</u> (infinitesimal) element of the tether remains constant in time. Note that the constant in equation (11) is only "constant" in time, but in general is a function of s; the value of the "constant" C(s) depends on the initial conditions. Of course the reason that we have this result is that we set $Q = M_B = M_T = 0$, which are all the moments about the center of mass of ds.

Equation (9) is the translational equation of motion for <u>each</u> infinitesimal element, ds, for a perfectly flexible, inelastic tether. In equation (9), replace $\overline{r}^{*}_{cm}$ by $\overline{r}(s,t)^{*}$, $\overline{T}$ by $\overline{T}(s,t)$ and $\overline{K}(s)$ by $\overline{K}(s,t)$. And to denote that the total derivatives with respect to time are to be taken with s held constant, the total derivatives are replaced by partial derivatives. Equation (9) becomes

$$\rho\, \frac{\partial^2 \overline{r}(s,t)^{*}}{\partial t^2} = -\frac{\partial \overline{T}(s,t)}{\partial s} + \overline{K}(s,t) \tag{12}$$

where $\overline{K}(s,t)$ represents the external forces per unit distance along the tether.

4. The Inelasticity Constraint

The assumption that the tether is inelastic (no stretching) can be written as

$$\left|\frac{\partial \bar{r}^*}{\partial s}\right| = 1 \text{ or}$$

$$\frac{\partial \bar{r}^*}{\partial s} \cdot \frac{\partial \bar{r}^*}{\partial s} = 1 \tag{13}$$

Moreover, the vector $\Delta \bar{r}^* \Delta S$, in the limit as ΔS goes to zero, along with constraint (13), coincides with the unit vector along the tangent of the tether at the point s. Denoting this unit vector by $\hat{\ell}$

$$\bar{T} = T \cdot \hat{\ell} = T \cdot \frac{\partial \bar{r}^*}{\partial s} \tag{14}$$

Equation (12) then becomes

$$\rho \frac{\partial^2}{\partial t^2} \bar{r}(s,t)^* = - \frac{\partial}{\partial s}\left[T(s,t)\frac{\partial \bar{r}(s,t)^*}{\partial s}\right] + \bar{K}(s,t) \tag{15}$$

The inelasticity contraint provides a differential equation for the determination of $\frac{\partial \bar{r}^*}{\partial s}$ and $\bar{T}$. Differentiating equation (13) twice with respect to time gives

$$\frac{\partial \bar{r}^*}{\partial s} \cdot \frac{\partial}{\partial t}\frac{\partial \bar{r}^*}{\partial s} = 0 \tag{16}$$

and

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$$\frac{\partial}{\partial t}\left[\frac{\partial \bar{r}^*}{\partial s}\right] \cdot \frac{\partial}{\partial t}\left[\frac{\partial \bar{r}}{\partial s}\right] + \frac{\partial \bar{r}^*}{\partial s} \cdot \frac{\partial^2}{\partial t^2}\left[\frac{\partial \bar{r}^*}{\partial s}\right] = 0 \tag{17}$$

Also since $\bar{r}^*(s,t)$ and its derivatives are assumed to be continuous, then

$$\frac{\partial}{\partial t}\left[\frac{\partial \bar{r}^*}{\partial s}\right] = \frac{\partial}{\partial s}\left[\frac{\partial \bar{r}^*}{\partial t}\right], \tag{18}$$

and Equation (17) can be written as

$$\frac{\partial}{\partial s}\left[\frac{\partial \bar{r}^*}{\partial t}\right] \cdot \frac{\partial}{\partial s}\left[\frac{\partial \bar{r}}{\partial t}\right]^* + \frac{\partial \bar{r}^*}{\partial s} \cdot \frac{\partial}{\partial s}\left[\frac{\partial^2 \bar{r}^*}{\partial t^2}\right] = 0 \tag{19}$$

Substituting equation (15) into equation (19) gives:

$$\frac{\partial}{\partial s}\left[\frac{\partial \bar{r}}{\partial t}\right]^* \cdot \frac{\partial}{\partial s}\left[\frac{\partial \bar{r}}{\partial t}\right] + \frac{1}{\rho}\frac{\partial \bar{r}^*}{\partial s} \cdot \frac{\partial}{\partial s}\left[- \frac{\partial}{\partial s}\left[T\frac{\partial \bar{r}^*}{\partial s}\right] + \bar{K}\right] = 0$$

or

$$\frac{\partial}{\partial s}\left[\frac{\partial \bar{r}^*}{\partial s}\right] \cdot \frac{\partial}{\partial s}\left[\frac{\partial \bar{r}^*}{\partial t}\right] - \frac{1}{\rho}\frac{\partial \bar{r}^*}{\partial s} \cdot \frac{\partial^2}{\partial s^2}\left[T\frac{\partial \bar{r}^*}{\partial s}\right] + \frac{1}{\rho}\frac{\partial \bar{r}^*}{\partial s} \cdot \frac{\partial \bar{K}}{\partial s} = 0 \tag{20}$$

5. The Equations of Motion for the Satellite and the Space Vehicle

In this section, differential equations will be developed for the satellite and the Space Vehicle in order to determine the motion of the entire satellite - tether - vehicle system as well as to provide information concerning the boundary conditions for the tension at each end of the tether.

The forces acting on the satellite are the tension due to the tether $T(1,t)$; and other external forces, $\overline{K}^P(L,t)$, where L denotes the value for s at the end of the tether, and $\overline{K}^P(L,t)$, denote forces acting on a point mass as opposed to K(s,t) which is a force per length.

Hence,

$$\frac{\partial^2 \overline{r}(1,t)^*}{\partial t^2} = \frac{-1}{m_{SAT}} \; T(L,t) \; \frac{\partial \overline{r}(L,t)^*}{\partial s} \; + \frac{1}{m_{SAT}} \overline{K}^P(L,t) \tag{21}$$

or

$$\frac{\partial^2 \overline{r}(L,t)^*}{\partial t^2} = \frac{-1}{m_{SAT}} \overline{T}(L,t) \; + \overline{K}^P(L,t) \tag{22}$$

where, as discussed in section 4,

$$\overline{T}(L,t) = T(L,t)\,\hat{\ell} = T(L,t)\frac{\partial \overline{r}^*}{\partial s} \tag{23}$$

and where the negative sign on the first term on the right in equation (22) indicates that the force of tension on the satellite is in the negative s direction, toward the tether.

For the other end of the tether , attached to the space vehicle, make the following assumptions. Assume the tether is attached to the space vehicle at some point 0'. The point 0' is in general not an inertial point. Let 0' be the origin of our coordinate system with rectangular Cartesian coordinate axes defined as follows. The z-axis is along the line from the center of the earth to the point 0', positive upwards. The x-axis is in a direction such that the x - z plane contains the velocity vector of the point 0', with positive direction in the direction of the velocity vector. And y-axis is defined by the right-hand rule.

This definition of the x-axis is convenient, since it is in the direction of (but not necessarily coincident with - allowing for non-zero eccentricities of the space vehicle's orbit) the velocity vector of the point of attachment at the space vehicle. For zero-eccentricities and assuming 0' at the center of the space vehicle, the velocity vector will, of course, lie along the x-axis. And the y-axis (determined by right-hand rule) will be normal to the orbital plane. For the actual situation, assuming the space vehicle is rotating at a reasonably slow rate, and for a non-zero eccentricity, and including perturbations, the x-axis should be close to the plane containing the position and velocity of the space vehicle. Let the unit vectors along the x,y,z axis be $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively.

The coordinate axes will be rotating with angular velocity $\bar{\omega}$. The point 0' will be located at s = 0 or at $\bar{r}(0,t)^*$. The velocity and acceleration of the point of attachment will be denoted by

$$\frac{\partial r(0,t)^*}{\partial t} \text{ and } \frac{\partial^2 \overline{r}(0,t)^*}{\partial t^2},$$

respectively. Let $\overline{r}_{s.v.}(t)^*$ be the vector from the center of the earth to a reference point within the space vehicle. The acceleration of the vehicle is

$$\frac{\partial^2 \overline{r}_{s.v.}(t)^*}{2t^2} = \frac{1}{m_{s.v}} \overline{T}(0,t) \Bigg|_{\text{acting at } 0'}$$

$$+ \frac{1}{m_{s.v.}} \overline{K}_{s.v.}(t) \tag{24}$$

where $\overline{T}(0,t)$ is the force on the vehicle due to the tether at $s = 0$ or at $0'$ and $\overline{K}_{s.v.}(t)$ are all <u>other</u> external forces such as gravity, drag, thrusting, etc. acting on the space vehicle, and located at $\overline{r}_{s.v.}(t)^*$.

The acceleration at the point of attachment, 0', is

$$\frac{\partial^2 \overline{r}(0,t)^*}{\partial t^2} = \frac{\partial^2 \overline{r}_{s.v.}(t)^*}{\partial t^2} + \frac{\partial^2 \overline{r}_{rel}(t)^*}{\partial t^2} \tag{25.1}$$

where

$$\frac{\partial^2 \overline{r}_{rel}(t)^*}{\partial t^2} = \dot{\Omega}_{s.v.} \, x(\overline{r}(0,t)^* - \overline{r}_{s.v.}(t)^*)$$

$$+ \bar{\Omega}_{s.v.} \times (\bar{\Omega}_{s.v.} \times (\bar{r}(0,t)^{*} - \bar{r}_{s.v}(t)^{*})) \qquad (25.2)$$

where Ω and $\dot{\Omega}$ are the angular velocity and angular acceleration of the space vehicle, respectively.

In some cases, for approximate solutions, some assumptions can be made to decrease the time of computation. For example, assume that $m_{s.v.}$ is very large compared to the tether. Let this be assumption A. And assume that

$$\frac{1}{m_{s.v.}} \overline{T}(0,t) << \frac{1}{m_{s.v.}} \overline{K}_{s.v.}(t),$$

Let this be assumption B.

And if one assumes two-body motion for the space vehicle's center of mass (assumption C), then equation (24) becomes

$$\frac{\partial^2 \overline{r}_{s.v.}(t)^*}{\partial t^2} = \frac{-Gm_{earth}\overline{r}_{s.v.}(t)^*}{\left|\overline{r}_{s.v.}(t)^*\right|^3} \tag{26}$$

The assumptions that the mass of the tether and that $\overline{T}(0,t)$ on the space vehicle can be neglected for the motion of the space vehicle (assumptions A+B) allow us to be able to determine <u>a priori</u> the acceleration of the space vehicle, and, as will be seen later, allows us to determine T(0,t) readily, for the tether.

The assumption that $\overline{K}(0,t)$ contains only two-body forces (assumption C) allows us to obtain the solution for the motion of the space vehicle analyticaly, for two-body motion. Hence,

$$r(0,t)^*, \frac{\partial \overline{r}}{\partial t}(0,t)^*, \frac{\partial^2 \overline{r}}{\partial t^2}(0,t)^*,$$

as well as $\overline{\omega}(t)$, $\dot{\overline{\omega}}(t)$, are known a priori, <u>with these assumptions</u>.

Another assumption can be made by placing the point of attachment 0', at the center of the space vehicle, at $\overline{r}^*_{s.v.}$ (assumption D). If the errors made by the above assumptions are larger than desirable, then the determination of $\frac{\partial^2 r}{\partial t}(0,t)^*$ will not be known a priori and the motion of the satellite - tether - space vehicle system must be solved for simultaneously (see for example report #1 of this contract). The developments, equations and algorithms given in this report do not contain any of the assumptions just discussed, but are for the general situation, where the motion of the space vehicle and the tether are solved for simultaneously and all external forces on the tether and space vehicle are included.

Let $\overline{r}(s,t)^*$ be from 0 to the point s on the tether, then

$$\overline{r}(s,t) = \overline{r}^*(s,t) - \overline{r}^*(s=0,t) \tag{27}$$

where $\overline{r}^*(s = 0,t)$ is to the point of attachment 0', since s=0 at that point. Hence $\overline{r}(s,t)$ is the vector from the 0', at the vehicle to the point s on the tether. And, for a non-inertial, rotating reference frame:

$$\frac{\partial \overline{r}(s,t)^*}{\partial t} = \frac{\partial \overline{r}(s,t)}{2t} + \omega \times \overline{r}(s,t) \tag{28}$$

And, for the acceleration in the frame;

$$\frac{\partial^2\bar{r}(s,t)^*}{\partial t^2} = \frac{\partial^2\bar{r}(0,t)^*}{\partial t^2} + \frac{\partial^2\bar{r}(s,t)}{\partial t^2}$$

$$+ \ \bar{\omega} \times (\bar{\omega} \times \bar{r}(s,t)) + \dot{\bar{\omega}} \times \bar{r}(s,t)$$

$$+ \ 2\,\bar{\omega} \times \frac{\partial\bar{r}(s,t)}{\partial t} \qquad (29)$$

The first term on the right can be given by equations (25.1) and (25.2), the fourth term would be included for non-zero eccentricity orbits of the space vehicle, the third term is the centripetal acceleration, and the last is the coriolis acceleration. The second term, $\frac{\partial^2\bar{r}(s,t)}{\partial s^2}$ is the acceleration of the element at s <u>with respect to the point of attachment,</u> <u>0'</u>. Notice that the last term contains a velocity <u>with respect to</u> the point of attachment, 0', $\frac{\partial\bar{r}(s,t)}{\partial t}$, that is, the relative velocity, defined in equation (28).

If an additional assumption of zero eccentricity is imposed (assumption E) along with assumptions A,B, & C, and also with the point of attachment placed at the c.m. of the space vehicle (assumption D) then :

$$\bar{r}(0,t)^* = \text{constant} \; . \; R \qquad (30)$$

and

$$\frac{\partial \bar{r}(0,t)^*}{\partial t} = \sqrt{\frac{Gm_{earth}}{r(0,t)^*}}\ \hat{i} \tag{31}$$

$$= v_{circular}\ \hat{i} \tag{32}$$

and

$$\bar{\omega} = \frac{v^2_{circular}}{r^*(0,t)}\ \hat{j} \tag{33}$$

$$\dot{\bar{\omega}} = 0 \tag{34}$$

and eqt. 25.1 becomes simply

$$\frac{\partial^2 \bar{r}(0,t)^*}{\partial t^2} = \frac{\partial^2 \bar{r}_{s.v.}(t)}{\partial t^2}$$

In general (with no assumptions) and with the reference frame defined above, equation (22) becomes, using equation (29)

$$\frac{\partial^2}{\partial t^2}\bar{r}(L,t) = \frac{-1}{m_{SAT}}\bar{T}(L,t) + \frac{1}{m_{SAT}}\bar{K}^P(L,t)$$

$$- \frac{\partial^2 \bar{r}(0,t)^*}{\partial t^2} - \bar{\omega} \times \bar{\omega} \times \bar{r}(L,t)$$

$$- \dot{\bar{\omega}} \times \bar{r}(L,t) - 2\bar{\omega} \times \frac{\partial \bar{r}(L,t)}{\partial t} \tag{35}$$

with assumptions A,B,C,D,E, (neglecting tension on the space vehicle, placing 0' at the center of mass of the space vehicle, and assuming circular motion reduces) equation (35) to

$$\frac{\partial^2}{\partial t^2}\bar{r}(L,t) = \frac{-1}{m_{SAT}}\bar{T}(L,t) + \frac{1}{m_{SAT}}\bar{K}^P(1,t)$$

$$+ \frac{Gm_{earth}\bar{r}(0,t)^*}{[r(0,t)^*]^3} - \omega \times (\omega \times \bar{r}(L,t))$$

$$- 2\,\omega \times \frac{\partial \bar{r}(L,t)}{\partial t} \tag{36}$$

where $\bar{r}(L,t)$, $\frac{\partial \bar{r}(L,t)}{\partial t}$, $\frac{\partial}{\partial t^2} r^{-}(L,t)$ are with respect to the rotating coordinate system with origin at the center of mass of the space vehicle. Note that in equation (36), the term with $\dot{\bar{\omega}}$ does not appear since with the assumptions made, $\dot{\bar{\omega}} = 0$.

6. Equations of Motion in the Rotating Reference Frame

In the rotating coordinate system, equations (12) (or (15)) and equation (20) must be modified.

In equation (12), the acceleration is replaced using equation (29). The result is, using $\overline{T} = T \frac{\partial \overline{r}}{\partial s}$ and $\frac{\partial \overline{T}}{\partial s} = \frac{\partial T}{\partial s} \cdot \frac{\partial \overline{r}}{\partial s} + T \frac{\partial^2 \overline{r}}{\partial s^2}$,

$$\frac{\partial^2 \overline{r}(s,t)}{\partial t^2} = \frac{-1}{\rho} \frac{\partial T}{\partial s} \cdot \frac{\partial \overline{r}}{\partial s} - \frac{1}{\rho} T \frac{\partial \overline{r}}{\partial s^2} + \frac{1}{\rho} \overline{K}(s,t)$$

$$- \frac{\partial^2 \overline{r}(0,t)^*}{\partial t^2} - \overline{\omega} \times (\overline{\omega} \times \overline{r}(s,t)) \tag{37}$$

$$- \dot{\overline{\omega}} \times \overline{r}(s,t) - 2 \overline{\omega} \times \frac{\partial \overline{r}}{\partial t}(s,t)$$

For equation (20), rederive equation (19) by taking partial derivatives of equation (13) with respect to the rotating reference frame centered on O'. The result is

$$\frac{\partial}{\partial s} \left[\frac{\partial \overline{r}}{\partial t}\right] \cdot \frac{\partial}{\partial s}\left[\frac{\partial \overline{r}}{\partial t}\right] + \frac{\partial \overline{r}}{\partial s} \cdot \frac{\partial}{\partial s} \left[\frac{\partial^2 \overline{r}}{\partial t^2}\right] = 0 \tag{37.1}$$

Substituting equation (37) into this result gives

$$\frac{\partial}{\partial s} \frac{\partial \overline{r}}{\partial t} \frac{\partial}{\partial s} \frac{\partial \overline{r}}{\partial t} + \frac{\partial \overline{r}}{\partial s} \frac{\partial}{\partial s} [- \frac{1}{\rho} \frac{\partial}{\partial s} \overline{T} + \frac{1}{\rho} \overline{K} - \frac{\partial^2 \overline{r}}{\partial t^2} (0,t)^*$$

$$- \overline{\omega} \times (\overline{\omega} \times \overline{r}) - \dot{\overline{\omega}} \times \overline{r} - 2\overline{\omega} \times \frac{\partial \overline{r}}{\partial t}] = 0$$

and taking the partial derivative of the terms within the brackets provides:

$$\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right)\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) - \frac{1}{\rho}\frac{\partial \bar{r}}{\partial s} \cdot \frac{\partial^2}{\partial s^2}\bar{T} + \frac{1}{\rho}\frac{\partial \bar{r}}{\partial s} \cdot \frac{\partial \bar{K}}{\partial s} - \frac{\partial \bar{r}}{\partial s}\frac{\partial}{\partial s}\frac{\partial^2 \bar{r}}{\partial t^2}(0,t)^*$$

$$- \frac{\partial \bar{r}}{\partial s} \cdot \bar{\omega} \times \left(\bar{\omega} \times \frac{\partial \bar{r}}{\partial s}\right) - \frac{\partial \bar{r}}{\partial s} \cdot \dot{\bar{\omega}} \times \frac{\partial \bar{r}}{\partial s} - \frac{\partial \bar{r}}{\partial s} \cdot 2\,\bar{\omega} \times \frac{\partial}{\partial s}\frac{\partial \bar{r}}{\partial t} = 0$$

The term $\frac{\partial \bar{r}}{\partial s} \cdot \dot{\bar{\omega}} \times \frac{\partial \bar{r}}{\partial s}$ upon rearrangement, vanishes. Using $\bar{T} = T\frac{\partial \bar{r}}{\partial s}$ and since $\frac{\partial}{\partial s}\ddot{\bar{r}}\,(0,t) = 0$, gives

$$\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right)\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) - \frac{1}{\rho}\frac{\partial^2 T}{\partial s^2}\frac{\partial \bar{r}}{\partial s} \cdot \frac{\partial \bar{r}}{\partial s} - \frac{2}{\rho}\frac{\partial \bar{r}}{\partial s} \cdot \frac{\partial^2 \bar{r}}{\partial s^2}\frac{\partial T}{\partial s}$$

$$- \frac{1}{\rho}T\frac{\partial^3 \bar{r}}{\partial s^3} \cdot \frac{\partial \bar{r}}{\partial s} + \frac{1}{\rho} \cdot \frac{\partial \bar{r}}{\partial s} \cdot \frac{\partial \bar{K}}{\partial s}$$

$$- \frac{\partial \bar{r}}{\partial s} \cdot \bar{\omega} \times \left(\bar{\omega} \times \frac{\partial \bar{r}}{\partial s}\right) - \frac{\partial \bar{r}}{\partial s} \cdot 2\bar{\omega} \times \frac{\partial}{\partial s}\frac{\partial \bar{r}}{\partial t} = 0 \qquad (38)$$

Either equation (20) or (38) can be used. However, if equation (20) is used, $\frac{\partial \bar{r}^*}{\partial t}$ must be obtained using equation (28). Since relative velocities and accelerations will be used here, equation (38) is more convenient.

For the satellite, the equation corresponding to equation (37) is more complex since the quantity $\frac{\partial^2 \bar{r}}{\partial t^2}$ is difficult for the satellite and the adjoining element of the tether. The term $\frac{\partial}{\partial s}\,\frac{\partial \bar{r}}{\partial t^2}$ becomes

$$\frac{\partial}{\partial s} \; \frac{\partial^2 \bar{r}(L,t)}{\partial t^2} = \lim_{s \to L} \left(\frac{\partial^2 \bar{r}(L,t)}{\partial t^2} - \frac{\partial^2 \bar{r}(s,t)}{\partial t^2} \right) \; (L-s)$$

Substituting this definition into equation (19), along with equations (35) and (37), gives

$$\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) \cdot \frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) + \frac{\partial \bar{r}}{\partial s} \cdot \lim_{s \to L} \left(\frac{1}{L-s} \quad \frac{-1}{m_{SAT}} \; \bar{T}(L,t)\right.$$

$$\frac{1}{mSAT} \bar{K}^p \; (L,t) \; - \bar{\omega} \times (\bar{\omega} \times \bar{r}(2,t)) \; - 2 \; \bar{\omega} \times \frac{\partial \bar{r}(L,t)}{\partial t} \; - \dot{\bar{\omega}} \times \bar{r}(L,t)$$

$$+ \frac{1}{\rho} \frac{\partial}{\partial s} \bar{T}(s,t) - \frac{1}{\rho} \bar{K}(s,t) \; + \bar{\omega} \times (\bar{\omega} \times \bar{r}(s,t))$$

$$+ \dot{\bar{\omega}} \times \bar{r}(s,t) + 2 \; \bar{\omega} \times \frac{\partial \bar{r}(s,t)}{\partial t} = 0$$

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rearranging terms gives

$$\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) \cdot \frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) + \frac{\partial \bar{r}}{\partial s} \cdot \lim_{s\ 1}\left(\frac{1}{1-s}\right) \left[- \frac{1}{m_{SAT}} \bar{T}(L,t)\right.$$

$$+ \frac{1}{\rho} \frac{\partial}{\partial s} \bar{T}(s,t) + \frac{1}{m_{SAT}} \bar{K}^{*}(L,t) - \frac{1}{\rho} \bar{K}(s,t)$$

$$- \bar{\omega} \times (\bar{\omega} \times [r(L,t) - r(s,t)]) - 2\, \bar{\omega} \times$$

$$\left[\frac{\partial \bar{r}(L,t)}{\partial t} - \partial \bar{r}(s,t)\partial t\right] - \dot{\bar{\omega}} \times [\bar{r} \times (L,t) - \bar{r}(s,t)] = 0$$

Taking the limit, where feasible, gives

$$\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) \cdot \frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) + \frac{\partial \bar{r}}{\partial s} \cdot \lim_{s \to L}\left(\frac{1}{L-s}\right)\left[\frac{1}{m_{SAT}} \bar{T}(L,t)\right.$$

$$\left. + \frac{1}{\rho} \frac{\partial}{\partial s} T(s,t) + \frac{1}{m_{SAT}} \bar{K}^{*}(L,t) - \frac{1}{\rho} \bar{K}(s,t)\right]$$

$$- \frac{\partial \bar{r}}{\partial s} \cdot \omega \times \left(\omega \times \frac{\partial \bar{r}(L,t)}{\partial s}\right) - \frac{\partial \bar{r}}{\partial s} \cdot \partial \bar{\omega} \times \frac{\partial}{\partial s} \frac{\partial \bar{r}(L,t)}{\partial t}\ \frac{\partial \bar{r}}{\partial s}\ \dot{\bar{\omega}} \times \frac{\partial \bar{r}(L,t)}{\partial s} = 0$$

The last term vanishes, and the final result is,

$$\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) \cdot \frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) + \frac{\partial \bar{r}}{\partial s} \cdot \lim_{s \to L}\left(\frac{1}{L-s}\right) \cdot \left[\frac{-1}{m_{SAT}} \bar{T}(L,t) + \frac{1}{\rho}\frac{\partial}{\partial s} \bar{T}(s,t)\right.$$

$$\left. + \frac{1}{m_{SAT}} K^{*}(L,t) - \frac{1}{\rho}\bar{K}(s,t)\right] - \frac{\partial \bar{r}(L,t)}{\partial s}(L,t) \cdot \bar{\omega} \times \left(\bar{\omega} \times \frac{\partial \bar{r}}{\partial s}(L,t)\right.$$

$$\cdot \frac{\partial \bar{r}}{\partial s}(L,t) \cdot \partial \bar{\omega} \times \frac{\partial}{\partial s} \frac{\partial \bar{r}(L,t)}{\partial t} = 0$$

or,

$$\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) \cdot \frac{\partial}{\partial s}\frac{\partial \bar{r}}{\partial t} + \frac{\partial \bar{r}}{\partial s} \cdot \lim_{s \to L}\left(\frac{1}{L-s}\right) \left[\frac{-1}{\partial m_{SAT}} \bar{T}(L,t)\right.$$

$$+ \frac{1}{\rho}\frac{\partial T}{\partial s} \cdot \frac{\partial \bar{r}}{\partial s} + \frac{1}{\rho} T \frac{\partial^2 \bar{r}}{\partial s^2} + \frac{1}{m_{SAT}} K^{*}(L,t) - \frac{1}{\rho}\bar{K}(s,t)$$

$$- \frac{\partial \bar{r}(L,t)}{\partial s} \cdot \bar{\omega} \times \frac{\partial \bar{r}(L,t)}{\partial s} - \frac{\partial \bar{r}(L,t)}{\partial s} \cdot 2\,\bar{\omega} \times \frac{\partial}{\partial s}\frac{\partial \bar{r}(L,t)}{\partial t} = 0 \qquad (39)$$

For the space vehicle, the corresponding quantity is:

$$\frac{\partial}{\partial s}\frac{\partial^2 \bar{r}(0,t)}{\partial t} = \lim_{s \to 0} \frac{1}{s} \left| \frac{\partial^2 \bar{r}(s,t)}{\partial t^2} - \frac{\partial^2 \bar{r}(0,t)}{\partial t^2} \right|$$

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Substituting this expression into equation 37.1 using equation (37) gives

$$\frac{\partial}{\partial s} \cdot \frac{\partial \bar{r}}{\partial t} \cdot \frac{\partial}{\partial s} \frac{\partial \bar{r}}{\partial t} + \frac{\partial r}{\partial s} \cdot \left\{ \lim_{s \to 0} \frac{1}{s} \left[\frac{-1}{\rho} \frac{\partial}{\partial s} \bar{T}(s,t) \right.\right.$$

$$\left.\left. + \frac{1}{\rho} \bar{K}(s,t) \right] \right\} - \bar{\omega} \times (\bar{\omega} \times \bar{r}(s,t) \quad - \dot{\omega} \times \bar{r}(s,t)$$

$$- 2 \bar{\omega} \times \frac{\partial r}{\partial t} (s,t) \quad - \frac{\partial^2 \bar{r}(0,t)}{\partial t^2} \quad - \frac{\partial^2 r^*(0,t)}{\partial t^2} = 0 \qquad (40)$$

The quantity $\frac{\partial^2 \bar{r}(0,t)}{\partial t^2}$ is the relative acceleration of 0' at 0' and hence is zero.

Use the following:

$$\lim_{S \to 0} \frac{1}{s} [\bar{\omega} \times \omega \times \bar{r}(s,t)] = \omega \times \omega \times \frac{\partial \bar{r}(0,t)}{\partial t}$$

since $\bar{r}(0,t) = 0$, and similarly

$$\lim_{s \to 0} [\dot{\omega} \times \bar{r}(s,t)] = \dot{\omega} \times \frac{\partial \bar{r}(0,t)}{\partial s}$$

$$\lim_{s \to 0} \frac{1}{s} [2\omega \times \frac{\partial r(s,t)}{\partial t}] = 2 \bar{\omega} \times \frac{\partial}{\partial t} \lim_{s \to 0} \frac{1}{s} \bar{r}(s,t)$$

$$= 2 \bar{\omega} \times \frac{\partial}{\partial t} \frac{\partial \bar{r}(0,t)}{\partial s} = 2 \omega \times \frac{\partial}{\partial s} \frac{\partial \bar{r}(0,t)}{\partial t}$$

Equation (40 becomes)

$$\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) \cdot \frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial t}\right) + \frac{\partial \bar{r}}{\partial s} \cdot \lim_{s \to 0}\left(\frac{1}{s}\right) \quad \frac{1}{\rho}\frac{\partial}{\partial s}\bar{T}(s,t) + \frac{1}{\rho}\bar{K}(s,t)$$

$$- \frac{\partial \bar{r}}{\partial s} \cdot \omega \times \omega \times \frac{\partial \bar{r}(0,t)}{\partial s} - \frac{\partial \bar{r}}{\partial s} \cdot 2\,\omega \times \frac{\partial}{\partial s}\frac{\partial \bar{r}(0,t)}{\partial t}$$

$$- \frac{\partial \bar{r}}{\partial s} \cdot \dot{\bar{\omega}} \times \frac{\partial \bar{r}(0,t)}{\partial s} - \frac{\partial \bar{r}}{\partial s} \cdot \frac{\partial^2 \bar{r}^*(0,t)}{\partial t^2} = 0 \qquad (41)$$

where the last term vanishes.

7. Numerical Solution Algorithm

A. Relations and Definitions

Define the following quantities:

$\bar{r}(s,t_i)$ = Position of element at s at time t_i relative to space vehicle (0')

$\frac{\partial\bar{r}(s,t_i)}{\partial t}$ = Velocity of element at s at time t_i relative to space vehicle (0')

$\frac{\partial\bar{r}(s,t_i)}{\partial s}$ = Unit vector having a direction of the tangent at s at t_i

$\frac{\partial^2\bar{r}(s,t_i)}{\partial s^2}$, $\frac{\partial^3\bar{r}(s,t_1)}{\partial s^3}$ - First and second derivatives of $\frac{\partial\bar{r}}{\partial s}$

$\frac{\partial}{\partial t}\frac{\partial\bar{r}(s,t_i)}{\partial s}$ $(=\frac{\partial}{\partial s}\frac{\partial\bar{r}(s,t_i)}{\partial t})$ - rate of change of the unit vector $\frac{\partial\bar{r}}{\partial s}$ or with respect to time or rate of change of the velocity vector $\frac{\partial\bar{r}}{\partial t}$ with respect to s - all in the rotating coordinate system.

$\bar{K}(s,t_i)$ = all external forces on s except for tension, per unit length.

$\bar{K}(0,t_i)^*$ and $\bar{k}(1,t_i)^*$ = external forces on the space vehicle and satellite, respectively

$$\bar{A}(s,t_i) = -\frac{1}{\rho}\frac{\partial\bar{r}(s,t_i)}{\partial s} \tag{46}$$

$$\overline{AA}(s,t_i) = \frac{1}{\rho}\frac{\partial^2\overline{r}(s,t_i)}{\partial s^2} \tag{47}$$

$$\overline{B}(s,t_i) = \frac{1}{\rho}\overline{K}(s,t_i) - \overline{\omega} \times (\omega \times \overline{r}(s,t_i)) - 2\,\overline{\omega} \times \frac{\partial\overline{r}(s,t_i)}{\partial t} - \dot{\overline{\omega}} \times \overline{r}(s,t_i) \tag{48}$$

$$C(s,t_i) = \frac{\partial}{\partial s}\left(\frac{\partial\overline{r}(s,t,_i)}{\partial t}\right) \cdot \frac{\partial}{\partial s}\left(\frac{\partial\overline{r}(s,t_i)}{\partial t}\right) \tag{49}$$

$$= \frac{\partial}{\partial t}\,\frac{\partial\overline{r}(s,t_i)}{\partial s} \cdot \frac{\partial}{\partial t}\,\frac{\partial\overline{r}(s,t_i)}{\partial s} \tag{50}$$

$$D(1,t_i) = \frac{1}{m_{SAT}}\,\overline{k}(1,t_i) - 2\,\omega \times \frac{\partial\overline{r}}{\partial t} - \overline{\omega} \times (\overline{\omega} \times \overline{r}) - \dot{\overline{\omega}} \times \overline{r}(s,t_i) \tag{51}$$

$$E(s,t_i) = +\frac{1}{\rho}\frac{\partial\overline{k}(s,t_i)}{\partial s} \cdot \frac{\partial\overline{r}(s,t_i)}{\partial s}$$

$$- \frac{\partial\overline{r}(s,t_i)}{\partial s} \cdot \overline{\omega} \times \left(\overline{\omega} \times \frac{\partial\overline{r}}{\partial s}\right) \tag{52}$$

$$- \frac{\partial\overline{r}(s,t_i)}{\partial s} \cdot 2\,\overline{\omega} \times \frac{\partial}{\partial s}\frac{\partial\overline{r}}{\partial t} \tag{53}$$

$$G(s,t_i) = \frac{1}{\rho}\frac{\partial\overline{r}(s,t_i)}{\partial s} \cdot \frac{\partial\overline{r}(s,t_i)}{\partial s} \tag{54}$$

$$H(s,t_i) = \frac{-1}{\rho}\frac{\partial\overline{r}(s,t_i)}{\partial s} \cdot \frac{\partial^3\overline{r}(s,t_i)}{\partial s^3} \tag{55}$$

$$EE(L,t_i) = -\frac{\partial\bar{r}(L,t_i)}{\partial s} \cdot \bar{\omega} \times \left(\bar{\omega} \times \frac{\partial\bar{r}(L,t_i)}{\partial s}\right)$$

$$-\frac{\partial\bar{r}(L,t_i)}{\partial s} \cdot 2\,\bar{\omega} \times \frac{\partial}{\partial s}\frac{\partial\bar{r}(L,t)}{\partial t} \tag{55.1}$$

$$FF(s,t_i) = \frac{-2}{\rho} \cdot \frac{\partial\bar{r}}{\partial s} \cdot \frac{\partial^2\bar{r}}{\partial s^2} \tag{55.2}$$

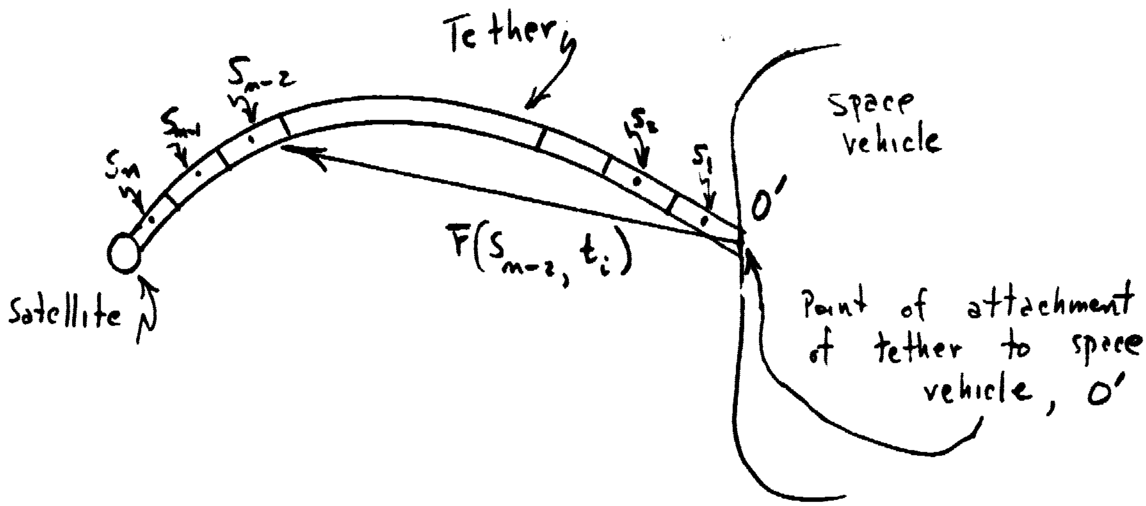

Figure 4

At time step t_i, assume the following are known from the previous integration step, or given as initial conditions and let L be the length of the tether:

$$\bar{r}(s_j,t_i), \quad \frac{\partial \bar{r}(s_j,t_i)}{\partial t}$$

for $0 \le s_j \le L; j = 1,2,\ldots,n$

The quantity s_j indicates the center of the j^{th} element, as shown in figure 4. The integer j indicates the segment, beginning with s, adjacent to the space vehicle, with n elements. The vector $\bar{r}(s_j,t_i)$ is the position vector to s_j from the point of attachment, 0'. The vector $\frac{\partial \bar{r}}{\partial t}(s_j,t_i)$ is the velocity of the middle of segment s_i with respect to the point of attachment, 0'.

The point of attachment, 0' as discussed above, is placed at the beginning of segment s_j, at $\bar{r}(s=0,t_i) = 0$.

The center of mass of the satellite is placed at the end of segment s_n, at $\bar{r}(s = L,t_i)$. In all of the following, the difference $s_{j+1} - s_j$ can vary with j, allowing for variable integration step size in s, along the tether.

Develop a subroutine to calculate $(\bar{K}(s,t)$ for any value of s and t. Develop a subroutine that can calculate the following quantities, for any value s, at time t_i, when given $\bar{K}(s_j,t_i)$, $\bar{r}(s_j,t_i),\frac{\partial \bar{r}}{\partial t}(s_j,t_i)$ and $s_j; j = 1,2,\ldots n$, for s between s_{j+1} and s_j.

For j = 1,2, ..., n-1:

$$\overline{r}(s,t_i) = \overline{r}(s_j t_i) = \overline{r}(s_{j+1}),t_i) + \frac{\overline{r}(s_{j+1},t_i) - r(s_j,t_i)}{s_{j+1} - s_j} \cdot (s - s_j) \qquad (56)$$

$$\frac{\partial\overline{r}}{\partial t}(s,t_i) = \frac{\partial\overline{r}}{\partial t}(s_j,t_i) + \frac{\dfrac{\partial\overline{r}(s_{j+1},t_i)}{\partial t} - \dfrac{\partial\overline{r}(s_j,t_i)}{\partial t}}{s_{j+1} - s_j} \cdot (s-s_j) \qquad (57)$$

$$\overline{K}(s,t_j) = \overline{K}(s_j t_i) + \frac{\overline{K}(s_j+1) - \overline{K}(s,t_i)}{s_{j+1} - s_j} \cdot (s-s_j) \qquad (58)$$

$$\frac{\partial K}{\partial s}(s,t_i) = \frac{\overline{K}(s,t_i) - \overline{K}(s_j,t_i)}{s-s_j} \qquad (59)$$

$$\frac{\partial\overline{r}}{\partial s}(s,t_i) = \frac{\overline{r}(s,t_i) - \overline{r}(s_{j,},t_i)}{s - s_j} \qquad (60)$$

$$\frac{\partial^2\overline{r}(s,t_i)}{\partial s^2} = \frac{\dfrac{\partial\overline{r}(s,t_i)}{\partial s} - \dfrac{\partial\overline{r}\ (s_j t_i)}{\partial s}}{s-s_j} \qquad (61)$$

$$\frac{\partial^3\overline{r}(s,t_i)}{\partial s^2} = \frac{\dfrac{\partial^2\overline{r}(s,t_i)}{\partial s^2} - \dfrac{\partial^2\overline{r}(s_j,t_i)}{\partial s^2}}{s-s_j} \qquad (62)$$

$$\frac{\partial}{\partial s}\left[\frac{\partial}{\partial t}\overline{r}(s,t_i)\right] = \left[\frac{\partial r}{\partial t}(s,t_i) - \frac{\partial r}{\partial t}(s_j,t_i)\right]\ (s-s_j) \qquad (63)$$

For j = n ($s_n < s \leq 1$):

$$\overline{r}(s,t_i)=\overline{r}(s_n,t_i) + \frac{r(s_n,t_i) - r(s_{n-1}t_i}{s_n-s_{n-1}}(s-s_n) \tag{64}$$

$$\frac{\partial\overline{r}(s,t_i)}{\partial t} = \frac{\partial\overline{r}(s_n,t_i)}{\partial t}$$

$$+ \frac{\frac{\partial r(s_n, t_i)}{\partial t} - \partial r(s_n-1,t_i)}{s_n - s_{n-1}} \cdot (s-s_n) \tag{65}$$

$$\overline{K}(s,t_i) = \overline{K}(s_nt_i) + \frac{\overline{K}(s_n,t_i) - \overline{K}(s_n-1,t_i)}{s_n-S_{n-1}} (s-s_n) \tag{66}$$

$$\frac{\partial K}{\partial s}(s,t_i) = \frac{K(s,t_i) - \overline{K}(s_nt_i)}{s-s_n} \tag{67}$$

$$\frac{\partial\overline{r}}{\partial s}(s,t_i) = \frac{\overline{r}(s,t_i) - \overline{r}(s_n,t_i)}{s-s_n} \tag{68}$$

$$\frac{\partial^2\overline{r}}{\partial s^2} = \frac{\frac{\partial\overline{r}(s,t,_i)}{\partial s} - \frac{\partial\overline{r}(s_nt_i)}{\partial s}}{s-s_n} \tag{69}$$

$$\frac{\partial^3\overline{r}\ (s,t_i)}{\partial s^2} = \frac{\frac{\partial^2 r(s,t_i)}{\partial s^2} - \frac{\partial^2 r(s,t_i)}{\partial s^2}}{s - s_n} \tag{70}$$

$$\frac{\partial}{\partial s}\frac{\partial r(s,t_i)}{\partial t} = \frac{\frac{\partial\overline{r}(s,t_i)}{\partial t} - \frac{\partial\overline{r}(s_nt_i)}{\partial t}}{s - s_n} \tag{71}$$

calculate $\dfrac{\partial^2 \bar{r}(0,t_j)}{\partial t^2}$ from eqts (25.1) and (25.2) and calculate

$$\frac{\partial \bar{r}^*}{\partial t}(0,t_i), \tag{72}$$

$\bar{r}(0,t_i)^*$ $\omega\ (t_i)$ and $\dot{\bar{\omega}}\ (t_i)$.

If circular motion is defined, then

$$\bar{r} = a\bar{k} \tag{73}$$

$$\dot{\bar{r}} = 0 \tag{74.1}$$

$$\dot{\omega} = 0 \tag{74.2}$$

and $\bar{\omega}$ is given by Eqt. (33)

Then,

- with (60), calculate (46)
- with (56,(57),(64),(58) or (67) calculate (48),
- with (63) or (71) calculate (49),
- with (58) or (66), (57) or (65), calculate (51),
- with (59) or (67), (60), (57) or (64), (71) or (63), or calculate (52)
- with (60) or (68),(65) or (72) calculate (53)
- with (60) or (68), calculate (54)
- with (62) or (70) calculate (55)
- with (61) calculate (47)
  (end of subroutine)

B. Differential Equations

The differential equations are as follows. From Equation (37):

$$\frac{\partial \bar{r}(s,t)}{\partial t} = \bar{v}(s,t) \tag{75}$$

$$\frac{\partial \bar{v}(s,t)}{\partial t} = \bar{A}(s,t)\ u(s,t) + \bar{AA}(s,t)\ T(s,t) \quad - \frac{\partial^2 \bar{r}^*(0,t)}{\partial t^2} + \bar{B}(s,t_i) \tag{76}$$

From equation (35):

$$\frac{\partial \bar{r}(1,t)}{\partial t} = \bar{v}(1,t) \tag{77}$$

$$\frac{\partial \bar{v}(1,t)}{\partial t} = \frac{-1}{m_{SAT}}\ T(L,t)\ .\ \frac{\partial \bar{r}(L,t)}{\partial s} - \frac{\partial^2 \bar{r}^*(0,t)}{\partial t^2} + \bar{D}(L,t) \tag{78}$$

where, from eqts. (24), (25.1) and (25.2),

$$\frac{\partial^2 \bar{r}^*(0,t)}{\partial t^2} = \frac{\partial^2 r^*_{s.v.}}{\partial t^2} + \frac{\partial^2 \bar{r}^*_{rel}}{\partial t^2} \tag{79}$$

$$= \frac{1}{m_{s.v.}}\ \bar{T}(0,t) + \frac{1}{m_{s.v.}} K^*_{s.v.}(t)$$

$$+ \dot{\bar{\Omega}}_{s.v.}\ x\ (\bar{r}^*(0,t) - \bar{r}^*_{s.v.}(t))$$

$$+ \bar{\Omega}_{s.v.}\ x\ (\bar{\Omega}_{s.v.}\ x\ (\bar{r}^*(0,t) - \bar{r}^*_{s.v.}(t)) \tag{80}$$

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If Assumption A,B,C,D, and E of Section 5 are made for two-body motion, we have

$$\frac{\partial^2 \bar{r}^*(0,t)}{\partial t^2} = \frac{-G\, m_0\, \bar{r}^*(0,t)}{|\bar{r}^*\ (0,t)|^3}$$

And for circular motion with radius a

$\frac{\partial^2 \bar{r}^*(0,t)}{\partial t^2} = \frac{-G\, m_0}{a^2} R$ -- with $\dot{\bar{\omega}} = 0$ and $\bar{\omega}$ given by equation (33), with $\frac{\partial \bar{r}^*(0,t)}{\partial t}$ given by equations (31) or (32). Also, with the above assumptions, A,B,C,D and E, the solution for $\bar{r}^*(0,t)$ is known analytically from the two-body problem.

If these assumptions are not made, but complete generality is retained, eqt. (80) and eqts for Ω vec, $\dot{\bar{\Omega}}$, $\bar{r}_{s.v.}$ must be integrated simultaneously with eqts. (75) & (76), to provide $\partial^2 \bar{r}^* \frac{(0,t)}{\partial t^2}$

From equation (38)

$$\frac{\partial T}{\partial s}(s,t) = U(s,t) \tag{81}$$

$$\frac{\partial U}{\partial s}(s,t) = \frac{C(s,t)}{G(s,t)} + \frac{H(s,t)}{G(s,t)}\, T(s,t) + \frac{FF(s,t)}{G(s,t)}\, U(s,t) + \frac{F(s,t)}{G(s,t)} \tag{82}$$

For the end of the tether, adjacent to the satellite, assuming the satellite is a point mass at the end of the satellite, from equation (39), letting $T_L = T(s=L,t), U_s=U(s,t)$:

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$$C(L,t) + EE(L,t) + \frac{\partial \bar{r}}{\partial s} \cdot \left\{ \frac{\lim}{s \to L} \left(\frac{1}{L-s}\right) \left[\right.\right.$$

$$- \frac{1}{m_{SAT}} \cdot T_e \cdot \frac{\partial \bar{r}}{\partial s} + \frac{1}{\rho} U_s \cdot \frac{\partial \bar{r}}{\partial s} + T_s \frac{\partial^2 \bar{r}}{\partial s^2}$$

$$\left.\left. + \frac{1}{m_{SAT}} K^*(1,t) - \frac{1}{\rho} \bar{K}(s,t) \right] \right\} = 0 \tag{83.1}$$

Discretizing equation 83.1 for the numerical integratin, and solving for T_L gives:

$$\frac{-1}{m_{SAT}} T_L \frac{\partial \bar{r}}{\partial s} \cdot \frac{\partial \bar{r}}{\partial s} = \frac{-1}{\rho} U_s \frac{\partial \bar{r}}{\partial s} \cdot \partial \bar{r} \partial s - \frac{1}{\rho} T_s \frac{\partial^2 \bar{r}}{\partial s^2} \cdot \frac{\partial \bar{r}}{\partial s}$$

$$\frac{-1}{m_{SAT}} T_L \frac{\partial \bar{r}}{\partial s} \cdot \bar{K}^*(L,t) + \frac{1}{\rho} \frac{\partial \bar{r}}{\partial s} \cdot \bar{K}(s,t)$$

$$- (L-s)\, C\,(L,t) - (L-s) EE(L,t) \tag{83.2}$$

For the end of the tether, adjacent to the space vehicle at the point of attachment, 0', from equation (41)

$$C(0,t) + EE(0,t) + \frac{\partial \bar{r}(0)}{\partial s} \cdot \left\{ \frac{\lim}{s \to 0} \frac{1}{s} \right.$$

$$\left. \left[- \frac{1}{\rho} \frac{2}{\partial s} T(s) \cdot \frac{\partial \bar{r}(s)}{\partial s} - \frac{1}{\rho} T(s) \frac{\partial^2 \bar{r}(s)}{\partial s^2} + \frac{1}{\rho} \bar{K}(s,t) \right] \right\}$$

$$- \frac{\partial \bar{r}}{\partial s} \cdot \frac{\partial^2 \bar{r}^*(0,t)}{\partial t^2} = 0 \tag{84.1}$$

Discretizing equation (84.1) for the numerical integration, and solving for $U(s) = \frac{\partial T(s)}{\partial s}$ gives:

$$- C(0,t) - EE(0,t) + \frac{\partial \bar{r}(0)}{\partial s} \cdot \frac{\partial^2 \bar{r}^*(0,t)}{\partial t^2} - \left(\frac{1}{s} \right) \frac{\partial r(s)}{\partial s} \cdot \frac{1}{\rho} \bar{K}(s,t)$$

$$+ \frac{1}{\rho} \left(\frac{1}{s} \right) \frac{\partial r(s)}{\partial s} \frac{\partial^2 \bar{r}(s)}{\partial s^2} T(s) = \frac{1}{s} \left(\frac{-1}{\rho} \right) \cdot \frac{\partial \bar{r}(s)}{\partial s} \frac{\partial T(s)}{\partial s} \cdot \frac{\partial \bar{r}(s)}{\partial s}$$

Using eqts. (46) thru (55.1)

$$\frac{\partial T(s)}{\partial s} = - \rho \cdot \frac{AA(s)\ A(s)}{G(s)} T(s) + \frac{A(s)}{G(s)} \bar{K}(s) - (s) \frac{C(0)}{G(s)} - (s) \frac{EE(0)}{G(s)}$$

$$+ (s) \frac{\partial \bar{r}(0)}{\partial s} \cdot \frac{1}{G(S)} \cdot \frac{\partial^2 \bar{r}(0_2 t)}{\partial t^2} \qquad (84.2)$$

In equation (80), define a function

$$S(\Omega, \bar{\Omega}, K^*) = \frac{1}{m_{s.v.}} K^*_{s.v.} (t)$$

$$+ \dot{\Omega}_{s.v.} \times (\bar{r}^*(0,t) - \bar{r}^*_{s.v.}(t))$$

$$+ \Omega_{s.v.} \times \Omega_{s.v.} \times (\bar{r}^*(0,t) - \bar{r}^*_{s.v.}(t))$$

Then

$$\frac{\partial^2 \bar{r}^*(0,t)}{\partial t^2} = \bar{S}(\Omega, \dot{\Omega}, K^*) + \frac{1}{m_{s.v.}} \bar{T}(0,t) \tag{84.4}$$

Substituting equation 84.4 into 84.2 gives

$$\frac{\partial T(s)}{\partial s} = - \rho \frac{\overline{AA}\ (s)\ .\ \bar{A}(s)}{G(s)}\ T(s) \quad + \frac{A(s)}{G(s)} \bar{K}(s)$$

$$- s\frac{c(0)}{G(s)} - s\frac{EE(0)}{G(s)}$$

$$+ s\ \frac{\partial \bar{r}(0)}{\partial s}\ \frac{1}{G(s)}\ S(\Omega, \dot{\Omega}, k^*) \quad + \frac{1}{m_{s.v.}}\ s\ \frac{1}{G(s)}\ \frac{\partial \bar{r}(0)}{\partial s}\ T(0) \tag{84.5}$$

C Boundary Conditions at s = L

Near the satellite s ~ 1, we need to impose certain constraints. Let the satellite be a point mass, m_{SAT}, placed at the end of the tether, at point s = L. Let s_L denote this point ($s_L = L$).

Make the following assumptions. A. Assume that close to the satellite, the quantity

$$U(s,t_i) = \frac{\partial T}{\partial s}\ (s,t_i) = c(t) \tag{85}$$

where c is a constant, with respect to s (but can be a variable in time).

B. Assume that close to the satellite the quantity $\frac{\partial}{\partial s}\left(\frac{\partial \bar{r}}{\partial s} \right)$ is a constant, also with respect to s only.

With these constraints, a boundary condition can be derived for the tension of the tether at s_L. The constraints imposed by A and B, are approximate. The constraints are obtained by examination of one equilibrium solution near the satellite. Hence when $s*s_L$ but $s-S_L$ small, the error due to constraints A and B should be small when the acceleration of the tether and satellite is small and close to an equilibrium solution.

If the acceleration of m_{SAT} becomes significant, than smaller step-sizes in s near the satellite should be chosen for the following.

Let $U = U(s,t_i)$, $T_n = T(s_n,t_i)$, $U_n = U(s_n,t_i)$, $T_L = T(s_L,t_i)$, $T_{n-1} = T(s_{n-1},t_i)$, $U_{n-1} = U(s_{n-1},t_i)$ and so on, for brevity. The quantities $T_n, U_n, T_L, U, T_{n-1}, U_{n-1}$ are depicted in figure 5.

Applying equation (8~~4~~5) to segments s_n and s_{n-1} gives

$$U_{n-1} = \frac{T_n - T_{n-1}}{s_n - s_{n-1}} = \text{Constant} = c_1 \tag{86}$$

The quantity T_L is the tension at the point of attachment of the point mass m_{SAT} to the tether at T_L.

For the segment s_n :

$$U_n \frac{T_L - T_n}{s_L - s_n} = \text{constant} = C_1 \tag{87}$$

Setting (86) and (87) equal gives

$$\frac{T_L - T_n}{s_L - s_n} = \frac{T_n - T_{n-1}}{s_n - s_{n-1}} \tag{88}$$

And solving for T_L in (88) gives

$$T_L = T_n + \frac{s_L - s_n}{s_n - s_{n-1}} (T_n - T_{n-1}) \tag{89}$$

Similarly, from Assumption B:

$$\frac{\partial r(\ell, t)}{\partial s} = \frac{\partial r(s_n, t)}{\partial s} + \frac{s_\ell - s_n}{s_n - s_{n-1}} \left(\frac{\partial \bar{r}(s_n, t)}{\partial s} - \frac{\partial \bar{r}(s_{n-1}, t)}{\partial s} \right) \tag{90}$$

Since the slope, $\frac{\partial T}{\partial s}$, is assumed constant near $s = \ell$, other equations analogous to equation (89) are :

$$T_n = T_{n-2} + \frac{s_n - s_{n-2}}{s_{n-1} - s_{n-2}} (T_{n-1} - T_{n-2}) \tag{91}$$

or

$$T_n = T_{n-1} + \frac{s_n - s_n{}^{-1}}{s_{n-1} - s_{n-2}} (T_{n-1} - T_{n-2}) \tag{92}$$

The numerical solution procedure is as follows:

The numerical integration begins at $s = 1$, with T_1 approximated, and U_1 given in the following section of this report. The integration of equations (81) and (82) procedes to S_n. Then T_L is obtained from equation (83.2), and from (89).

The value T_1 is adjusted iteratively so that the two values of T_n coincide to the desired accuracy.

Equation (90) can be used in two ways. First, the position vector along the cable can be initially set so that equation (90) is satisfied. Then r(L,t) is allowed to be determined from equations (77) and (78).

Or Equation (90) can be improved continually as a constraint on the solution for $\overline{r}(L,t)$ obtained from equations (77), (78).

D. Boundary Condition at s=0

At s=0, near the point of attachment to the space vehicle, it will be assumed that T(s,t) is a constant, or that $U(s,t)_{s=0} = 0$. This is the case for an equilibrium solution, and we assume that we are close to an equilibrium solution. Let this boundary condition be denoted as boundary condition II.

E. Algorithm for the Solution

The boundary conditions can be satisfied by various boundary value solution methods. We will discuss a standard method often referred to as a boundary value solution method using successive solutions of initial value problems. Sometimes the method is informally referred to as the "shooting method of solution." The first step is to approximate the tension at the point of attachment, T(0,t). Then with the boundary condition II, U(0,t) = 0, equations (81) and (82) can be integrated from s=0 to $s=s_1$. $\bar{T}(0,t)$ is known and U(0,t)=0 is the initial condition for the integration.

The equations (81), (82) and (83) can be integrated for $U(s_j\ t_i)$ and $T(s_j,\ t_i)$, for all s at a fixed time t_i.