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PERFORMANCE ANALYSIS OF THE WORD SYNCHRONIZATION PROPERTIES OF THE OUTER CODE IN A TDRSS DECODER

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PERFORMANCE ANALYSIS OF THE WORD SYNCHRONIZATION PROPERTIES

OF THE OUTER CODE IN A TDRSS DECODER

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In [1], a self-synchronizing coding system for NASA's TDRSS satellite system was described. The coding system used is a concatenation of a (2,1,7) inner convolutional code with a (255,223) Reed-Solomon outer code. The scheme described in [1] achieves both symbol and word synchronization without requiring that any additional symbols be transmitted. In this report we discuss the performance of the word synchronization properties of this scheme.

The outer code used is a (255,223) Reed-Solomon code over GF(q), where $q=2^8$. It has a minimum distance d=33, and therefore can be used to correct t=16 or fewer errors. Suppose that the code word $\underline{v}$ is transmitted. At the receiver, misframing is either due to a synchronization loss of ℓ symbols of $\underline{v}$, as shown in Figure 1, or to a synchronization gain of ℓ symbols from the preceding code vector, as shown in Figure 2, where $\underline{r}$ is the received word.

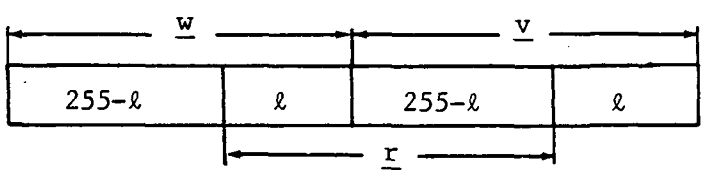

Figure 1

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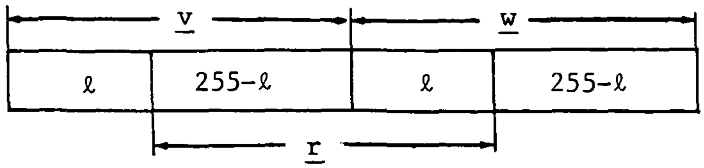

Figure 2

It was shown in [1] that the RS outer code can be used for simultaneously recovering from a sync loss (or gain) of ℓ ($\ell \leq 16$) symbols and any combination of $16-\ell$ symbol errors due to noise.

At the receiver, three outcomes are possible: (1) decoding is correct and word synchronization is acquired; (2) the decoder fails, indicating a lack of synchronization; or (3) decoding is completed but is incorrect, resulting in a false declaration of synchronization. This corresponds to an undetected error. If outcome (2) occurs, the received word is shifted 32 positions and decoded again. This procedure is repeated until correct word synchronization is acquired.

Let $P_c(\ell)$, $P_f(\ell)$, and $P_u(\ell)$ be the probabilities corresponding to the three outcomes described above, respectively, where ℓ, $0 \leq \ell \leq 255$, denotes the number of symbols gained (or lost). Obviously we have

$$P_c(\ell) + P_f(\ell) + P_u(\ell) = 1. \tag{1}$$

First, for simplicity, assume that the channel error rate $\varepsilon=0$. Then we have

$$P_f(\ell) = 0 \qquad \text{for} \quad 0 \leq \ell \leq 16 \tag{2.1}$$

$$P_f(\ell) = \sum_{i=0}^{\ell-17} \binom{\ell}{i} \left(\frac{1}{q}\right)^i \left(\frac{q-1}{q}\right)^{\ell-i} \qquad \text{for} \quad 17 \leq \ell \leq 128 \tag{2.2}$$

$$P_f(\ell) = P_f(256-\ell) \qquad \text{for} \quad 129 \leq \ell \leq 255 \tag{2.3}$$

$$P_c(\ell) = 1 \qquad \text{for} \quad 0 \leq \ell \leq 16 \tag{3.1}$$

$$P_c(\ell) = 0 \qquad \text{for} \quad 17 \leq \ell \leq 128 \tag{3.2}$$

$$P_c(\ell) = P_c(256-\ell) \qquad \text{for} \quad 129 \leq \ell \leq 255 \tag{3.3}$$

$$P_u(\ell) = 0 \qquad \text{for} \quad 0 \leq \ell \leq 16 \tag{4.1}$$

$$P_u(\ell) = 1 - P_f(\ell) \qquad \text{for} \quad 17 \leq \ell \leq 128 \tag{4.2}$$

$$P_u(\ell) = P_u(256-\ell) \qquad \text{for} \quad 129 \leq \ell \leq 255 \tag{4.3}$$

$P_f(\ell)$ and $P_u(\ell)$, based on (2) and (4), are given in table 1 and table 2, respectively.

Now consider the effect of channel errors due to noise. Suppose that any symbol which is transmitted has a probability $(1-\varepsilon)$ of being received correctly and a probability $\varepsilon/(q-1)$ of being transformed into each of the $q-1$ other symbols. Furthermore, assume that successive symbols incur errors independently. Thus, the probability that the received word differs from the transmitted word in exactly i positions is given by the expression

$$\binom{n}{i} (q-1)^i \left(\frac{\varepsilon}{q-1}\right)^i (1-\varepsilon)^{n-i} = \binom{n}{i} \varepsilon^i (1-\varepsilon)^{n-i} \tag{5}$$

Then the probabilities $P_f(\ell)$, $P_c(\ell)$, and $P_u(\ell)$ are given by

$$P_f(\ell) = \sum_{i=17-\ell}^{16} \binom{255-\ell}{i} \varepsilon^i (1-\varepsilon)^{255-\ell-i} \left\{ \sum_{m=0}^{i+\ell-17} \binom{\ell}{m} \left(\frac{1}{q}\right)^m \left(\frac{q-1}{q}\right)^{\ell-m} \right\}$$

$$+ \sum_{i=17}^{255-\ell} \binom{255-\ell}{i} \varepsilon^i (1-\varepsilon)^{255-\ell-i} \qquad \text{for} \quad 1 \leq \ell \leq 16 \tag{6.1}$$

$$P_f(0) = \sum_{i=17}^{255} \binom{255}{i} \varepsilon^i (1-\varepsilon)^{255-i} \qquad \text{for} \quad \ell=0 \tag{6.2}$$

$$P_f(\ell) = \sum_{i=0}^{17} \binom{255-\ell}{i} \varepsilon^i (1-\varepsilon)^{255-\ell-i} \left\{ \sum_{m=0}^{i+\ell-17} \binom{\ell}{m} \left(\frac{1}{q}\right)^m \left(\frac{q-1}{q}\right)^{\ell-m} \right\}$$

$$+ \sum_{i=17}^{255-\ell} \binom{255-\ell}{i} \varepsilon^i (1-\varepsilon)^{255-\ell-i} \qquad \text{for} \quad 17 \leq \ell \leq 128 \tag{6.3}$$

$$P_f(\ell) = P_f(256-\ell) \quad \text{for } 129 \le \ell \le 255 \tag{6.4}$$

$$P_c(\ell) \approx 1-P_f(\ell) \quad \text{for } 0 \le \ell \le 16 \tag{7.1}$$

$$P_c(\ell) = 0 \quad \text{for } 17 \le \ell \le 128 \tag{7.2}$$

$$P_c(\ell) = P_c(256-\ell) \quad \text{for } 129 \le \ell \le 255 \tag{7.3}$$

$$P_u(\ell) \approx 0 \quad \text{for } 0 \le \ell \le 16 \tag{8.1}$$

$$P_u(\ell) = 1-P_f(\ell) \quad \text{for } 17 \le \ell \le 128 \tag{8.2}$$

$$P_u(\ell) = P_u(256-\ell) \quad \text{for } 129 \le \ell \le 255 \tag{8.3}$$

If we let $\varepsilon=0$, (6)-(8) become (2)-(4). Evaluation of $P_f(\ell)$ based on (6.1)-(6.3) is given in Table 3.

In the following we evaluate the average number of decoding trials, and the probability of false declaration of sync. Let P(m) be the probability of performing exactly m decoding trials before word sync is acquired. Then the average number of decoding trials, E(m), is given by

$$E(m) = \sum_{m=1}^{\infty} mp(m) \tag{9}$$

where we can show (see Appendix A) that

$$P(1) = \frac{1}{256}\{P_c(0) + 2\sum_{\ell=1}^{16} P_c(\ell)\}, \tag{10.1}$$

$$P(m) \approx \frac{1}{256}\{\sum_{\ell=17}^{32} P_f(\ell)P_c(32-\ell) + \sum_{\ell=1}^{16} P_c(\ell) + [1-P_c(16)]P_c(16)\},$$

$$\text{for } m = 2,3,4,5,6, \tag{10.2}$$

$$P(7) \approx P(2) + \frac{1}{256}[1-P_c(16)]P_c(16), \tag{10.3}$$

$$P(8) \approx P(2) + \frac{1}{256}\{[1-P_c(16)]^2 P_c(16)-P_c(16)\}. \tag{10.4}$$

$$P(m) \approx \frac{1}{256}\{\sum_{\ell=17}^{32} [P_f(\ell)]^{K+1}[1-P_c(32-\ell)]^K P_c(32-\ell)$$

$$+ \sum_{\ell=1}^{16} [1-P_c(\ell)]^K P_c(\ell) + [1-P_c(16)]^{K+1} P_c(16)\},$$

$$\text{for} \quad 8K+1 \leq m \leq 8(K+1), \quad K = 1,2,3,\ldots. \tag{10.5}$$

Evaluation of $E(m)$ based on (6), (7), (9), and (10) is given in table 4 and shown in Figure 3.

Let $P^u(m)$ be the probability of a false declaration of synchronization on the m^{th} decoding trial, and let R denote the total probability of false declaration of sync. Then we have

$$R = \sum_{m=1}^{\infty} P^u(m), \tag{11}$$

where we can show (see Appendix B) that

$$P^u(1) \approx 0, \tag{12.1}$$

$$P^u(m) \approx \frac{1}{256}\{\sum_{\ell=17}^{20} P_u(\ell)\}, \qquad \text{for} \quad m = 2,3,\ldots,8, \tag{12.2}$$

$$P^u(m) \approx \frac{1}{256}\{\sum_{\ell=17}^{20} [P_f(\ell)]^K[1-P_c(32-\ell)]^K P_u(\ell)\},$$

$$\text{for} \quad 8K + 1 \leq m \leq 8(K+1), \quad \text{where} \quad K = 1,2,\ldots. \tag{12.3}$$

Evaluation of R based on (6), (8), (11), and (12) is given in table 5 and shown in Figure 4.

From Figure 3 we see that $E(m)$ increases very quickly when $\varepsilon > 10^{-3}$. When the channel becomes too noisy ($\varepsilon \geq 5 \times 10^{-3}$), it is possible that word sync may never be acquired. In this case the best strategy to achieve word sync would be to reduce the number of positions shifted between decoding trials to 16 from 32.

From the discussion above we see that an important parameter which determines the performance of the word sync procedure is the ratio of the decoding failure probability $P_f(\ell)$ to the undetected error probability $P_u(\ell)$. Ideally, $P_u(\ell)$ should be as small as possible compared to $P_f(\ell)$ when the error-correcting-capability of the code is exceeded ($\ell > 16$). A computer simulation of a (255,223) Reed-Solomon code was carried out, and results for $P_f(\ell)$ and $P_u(\ell)$ are given in tables 6 and 7, respectively. Comparing them with the formula results in tables 1-3, we see that they are very close.

Reference

[1] S. Lin, D.J. Costello, W. Miller, and J. Morakis, "Self-Synchronizing Outer Codes for the TDRSS Decoder," Proceedings IEEE Global Telecommunications Conference, pp. 30.7.1-30.7.4, San Diego, CA, November 1983.

APPENDIX A

From (7.1) -(7.3) we see that correct decoding and word sync acquisition is possible only when ℓ, the amount of sync loss (or gain) satisfies the following inequalities:

$$0 \leq \ell \leq 16 \tag{A1.1}$$

or

$$240 \leq \ell \leq 255. \tag{A1.2}$$

Note that it is possible that the decoder cannot achieve correct decoding due to noise even though ℓ satisfies (A1.1) or (A1.2), and that decoding is certainly incorrect if ℓ is not within the region specified by (A1.1) -(A1.2). In either case the received word must be shifted and decoded a number of times in order to acquire word sync, with each shift equal to 32 symbol positions. Let P(m) denote the probability of exactly m decoding trials. Then we have

$$P(1) = \frac{1}{256} \{ \sum_{\ell=0}^{16} P_c(\ell) + \sum_{\ell=240}^{255} P_c(256-\ell)\}$$

$$= \frac{1}{256} \{P_c(0) + 2 \sum_{\ell=1}^{16} P_c(\ell)\}. \tag{A2}$$

$$P(2) = \frac{1}{256} \{ \sum_{\ell=17}^{32} P_f(\ell)P_c(32-\ell) + \sum_{\ell=33}^{48} P_f(\ell)P_c(\ell-32)$$

$$+ [1-P_c(16)]P_c(256-240)\}. \tag{A3}$$

$$P(3) = \frac{1}{256} \{ \sum_{\ell=49}^{64} P_f(\ell)P_f(\ell-32)P_c(64-\ell) + \sum_{\ell=65}^{80} P_f(\ell)P_f(\ell-32)P_c(\ell-64)$$

$$+ P_f(48)[1-P_c(48-32)]P_c(256-240)\}. \tag{A4}$$

From Tables 1 and 3 we see that $P_f(\ell)$ is very close to 1 for $\ell \geq 20$. So (3) and (4) can be reduced to

$$P(2) \approx P(3) \approx \frac{1}{256} \{ \sum_{\ell=17}^{32} P_f(\ell)P_c(32-\ell) + \sum_{\ell=1}^{16} P_c(\ell) + [1-P_c(16)]P_c(16)\}. \tag{A5}$$

In the same way we can obtain

$$P(m) \approx P(2), \quad \text{for} \quad m = 4,5,6, \tag{A6}$$

$$P(7) \approx P(2) + \frac{1}{256}[1-P_c(16)]P_c(16). \tag{A7}$$

$$P(8) \approx P(2) + \frac{1}{256}\{[1-P_c(16)]^2 P_c(16)-P_c(16)\}. \tag{A8}$$

For $8K + 1 \leq m \leq 8(K+1)$, where $K = 1,2,3,\ldots$, we have

$$P(m) \approx \frac{1}{256}\{\sum_{\ell=17}^{32} [P_f(\ell)]^{K+1} [1-P_c(32-\ell)]^K P_c(32-\ell)$$

$$+ \sum_{\ell=1}^{16} [1-P_c(\ell)]^K P_c(\ell) + [1-P_c(16)]^{K+1} P_c(16)\}. \tag{A9}$$

APPENDIX B

Let $P^u(m)$ denote the probability of a false declaration of word sync on the m^{th} decoding trial. From (8.1)-(8.3) and tables 1-3 we see that $P_u(\ell) \approx 0$ for $0 \le \ell \le 16$, $240 \le \ell \le 255$, and $20 \le \ell \le 236$. Using this fact we have

$$P^u(1) = \frac{1}{256}\left\{\sum_{\ell=0}^{16} P_u(\ell) + \sum_{\ell=240}^{255} P_u(\ell)\right\} \approx 0 \tag{B1}$$

$$P^u(2) = \frac{1}{256}\left\{\sum_{\ell=17}^{32} P_u(\ell) + \sum_{\ell=33}^{48} P_u(\ell)\right\} \approx \frac{1}{256}\left\{\sum_{\ell=17}^{20} P_u(\ell)\right\}. \tag{B2}$$

$$P^u(3) = \frac{1}{256}\left\{\sum_{\ell=49}^{64} P_f(\ell)P_u(\ell-32) + \sum_{\ell=65}^{80} P_f(\ell)P_u(\ell-32)\right\}$$

$$\approx \frac{1}{256}\left\{\sum_{\ell=17}^{20} P_u(\ell)\right\}. \tag{B3}$$

Similarly, we can obtain

$$P^u(m) \approx \frac{1}{256}\left\{\sum_{\ell=17}^{20} P_u(\ell)\right\}, \qquad \text{for} \quad m = 4,5,6,7,8 \tag{B4}$$

$$P^u(m) \approx \frac{1}{256}\left\{\sum_{\ell=17}^{20} [P_f(\ell)]^K [1-P_c(32-\ell)]^K \, P_u(\ell)\right\},$$

$$\text{for} \quad 8K + 1 \le m \le 8(K+1), \quad \text{where } K = 1,2,3,\ldots \tag{B5}$$

Table 1. Probability of decoding failure $P_f(\ell)$ with $\varepsilon = 0$.

ℓ	$P_f(\ell)$
≤ 16	0
17	.93562896
18	.99776057
19	.99994449
20	.99999893
≥ 21	$>$.99999900

Table 2. Probability of undetected error $P_u(\ell)$ with $\varepsilon = 0$.

ℓ	$P_u(\ell)$
≤ 16	0
17	$.643710 \times 10^{-1}$
18	$.223943 \times 10^{-2}$
19	$.551120 \times 10^{-4}$
20	$.107302 \times 10^{-5}$
≥ 21	$\leq .175672 \times 10^{-7}$

Table 3 (1) Probability of Decoding Failure $P_f(\ell)$

ℓ \ ε	10^{-7}	10^{-6}	10^{-5}
1	$.882116 \times 10^{-87}$	$.881927 \times 10^{-71}$	$.88004 \times 10^{-55}$
2	$.553494 \times 10^{-81}$	$.553383 \times 10^{-66}$	$.552273 \times 10^{-51}$
3	$.326877 \times 10^{-75}$	$.326811 \times 10^{-61}$	$.326159 \times 10^{-47}$
4	$.180889 \times 10^{-69}$	$.180853 \times 10^{-56}$	$.180494 \times 10^{-43}$
5	$.933214 \times 10^{-64}$	$.93303 \times 10^{-52}$	$.93119 \times 10^{-40}$
6	$.446193 \times 10^{-58}$	$.446106 \times 10^{-47}$	$.445233 \times 10^{-36}$
7	$.196344 \times 10^{-52}$	$.196305 \times 10^{-42}$	$.195925 \times 10^{-32}$
8	$.788615 \times 10^{-47}$	$.788464 \times 10^{-38}$	$.786951 \times 10^{-29}$
9	$.286227 \times 10^{-41}$	$.286173 \times 10^{-33}$	$.285631 \times 10^{-25}$
10	$.927185 \times 10^{-36}$	$.927012 \times 10^{-29}$	$.925286 \times 10^{-22}$
11	$.263875 \times 10^{-30}$	$.263827 \times 10^{-24}$	$.263347 \times 10^{-18}$
12	$.646339 \times 10^{-25}$	$.646225 \times 10^{-20}$	$.645084 \times 10^{-15}$
13	$.132472 \times 10^{-19}$	$.13245 \times 10^{-15}$	$.13226 \times 10^{-11}$
14	$.218107 \times 10^{-14}$	$.218073 \times 10^{-11}$	$.217729 \times 10^{-8}$
15	$.270443 \times 10^{-9}$	$.270405 \times 10^{-7}$	$.270031 \times 10^{-5}$
16	$.22449 \times 10^{-4}$	$.224467 \times 10^{-3}$	$.224242 \times 10^{-2}$
17	.93563045	.93564381	.93577725
18	.99776062	.99776109	.99776576
19	.99994489	.99994490	.99994502
20	.99999893	.99999893	.99999893
≥ 21	$\geq .99999990$	$\geq .99999990$	$\geq .99999990$

Table 3(2) Probability of Decoding Failure $P_f(\ell)$

ℓ \ ε	10^{-4}	10^{-3}	5×10^{-3}	10^{-2}
1	$.861389 \times 10^{-39}$	$.695220 \times 10^{-23}$	$.408272 \times 10^{-12}$	$.806700 \times 10^{-8}$
2	$.541298 \times 10^{-36}$	$.442769 \times 10^{-21}$	$.550947 \times 10^{-11}$	$.582823 \times 10^{-7}$
3	$.319709 \times 10^{-33}$	$.261821 \times 10^{-19}$	$.657607 \times 10^{-10}$	$.354656 \times 10^{-6}$
4	$.176944 \times 10^{-30}$	$.145067 \times 10^{-17}$	$.732838 \times 10^{-9}$	$.199566 \times 10^{-5}$
5	$.912989 \times 10^{-28}$	$.74947 \times 10^{-16}$	$.761881 \times 10^{-8}$	$.104689 \times 10^{-4}$
6	$.436594 \times 10^{-25}$	$.358935 \times 10^{-14}$	$.734996 \times 10^{-7}$	$.510228 \times 10^{-4}$
7	$.192157 \times 10^{-22}$	$.158256 \times 10^{-12}$	$.653655 \times 10^{-6}$	$.229693 \times 10^{-3}$
8	$.771928 \times 10^{-20}$	$.637136 \times 10^{-11}$	$.531743 \times 10^{-5}$	$.948368 \times 10^{-3}$
9	$.280268 \times 10^{-17}$	$.231915 \times 10^{-9}$	$.392017 \times 10^{-4}$	$.356140 \times 10^{-2}$
10	$.908203 \times 10^{-15}$	$.753963 \times 10^{-8}$	$.2589691 \times 10^{-3}$	$.120448 \times 10^{-1}$
11	$.258591 \times 10^{-12}$	$.215575 \times 10^{-6}$	$.15116574 \times 10^{-2}$	$.362575 \times 10^{-1}$
12	$.63378 \times 10^{-10}$	$.531314 \times 10^{-5}$	$.7659994 \times 10^{-2}$	$.957902 \times 10^{-1}$
13	$.130009 \times 10^{-7}$	$.109847 \times 10^{-3}$	$.32929854 \times 10^{-1}$	.21845707
14	$.214324 \times 10^{-5}$	$.183229 \times 10^{-2}$	.116471360	.42203251
15	$.26632 \times 10^{-3}$	$.23218 \times 10^{-1}$	.325018357	.67780659
16	$.22201 \times 10^{-1}$	.20120591	.675814520	.89572830
17	.93709658	.94889303	.979742897	.99366426
18	.99781195	.99822460	.999300290	.99978266
19	.99994615	.99995632	.999982800	.99999466
20	.99999895	.99999915	.999999665	.99999990
≥ 21	$\geq .99999990$	$\geq .99999990$	$\geq .999999995$	$\geq .99999999$

Table 4. Average Number of Decoding Trials E(m)

ε	E(m)
0	4.4635
10^{-7}	4.4635
10^{-6}	4.4637
10^{-5}	4.4648
10^{-4}	4.4722
10^{-3}	4.6804
5×10^{-3}	5.9659
10^{-2}	9.2061

Table 5. Probability of False Declaration of Synchronization R

ε	R
0	1.8229×10^{-3}
10^{-7}	1.8228×10^{-3}
10^{-6}	1.8224×10^{-3}
10^{-5}	1.8186×10^{-3}
10^{-4}	1.7813×10^{-3}
10^{-3}	1.4517×10^{-3}
5×10^{-3}	6.9107×10^{-4}
10^{-2}	3.9156×10^{-4}

Table 6. Probability of Decoding Failure $P_f(\ell)$
(computer simulation results)

ℓ \ ε	0	5×10^{-3}	10^{-2}
15	0	.29056	.64711
16	0	.66047	.88419
17	.93360	.98024	.99423
18	.99822	.99941	.99949
19	.99991	1	1
≥ 20	1	1	1

Table 7. Probability of Undetected Error $P_u(\ell)$
(computer simulation results)

ℓ \ ε	0	5×10^{-3}	10^{-2}
17	$.664 \times 10^{-1}$	$.1976 \times 10^{-1}$	$.5770 \times 10^{-2}$
18	$.178 \times 10^{-2}$	$.5900 \times 10^{-3}$	$.5100 \times 10^{-3}$
19	$.900 \times 10^{-4}$	0	0
≥ 20	0	0	0

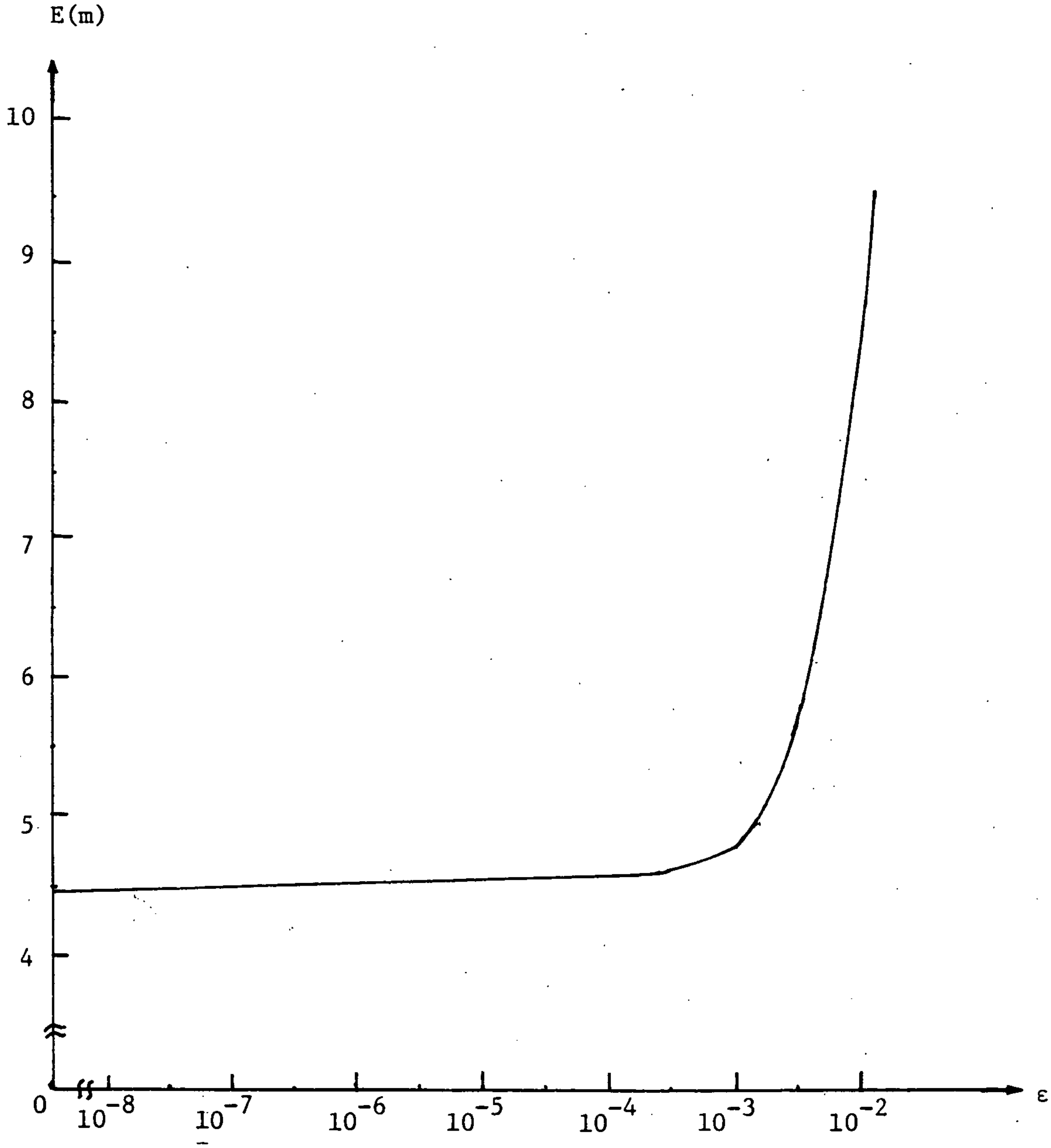

Figure 3. The Average Number of Decoding Trials

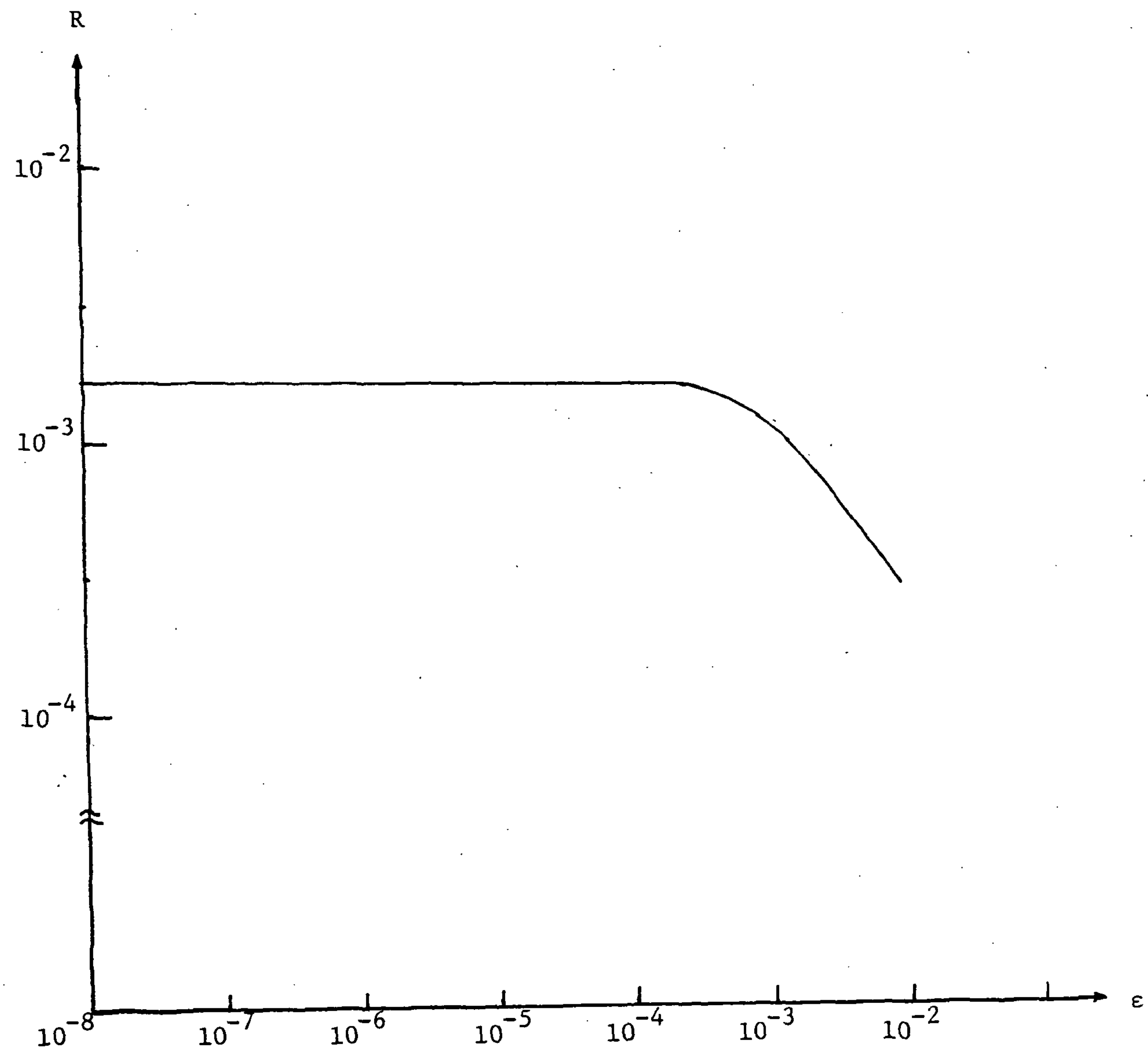

Figure 4. The Probability of False Declaration of Synchronization