NASA-TM-85812

NASA Technical Memorandum 85812

NASA-TM-85812 19840017879

CANONICAL SOLUTIONS FOR UNSTEADY FLOW FIELDS

GRACE C. LIU

MAY 1984

NASA
National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

TABLE OF CONTENTS

N84-25947#

SYMBOLS

a	acceleration of flow
A	variable used to define equation (3.29)
B	variable used to define equation (3.23)
c	speed of sound, $c^2 = \gamma p/\rho$
c_0	speed of sound at $x = 0$, $t = 0$
C^-	characteristic line with $dx/dt = u - c$
C^+	characteristic line with $dx/dt = u + c$
D	variable used to define equation (2.32)
E	variable used to define equation (2.33)
h	enthalpy
$k(\eta)$	acceleration of a characteristic line
m	mass flow rate
M	Mach number
OA	head characteristic of the expansion fan
OB	tail characteristic of the expansion fan
p	pressure
q	energy formation
Q	flow properties
R	universal gas constant R_0 divided by the effective molecular weight of the particular gas
T	temperature
u	velocity of flow
v	relative velocity
x	displacement
α	dimensionless heat of reaction, $\alpha = q\rho_2/p_2$

γ	ratio of specific heat, $\gamma = c_p/c_\nu$
η	angular coordinate, $\eta = x/c_0 t$
ξ	slope of the characteristic line near the origin
ξ_1	slope of the tail characteristic line OB at origin
ξ_2	slope of the head characteristic line OA at origin
π	dimensionless final pressure ratio, p_1/p_2
ρ	density
ν	specific volume ratio, ρ_2/ρ_1

Subscripts

d	reaction front
p	piston
s	shock
t	partial derivative with respect to time
x	partial derivative with respect to space
1	region behind the shock wave or region behind the expansion fan
2	region ahead of the wave, i.e., free stream
3	region inside the expansion fan
4	region between the Chapman-Jouguet detonation and expansion fan
5	region between precompression wave and Chapman-Jouguet deflagration wave

Superscripts

(0)	leading term
(1)	first order term
–	normalized quantity
~	flow properties on higher pressure side
*	Chapman-Jouguet reaction front

CHAPTER 1

INTRODUCTION

1.1 Purpose of Paper

This paper presents derivations of canonical solutions for the initial value problem of one-dimensional unsteady gas dynamics with discontinuous nonuniform initial data. The canonical solutions are constructed by assuming (i) regular power series expansions for regions ahead of and behind each discontinuity wave, namely, a shock wave (with or without chemical reaction) or a contact line, and (ii) special expansions within a centered expansion fan. The solutions define not only the speeds but also the accelerations of the discontinuity waves and the expansion waves. These canonical solutions are intended to be employed as the building blocks in a numerical scheme for the solution of a general initial value problem. To understand the need for, and the importance of, this work, a brief review of related methods is included in the next section.

1.2 Review of Work

A number of fluid dynamics problems as well as other physical problems are dominated by the presence of strong discontinuities. In order to compute the solutions to these problems accurately, the discontinuities have to be understood and specific information about their generation and propagation must be included in numerical solution schemes. Many computational methods for gas flow are based on approximating the problem with a number of elementary or canonical flow problems called Riemann problems. An example of a Riemann problem is the simple shock tube problem for which initial data, discontinuous but otherwise uniform, is given. The canonical problem provides an explicit and elementary class of solutions which contain extensive information about wave interaction. The existence and uniqueness of the solution of the Riemann problem for gas dynamics, subject to an appropriate formulation of the entropy condition, was established by Liu [1]. Glimm recently developed a front tracking code with the aim of providing a general and flexible method for obtaining accurate solutions to problems which are piecewise smooth [2]. The Riemann problem provides the key input to this method. Chorin introduced the random choice method as a numerical tool for solving hyperbolic systems [3]. The method is based on a constructive existence proof due to Glimm [4]. The solution is advanced in time by a sequence of operations which includes the solution of Riemann problems and a sampling procedure. Chorin further illustrated the usefulness of the method in the analysis of reacting gas flow [5]. Examples were given of time-dependent detonation and deflagration waves, with infinite and finite reaction rates.

For a steady supersonic flow, the flow quantities and their gradients behind a shock can be related to the flow quantities ahead of the shock and to the slope and the curvature of the shock [6]. These relationships can be used in the numerical computation of steady flow fields with shock waves, and because they explicitly account for nonuniformities on either side of the shocks, they are inherently superior to uniform flow relations. A more extensive application of these ideas was carried out by Darden [7] in conjunction with calculation of sonic boom in the vertical plane of symmetry near a point of coalescence of two shocks. Near such a point, there exist regions bounded by shocks or by a shock and a slip stream or an expansion wave. To include asymmetric lifting effects, the second circumferential derivatives of the flow quantities in each region in the vertical plane are needed. Explicit relationships between the various derivatives of the flow quantities at the point of coalescence and the slopes and curvatures of the shocks, the slip streams, and the expansion fans were derived in reference 7.

In the current paper, canonical solutions for one-dimensional gas dynamics problems with discontinuous and nonuniform initial data are presented. They are valid in a small space-time region around the discontinuities. The canonical solutions relate the first partial derivatives of the flow quantities with respect to t and x to the velocities and accelerations of the resulting discontinuity waves. These local solutions can then be used to replace the Riemann problem as the building block in a numerical scheme for a general initial value problem. The advantage of the new local solutions is that they permit the use of a much larger step size than methods which assume uniform flow in each mesh.

1.3 Outline of Discussion

The general initial value problem is treated by decomposing it into elementary problems which involve a shock, an expansion fan, and a contact line. Chapter 2 is devoted to the problem with one shock wave. A typical case is that in which a piston moves into $x > 0$ with given initial data for $x > 0$. The piston can later be replaced by a contact line in the initial value problem. However, it is found that the system of equations for the derivatives obtained by perturbing the shock conditions becomes singular when the shock strength approaches zero and, therefore, they are unsuitable for use with weak shocks. The system of equations can be regrouped such that a common factor which vanishes for zero shock strength is cancelled analytically. For shocks of zero strength, the curvature of the shock differs from that of the characteristic ahead and behind the shock because of a jump in the curvature of the streamline which remains continuous in slope. Solutions are constructed by assuming regular power series expansions for regions ahead of and behind each discontinuity wave.

Chapter 3 describes results for problems involving an expansion fan. The typical problem is that of a piston at $x = 0$, $t = 0$, which moves away from the gas lying in $x > 0$. As before, regular power series are used for the regions ahead of and behind the expansion fan, and a special expansion is employed for the centered expansion fan region. In this region, the usual governing equations for one-dimensional flow are rewritten in Riemann invariant form in order to uncouple the variables.

For an arbitary initial value problem, a shock wave, a centered expansion wave, and a contact line will exist simultaneously. The preceding two solutions for shock waves and centered expansion waves can be combined by regarding the piston path as a contact line. The results of this generalization are presented in Chapter 4.

Elementary solutions are given in Chapter 5 which include flow with chemical reactions. The cases of a strong detonation wave, a Chapman-Jouguet detonation wave followed by an expansion fan, and a Chapman-Jouguet deflagration with a precompression shock wave are studied. These solutions can be combined with the results in Chapters 2 and 3 to describe the complete solution for initial value problems including chemical reactions.

Except for the material in Chapter 5, explicit formulas for the speed and acceleration of the discontinuity waves and the spatial and time derivatives of the flow properties behind the waves are obtained in each chapter. In the cases including chemical reaction, the appropriate systems of equations are established, but explicit evaluation of flow properties is omitted for brevity.

CHAPTER 2

RELATIONSHIP ACROSS ONE-DIMENSIONAL UNSTEADY SHOCK WAVES

Figure 1 shows the initial value problem induced by a forward moving piston $x_p = F(t)$ with $\dot{F}(0) = u_p > 0$ and $\ddot{F}(0) = a_p$. Initial data which have continuous right-sided first derivatives at $t = 0$ for $x > 0$ are prescribed. The problem here is to determine the flow field for small t near $x = 0$; that is, to find the flow quantities and their first derivatives behind the shock, region I, and the shock path. Quantities in region II ahead of the shock wave are completely defined by the initial data and can be determined independently of the motion of the piston.

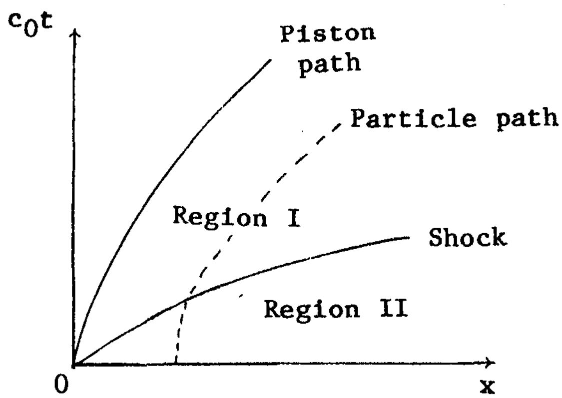

Figure 1.- Shock wave.

2.1 Governing Equations

The differential equations governing one-dimensional, isentropic flow of the medium, except at discontinuities, are as follows:

$$\rho_t + \rho u_x + u\rho_x = 0 \qquad \text{conservation of mass} \tag{2.1}$$

$$\rho(u_t + uu_x) + p_x = 0 \qquad \text{conservation of momentum} \tag{2.2}$$

$$\rho_t + u\rho_x = \frac{1}{c^2}(p_t + up_x) \qquad \text{conservation of energy} \tag{2.3}$$

where ρ is the density, u is the velocity, p is the pressure, c is the speed of sound, and where x, t are independent distance and time variables, respectively. Subscripts denote partial derivatives with respect to x or t.

2.2 Shock Conditions

Across a shock, the jump conditions are given as follows [8]:

$$\rho_1 v_1 = \rho_2 v_2 = m \tag{2.4}$$

$$p_1 + mv_1 = p_2 + mv_2 \tag{2.5}$$

$$h_1 + \frac{1}{2} v_1^2 = h_2 + \frac{1}{2} v_2^2 \tag{2.6}$$

where the subscripts 1 or 2 indicate the region behind or ahead of the shock. The notation v_j, $j = 2,1$ stands for the velocity of the fluid relative to the shock in front of and behind the shock, respectively; that is, $v_j = u_j - u_s(t)$, $j = 1,2$. Enthalpy is denoted by h and m is the mass flow rate.

Before perturbing the shock conditions, note that for weak shocks these equations become linear equations for $u_2 - u_1$, $p_2 - p_1$, $\rho_2 - \rho_1$. The determinant of the system of linear equations vanishes as the shock strength goes to zero. Therefore, for weak shocks the solution of these equations will be in terms of ratios of small numbers and will induce inaccuracy in any numerical scheme. In order to avoid this difficulty, the basic equations should be recombined to cancel out a common factor which vanishes as the shock strength goes to zero. With the ideal gas relationship $h = \gamma RT/(\gamma - 1)$, equation (2.6) becomes

$$v_2^2 - v_1^2 = \frac{2\gamma R}{\gamma - 1}(T_1 - T_2) \tag{2.7}$$

Rewriting equation (2.5) using $p = \rho RT$ and equation (2.4) yields

$$v_2 - v_1 = \frac{p_1 - p_2}{m} = \frac{R}{m}(\rho_1 T_1 - \rho_2 T_2) = (v_2 - v_1)\left[\frac{v_2 + v_1}{v_1}\left(\frac{\gamma - 1}{2\gamma}\right) + \frac{RT_2}{v_1 v_2}\right]$$

where equation (2.7) has been used to substitute for T_1. Now the factor $(v_2 - v_1)$ can be cancelled to obtain

$$2\gamma v_1 v_2 - (\gamma - 1)(v_2 + v_1)v_2 = 2\gamma RT_2 = 2c_2^2$$

Finally, the speed of sound relation $c_2^2 = \gamma p_2/\rho_2 = \gamma RT_2$ is used to arrive at

$$(\gamma - 1)v_2^{\,2} - (\gamma + 1)v_1 v_2 + 2c_2^{\,2} = 0 \tag{2.8}$$

Equations (2.4), (2.5), and (2.8) are regarded as the basic shock conditions. They are, of course, valid for finite shocks, but the determinant of the system of linearized equations will now remain finite as the shock strength approaches zero.

2.3 Method of Solution

The flow quantities for small t behind the shock are related, not only to the slope and curvature of the shock and to the initial velocity and acceleration of the piston, but also to the flow properties ahead of the shock. The following data are provided:

1. The flow properties for $x > 0$ at $t = 0$. Therefore, the first partial derivatives with respect to x of the properties are known at $t = 0$.
2. The velocity u_p and acceleration a_p of the piston at $t = 0$.

The piston path $x_p(t)$ is then, to second order in t,

$$x_p(t) = u_p t + \frac{1}{2} a_p t^2 \tag{2.9}$$

In regions I and II, the solutions are regular. Hence, their spatial and time derivatives exist as x and t approach 0^+ in each region. If Q

represents any flow quantity and $j = 1,2$ represents the regions, then the power series expansion about (0,0) of any quantity is

$$Q_j(x,t) = Q_j(0,0) + x\frac{\partial}{\partial x} Q_j(0,0) + c_0 t \frac{\partial}{c_0 \partial t} Q_j(0,0)$$

$$+ \text{ higher order terms}$$

or

$$Q_j(x,t) = Q_j^{(0)} + xQ_{jx} + c_0 t\, Q_{jt} + \text{ higher order terms} \tag{A}$$

where $Q_j^{(0)} = Q_j(0,0)$, $Q_{jx} = \frac{\partial}{\partial x} Q_j(0,0)$, and $Q_{jt} = \frac{1}{c_0}\frac{\partial Q_j}{\partial t}$ at (0,0). The speed of sound at (0,0) in region II is taken as the reference speed c_0. Let the shock front velocity and acceleration at time $t = 0$ be denoted by u_s and a_s, respectively. One has

$$x_s(t) = \dot{x}_s(0)t + \frac{1}{2}\ddot{x}_s(0)t^2 + 0(t^3),$$

and

$$u_s(t) = u_s + a_s t + 0(t^2) \tag{B}$$

The procedure used in the following derivation is outlined as follows:

1. Substitution of the regular power series expansion into the governing equations (2.1) to (2.3) and use of the initial data $u_2^{(0)}$, $p_2^{(0)}$, $\rho_2^{(0)}$, u_{2x}, p_{2x}, and ρ_{2x}, to define the flow properties in region II, the region ahead of the shock.
2. Substitution of the expansion series (A) and (B) into the shock conditions (2.4), (2.5), and (2.8) and the governing equations (2.1) to (2.3), and then comparison of the coefficients of like powers of

t to define the velocity and acceleration of the shock and the flow properties behind the shock. The boundary condition on the piston path which must be satisfied is

$$u_1(x_p,t) = \frac{dx_p}{dt} = u_p + a_p t \tag{2.10}$$

The results obtained for the leading terms should be exactly the same as those for a shock separating two uniform regions. The detailed results for the acceleration of the shock and the derivatives of the flow properties are derived in the next section.

2.4 Explicit Solutions

The spatial derivatives of the flow properties for $x = 0^+$ at $t = 0$, that is, $Q_2^{(0)}$ and Q_{2x} in the expansion series (A), are given. Substituting the power series expansion (A) into equations (2.1) to (2.3) defines the time derivatives in region II in the form

$$\rho_{2t} = \frac{-1}{c_0}\left(\rho_2^{(0)}u_{2x} + \rho_{2x}\, u_2^{(0)}\right) \tag{2.11}$$

$$u_{2t} = \frac{-1}{c_0}\left(\frac{p_{2x}}{\rho_2^{(0)}} + u_2^{(0)}u_{2x}\right) \tag{2.12}$$

$$p_{2t} = \frac{-1}{c_0}\left(c_2^{(0)^2}\rho_2^{(0)}u_{2x} + u_2^{(0)}p_{2x}\right) \tag{2.13}$$

Thus the flow properties and all their x- and t- derivatives in region II are defined.

Now we proceed to the determination of quantities in region I. First, the expansion series (A) are substituted into the shock conditions (2.4), (2.5) and (2.8), and the results are evaluated on the shock path (B). Then the constant terms and the coefficients of t are equated to zero in each equation. If it is noted that the relative velocities v_j are

$$v_2 = \left(u_2^{(0)} - u_s\right) + \left(u_{2x}u_s + u_{2t}c_0 - a_s\right)t \tag{2.14}$$

$$v_1 = \left(u_1^{(0)} - u_s\right) + \left(u_{1x}u_s + u_{1t}c_0 - a_s\right)t \tag{2.15}$$

then the constant terms yield the standard equations for a straight shock front:

$$\rho_1^{(0)}\left(u_1^{(0)} - u_s\right) = \rho_2^{(0)}\left(u_2^{(0)} - u_s\right) \tag{2.16}$$

$$p_1^{(0)} + \rho_2^{(0)}\left(u_1^{(0)} - u_s\right)\left(u_2^{(0)} - u_s\right) = p_2^{(0)} + \rho_2^{(0)}\left(u_2^{(0)} - u_s\right)^2 \tag{2.17}$$

$$(\gamma + 1)\left(u_1^{(0)} - u_s\right)\left(u_2^{(0)} - u_s\right) - (\gamma - 1)\left(u_2^{(0)} - u_s\right)^2 = 2c_2^{(0)^2} \tag{2.18}$$

The coefficients of t yield three more equations relating the derivatives of the quantities ahead and behind the shock front to its initial acceleration:

$$(\rho_{1x}u_s + \rho_{1t}c_0)\left(u_1^{(0)} - u_s\right) + \rho_1^{(0)}(u_{1x}u_s + u_{1t}c_0 - a_s)$$

$$= (\rho_{2x}u_s + \rho_{2t}c_0)\left(u_2^{(0)} - u_s\right) + \rho_2^{(0)}\ (u_{2x}u_s + u_{2t}c_0 - a_s) \qquad (2.19)$$

$$(p_{1x}u_s + p_{1t}c_0) + \rho_2^{(0)}\left(u_1^{(0)} - u_s\right)(u_{2x}u_s + u_{2t}c_0 - a_s)$$

$$+ \rho_2^{(0)}\left(u_2^{(0)} - u_s\right)(u_{1x}u_s + u_{1t}c_0 - a_s)$$

$$+ (\rho_{2x}u_s + \rho_{2t}c_0)\left(u_1^{(0)} - u_s\right)\left(u_2^{(0)} - u_s\right)$$

$$= (p_{2x}u_s + p_{2t}c_0)$$

$$+ 2\rho_2^{(0)}\left(u_2^{(0)} - u_s\right)(u_{2x}u_s + u_{2t}c_0 - a_s)$$

$$+ (\rho_{2x}u_s + \rho_{2t}c_0)\left(u_2^{(0)} - u_s\right)^2 \qquad (2.20)$$

$$(\gamma + 1)\left(u_1^{(0)} - u_s\right)(u_{2x}u_s + u_{2t}c_0 - a_s)$$

$$+ (\gamma + 1)\left(u_2^{(0)} - u_s\right)(u_{1x}u_s + u_{2t}c_0 - a_s)$$

$$- 2(\gamma - 1)\left(u_2^{(0)} - u_s\right)(u_{2x}u_s + u_{2t}c_0 - a_s)$$

$$= 2c_2{}^{(0)2}\left[\frac{p_{2x}u_s + p_{2t}c_0}{p_2^{(0)}} - \frac{\rho_{2x}u_s + \rho_{2t}c_0}{\rho_2^{(0)}}\right] \qquad (2.21)$$

Now the boundary condition, equation (2.10), at $t = 0$, requires that

$$u_1^{(0)} = u_p \tag{2.22}$$

Then equation (2.18) defines the velocity of the shock as a function of the piston velocity and flow properties at (0,0). This is the well-known result for a shock separating two uniform regions. A quadratic equation on u_s is obtained in the form

$$\begin{aligned} 2u_s^2 + \left[(\gamma - 3)u_2^{(0)} - (\gamma + 1)u_p\right]u_s + (\gamma + 1)u_p u_2^{(0)} \\ - (\gamma - 1)u_2^{(0)^2} - 2c_2^{(0)^2} = 0 \end{aligned} \tag{2.23}$$

from which u_s is determined as

$$u_s = \frac{\gamma + 1}{4} u_p - \frac{\gamma - 3}{4} u_2^{(0)} \pm \frac{\gamma + 1}{4} \sqrt{(u_2^{(0)} - u_p)^2 + \left(\frac{4c_2^{(0)}}{\gamma + 1}\right)^2} \tag{2.24}$$

In equation (2.24), the + sign is for a forward facing shock, that is, with the gas on the right-hand side, $x > 0$, and the - sign is for a backward facing shock, with the gas on the left-hand side, $x < 0$. Now using the shock speed from equation (2.24) in equations (2.16) and (2.17) with $c^2 = \gamma p/\rho$, the leading terms of the flow properties behind the shock can be found. It is easily seen that those results are exactly the same as the well-known results for a shock separating two uniform regions and are given by:

$$\rho_1^{(0)} = \frac{\rho_2^{(0)}\left(u_2^{(0)} - u_s\right)}{(u_p - u_s)} \tag{2.25}$$

$$p_1^{(0)} = p_2^{(0)} + \rho_2^{(0)}\left(u_2^{(0)} - u_s\right)\left(u_2^{(0)} - u_p\right) \tag{2.26}$$

At this stage of the analysis all the leading terms $Q_1^{(0)}$ are known.

To determine the first-order terms Q_{1x} and Q_{1t}, the expansion series (A) are substituted into equations (2.1) to (2.3). The resulting three equations are evaluated in region I using the fact that $u_1^{(0)} = u_p$ (eq. 2.22). Similarly, the series (A) is substituted into equation (2.10). This yields

$$\rho_{1x}u_p + \rho_{1t}c_0 = -\rho_1^{(0)}u_{1x} \tag{2.27}$$

$$u_{1x}u_p + u_{1t}c_0 = \frac{-p_{1x}}{\rho_1^{(0)}} \tag{2.28}$$

$$p_{1x}u_p + p_{1t}c_0 = -{c_1^{(0)}}^2\rho_1^{(0)}u_{1x} \tag{2.29}$$

$$u_{1x}u_p + u_{1t}c_0 = a_p \tag{2.30}$$

Equations (2.19) to (2.21) and (2.27) to (2.30) represent a system of seven equations for the six derivatives of the flow quantities in region I at the point (0,0) and the acceleration of the shock a_s.

To solve these seven equations, it is noted that equations (2.27) to (2.30) can be used in equations (2.20) and (2.21) to derive a set of linear equations on a_s and u_{1x} as follows:

$$\begin{bmatrix} -(u_2^{(0)} - u_p) & \dfrac{u_2^{(0)} - u_s}{u_p - u_s}\left[c_1^{(0)^2} + (u_p - u_s)^2\right] \\ -(u_2^{(0)} - u_p) + \dfrac{4}{\gamma + 1}(u_2^{(0)} - u_s) & (u_2^{(0)} - u_s)(u_p - u_s) \end{bmatrix} \begin{bmatrix} a_s \\ u_{1x} \end{bmatrix} = \begin{bmatrix} 2a_p(u_2^{(0)} - u_s) + D \\ (u_2^{(0)} - u_s)a_p + E \end{bmatrix} \tag{2.31}$$

The quantities D and E in equation (2.31) are defined as

$$D = \frac{-1}{\rho_2^{(0)}}\Big\{(\rho_{2x}u_s + \rho_{2t}c_0) + \rho_2^{(0)}(u_{2x}u_s + u_{2t}c_0)\left[(u_2^{(0)} - u_s) + (u_2^{(0)} - u_p)\right]$$

$$+ (\rho_{2x}u_s + \rho_{2t}c_0)(u_2^{(0)} - u_s)(u_2^{(0)} - u_p)\Big\} \tag{2.32}$$

$$E = \frac{1}{\gamma + 1}\Big\{(u_{2x}u_s + u_{2t}c_0)\left[(\gamma + 1)(u_p - u_s) - 2(\gamma - 1)(u_2^{(0)} - u_s)\right]$$

$$- 2c_2^{(0)2}\left[\frac{p_{2x}u_s + p_{2t}c_0}{p_2^{(0)}} - \frac{\rho_{2x}u_s + \rho_{2t}c_0}{\rho_2^{(0)}}\right]\Big\} \tag{2.33}$$

Then the shock curvature a_s follows as

$$a_s = \frac{a_p(u_2^{(0)} - u_s)\left[(u_p - u_s)^2 - c_1^{(0)^2}\right] + (D-E)(u_p - u_s)^2 - c_1^{(0)^2}E}{(u_2^{(0)} - u_p)c_1^{(0)^2} - \frac{4}{\gamma + 1}(u_2^{(0)} - u_s)\left[(u_p - u_s)^2 + c_1^{(0)^2}\right]} \tag{2.34}$$

and u_{1x} is

$$u_{1x} = \frac{a_p(u_p - u_s)\left[(u_2^{(0)} - u_p) - \frac{8}{\gamma + 1}(u_2^{(0)} - u_s)\right] + \frac{(u_2^{(0)} - u_p)(u_p - u_s)}{u_2^{(0)} - u_s}(D - E) - \frac{4}{\gamma + 1}D(u_p - u_s)}{(u_2^{(0)} - u_p)c_1^{(0)^2} - \frac{4}{\gamma + 1}(u_2^{(0)} - u_s)\left[(u_p - u_s)^2 + c_1^{(0)^2}\right]} \tag{2.35}$$

The denominator in these expressions is rewritten in the form

$$- \{c_1^{(0)^2}[(u_p - u_s) + \frac{3 - \gamma}{\gamma + 1}(u_2^{(0)} - u_s)] + \frac{4}{\gamma + 1}(u_2^{(0)} - u_s)(u_p - u_s)^2\}$$

Now, $u_p - u_s = v_1$ and $u_2^{(0)} - u_s = v_2$. Therefore, since $\rho_1 v_1 = \rho_2 v_2$, $u_p - u_s$ and $u_2^{(0)} - u_s$ have the same sign, this denominator cannot vanish.

Now use of equations (2.27) to (2.30) in equation (2.19) allows ρ_{1x} to be determined as

$$\rho_{1x} = \frac{(\rho_{2x}u_s + \rho_{2t}c_0)(u_2^{(0)} - u_s) + \rho_2^{(0)}(u_{2x}u_s + u_{2t}c_0) + (\rho_1^{(0)} - \rho_2^{(0)})a_s - \rho_1^{(0)}a_p + 2\rho_2^{(0)}(u_2^{(0)} - u_s)u_{1x}}{(u_p - u_s)^2} \tag{2.36}$$

and the solution for p_{1x} is directly expressed with equations (2.28) and (2.30) as

$$p_{1x} = -a_p \rho_1^{(0)} \tag{2.37}$$

Finally, equations (2.27) to (2.29) are used to define the time derivatives of the flow properties at (0,0) in the form

$$u_{1t} = \frac{a_p - u_{1x}u_p}{c_0} \tag{2.38}$$

$$\rho_{1t} = \frac{-(\rho_1^{(0)}u_{1x} + \rho_{1x}u_p)}{c_0} \tag{2.39}$$

$$p_{1t} = \frac{\rho_1^{(0)}\left(a_p u_p - c_1^{(0)^2} u_{1x}\right)}{c_0} \tag{2.40}$$

The shock problem with a piston moving into the flow has now been solved completely for small t, near $x = 0$. First, equations (2.11) to (2.13) give the time derivatives for the region ahead of the shocks, which are completely defined by the initial data, $Q_2^{(0)}$ and Q_{2x}, independent of the piston motion. Next, equation (2.24) gives the velocity of the shock, and then equations (2.22), (2.25), and (2.26) define the leading terms of the flow properties behind the shock; they are exactly the same as the well-known results for a shock separating two uniform regions. Equation (2.34) yields the acceleration of the shock. The derivatives of the flow properties behind the shock at the point (0,0), that is, u_{1x}, ρ_{1x}, p_{1x}, u_{1t}, ρ_{1t}, p_{1t}, are determined in equations (2.35) to (2.40), respectively. Thus, the velocity and acceleration of the shock as well as the first three terms in the power series expansion of all the flow quantities behind the shock have been defined in this chapter.

CHAPTER 3

CENTERED EXPANSION WAVES

The typical problem involving a centered expansion wave is that in which a piston starts to move at $t = 0$ away from a gas which lies in $x > 0$. The problem in this investigation is to determine the relation of the initial flow properties in $x > 0$ and the velocity and acceleration of the piston to the flow properties in the expansion fan, the boundary of the fan, and the flow properties behind the fan.

3.1 Governing Equations

The governing equations for one-dimensional flow are equations (2.1) to (2.3). These equations can be rewritten in order to uncouple the variables. By using Riemann's invariant approach, the characteristics of the differential equations of one-dimensional isentropic flow can be found. Combining equations (2.1), (2.3) and $c^2 = \gamma p/\rho$, yields

$$\gamma p u_x + p_t + u p_x = 0$$

Using equation (2.2) one has

$$c(\rho u u_x + \rho u_t + p_x) = 0$$

Adding and subtracting these two equations yields

$$p_t + (u + c)p_x + \rho c[u_t + (u + c)u_x] = 0 \tag{3.1}$$

$$p_t + (u - c)p_x - \rho c[u_t + (u - c)u_x] = 0 \tag{3.2}$$

Total differentiation of both sides of $c^2 = \gamma p/\rho$ with respect to t and use of equation (2.3) gives

$$2c\,\frac{dc}{dt} = \frac{\gamma - 1}{\rho}\,\frac{dp}{dt} \tag{3.3}$$

This relationship can be applied in equations (3.1) and (3.2), to rewrite the system as

$$\left(\frac{\partial}{\partial t} + (u + c)\,\frac{\partial}{\partial x}\right)\left(u + \frac{2}{\gamma - 1}\,c\right) = 0 \tag{3.4}$$

$$\left(\frac{\partial}{\partial t} + (u - c)\,\frac{\partial}{\partial x}\right)\left(u - \frac{2}{\gamma - 1}\,c\right) = 0 \tag{3.5}$$

Equation (3.3), with the relationship $c^2 = \gamma p/\rho$, can be written as

$$2p\,\frac{dc}{dt} = \frac{\gamma - 1}{\gamma}\,c\,\frac{dp}{dt}$$

or

$$2p\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right)c = \frac{\gamma - 1}{\gamma}\,c\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right)p \tag{3.6}$$

Now equations (3.4) and (3.5) express the relationship between u and c, and equation (3.6) can be used to define the pressure p. These three equations can then be used to replace the original system of equations (2.1) to (2.3).

3.2 Method of Solution

Define region I as the region behind the expansion fan. Region II is the region ahead of the expansion fan where the flow properties are not affected by the motion of the piston. The expansion fan itself is called region III (Fig. 2).

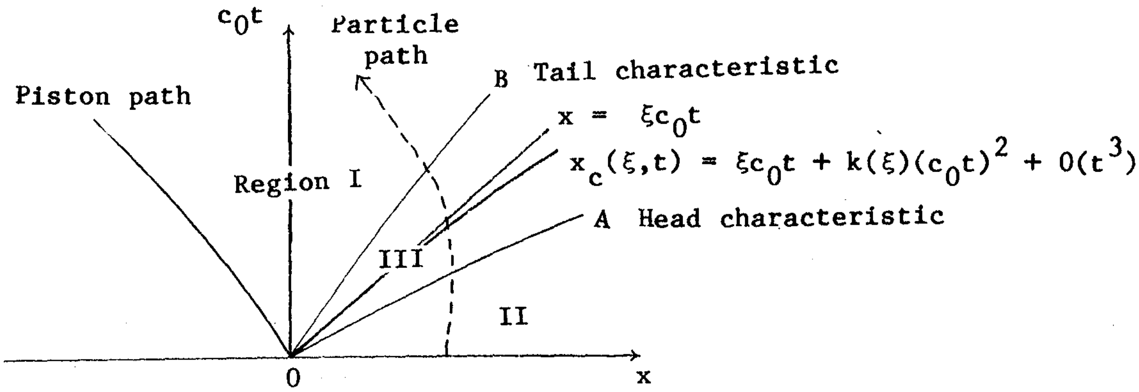

Figure 2.- Expansion fan.

OA is the characteristic line that separates regions III and II, and OB is the characteristic line that separates regions III and I. For a forward facing expansion wave, OA and OB are forward characteristic lines C^+, on which $dx/dt = u + c$. For a backward-facing expansion wave, they are C^- characteristic lines with $dx/dt = u - c$.

Near the center of the expansion fan, the flow quantities must jump from their values at OA to those at OB. Locally, they behave as a simple expansion wave and are functions of the slope of the characteristic lines at the center (in the x, c_0t plane). It is convenient to introduce the angular coordinate

$\eta = x/c_0 t$ as an independent variable to replace x. With η and t as independent variables in region III, any flow quantity, $Q_3(x,t)$, can be expressed as a power series in t with coefficients which are functions of η:

$$Q_3(x,t) = Q_3^{(0)}\left(\frac{x}{c_0 t}\right) + Q_3^{(1)}\left(\frac{x}{c_0 t}\right)c_0 t + \text{higher order terms}$$

$$= Q_3^{(0)}(\eta) + Q_3^{(1)}(\eta)\ c_0 t + \text{higher order terms} \tag{C}$$

A characteristic curve will be specified by a parameter ξ, where ξ is the slope of the curve at the origin (see fig. 3).

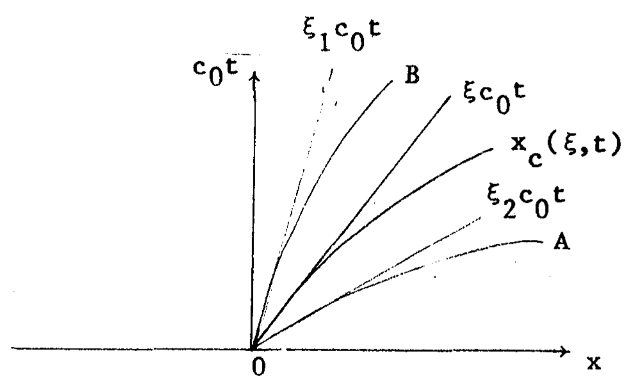

Figure 3.- Geometry of the expansion.

Thus, the family of characteristics is given by $x = x_c(\xi,t)$ where $x_c(\xi,t) = \xi c_0 t + k(\xi)(c_0 t)^2 + O(t^3)$ in which $k(\xi) = d^2x_c/2d(c_0 t)^2\big|_{t=0}$ is to be determined in the course of the analysis. If ξ_2 and ξ_1 are the slopes of the first (OA) and last (OB) characteristics at the origin, then the family of characteristics is given by $\xi_1 < \xi < \xi_2$. Note that, by definition, the angular coordinate is related to the parameter ξ on a characteristic through

$$\eta = \frac{x_c(\xi,t)}{c_0 t} = \xi + k(\xi)c_0 t + \text{higher order terms} \tag{3.7}$$

3.3 Outline of the Steps

The mathematical procedure is to substitute expansions (A), (B) and (C) into the governing equations and then to compare the coefficients of like powers of t. The outline of the steps follows:

1. Determination the flow properties in region II by substituting series (A) into equations (2.1) to (2.3). This will then determine the first characteristic line in region III, OA, and its slope and curvature, by use of the boundary condition which requires that the flow properties are continuous across OA.
2. Determination of the flow properties in the expansion fan, region III, by substitution of series (C) into the governing equations (3.4) to (3.6).
3. Determination of the last characteristic line in III, OB, and its slope and curvature.
4. Application of the continuity condition along OB and determination of the first derivatives of the flow properties in region I, which must be compatible with the velocity and acceleration of the piston.

The results for centered expansion waves, the velocity and acceleration of the waves, and the gradients of the flow properties are derived in next section.

3.4 Explicit Solutions

The time derivatives for flow properties in region II are obtained the same way as for the shock problem in Chapter II. The equation of the characteristic OA is

$$x = x_c(\xi_2,t) = \xi_2 c_0 t + k(\xi_2)(c_0 t)^2 + \dots \tag{3.7a}$$

The flow properties are required to be continuous across OA so that

$$Q_2^{(0)} + Q_{2x} x_c(\xi_2,t) + Q_{2t} c_0 t = Q_3(x_c(\xi_2,t),t)$$

$$= Q_3^{(0)}(\eta_2) + Q_3^{(1)}(\eta_2) c_0 t$$

where $\eta_2 = x_c(\xi_2,t)/c_0 t = \xi_2 + k(\xi_2) c_0 t + 0(t^2)$ from equation (3.7). Thus, using Taylor series expansion, this equation becomes (up through linear terms in t),

$$Q_2^{(0)} + Q_{2x}\xi_2 c_0 t + Q_{2t} c_0 t$$

$$= Q_3^{(0)}(\xi_2) + k(\xi_2) c_0 t \; Q_3^{(0)\prime}(\xi_2) + Q_3^{(1)}(\xi_2) c_0 t \tag{3.8}$$

Now, comparing the coefficients of like powers of t in equation (3.8), it is found that

$$Q_3^{(0)}(\xi_2) = Q_2^{(0)} \tag{3.9}$$

and

$$Q_3^{(1)}(\xi_2) = Q_{2x}\xi_2 + Q_{2t} - k(\xi_2)Q_3^{(0)\prime}(\xi_2) \tag{3.10}$$

Equations (3.9) and (3.10) represent boundary conditions on the functions $Q_3^{(0)}(\eta)$ and $Q_3^{(1)}(\eta)$ at the point $\eta = \xi_2$ and will be used in the following steps.

By the definition of the characteristic lines, $dx/dt = u + c$ for forward-facing expansion waves, and $dx/dt = u - c$ for backward-facing expansion waves. Thus

$$\begin{aligned} \xi_2 c_0 + 2k(\xi_2)c_0^2 t = u_2^{(0)} + u_{2x}\xi_2 c_0 t + u_{2t}c_0 t \\ \pm (c_2^{(0)} + c_{2x}\xi_2 c_0 t + c_{2t}c_0 t) \end{aligned} \tag{3.11}$$

Comparing coefficients of the same power of t, one obtains

$$\xi_2 = \frac{u_2^{(0)} \pm c_2^{(0)}}{c_0} \tag{3.12}$$

and

$$k(\xi_2) = \frac{u_{2x} \pm c_{2x}}{2c_0}\xi_2 + \frac{u_{2t} \pm c_{2t}}{2c_0} \tag{3.13}$$

Hence, the slope and curvature of the head characteristic OA are defined.

To proceed with the second step in the analysis, the special expansion (C) is used in region III. The spatial and time derivatives are as follows:

$$Q_{3t}(\eta,t) = -Q_3^{(0)\prime}(\eta)\frac{\eta}{t} - Q_3^{(1)\prime}(\eta)\eta c_0 + Q_3^{(1)}(\eta)c_0 \tag{3.14}$$

and

$$Q_{3x}(\eta,t) = Q_3^{(0)\prime}(\eta)\,\frac{1}{c_0 t} + Q_3^{(1)\prime}(\eta) \tag{3.15}$$

Substituting these equations into the governing equations (3.4) and (3.5) and then comparing the coefficients of the same power of t, one gets

$$\left[-c_0\eta + u_3^{(0)}(\eta) + c_3^{(0)}(\eta)\right]\left[u_3^{(0)\prime}(\eta) + \frac{2}{\gamma - 1}\, c_3^{(0)\prime}(\eta)\right] = 0 \tag{3.16}$$

$$\left[-c_0\eta + u_3^{(0)}(\eta) - c_3^{(0)}(\eta)\right]\left[u_3^{(0)\prime}(\eta) - \frac{2}{\gamma - 1}\, c_3^{(0)\prime}(\eta)\right] = 0 \tag{3.17}$$

$$\begin{aligned}&\left[-c_0\eta + u_3^{(0)}(\eta) - c_3^{(0)}(\eta)\right]\left[u_3^{(1)\prime}(\eta) - \frac{2}{\gamma - 1}\, c_3^{(1)\prime}(\eta)\right] + u_3^{(1)}(\eta)\left[c_0 + u_3^{(0)\prime}(\eta)\right.\\ &\quad \left. - \frac{2}{\gamma - 1}\, c_3^{(0)\prime}(\eta)\right] - c_3^{(1)}(\eta)\left[\frac{2}{\gamma - 1}\, c_0 + u_3^{(0)\prime}(\eta) - \frac{2}{\gamma - 1}\, c_3^{(0)\prime}(\eta)\right] = 0\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\left[-c_0\eta + u_3^{(0)}(\eta) + c_3^{(0)}(\eta)\right]\left[u_3^{(1)\prime}(\eta) + \frac{2}{\gamma - 1}\, c_3^{(1)\prime}(\eta)\right] + u_3^{(1)}(\eta)\left[c_0 + u_3^{(0)\prime}(\eta)\right.\\ &\quad \left. + \frac{2}{\gamma - 1}\, c_3^{(0)\prime}(\eta)\right] + c_3^{(1)}(\eta)\left[\frac{2}{\gamma - 1}\, c_0 + u_3^{(0)\prime}(\eta) + \frac{2}{\gamma - 1}\, c_3^{(0)\prime}(\eta)\right] = 0\end{aligned} \tag{3.19}$$

From equations (3.16) and (3.17), two sets of results are obtained:

$$\eta = \frac{u_3^{(0)}(\eta) \pm c_3^{(0)}(\eta)}{c_0} \tag{3.20}$$

$$u_3^{(0)\prime}(\eta) \mp \frac{2}{\gamma - 1}\, c_3^{(0)\prime}(\eta) = 0 \tag{3.21}$$

in which the upper signs correspond to forward-facing expansion waves and the lower signs correspond to backward-facing expansion waves.

Integration of equation (3.21) and application of boundary condition (3.9) at $\eta = \xi_2$ yields

$$u_3^{(0)}(\eta) \mp \frac{2}{\gamma - 1} c_3^{(0)}(\eta) = \text{const} = u_2^{(0)} \mp \frac{2}{\gamma - 1} c_2^{(0)} \tag{3.22}$$

If a constant B is defined by

$$B = \mp \frac{u_2^{(0)}}{c_0} + \frac{2}{\gamma - 1} \frac{c_2^{(0)}}{c_0} \tag{3.23}$$

then the terms $c_3^{(0)}(\eta)$ and $u_3^{(0)}(\eta)$ can be written, using equations (3.20) and (3.22) as

$$c_3^{(0)}(\eta) = \pm \frac{\gamma - 1}{\gamma + 1} (\eta \pm B) c_0 \tag{3.24}$$

$$u_3^{(0)}(\eta) = \frac{2}{\gamma + 1} \left(\eta \mp \frac{\gamma - 1}{2} B\right) c_0 \tag{3.25}$$

Equations (3.20) and (3.22) are now substituted into equations (3.18) and (3.19), which yields

$$u_3^{(1)}(\eta)\left[c_0 \pm \frac{4}{\gamma - 1} c_3^{(0)'}(\eta)\right] \pm \frac{2}{\gamma - 1} c_3^{(1)}(\eta)\left[c_0 \pm 2c_3^{(0)'}(\eta)\right] = 0 \tag{3.26}$$

$$\mp 2c_3^{(0)}(\eta)\left[u_3^{(1)'}(\eta) \mp \frac{2}{\gamma - 1} c_3^{(1)'}(\eta)\right] + \left[u_3^{(1)}(\eta) \mp \frac{2}{\gamma - 1} c_3^{(1)}(\eta)\right] c_0 = 0 \tag{3.27}$$

Substitution of equations (3.24) and (3.25) into equation (3.26) results in

$$u_3^{(1)}(\eta) = \pm Ac_3^{(1)}(\eta) \tag{3.28}$$

where

$$A = \frac{-2(3\gamma - 1)}{(\gamma - 1)(\gamma + 5)} \tag{3.29}$$

It follows that

$$u_3^{(1)'}(\eta) = \pm Ac_3^{(1)'}(\eta)$$

which can then be substituted into equation (3.27) to obtain

$$2c_3^{(0)}(\eta)c_3^{(1)'}(\eta) \mp c_3^{(1)}(\eta)c_0 = 0$$

or

$$c_3^{(1)'}(\eta) - \frac{\gamma + 1}{2(\gamma - 1)} \frac{1}{\eta \pm B} c_3^{(1)}(\eta) = 0$$

The general solution for $c_3^{(1)}(\eta)$ is given by

$$c_3^{(1)}(\eta) = c_3^{(1)}(\xi_2)\left(\frac{\eta \pm B}{\xi_2 \pm B}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{3.30}$$

where, using the boundary condition (3.10),

$$c_3^{(1)}(\xi_2) = \xi_2 c_{2x} + c_{2t} \mp \frac{\gamma - 1}{\gamma + 1} k(\xi_2)c_0 \tag{3.31}$$

and then $u_3^{(1)}(\eta)$ is known from equation (3.28). Of course, from equation (3.28) it follows that $u_3^{(1)}(\xi_2) = \pm Ac_3^{(1)}(\xi_2)$, and this must be consistent with the boundary condition (3.10) on $u_3^{(1)}(\xi_2)$ and $c_3^{(1)}(\xi_2)$. This has been proved by the author by applying the characteristic equations (3.4) and (3.5) in region II and using equation (3.12) for ξ_2. This yields a relationship between $\xi_2 u_{2x} + u_{2t}$ and $\xi_2 c_{2x} + c_{2t}$, from which it follows that $u_3^{(1)}(\xi_2) = \pm A c_3^{(1)}(\xi_2)$.

Now equation (3.6) represents the relationship between u, c, and p. Since u and c are already known, this equation can be used to calculate pressure. Substituting the expansion series (C) into equation (3.6) and comparing the coefficients of like powers of t yields two equations. The first one is

$$\frac{\gamma - 1}{\gamma} c_3^{(0)}(\eta) p_3^{(0)'}(\eta)\left[-\eta + \frac{u_3^{(0)}(\eta)}{c_0}\right] = 2p_3^{(0)}(\eta) c_3^{(0)'}(\eta)\left[-\eta + \frac{u_3^{(0)}(\eta)}{c_0}\right] \tag{3.32}$$

and the second one is

$$\begin{aligned} & p_3^{(1)'}(\eta)\left[\mp \frac{\gamma - 1}{\gamma} c_3^{(0)^2}(\eta)\right] + p_3^{(1)}(\eta)\left[\pm 2c_3^{(0)'}(\eta)c_3^{(0)}(\eta) + \frac{\gamma - 1}{\gamma} c_3^{(0)}(\eta)c_0\right] \\ & \quad + p_3^{(0)'}(\eta)\left[\mp \frac{\gamma - 1}{\gamma} c_3^{(1)}(\eta)c_3^{(0)}(\eta) + \frac{\gamma - 1}{\gamma} u_3^{(1)}(\eta)c_3^{(0)}(\eta)\right] \\ & \quad - 2p_3^{(0)}(\eta)\left[\mp c_3^{(1)'}(\eta)c_3^{(0)}(\eta) + c_3^{(1)}(\eta)c_0 + u_3^{(1)}(\eta)c_3^{(0)'}(\eta)\right] = 0 \end{aligned} \tag{3.33}$$

Since the factor $-\eta + u_3^{(0)}(\eta)/c_0 = \mp c_3^{(0)}/c_0$ is not zero, it can be cancelled in equation (3.32) resulting in a first-order ordinary differential equation for $p_3^{(0)}(\eta)$:

$$p_3^{(0)'}(\eta) - \frac{2\gamma}{\gamma - 1} \frac{1}{\eta \pm B} p_3^{(0)}(\eta) = 0$$

Integrating this differential equation and using the boundary condition (3.9), the solution for $p_3^{(0)}(\eta)$ can be obtained as

$$p_3^{(0)}(\eta) = p_2^{(0)}\left(\frac{\eta \pm B}{\xi_2 \pm B}\right)^{\frac{2\gamma}{\gamma-1}} \tag{3.34}$$

A differential equation for $p_3^{(1)}(\eta)$ can also be obtained by rewriting equation (3.33) with the aid of equations (3.24), (3.28), (3.30) and (3.34) in the form:

$$p_3^{(1)\prime}(\eta) - \frac{3\gamma+1}{\gamma-1}\frac{1}{\eta \pm B}p_3^{(1)}(\eta) - \frac{\gamma(\gamma+1)(\gamma+5)}{2(\gamma-1)^2}u_3^{(1)}(\xi_2)\frac{p_2^{(0)}}{c_0}\frac{1}{(\eta \pm B)^2}\left(\frac{\eta \pm B}{\xi_2 \pm B}\right)^{\frac{5\gamma+1}{2(\gamma-1)}} = 0$$

The general solution of this inhomogeneous equation is

$$p_3^{(1)}(\eta) = \frac{-\gamma(\gamma+1)(\gamma+5)}{(\gamma-1)(3\gamma-1)}\frac{u_3^{(1)}(\xi_2)}{c_0}p_2^{(0)}\frac{(\eta \pm B)^{\frac{3(\gamma+1)}{2(\gamma-1)}}}{(\xi_2 \pm B)^{\frac{(5\gamma+1)}{2(\gamma-1)}}} + G(\eta \pm B)^{\frac{3\gamma+1}{\gamma-1}}$$

The integration constant, G, can be written as

$$G = p_3^{(1)}(\xi_2)\frac{1}{(\xi_2 \pm B)^{\frac{3\gamma+1}{\gamma-1}}} + \frac{\gamma(\gamma+1)(\gamma+5)}{(\gamma-1)(3\gamma-1)}\frac{u_3^{(1)}(\xi_2)}{c_0}p_2^{(0)}\frac{1}{(\xi_2 \pm B)^{\frac{4\gamma}{\gamma-1}}}$$

so that solution $p_3^{(1)}(\eta)$ is represented by

$$p_3^{(1)}(\eta) = \left\{\frac{\gamma(\gamma+1)(\gamma+5)}{(\gamma-1)(3\gamma-1)}\frac{u_3^{(1)}(\xi_2)}{c_0}\frac{p_2^{(0)}}{\xi_2 \pm B}\left[\left(\frac{\eta \pm B}{\xi_2 \pm B}\right)^{\frac{-3\gamma+1}{2(\gamma-1)}} + 1\right] + p_3^{(1)}(\xi_2)\right\}\left(\frac{\eta \pm B}{\xi_2 \pm B}\right)^{\frac{3\gamma+1}{\gamma-1}} \tag{3.35}$$

and $p_3^{(1)}(\xi_2)$ is defined by the boundary condition (3.10) in terms of known quantities as

$$p_3^{(1)}(\xi_2) = p_{2x}\xi_2 + p_{2t} - k(\xi_2)\, p_3^{(0)'}(\xi_2)$$

Proceeding further, the continuity equation (2.1) can be used to define the density. The procedure is the same as before: substituting expansion series (C) and its derivatives with respect to x and t into equation (2.1) and then comparing coefficients of like powers of t. Two equations are obtained:

$$\rho_3^{(0)'}(\eta)\left[-\eta + \frac{u_3^{(0)}(\eta)}{c_0}\right] + \rho_3^{(0)}(\eta)\, \frac{u_3^{(0)'}(\eta)}{c_0} = 0 \tag{3.36}$$

and

$$\rho_3^{(1)'}(\eta)\left[-\eta c_0 + u_3^{(0)}(\eta)\right] + \rho_3^{(1)}(\eta)\left[c_0 + u_3^{(0)'}(\eta)\right] + u_3^{(1)'}(\eta)\rho_3^{(0)}(\eta)$$
$$+ u_3^{(1)}(\eta)\rho_3^{(0)'}(\eta) = 0 \tag{3.37}$$

Now, solving for $\rho_3^{(0)}(\eta)$ from equation (3.36), with the help of equation (3.25) and the boundary condition (3.9), one obtains

$$\rho_3^{(0)}(\eta) = \rho_2^{(0)}\left(\frac{\eta \pm B}{\xi_2 \pm B}\right)^{\frac{2}{\gamma-1}} \tag{3.38}$$

Similarily, solving equation (3.37) using equations (3.25), (3.28), (3.30), and (3.38), the solution of $\rho_3^{(1)}(\eta)$ takes the form:

$$\rho_3^{(1)}(\eta) = \left\{\frac{(\gamma + 1)(\gamma + 5)}{(\gamma - 1)(-3\gamma + 1)} \frac{u_3^{(1)}(\xi_2)}{c_0} \frac{\rho_2(0)}{\xi_2 \pm B}\left[\left(\frac{\eta \pm B}{\xi_2 \pm B}\right)^{\frac{-3\gamma+1}{2(\gamma-1)}} - 1\right] + \rho_3^{(1)}(\xi_2)\right\}\left(\frac{\eta \pm B}{\xi_2 \pm B}\right)^{\frac{\gamma+3}{\gamma-1}} \qquad (3.39)$$

where $\rho_3^{(1)}(\xi_2)$ is known from the boundary condition (3.10).

At this stage, all the flow properties in the fan region III are defined. The next step is to define the tail of the fan OB; that is, its slope and curvature at $t = 0$. Since OB is also a characteristic line, with $dx/dt = u \pm c$ on OB, the flow properties must be continuous across it. The boundary conditions on OB are similar to those on OA. First denote the slope of the OB at origin by ξ_1. The characteristic line OB can be written as follows:

$$x = x_c(\xi_1, t) = \xi_1 c_0 t + k(\xi_1)(c_0 t)^2 \qquad (3.40)$$

Now, the boundary condition along OB is expressed as before as

$$Q_1^{(0)} + Q_{1x}\xi_1 c_0 t + Q_{1t} c_0 t$$

$$= Q_3^{(0)}(\xi_1) + k(\xi_1) c_0 t Q_3^{(0)\prime}(\xi_1) + Q_3^{(1)}(\xi_1) c_0 t$$

By comparing the coefficients of the same power of t, two equations are derived:

$$Q_1^{(0)} = Q_3^{(0)}(\xi_1) \qquad (3.41)$$

$$Q_{1x}\xi_1 + Q_{1t} = Q_3^{(1)}(\xi_1) + k(\xi_1) Q_3^{(0)\prime}(\xi_1) \qquad (3.42)$$

The first of these can be used to define the slope of OB; that is, if Q is identified as u, than equations (3.41) and (3.25) define ξ_1, using the fact that $u_1^{(0)} = u_p$ (see eq. 2.22):

$$\xi_1 = \frac{\gamma + 1}{2} \frac{u_p}{c_0} \pm \frac{\gamma - 1}{2} B \tag{3.43}$$

The constant B is defined in equation (3.23). Substituting ξ_1, into equation (3.20) and appling equation (3.41) yields

$$\xi_1 = \frac{u_3^{(0)}(\xi_1) \pm c_3^{(0)}(\xi_1)}{c_0} = \frac{u_1^{(0)} \pm c_1^{(0)}}{c_0} \tag{3.43a}$$

from which

$$c_1^{(0)} = \pm \frac{\gamma - 1}{2} (u_p \pm Bc_0) \tag{3.44}$$

By definition, on the characteristic line $dx/dt = u \pm c$. This implies that

$$\xi_1 c_0 + 2k(\xi_1)c_0^2 t = u_3^{(0)}(\xi_1) + k(\xi_1)c_0 t\ u_3^{(0)'}(\xi_1) + u_3^{(1)}(\xi_1)c_0 t$$
$$\pm \left(c_3^{(0)}(\xi_1) + k(\xi_1)c_0 t\ c_3^{(0)'}(\xi_1) + c_3^{(1)}(\xi_1)c_0 t\right)$$

With equations (3.24), (3.25), (3.28), (3.30) and (3.43), every term in this equation is known except $k(\xi_1)$. Therefore

$$k(\xi_1) = -\frac{(\gamma + 1)(\gamma - 3)}{2(3\gamma - 1)} \frac{u_3^{(1)}(\xi_1)}{c_0} \tag{3.45}$$

where $u_3^{(1)}(\xi_1)$ is defined in equation (3.28). This completes the third step in the analysis as outlined previously.

The final step consists in using the slope and curvature of OB and the boundary conditions on the piston path to define the flow properties in region I. First, substituting the expansion series (A) into the governing equations (3.4) and (3.5) yields

$$u_{1t}c_0 + \left(u_1^{(0)} + c_1^{(0)}\right)u_{1x} + \frac{2}{\gamma - 1} c_{1t}c_0 + \frac{2}{\gamma - 1} \left(u_1^{(0)} + c_1^{(0)}\right)c_{1x} = 0 \quad (3.46)$$

and

$$u_{1t}c_0 + \left(u_1^{(0)} - c_1^{(0)}\right)u_{1x} - \frac{2}{\gamma - 1} c_{1t}c_0 - \frac{2}{\gamma - 1} \left(u_1^{(0)} - c_1^{(0)}\right)c_{1x} = 0 \quad (3.47)$$

Adding and subtracting equation (3.47) from (3.46), this system of equations becomes

$$u_{1t}c_0 + u_1^{(0)}u_{1x} + \frac{2}{\gamma - 1} c_1^{(0)}c_{1x} = 0 \quad (3.48)$$

$$u_{1x}c_1^{(0)} + \frac{2}{\gamma - 1} c_{1t}c_0 + \frac{2}{\gamma - 1} u_1^{(0)}c_{1x} = 0 \quad (3.49)$$

along with these, the boundary conditions on OB , equation (3.42), imply that

$$u_{1t} + u_{1x}\xi_1 = k(\xi_1)u_3^{(0)'}(\xi_1) + u_3^{(1)}(\xi_1) \quad (3.50)$$

$$c_{1t} + c_{1x}\xi_1 = k(\xi_1)c_3^{(0)'}(\xi_1) + c_3^{(1)}(\xi_1) \quad (3.51)$$

However, it is easily seen using equation (3.43a) that the determinant of the system (3.48) to (3.51) vanishes. A fourth independent relation can be found in equation (2.30): $u_{1x}u_p + u_{1t}c_0 = a_p$. Using this yields

$$c_{1x} = \mp \frac{a_p}{u_p \pm Bc_0} \tag{3.52}$$

$$u_{1x} = \left\{\frac{4(\gamma + 1)c_0}{(3\gamma - 1)(\gamma - 1)} u_3^{(1)}(\xi_1) - \frac{2}{\gamma - 1} a_p\right\} \frac{1}{u_p \pm Bc_0} \tag{3.53}$$

and

$$c_{1t} = \mp \left\{\frac{(\gamma + 1)(\gamma - 1)}{(3\gamma - 1)} u_3^{(1)}(\xi_1) - \frac{a_p \xi_1}{u_p \pm Bc_0}\right\} \tag{3.54}$$

$$u_{1t} = - \left\{\frac{4(\gamma + 1)u_p}{(3\gamma - 1)(\gamma - 1)} u_3^{(1)}(\xi_1) - \frac{2}{\gamma - 1} a_p \xi_1\right\} \frac{1}{u_p \pm Bc_0} \tag{3.55}$$

Substituting expansion series (A) into equation (3.6) for region I, yields

$$p_{1t}c_0 + p_{1x}u_p = \frac{2\gamma}{\gamma - 1} \frac{p_1^{(0)}}{c_1^{(0)}} (c_{1t}c_0 + c_{1x}u_p) \tag{3.56}$$

From the boundary condition on OB

$$p_1^{(0)} = p_3^{(0)}(\xi_1) = p_2^{(0)}\left(\frac{\xi_1 \pm B}{\xi_2 \pm B}\right)^{\frac{2\gamma}{\gamma-1}} \tag{3.57}$$

and

$$p_{1t} + p_{1x}\xi_1 = p_3^{(0)'}(\xi_1)k(\xi_1) + p_3^{(1)}(\xi_1) \tag{3.58}$$

Therefore, equations (3.56) and (3.58) with equations (3.43) and (3.49) yield for p_{1x} the expression

$$p_{1x} = \frac{\gamma p_1^{(0)} u_{1x} + p_3^{(0)'}(\xi_1)k(\xi_1)c_0 + p_3^{(0)}(\xi_1)c_0}{\frac{\gamma - 1}{2}(u_p \pm Bc_0)} \tag{3.59}$$

Substitution of equation (3.59) back into equation (3.56) then defines p_{1t}.

The procedure that defines the density ρ in this region is the same as that which defines the pressure. The equations used are the continuity equation and the boundary condition, equations (2.1) and (3.42), with Q identified as ρ. They are as follows:

$$\rho_{1t}c_0 + \rho_{1x}u_p = -\rho_1^{(0)}u_{1x} \tag{3.60}$$

$$\rho_{1x}\xi_1 + \rho_{1t} = \rho_3^{(0)'}(\xi_1)k(\xi_1) + \rho_3^{(1)}(\xi_1) \tag{3.61}$$

Then the spatial derivative of ρ can be determined as

$$\rho_{1x} = \frac{\rho_1^{(0)}u_{1x} + \rho_3^{(0)'}(\xi_1)k(\xi_1)c_0 + \rho_3^{(1)}(\xi_1)c_0}{\frac{\gamma - 1}{2}(u_p \pm Bc_0)} \tag{3.62}$$

Substitution of equation (3.62) back into equation (3.60) then defines ρ_{1t}.

The problem involving a centered expansion wave, that is, in which a piston moves at $t = 0$ away from the gas, has now been completely solved.

The governing equations inside the fan are expressed in Riemann invariant form, equations (3.4) and (3.5), in order to uncouple the variables. This form relates u and c, and then the original differential equations are used to define p and ρ. The regular power series (A and B) are used for the regions ahead of and behind the expansion fan, and a special expansion (C) is employed for the centered expansion fan region.

The flow properties in the region ahead of the fan depend only upon the initial data of the flow and can be expressed in the same manner as for the region ahead of a shock. The head characteristic, OA, its slope and curvature, are defined by using the boundary conditions, equation (3.8), that require the flow properties to be continuous across OA; they are given in equations (3.12) and (3.13). The solutions for the flow properties inside the fan are in two parts. The leading terms, $c_3^{(0)}(\eta)$, $u_3^{(0)}(\eta)$, $p_3^{(0)}(\eta)$ and $\rho_3^{(0)}(\eta)$, are defined in equations (3.24), (3.25), (3.34), and (3.38), respectively. These are the same as the solutions for a centered expansion wave with uniform initial data, and they are functions only of the flow properties ahead of the fan. The first order terms $c_3^{(1)}(\eta)$, $u_3^{(1)}(\eta)$, $p_3^{(1)}(\eta)$, and $\rho_3^{(1)}(\eta)$, are defined in equations (3.30), (3.28), (3.35) and (3.39), respectively. They are also related only to the flow properties ahead of the fan. The piston velocity at $t = 0$ first comes into the picture when defining the slope of the tail characteristic in equation (3.43). The curvature of the tail characteristic, equation (3.45), can be expressed after its slope is defined. It is interesting to note that the acceleration of the piston does not affect the solution anywhere ahead of and within the expansion fan. The leading terms in the region behind the fan are defined by equation (3.41), with Q identified as u, c, p, or ρ. Initial spatial and time derivatives of c and u in region I are defined by equations (3.52) to

(3.55), and the initial spatial and time derivatives of p and ρ are defined by equations (3.59), (3.62), (3.56) and (3.60). Thus, the various series expansions of the flow properties in the fan and behind the fan have all been determined.

CHAPTER 4

INITIAL VALUE PROBLEM

The initial value problem with discontinuous initial data joining two uniform regions was studied by Riemann [9] and was used as the canonical solution, or the building block, by Chorin [3] for the numerical solution of a general initial value problem. In this section, an improved canonical solution for the propagation of discontinuous initial data joining two nonuniform regions is constructed by including the first order effects of the nonuniformities.

For such an initial value problem, a shock wave, a centered expansion wave, and a contact line all exist simultaneously. An expansion wave propagates into the higher pressure side, a shock wave propagates into the lower pressure side.

A contact line is a particle path separating the region behind the shock from the region behind the expansion wave. Across it the temperatures and the densities may be different, but it is necessary that the pressure and fluid velocity be the same. The problem is to relate the velocity and acceleration of each discontinuity wave (that is, the shock wave, the expansion fan, and the contact line), to the initial flow properties and their gradients (see fig. 4).

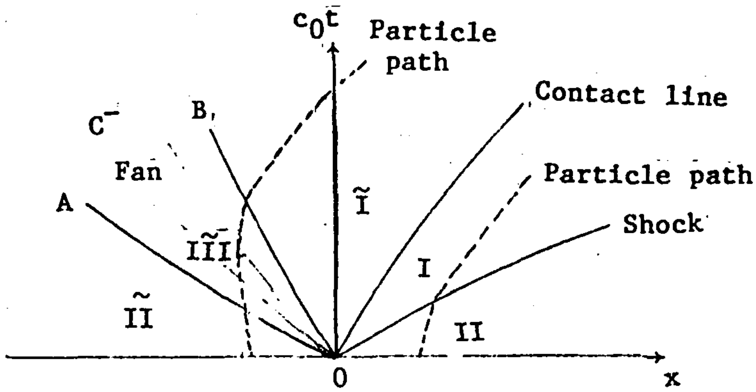

Figure 4.- Shock wave, expansion fan, and contact line.

The (~) indicates the flow properties on the higher pressure side, that is, the expansion wave side.

4.1 Method of Solution

The preceding two solutions for shock waves and for centered expansion waves are to be combined by considering the piston path as a contact line, so that the fluid velocity in the region behind the shock and in the region behind the expansion fan are the same along this line. The condition for continuity of the pressure across the contact line gives two equations, which define u_p and a_p, the speed and acceleration of the contact line.

4.2 Explicit Formulas

The condition for continuity of the pressure across the contact line gives

$$p_1^{(0)} + (p_{1x}u_p + p_{1t}c_0)t = \tilde{p}_1^{(0)} + (\tilde{p}_{1x}u_p + \tilde{p}_{1t}c_0)t \tag{4.1}$$

The constant term and the coefficient of t yield

$$p_1^{(0)} = \tilde{p}_1^{(0)} \tag{4.2}$$

and

$$p_{1x}u_p + p_{1t}c_0 = \tilde{p}_{1x}u_p + \tilde{p}_{1t}c_0 \tag{4.3}$$

By substituting equation (2.24), the solution for u_s, into equation (2.26), $p_1^{(0)}$ is related to u_p from the shock solutions

$$p_1^{(0)} = p_2^{(0)} + \frac{\gamma + 1}{4}\rho_2^{(0)}\left[(u_2^{(0)} - u_p) \mp \sqrt{(u_2^{(0)} - u_p)^2 + \left(\frac{4}{\gamma + 1}c_2^{(0)}\right)^2}\right](u_2^{(0)} - u_p) \tag{4.4}$$

where the upper sign applies for a forward-facing shock and the lower sign for backward-facing shock.

Similarly, by substituting equations (3.12) and (3.43), the slopes of the head and tail characteristics of the expansion fan, into the definition of $p_1^{(0)}$ given by equation (3.34), and by the use of (3.23), the definition of B, results in a relation between $\tilde{p}_1^{(0)}$ and u_p in the form

$$\tilde{p}_1{}^{(0)} = \tilde{p}_2{}^{(0)}\left(\mp \frac{\gamma - 1}{2} \frac{u_p \mp Bc_0}{\tilde{c}_2{}^{(0)}}\right)^{\frac{2\gamma}{\gamma-1}} \tag{4.5}$$

where the upper sign applies to a backward-facing expansion fan, and the lower sign to a forward-facing expansion fan. Equating the expression in equations (4.4) and (4.5) , results in a relation which determines u_p:

$$p_2{}^{(0)} + \frac{\gamma + 1}{4} \rho_2{}^{(0)}\left[(u_2{}^{(0)} - u_p) \mp \sqrt{(u_2{}^{(0)} - u_p)^2 + \left(\frac{4}{\gamma + 1} c_2{}^{(0)}\right)^2}\right](u_2{}^{(0)} - u_p)$$

$$= \tilde{p}_2{}^{(0)}\left(1 \mp \frac{\gamma - 1}{2} \frac{u_p - \tilde{u}_2{}^{(0)}}{\tilde{c}_2{}^{(0)}}\right)^{\frac{2\gamma}{\gamma-1}} \tag{4.6}$$

Equation (4.6) must be solved iteratively for u_p. Note that the upper sign holds for the case of a right shock/left expansion fan; that is, with the pressure in $x < 0$ higher than that in $x > 0$ at $t = 0$. The lower sign is for the case of a left shock/right expansion fan; that is, for the higher pressure in $x > 0$. The (~) indicates flow properties corresponding to the higher pressure side. After u_p has been calculated, all the leading terms in the expansions of the flow properties on either side of the contact line can be determined by using the equations in Chapters 2 and 3.

The curvature of the contact line can be found using equation (4.3). First, repeating equation (2.29), in the region behind the shock

$$p_{1x}u_p + p_{1t}c_0 = -c_1{}^{(0)^2}\rho_1{}^{(0)}u_{1x} \tag{4.7}$$

where $c_1^{(0)}$ can be determined by equaitons (3.25), (3.26), and with the relation $c^2 = \gamma p/\rho$, $\rho_1^{(0)}$ and u_{1x} are defined by equations (2.25) and (2.35). Similarly in the region behind the expansion fan,

$$\tilde{p}_{1x}u_p + \tilde{p}_{1t}c_0 = -\tilde{c}_1^{(0)^2}\tilde{\rho}_1^{(0)}\tilde{u}_{1x} \tag{4.8}$$

In consequence of equation (4.3), the right hand sides of equations (4.7) and (4.8) must be equal. Substitution of the expressions (2.35) for u_{1x} and (3.53) for $\tilde{u}_{1x}$ then gives

$$\begin{aligned} a_p = \Bigg\{ & \frac{c_1^{(0)^2}\rho_1^{(0)}(u_p - u_s)\left[(u_2^{(0)} - u_p) - \frac{8}{\gamma - 1}(u_2^{(0)} - u_s)\right]}{(u_2^{(0)} - u_p)c_1^{(0)^2} - \frac{4}{\gamma + 1}(u_2^{(0)} - u_s)\left[(u_p - u_s)^2 + c_1^{(0)^2}\right]} + \frac{2}{\gamma - 1}\frac{\tilde{c}_1^{(0)^2}\tilde{\rho}_1^{(0)}}{u_p \mp Bc_0}\Bigg\}^{-1} \\ & \times \Bigg\{ \frac{c_1^{(0)^2}\rho_1^{(0)}\left[-\frac{(u_2^{(0)} - u_p)}{u_2^{(0)} - u_s}(u_p - u_s)(D - E) + \frac{4}{\gamma + 1}D(u_p - u_s)\right]}{(u_2^{(0)} - u_p)c_1^{(0)^2} - \frac{4}{\gamma + 1}(u_2^{(0)} - u_s)\left[(u_p - u_s)^2 + c_1^{(0)^2}\right]} \\ & + \frac{4(\gamma + 1)c_0}{(3\gamma - 1)(\gamma - 1)}\frac{\tilde{c}_1^{(0)^2}\tilde{\rho}_1^{(0)}u_3^{(1)}(\xi_1)}{(u_p \mp Bc_0)}\Bigg\} \end{aligned} \tag{4.9}$$

This completes the required analysis of this chapter. The determination of the contact line has been accomplished by joining the solutions for the shock problem and the expansion wave problem together using the conditions of continuity of pressure and velocity across the contact line. The initial velocity and acceleration of the contact line are given by equations (4.6) and (4.9). Thus, the solution of the initial value problem with discontinuous initial data joining two uniform regions has been extended to the nonuniform case. The canonical solution for discontinuous initial data joining two uniform regions has been used as the basic building block for the random choice method and for front tracking. With the results of the present work, it is now possible to explicitly include information about nonuniform initial data in such a building block. This leads to a numerical procedure which can utilize a much larger step size in the vicinity of discontinuities in the initial data. The actual implementation of the current results into an improved numerical scheme for general initial data is beyond the scope of the present paper.

CHAPTER 5

UNSTEADY FLOW PROBLEMS WITH CHEMICAL REACTION

Flow problems with chemical reactions are discussed in this chapter. The problem is to relate the initial flow properties and their gradients to the speed and acceleration of the reaction front and other discontinuities. The piston moving into the combustible gas for $x > 0$ at $t = 0$ is treated first. These solutions can then be combined with the shock or expansion fan solutions in the preceding chapters using exactly the same procedures as in chapter 4 for the initial value problem without chemical reaction. The algebra involved in this combination is extensive for the initial value problem with chemical reaction; thus, it will not be carried out in the present work, and only the separate elementary problems will be treated. In addition, the elementary problems themselves will not be solved explicitly. For brevity, the solutions will be considered complete where an appropriate set of algebraic equations governing the unknown quantities is established.

The shock conditions across a reaction front are [10].

$$\rho_1 v_1 = \rho_2 v_2 = m \tag{5.1}$$

$$p_1 + m v_1 = p_2 + m v_2 \tag{5.2}$$

$$h_1 + \frac{1}{2} v_1^2 + q_1 = h_2 + \frac{1}{2} v_2^2 + q_2 \tag{5.3}$$

where ρ is the density, p is the pressure of the gas, v is the relative velocity of the flow with respect to the shock, h is the enthalpy, ($h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$ is assumed) and q is the energy formation which can be released through chemical reaction. The subscript 1 refers to burned gas and the subscript 2 refers to unburned gas (i.e., gas which has not yet undergone the chemical reaction). In the present section it is assumed that part of q is released instantaneously in an infinitely thin reaction zone and the unburned gas is on the right. For the sake of simplicity, it is assumed that $\gamma_1 = \gamma_2 = \gamma$. (The case $\gamma_1 \neq \gamma_2$ is more difficult only because of additional algebra). When $\gamma_1 = \gamma_2 = \gamma$ the reaction can be exothermic (i.e., can release energy) only if $q_2 > q_1$; that is, for the same pressure and density, the total energy and enthalpy of the unburned gas is always greater than that of the burned gas. In this model, viscous effects, heat conduction, and radiative heat transfer are neglected.

5.1 Chapman-Jouguet Waves

For completeness, a brief summary is given in this section of the elementary theory of one-dimensional detonation and deflagration waves, and a derivation of relations between the variables on the two sides of such waves is carried out for later use. The material presented here is essentially as found in references 9 to 11. Chapman-Jouguet waves have a special significance in many systems. It is therefore of interest to investigate the properties of these waves first. With the aid of equations (5.1) and (5.2), equation (5.3) can be written as

$$\frac{\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) - \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(p_1 - p_2) = q > 0 \tag{5.4}$$

where $q = q_2 - q_1$. A dimensionless final pressure ratio, a specific volume ratio, and a dimensionless heat of reaction are defined as follows:

$$\pi = \frac{p_1}{p_2}$$

$$\nu = \frac{\rho_2}{\rho_1}$$

$$\alpha = \frac{q\rho_2}{p_2}$$

Multiplying equation (5.4) by ρ_2/p_2 yields

$$\frac{\gamma}{\gamma - 1}(\pi\nu - 1) - \frac{1}{2}(\nu + 1)(\pi - 1) = \alpha$$

the solution of which is

$$\pi = \frac{\left(2\alpha + \frac{\gamma + 1}{\gamma - 1}\right) - \nu}{\left(\frac{\gamma + 1}{\gamma - 1}\right)\nu - 1} \tag{5.5}$$

The curve given by equation (5.5). is called a Hugoniot curve (Fig. 5).

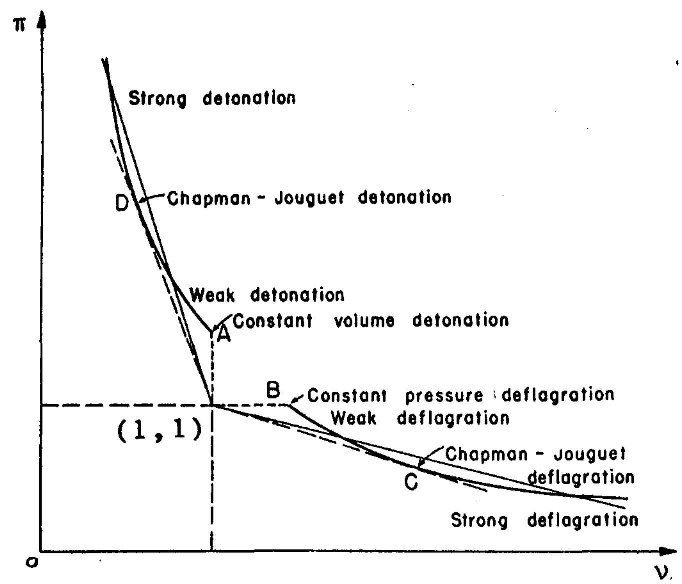

Figure 5.- Hugoniot curve for detonations and deflagrations.

The intersection between the Hugoniot curve and the straight line through the point (1,1) establishes the final state of the system. Each Hugoniot curve is therefore divided into two disconnected branches, an upper branch called the detonation branch, and a lower branch called the deflagration branch. Combustion waves are termed detonation waves or deflagration waves according to the branch of the Hugoniot curve upon which the final condition falls. In passing through a detonation wave, the gas is slowed down and its pressure and density increase; conversely, in going through a deflagration the gas speeds up and expands, and its pressure decreases.

There are two points on the Hugoniot curve at which the straight line from (1,1) is tangent to the curve (points D and C on Fig. 5). These are referred to as the Chapman-Jouguet points. The slope of the Hugoniot curves is given by

$$\frac{d\pi}{d\nu} = - \frac{\frac{\gamma + 1}{\gamma - 1}\pi + 1}{\frac{\gamma + 1}{\gamma - 1}\nu - 1} \tag{5.6}$$

while a straight line through the point (1,1) with this slope has the equation

$$- \frac{\frac{\gamma + 1}{\gamma - 1}\pi + 1}{\frac{\gamma + 1}{\gamma - 1}\nu - 1} = \frac{\pi - 1}{\nu - 1} \tag{5.7}$$

or

$$\nu = \frac{\gamma\pi}{(\gamma + 1)\pi - 1} \tag{5.8}$$

Simultaneous solution of equations (5.5) and (5.8) yields the ordinates of the Chapman-Jouguet points as

$$\pi_{\pm} = 1 + \alpha(\gamma - 1)\left\{1 \pm \sqrt{1 + \frac{2\gamma}{\alpha(\gamma^2 - 1)}}\right\} \tag{5.9}$$

and the abscissas of the points as

$$\nu_{\pm} = 1 + \alpha\left(\frac{\gamma - 1}{\gamma}\right)\left\{1 \mp \sqrt{1 + \frac{2\gamma}{\alpha(\gamma^2 - 1)}}\right\} \tag{5.10}$$

in which the upper signs correspond to the detonation branch and lower signs correspond to the deflagration branch. A Chapman-Jouguet wave is defined as one for which the pressure and density ratio ahead (region II) and behind (region I) are given by equations (5.9) and (5.10).

Using the continuity and momentum equations and the relation $c^2 = \gamma p/\rho$, it can be shown [10] that the Mach number of the flow ahead of the Chapman-Jouguet front (region II) relative to the front is given by

$$M_{2\pm} = \sqrt{1 + \frac{\alpha(\gamma^2 - 1)}{2\gamma}} \pm \sqrt{\frac{\alpha(\gamma^2 - 1)}{2\gamma}} \tag{5.11}$$

According to equation (5.11), the initial Mach number always exceeds unity for Chapman-Jouguet detonations and lies between zero and unity for Chapman-Jouguet deflagrations. In addition [9], it is known that Chapman-Jouguet waves correspond to the minimum possible propagation speed for detonations and the maximum possible propagation speed for deflagrations; therefore, all detonations propagate at supersonic velocities, and all deflagrations propagate at subsonic velocities.

5.2 Explicit Formulas for Chapman-Jouguet Waves

If the flow fields separated by the Chapman-Jouguet wave are nonuniform, the wave will not propagate with a constant speed. To relate the initial acceleration of the wave to the gradients of the initial data, we introduce again the power series expansion (A) for flow properties Q near the wave front which is taken at $x = 0$ for $t = 0$:

$$Q_j(x,t) = Q_j^{(0)} + xQ_{jx} + c_0 t Q_{jt} + \text{higher order terms} \tag{A}$$

Let u_d and a_d be the reaction front speed and acceleration at the origin, respectively. The reaction front path, x_d, is then defined as

$$x_d(t) = u_d t + \frac{1}{2} a_d t^2 + 0(t^3) \tag{5.12}$$

Substituting (A) into equations (5.9) to (5.11), evaluating along the reaction front, and then comparing the coefficients of like powers of t, the leading terms of the flow properties behind the reaction front are all defined explicitly:

$$p_1^{(0)} = p_2^{(0)}\left[1 + \frac{q\gamma(\gamma - 1)}{c_2^{(0)}}\left(1 \pm \sqrt{1 + \frac{2c_2^{(0)^2}}{q(\gamma^2 - 1)}}\right)\right] \tag{5.13}$$

$$\rho_1^{(0)} = \rho_2^{(0)}\left[1 + \frac{q(\gamma - 1)}{c_2^{(0)^2}}\left(1 \mp \sqrt{1 + \frac{2c_2^{(0)^2}}{q(\gamma^2 - 1)}}\right)\right]^{-1} \tag{5.14}$$

$$c_1^{(0)^2} = \frac{\gamma p_1^{(0)}}{\rho_1^{(0)}} \tag{5.15}$$

and the speed of the reaction front is found as

$$u_d = u_2^{(0)}(\pm)c_2^{(0)}\left(\sqrt{1 + \frac{q(\gamma^2 - 1)}{2c_2^{(0)^2}}} \pm \sqrt{\frac{q(\gamma^2 - 1)}{2c_2^{(0)^2}}}\right) \tag{5.16}$$

In equation (5.16) and elsewhere, the notation (±) is used to indicate the direction of propagation of the reaction front. The upper sign corresponds to a right-facing front, that is, with the unburned gas on the right, and the lower sign corresponds to a left-facing front. The upper and lower signs which appear without enclosing parentheses refer to the Chapman-Jouguet points for detonation and deflagration, respectively.

The coefficients of the linear terms in t from the foregoing substitution lead to a set of linear equations which couple the derivatives of the flow quantities; they are

$$\bar{p}_1^{(1)} = \bar{p}_2^{(1)} + q(\gamma - 1)(\bar{\rho}_2^{(1)} - \bar{p}_2^{(1)})\,\frac{\rho_2^{(0)}}{p_1^{(0)}}\left[1 \pm \frac{1 + \dfrac{c_2^{(0)^2}}{q(\gamma^2 - 1)}}{\sqrt{1 + \dfrac{2c_2^{(0)^2}}{q(\gamma^2 - 1)}}}\right] \tag{5.17}$$

$$\bar{\rho}_1^{(1)} = \bar{\rho}_2^{(1)} - \frac{q(\gamma - 1)}{\gamma}\,(\bar{\rho}_2^{(1)} - \bar{p}_2^{(1)})\,\frac{\rho_1^{(0)}}{p_2^{(0)}}\left[1 \mp \frac{1 + \dfrac{c_2^{(0)^2}}{q(\gamma^2 - 1)}}{\sqrt{1 + \dfrac{2c_2^{(0)^2}}{q(\gamma^2 - 1)}}}\right] \tag{5.18}$$

and

$$a_d = u_2^{(0)}\bar{u}_2^{(1)}(\mp)\,\frac{1}{2}\,c_2^{(0)}\left[\frac{\sqrt{1+\dfrac{q(\gamma^2-1)}{2c_2^{(0)^2}}}}{1+\dfrac{q(\gamma^2-1)}{2c_2^{(0)^2}}}\right](\bar{\rho}_2^{(1)} - \bar{p}_2^{(1)}) \qquad (5.19)$$

In equations (5.17) to (5.19), the following definitions have been introduced:

$$\bar{p}_1^{(1)} = \frac{p_{1x}u_d + p_{1t}c_0}{p_1^{(0)}} \qquad \bar{p}_2^{(1)} = \frac{p_{2x}u_d + p_{2t}c_0}{p_2^{(0)}}$$

$$\bar{\rho}_1^{(1)} = \frac{\rho_{1x}u_d + \rho_{1t}c_0}{\rho_1^{(0)}} \qquad \bar{\rho}_2^{(1)} = \frac{\rho_{2x}u_d + \rho_{2t}c_0}{\rho_2^{(0)}}$$

$$\bar{u}_1^{(1)} = \frac{u_{1x}u_d + u_{1t}c_0}{u_1^{(0)}} \qquad \bar{u}_2^{(1)} = \frac{u_{2x}u_d + u_{2t}c_0}{u_2^{(0)}}$$

Now it is known that a Chapman-Jouguet reaction front, when observed from the burned gas behind it, moves with speed of sound [9,11]. This is the famous conclusion made by Jouguet in 1905. This relation gives

$$u_1^{(0)} = u_d(\mp)c_1^{(0)} \qquad (5.20)$$

and

$$\bar{u}_1^{(1)} = \frac{1}{u_1^{(0)}}\left\{a_d(\pm)\,\frac{1}{2}\,c_1^{(0)}(\bar{\rho}_1^{(1)} - \bar{p}_1^{(1)})\right\} \qquad (5.21)$$

Since u_d and $c_1^{(0)}$ are already defined, the leading term of the flow velocity behind the Chapman-Jouguet reaction front $u_1^{(0)}$ is determined. Note that the region behind the reaction front (Fig. 6) is a regular region similar to the region behind the shock treated in Chapter 2.

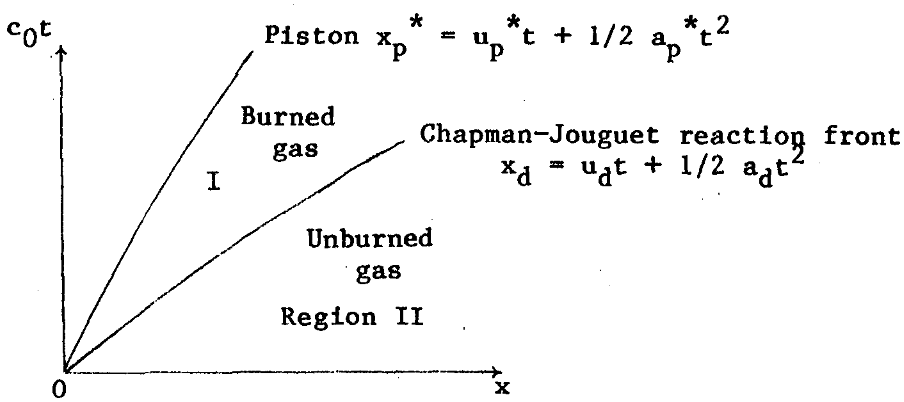

Figure 6.- Wave pattern of Chapman-Jouguet detonation front.

Suppose the Chapman-Jouguet wave is initiated by a piston motion,with piston initial velocity and acceleration u_p^* and a_p^*, respectively. The piston path is then $x_p^* = u_p^*t + 1/2\ a_p^*t^2$. The boundary condition on the piston path, equation (2.10), requires that

$$u_p^* = u_1^{(0)}$$

$$a_p^* = u_{1x}u_p^* + u_{1t}c_0 \tag{5.22}$$

To determine the first order terms of the flow properties behind the front, substitute of the expansion series (A) into equations (2.1) to (2.3) and use the relation $u_p^* = u_p$. This gives three equations identical to equations (2.27) to (2.29) except that u_p is changed to u_p^*. Equations (5.17), (5.18), (5.19), and (5.22) together with these three equations form a system which can be shown to be redundant in a manner analogous to the behavior of equations (3.48) to (3.51). Thus, equation (5.21), for example, can be omitted. The remaining six equations form a linear system from which the six spatial and time derivatives of the flow quantities behind the reaction front can be calculated. The initial acceleration of the front is expressed directly by equation (5.19); it is independent of the piston acceleration.

5.3 Solution of Flow Problems Involving a Detonation Process

The detonation is assumed to be initiated by moving a piston into a nonuniform region ($x > 0$) with initial speed and acceleration u_p and a_p. Depending on the initial velocity of the piston, two types of detonation can occur. First, if u_p is greater then or equal to the Chapman-Jouguet value u_p^* discussed in the previous paragraph, the detonation is termed a strong detonation. (Fig. 7)

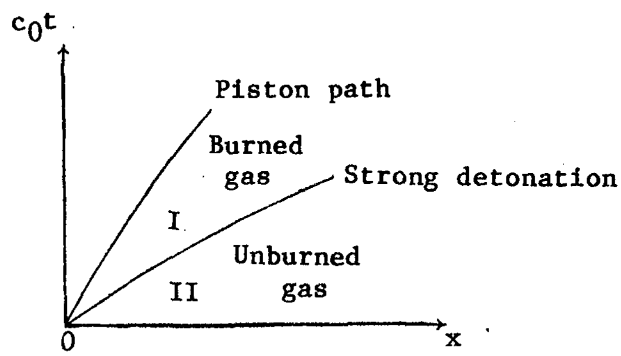

Figure 7.- Piston problem with strong detonation.

The flow relative to the reaction front is supersonic ahead and subsonic behind, which is the same as for an adiabatic shock. The differential equations of the flow field except at the reaction front are equations (2.1) to (2.3), and the jump conditions are equations (5.1) to (5.3). The only equation that is different from those describing the shock problem is the energy jump condition (5.3), which has an extra term to account for chemical energy due to burning.

The procedure used in solving the problem is exactly the same as that for the shock problem. First rewrite equation (5.3) using equations (5.1) and (5.2)

$$(\gamma - 1)v_2^{\,2} - (\gamma + 1)v_1 v_2 + 2c_2^{\,2} + \frac{2q}{v_2 - v_1}(\gamma - 1)v_2 = 0 \tag{5.23}$$

Now, substitution of the expansion series (A) into equation (5.23) and comparison of like powers of t gives, at the leading term

$$u_d - u_2^{(0)} = \left[\frac{\gamma - 1}{2} \frac{-q}{u_1^{(0)} - u_2^{(0)}} + \frac{\gamma + 1}{4} \left(u_1^{(0)} - u_2^{(0)}\right)\right]$$

$$\pm \sqrt{\left[\frac{\gamma - 1}{2} \frac{-q}{u_1^{(0)} - u_2^{(0)}} + \frac{\gamma + 1}{4} \left(u_1^{(0)} - u_2^{(0)}\right)\right]^2 + {c_2^{(0)}}^2} \tag{5.24}$$

which defines the initial speed of the reaction front. The first order term in t yields

$$\left[\left(u_2^{(0)} - u_p\right) - \frac{4}{\gamma + 1}\left(u_2^{(0)} - u_d\right) + 2\,\frac{\gamma - 1}{\gamma + 1} \frac{-q}{u_2^{(0)} - u_p}\right] a_d$$

$$- \left[2q\,\frac{\gamma - 1}{\gamma + 1} \frac{u_p - u_d}{u_2^{(0)} - u_d} + \left(u_2^{(0)} - u_d\right)\left(u_p - u_d\right)\right] u_{1x}$$

$$+ \left[\left(u_2^{(0)} - u_d\right) a_p + E\right] = 0 \tag{5.25}$$

where E is defined by equation (2.33) with u_s changed to u_d. With equation (2.21) replaced by equation (5.25), and by making use of equation (5.24) for the initial velocity of the reaction front, the problem can be solved using exactly the same procedure described in Chapter 2.

In the second case, in which u_p is less than or equal to u_p^*, a flow involving a Chapman-Jouguet detonation is possible. The flow immediately behind the Chapman-Jouguet front need not be equal to the piston velocity u_p. Adjustment in this case is effected by a centered expansion wave which follows the detonation front immediately , since both the front and the first wave in the expansion fan move with sound velocity relative to the gas behind the front [9]. This expansion wave drops out only if $u_p = u_p^*$. The problem is solved by combining the Chapman-Jouguet detonation and expansion wave results. First define the region ahead the detonation front as region II, the region behind the expansion fan as region I, and the fan itself as region III (Fig. 8).

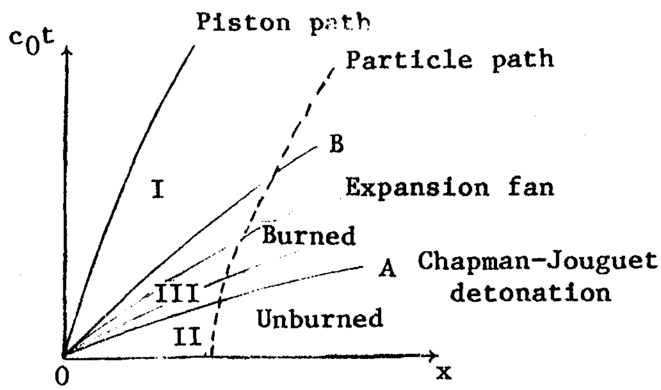

Figure 8.- Chapman-Jouguet detonation with expansion fan.

The procedure used is to solve the Chapman-Jouguet detonation with initial data in $x > 0$. Then the flow properties immediately behind the front are the initial conditions for the expansion fan problem. The solution in region II is completely specified by the initial data in $x > 0$. The Chapman-Jouguet detonation is solved with the results outlined in section 5.2. The leading terms of the flow properties immediately following the Chapman-Jouguet detonation will be denoted, $p_4^{(0)}$, $\rho_4^{(0)}$, and $u_4^{(0)}$. They are all defined in equations (5.13) , (5.14), and (5.20) by replacing subscript "1" by "4". The initial velocity of the front, u_d, is defined by equation (5.16). Since these solutions are the initial conditions for the expansion fan problem, the leading terms of the flow properties inside the fan and behind the fan can be worked out using the equations in Chapter 3.

To determine the first order terms of the flow quantities behind the front, we note that $\bar{p}_4^{(1)}$, $\bar{\rho}_4^{(1)}$, and $\bar{u}_4^{(1)}$ are now known from equations (5.17), (5.18), and (5.21). Because the front coincides with the leading characteristic of the expansion fan, x_d of equation (5.12) is equal to $x_c(\xi_2,t)$ of equation (3.7a). Therefore

$$\xi_2 = \frac{u_d}{c_0} \tag{5.26}$$

and

$$k(\xi_2) = \frac{1}{2}\frac{a_d}{c_0^2} \tag{5.27}$$

a_d is defined in equation (5.19), so $k(\xi_2)$ is known. Now, rewriting the boundary condition (3.10) using (5.26), yields

$$Q_3^{(1)}(\xi_2) = \frac{Q_{4x}u_d + Q_{4t}c_0}{c_0} - k(\xi_2)Q_3^{(0)\prime}(\xi_2) \tag{5.28}$$

The quantities $Q_{4x}u_d + Q_{4t}c_0$ are those known from equations (5.17), (5.18), and (5.21). Using the expression (5.27) for $k(\xi_2)$ and taking (5.28) as the boundary condition on OA, the first order terms of all the flow properties inside the fan and behind the fan can be calculated in the manner discussed in Chapter 3.

5.4 Solution of Flow Problems Involving a Deflagration Process

In deflagration processes the situation is in many respects quite different from that encountered with detonation processes. Suppose a weak deflagration wave begins at a piston at $x = 0$, $t = 0$. Then the velocity u

of the burned gas behind the deflagration front is negative, $u < 0$ [9]. This is compatible with the conditions of the problem only if the piston is withdrawn with a speed at least equal to u. What actually happens is that a precompression shock is sent out into the explosive. It pushes the explosive gas ahead with a velocity just sufficient to ensure that it may attain the same velocity as the piston when it is swept over and burned by the deflagration front. The occurrence of a precompression shock is in complete agreement with, or rather a consequence of, Jouguet's rule that the flow ahead of a deflagration is subsonic. Consequently, the deflagration influences the state of the gas ahead of it. Define the region ahead of the precompression shock as region II, the region behind the shock as region V, and region between the piston and the Chapman-Jouguet deflagration as region I (Fig. 9).

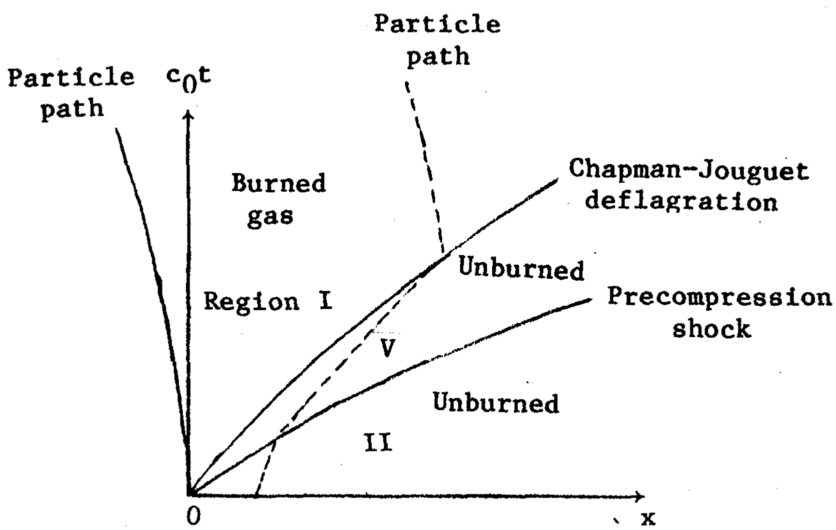

Figure 9.- Weak deflagration with precompression shock.

To determine the leading terms of the flow properties in region I and in region V and the initial speed of the deflagration (u_d) and of the shock (u_s), the results of the Chapman-Jouguet wave problem and the shock problem are used. First, an equation which relates the initial velocity of the piston (u_p) to the flow properties behind the shock is worked out. From equation (2.24), two relations can be established for $(u_5^{(0)} - u_s)$ and $(u_2^{(0)} - u_s)$ in terms of $(u_2^{(0)} - u_5^{(0)})$:

$$(u_5^{(0)} - u_s) = \frac{\gamma - 3}{4}\left(u_2^{(0)} - u_5^{(0)}\right) \mp \frac{\gamma + 1}{4}\sqrt{\left(u_2^{(0)} - u_5^{(0)}\right)^2 + \left(\frac{4c_2^{(0)}}{\gamma + 1}\right)^2} \tag{5.30}$$

and

$$(u_2^{(0)} - u_s) = \frac{\gamma + 1}{4}\left[\left(u_2^{(0)} - u_5^{(0)}\right) \mp \sqrt{\left(u_2^{(0)} - u_5^{(0)}\right)^2 + \left(\frac{4c_2^{(0)}}{\gamma + 1}\right)^2}\,\right] \tag{5.31}$$

Using $c^2 = \gamma p/\rho$ from equation (2.25) and (2.26),

$$c_5^{(0)2} = c_2^{(0)2}\,\frac{u_5^{(0)} - u_s}{u_2^{(0)} - u_s} + \gamma\left(u_2^{(0)} - u_5^{(0)}\right)\left(u_5^{(0)} - u_s\right) \tag{5.32}$$

From equations (5.13) and (5.14),

$$c_1^{(0)2} = c_5^{(0)2}\left[1 + \frac{q(\gamma - 1)}{c_5^{(0)2}}\,\frac{\gamma^2 + 1}{\gamma + 1} \pm \frac{q(\gamma - 1)^2}{c_5^{(0)2}}\sqrt{1 + \frac{2c_5^{(0)2}}{q(\gamma^2 - 1)}}\,\right] \tag{5.33}$$

Substituting equation (5.16) into (5.20) with the relation $u_1^{(0)} = u_p$, results in

$$u_p \pm c_1^{(0)} - u_5^{(0)} \mp c_5^{(0)} \left[\sqrt{1 + \frac{q(\gamma^2 - 1)}{2c_5^{(0)^2}}} \pm \sqrt{\frac{q(\gamma^2 - 1)}{2c_5^{(0)^2}}} \right] = 0 \qquad (5.34)$$

Equation (5.34) expresses the desired relation. The solution for $u_5^{(0)}$ can be determined iteratively by using equations (5.30) to (5.34). Once $u_5^{(0)}$ is determined, the velocity of shock, the leading terms of the flow properties behind the shock, the velocity of deflagration and the leading terms of the flow properties behind the deflagration can be calculated successively using the appropriate results from earlier chapters.

The first order terms of the flow properties in region V and region I, and the initial accelerations of the Chapman-Jouguet reaction front (a_d) and of compression shock (a_s), consist of 14 unknowns to be determined. The equations to be used are (i) the three jump conditions across the compression shock, equations (2.19), (2.20) (with region I there changed to region V), and equation (5.25); (ii) the conditions across the Chapman-Jouguet reaction front, equations (5.17), (5.18), and (5.19) (changing subscript 2 to 5 and and using the lower sign for deflagration); (iii) equations (2.27) to (2.29) for conditions inside region I; (iv) the counterparts of equations (2.27) to (2.29) for region 5, which are

$$\rho_{5x} u_5^{(0)} + \rho_{5t} c_0 = -\rho_5^{(0)} u_{5x}$$

$$u_{5x} u_5^{(0)} + u_{5t} c_0 = \frac{-p_{5x}}{\rho_5^{(0)}}$$

$$p_{5x} u_5^{(0)} + p_{5t} c_0 = -c_5^{(0)^2} \rho_5^{(0)} u_{5x}$$

(v) the boundary condition $a_p = u_{1x}u_p + u_{1t}c_0$; and (vi) equation (5.21) which follows from the fact that Mach number is unity behind the Chapman-Jouguet reaction front. The 14 unknowns can be determined by solving these 14 algebraic equations.

It is of interest to note that since the Mach number behind the Chapman-Jouguet reaction front is unity, this front is a characteristic line.
It was concluded in Chapter 3 that the initial acceleration of the piston, a_p, does not affect the flow properties ahead of the tail characteristic of an expansion fan, and does not affect the acceleration of the tail characteristic line itself. Since the Chapman-Jouguet reaction front is a characteristic line, the same situation would be expected to be true here; that is, the flow properties in region V and a_s and a_d should be determinable without knowing a_p. It is not obvious from the above set of 14 equations that this is so. It can be seen to be true by the following argument. Consider the 14 algebraic equations listed above, omit iii (equations 2.27 to 2.29) and v ($a_p = u_{1x}u_p + u_{1t}c_o$), and include the Riemann invariant relation (3.46) for the Chapman-Jouguet deflagration front. This results in 11 equations, which can be used to solve for the 11 unknowns, p_{5x}, p_{5t}, ρ_{5x}, ρ_{5t}, u_{5x}, u_{5t}, a_s, a_d, $\bar{p}_1^{(1)}$, $\bar{\rho}_1^{(1)}$, and $\bar{u}_1^{(1)}$. With these, all quantities in region V and a_d are known. Thus, the present case is consistent with previous result that the piston acceleration only affects the flow properties behind the tail characteristic, but not the initial acceleration of the tail characteristic, or the flow properties ahead of the tail characteristic.

This completes the discussion of the elementary piston problems with chemical reaction. First, the Chapman-Jouguet solution has been worked out in detail in section 1. The strong detonation problem, which is similar to the adiabatic shock problem, is discussed next. The Chapman-Jouguet detonation with an expansion fan following the detonation front and the Chapman-Jouguet deflagration with a precompression shock are discussed in sections 3 and 4, respectively. In these problems, explicit solutions are not worked out in detail because each involves extensive algebra; however, they can be determined numerically easily using the listed linear systems of equations. Once the separate elementary problems have been solved, they can be combined with the shock or expansion fan solutions in the preceeding chapters using exactly the same procedure as in chapter 4 to solve initial value problems with chemical reaction. Two examples of such problems are indicated in figures 10 and 11. Figure 10 shows a typical wave pattern including chemical reaction and an adiabatic expansion fan; Figure 11 illustrates a pattern with chemical reaction and a shock wave.

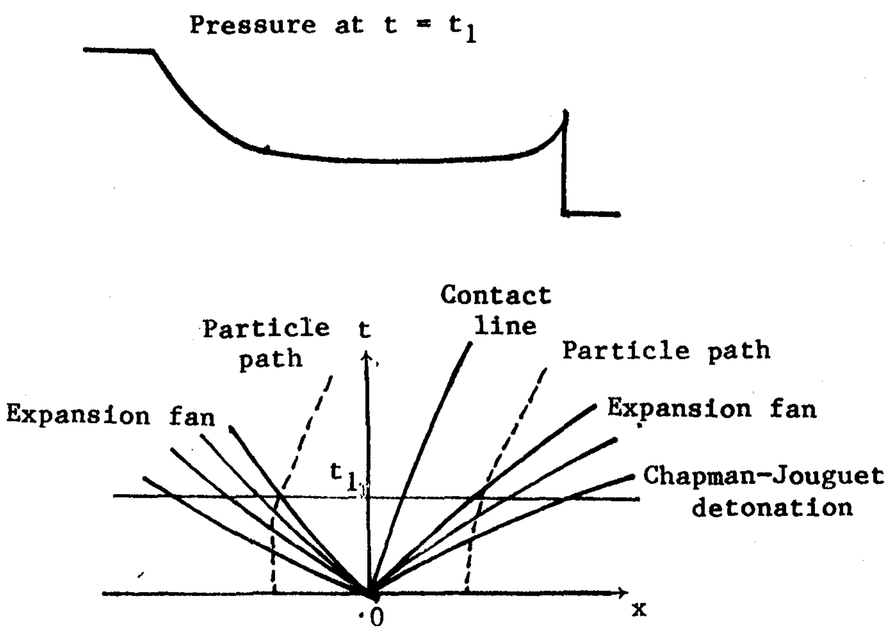

Figure 10.- Wave pattern including chemical reaction and adiabatic expansion fan.

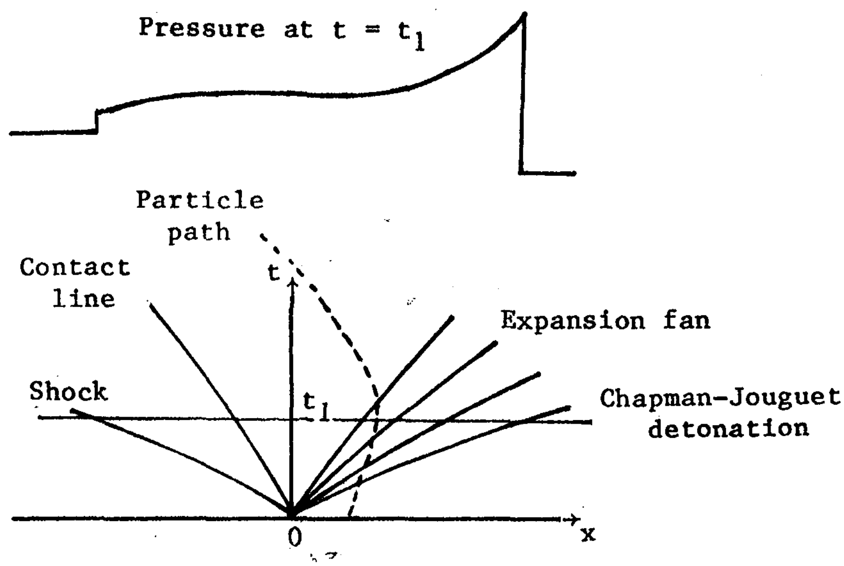

Figure 11.- Wave pattern with chemical reaction and shock wave.

CHAPTER 6

CONCLUDING REMARKS

The purpose of this work is to discuss the initial value problem of one-dimensional gas-dynamics involving discontinuous, nonuniform initial data. Canonical solutions which are valid in a small x, t region around a discontinuity, and which include the first order effects of nonuniformities, have been derived explicitly. Similar solutions corresponding to discontinuous but otherwise uniform initial data have been used in the past as building blocks for the numerical solution of general initial value problems. The results presented in the present work, in which the nonuniformities in the data are explicitly taken into account, are intended to be used as building blocks in an improved numerical scheme which will permit the use of much larger mesh sizes than do previous methods.

The theory has been derived by considering a group of elementary piston problems. Solutions with a shock or with a centered expansion wave have been worked out individually in order to relate initial flow properties and their gradients to the speed and acceleration of the discontinuity waves. Then they have been combined to represent the solution of an initial value problem by regarding the piston path as a contact line. In addition, problems with chemical reaction are discussed in terms of elementary piston problems which involve strong detonation waves, Chapman-Jouguet detonation waves, and deflagration waves. These solutions can be combined with the shock or expansion fan results to derive canonical solutions of the initial value problem with chemical reaction.

The actual implementation of the current results into an improved numerical scheme for general initial data is beyond the scope of the current paper.

BIBLIOGRAPHY

1. Liu, T. P.: The Riemann Problem for General Systems of Conservation Laws. Journal of Differential Equations, Vol. 18, 1975, pp. 218-234.

2. Glimm, J.: Tracking of Interfaces for Fluid Flow: Accurate Methods for Piecewise Smooth Problems. Transonic, Shock, and Multidimensional Flows: Advances in Scientific Computing, publication No. 47, Academic Press, Inc., 1982.

3. Chorin, A. J.: Random Choice Solution of Hyperbolic Systems. Journal of Computational Physics, Vol. 22, 1976, pp. 517-533.

4. Glimm, J.: Solutions in the Large for Nonlinear Hyperbolic Systems of Equations. Communications on Pure and Applied Mathematics, Vol. 18, 1965, pp. 697-715.

5. Chorin, A. J.: Random Choice Methods with Applications to Reacting Gas Flow. Journal of Computational Physics, Vol. 25, 1977, pp. 273-296.

6. Lin, C. C.; and Rubinov, S. I.: On the Flow Behind Curved Shocks. Journal of Mathematics and Physics, Vol. 27, No. 2, July 1948, pp. 105-129.

7. Darden, C. M.: An Analysis of Shock Coalescence Including Three-Dimensional Effects with Application to Sonic Boom Extrapolation. NASA TP 2214, 1984.

8. Liepmann, H. W.; and Roshko, A.: Elements of Gasdynamics. John Wiley and Sons, Inc., 1957.

9. Courant, R.; and Friedrichs, K. O.: Supersonic Flow and Shock Waves. Interscience Publishers, Inc., 1948.

10. Williams, F. A.: Combustion Theory: The Fundamental Theory of Chemically Reacting Flow Systems. Addison-Wesley Publishing Company, Inc., 1965.

11. Thompson, P. A.: Compressible-Fluid Dynamics. McGraw-Hill Book Company, 1972.

1. Report No. NASA TM-85812	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Canonical Solutions for Unsteady Flow Fields		5. Report Date May 1984
		6. Performing Organization Code 505-43-43-01
7. Author(s) Grace C. Liu		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address NASA Langley Research Center Hampton, VA 23665		11. Contract or Grant No.
		13. Type of Report and Period Covered Technical Memorandum
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20542		14. Sponsoring Agency Code

15. Supplementary Notes

Collateral publication of thesis submitted in partial fulfillment of the requirements for the Degree of Master of Science for the School of Engineering and Applied Science of the George Washington University.

16. Abstract

This work discusses the initial value problem of one-dimensional gas-dynamics involving discontinuous, nonuniform initial data. Canonical solutions which are valid in a small x, t region around a discontinuity, and which include the first order effects of nonuniformities in the data, are derived explicitly. The theory is derived by considering a group of elementary piston problems. Solutions with a shock or with a centered expansion wave are worked out individually in order to relate initial flow properties and their gradients to the speed and acceleration of the discontinuity waves. They are then combined to represent the solution of a general initial value problem by regarding the piston path as a contact line. In addition, problems with chemical reaction are discussed in terms of elementary piston problems which involve strong detonation waves, Chapman-Jouguet detonation waves, and deflagration waves.

17. Key Words (Suggested by Author(s)) Unsteady flow; Initial value problem; shock wave; Chemical reaction	18. Distribution Statement Unclassified - Unlimited; Subject Category 34

19. Security Classif. (of this report) UNCLASSIFIED	20. Security Classif. (of this page) UNCLASSIFIED	21. No. of Pages 73	22. Price A04

N-305