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SYMBOLS

a,b,c	components of $\mathbf{n}_{x3}$ in the x, y, and z-directions
$[B],[B_\phi],[B_\theta]$	transformation matrices describing the bending at P for the bend-twist sequence with no pretwist, for the twist-bend sequence with no pretwist, and for the pretwist-bend-twist sequence, respectively
$\mathbf{n}_R$	axis for the bending rotation
$\mathbf{n}_t$-$\mathbf{n}_n$-$\mathbf{n}_b$	tangential, normal, and binormal unit vectors to a space curve
$\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$	unit vectors in the x, y, and z-directions
$\mathbf{n}_{xB}$-$\mathbf{n}_{yB}$-$\mathbf{n}_{zB}$	intermediate triad of unit vectors corresponding to $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ after the pretwist and bend rotations of the pretwist-bend-twist sequence
$\mathbf{n}_{x\phi}$-$\mathbf{n}_{y\phi}$-$\mathbf{n}_{z\phi}$	intermediate triad of unit vectors corresponding to $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ after the twist rotation in the twist-bend sequence
$\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$	triad of mutually perpendicular unit vectors corresponding to $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ after deformation
$\mathbf{n}_\xi$-$\mathbf{n}_\eta$-$\mathbf{n}_\zeta$	triad of unit vectors corresponding to $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ after pretwist
P	arbitrary point on the elastic axis before deformation
P*	point P after deformation
$\mathbf{R}$	vector position of P
$\mathbf{R}^*$	vector position of P*
$\Delta\mathbf{R}$	vector difference between $\mathbf{R}$ and $\mathbf{R}^*$
[T]	transformation matrix between $\mathbf{n}_\xi$-$\mathbf{n}_\eta$-$\mathbf{n}_\zeta$ and $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$
$[\mathscr{T}],[\mathscr{T}_\theta],[\mathscr{T}_\alpha]$	transformation matrix between $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ and $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$ for no pretwist, initial pretwist, and combined twist cases
$[\mathscr{T}_B],[\mathscr{T}_{\theta B}]$	transformation matrix between $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ and $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$ determined with the two-term sequence for the zero and nonzero-pretwist cases
u,v,w	displacements at P in the x, y, and z-directions
x-y-z	axes characterizing blade geometry before pretwist and deformation where x is directed along the elastic axis
x_3-y_3-z_3	curvilinear coordinates of deformed blade where x_3 is directed along the elastic axis
α	sum of θ and ϕ_α

$\beta, \beta_\theta, \beta_\alpha$	flap rotation about $-y$ or updated $-y$-axis for the no pretwist, for the initial pretwist, and for the combined twist cases, respectively
ε	some small nondimensional value corresponding in magnitude to the slope of the deformed blade
ξ-η-ζ	curvilinear coordinates of a pretwisted blade with ξ directed along the elastic axis
$\zeta, \zeta_\theta, \zeta_\alpha$	lag rotation about the z-axis or updated z-axis for the no pretwist, for the initial pretwist, and for the combined twist cases, respectively
θ	pretwist angle
κ	bending curvature of a space curve
τ	torsional curvature for a space curve
Φ	bending rotation angle
$\phi, \phi_\theta, \phi_\alpha$	deformation twist variable defined in terms of Euler-like sequence for the no pretwist, for the initial pretwist, and for the combined twist cases, respectively
ϕ_e	elastic twist
$\phi_T, \phi_{T\theta}$	deformation twist variable defined in terms of the two-term sequence for the zero- and nonzero-pretwist cases
$\omega_{x_3}, \omega_{y_3}, \omega_{z_3}$	components of $\boldsymbol{\omega}$ in the x_3, y_3, and z_3-directions
$\boldsymbol{\omega}$	curvature vector of the blade
Ω	angular velocity of rotor
$\boldsymbol{\Omega}$	curvature vector of space curve

Subscripts:

1,2,3,4,5,6	refer to the flap-lag-twist (1), flap-twist-lag (2), twist-flap-lag (3), lag-flap-twist (4), lag-twist-flap (5), and twist-lag-flap sequences (6)

SUMMARY

Adequate modeling of elastic helicopter rotor blades experiencing moderately large deformations requires a nonlinear analysis. This analysis must be based on an appropriate description of the blade's deformation geometry, including elastic bending and twist. Built-in pretwist angles complicate the deformation process and its definition. In this study, relationships between the twist variables associated with different rotation sequences are listed, as well as corresponding forms of the transformation matrix. Included are relationships between the twist variables associated with, first, the pretwist applied initially and, second, the pretwist combined with the deformation twist. Many of the corresponding forms of the transformation matrix for the two cases are also listed. Moreover, it is shown that twist variables connected with the combined-twist treatment can be related to those in which the pretwist is applied initially. A method is outlined for determining these relationships, and some results are given. Additionally, a procedure is demonstrated for evaluating the transformation matrix that eliminates the Euler-like sequence altogether. The resulting form of the transformation matrix is unaffected by rotation sequence or pretwist treatment.

INTRODUCTION

To derive the equations of motion for an elastic rotor blade, one usually begins by relating the blade's instantaneous deformed geometry to that of the undeformed blade. It is common practice in the helicopter literature to evaluate the transformation matrix between the deformed and undeformed states using an Euler-like sequence of three successive rotations (e.g., refs. 1-4). Furthermore, in the helicopter literature the built-in twist of the undeformed blade is often included with the deformation twist in the rotation sequence (e.g., refs. 1 and 2). For linear mathematical models (ref. 5), the order of rotation and the handling of the pretwist do not affect the final form of the transformation matrix. However, in nonlinear analyses, the final form of the transformation matrix — and subsequently the derived equations of motion — will vary depending on the deformation sequence and pretwist treatment.

Adequate modeling of elastic rotor blades experiencing moderately large deformations requires a nonlinear analysis. In basing that analysis on an appropriate description of the deformation geometry, the question arises of what form to use for the transformation matrix and how it relates to other forms. Since the built-in pretwist angle is physically present before the deformation, applying the pretwist before the blade bending and twisting deformations is unquestionably correct. However, combining the pretwist with the deformation twist has not been shown to be less correct. This combination of twists usually simplifies the derivation and resulting expressions — but the validity of the approach should be examined.

Kaza and Kvaternik (refs. 2, 6, and 7) developed the curvatures, transformation matrices, and equations of motion for a rotating elastic blade, using the flap-lag-twist and lag-flap-twist rotation sequences concurrently. The authors assumed that the twist variables used in the two sequences were identical and equal to the elastic twist. Note that the elastic twist is that part of the deformation twist associated with the torsion of the blade. Bending can also contribute to the deformation twist. In not distinguishing between the elastic and deformation twists, Kaza and Kvaternik concluded that the two different sets of equations were not equivalent. The authors suggested interpreting the results as "two different nonlinear approximations of a given physical system" (ref. 2, p. 81).

Rosen and Friedmann (ref. 8, p. 163) concluded that the orientation of the triad of unit vectors characterizing the deformed blade can differ by second-order terms that depend on the rotation sequence. Therefore, these authors chose one sequence and were consistent throughout the derivation (ref. 3, p. 45 or ref. 8, p. 163).

Petersen (ref. 9) showed that the change in orientation of a rod with zero elastic twist could not be properly described by either a flap-lag or a lag-flap rotation sequence. Since zero elastic twist does not mean zero deformation twist, it is not surprising that the flap-lag and lag-flap sequences are inadequate. The resulting orientations differed from each other and from the actual orientation by rotations about the deformed elastic axis. The actual orientation could be determined by either a flap-lag-twist or a lag-flap-twist sequence in which two different twist variables were evaluated with the zero elastic twist condition. Petersen extended the results to include elastic and built-in twist.

Hodges et al. (ref. 10) showed that by regarding the twist variable associated with each Euler-like sequence as different, it was possible to find relationships between the twist variables for the different sequences. Using these relationships, it was then possible with the appropriate change of variable to make any one form of the transformation matrix identical to any other. The authors reasoned that regardless of the rotation sequence used in the analysis, an elastic beam with a given set of end conditions and loads is a single physical system. It was argued that the transformation matrix and curvatures were unique, but that the form varied according to the choice of variables — specifically, the choice of the twist variable. Two methods for determining the relationships between the different twist variables were employed. The principal approach used in reference 10 involved equating expressions for an integrated form of the torsional curvature. Another was also outlined (ref. 10, pp. 42 and 46) — to equate components of the various forms of the transformation matrix. Hodges et al. determined a relationship between the twist variables associated with the flap-lag-twist and the lag-flap-twist rotation sequences.

Several authors have made it a point to apply the pretwist before the deformation rotations (refs. 3 and 4). However, the arguments presented seem to rest more on "realism" than on the inappropriateness of combining the twists (ref. 4, pp. 40 and 41). In response, Hodges et al. (ref. 10, p. 20) claimed that the pretwist could be combined with the deformation twist in the flap-lag-twist and lag-flap-twist, but not in the flap-twist-lag, twist-flap-lag, lag-twist-flap, or twist-lag-flap rotation sequences. The authors argued that the pretwist, elastic twist, and the deformation twists are all rotations about the same axis for the flap-lag-twist and lag-flap-twist sequences.

The results of reference 10 for the case of zero pretwist are generalized and extended in this paper. Relationships between the twist variables associated with different rotation sequences are listed, as well as corresponding forms of the transformation matrix. Included are relationships between the twist variables associated with, first, the pretwist applied initially and, second, the pretwist combined with the deformation twist. Many of the corresponding forms of the transformation matrix for the two cases are also listed. Moreover, it is shown that twist variables connected with the combined-twist treatment can be related to those in which the pretwist is applied initially. A method is outlined for determining these relationships and some results are given. Additionally, a procedure is demonstrated for evaluating the transformation matrix that eliminates the Euler-like sequence altogether. The resulting form of the transformation matrix is unaffected by rotation sequence or pretwist treatment.

The emphasis in this paper is on generalizing some of the arguments presented in reference 10, validating the approach of lumping the pretwist with the deformation twist, offering an alternative approach for evaluating the transformation matrix, and providing the means to compare sets of equations that have been derived using different rotation sequences and different pretwist treatments.

DEFORMATION GEOMETRY

Assume that the elastic axis of the undeformed blade lies along the x-axis. Now, consider the blade frozen in some deformed state. At that instant, the elastic axis lies along a unique curve in three-dimensional space (fig. 1). Before deformation, the position of a point, P, on the elastic axis is

$$\mathbf{R} = x\mathbf{n}_x$$

After deformation, the position of P, now P*, is

$$\begin{aligned}\mathbf{R}^* &= \mathbf{R} + \Delta\mathbf{R} \\ &= (x + u)\mathbf{n}_x + v\mathbf{n}_y + w\mathbf{n}_z\end{aligned}$$

At any given instant, the displacement variables, u-v-w, are functions of position along the undeformed elastic axis, x. As a result,

$$d\mathbf{R}^* = dx[(1 + u')\mathbf{n}_x + v'\mathbf{n}_y + w'\mathbf{n}_z$$

where ( )' indicates differentiation with respect to x.

The curvilinear coordinate, x_3, is measured along the deformed elastic axis; $\mathbf{n}_{x3}$ is the unit vector tangential to the deformed elastic axis. Therefore,

$$d\mathbf{R}^* = dx_3\mathbf{n}_{x3}$$

The magnitude of $d\mathbf{R}^*$ is equal to dx_3 and also to

$$\begin{aligned}\|d\mathbf{R}^*\| &= (d\mathbf{R}^* \cdot d\mathbf{R}^*)^{1/2} \\ &= dx[(1 + u')^2 + (v')^2 + (w')^2]^{1/2}\end{aligned}$$

Equating expressions for $\|d\mathbf{R}^*\|$ yields a relationship between dx and dx_3:

$$\frac{dx}{dx_3} = [(1 + u')^2 + (v')^2 + (w')^2]^{-1/2}$$

Then, $\mathbf{n}_{x_3}$ can be expressed as

$$\mathbf{n}_{x_3} = \frac{dx}{dx_3} [(1 + u')\mathbf{n}_x + v'\mathbf{n}_y + w'\mathbf{n}_z]$$

Let

$$\mathbf{n}_{x_3} = a\mathbf{n}_x + b\mathbf{n}_y + c\mathbf{n}_z$$

where a, b, and c represent the previously derived components of $\mathbf{n}_{x_3}$. The relation

$$a^2 + b^2 + c^2 = 1$$

will be used repeatedly throughout the paper.

In helicopter-blade aeroelastic analyses, it is usually assumed that although deflections v and w can be moderately large, slopes v' and w' are small — of the order of some ε in size. In general, since rotor blades are long and slender, it is assumed that u' is smaller — of the order of ε^2. Using these guidelines and retaining terms through $O(\varepsilon^2)$,

$$a \cong 1 - \frac{1}{2} (v')^2 - \frac{1}{2} (w')^2$$

$$b \cong v'$$

$$c \cong w'$$

Thus, a is O(1), and b and c are $O(\varepsilon)$. To $O(\varepsilon^2)$, a can also be expressed as

$$a \cong 1 - \frac{b^2}{2} - \frac{c^2}{2}$$

TRANSFORMATION MATRIX

In general, fibers at P along the x-, y-, and z-directions do not remain mutually perpendicular, but distort in proportion to the shear strains (ref. 11, p. 23). Shear deformations are neglected in helicopter-blade analyses, however, and the local $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ triad is assumed to rotate into another triad of mutually perpendicular unit vectors, $\mathbf{n}_{x_3}$-$\mathbf{n}_{y_3}$-$\mathbf{n}_{z_3}$. Note that if shear effects are included, the change in orientation at P can be broken into a rigid body part plus a distortion part due to the shear effects. The rigid-body rotation is described in terms of a transformation matrix, $[\mathscr{T}]$, where

$$\begin{Bmatrix} \mathbf{n}_{x3} \\ \mathbf{n}_{y3} \\ \mathbf{n}_{z3} \end{Bmatrix} = [\mathscr{T}]\begin{Bmatrix} \mathbf{n}_{x} \\ \mathbf{n}_{y} \\ \mathbf{n}_{z} \end{Bmatrix} = \begin{bmatrix} a & b & c \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}\begin{Bmatrix} \mathbf{n}_{x} \\ \mathbf{n}_{y} \\ \mathbf{n}_{z} \end{Bmatrix}$$

Three of the nine components of $[\mathscr{T}]$ are known; the other six must be determined.

The variables that describe the displacement at P specify the direction of $\mathbf{n}_{x3}$ and the plane in which $\mathbf{n}_{y3}$ and $\mathbf{n}_{z3}$ fall. If the first row of $[\mathscr{T}]$ is as shown, the final orientation described by any orthonormal $[\mathscr{T}]$ will differ at most by a rotation about $\mathbf{n}_{x3}$. Therefore, whatever the scheme used to evaluate the remaining components of $[\mathscr{T}]$, if orthonormality is maintained, $[\mathscr{T}]$ will describe the desired orientation to within some twist angle.

For the analytical model to include the effects of elastic torsional deformation, an additional degree of freedom is required. Consider incorporating a rotational degree of freedom about the elastic axis — one that is defined in the process of evaluating the transformation matrix. This deformation-twist variable includes the elastic twist along with some part of the bending twist. The part of the bending twist that is included depends on the sequence used to evaluate the transformation matrix. By including a deformation-twist variable, the analytical model is given sufficient flexibility to properly orient $\mathbf{n}_{y3}$ and $\mathbf{n}_{z3}$. Twists like those developed in reference 9 to correct the lag-flap and flap-lag results for a rod with zero elastic twist are simply lumped into the deformation-twist variables. For the six Euler-like sequences examined here and in reference 10, along with the alternative approach developed here, the different twist variables and associated forms of the transformation matrix should offer equivalent descriptions of the deformed blade. In this context, the reasoning of Hodges et al. (ref. 10) becomes appropriate.

Whether or not the blade has built-in twist, the orientation of $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$ is still described in terms of $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ by an orthonormal matrix with a, b, and c in the top row. Although this transformation matrix includes a pretwist contribution, the conclusions of the last two paragraphs remain valid.

EVALUATING TRANSFORMATION MATRICES AND RELATING TWIST VARIABLES

No Pretwist

The lag-flap-twist sequence corresponds to the series of rotations (fig. 2):

1. ζ_4 about the z-axis

2. β_4 about the updated y-axis

3. ϕ_4 about the twice updated x-axis

The transformation matrix associated with the lag-flap-twist sequence is

$$[\mathcal{T}_4] = \begin{bmatrix} \cos\beta_4 \cos\zeta_4 & \cos\beta_4 \sin\zeta_4 & \sin\beta_4 \\ -\cos\phi_4 \sin\zeta_4 - \sin\phi_4 \sin\beta_4 \cos\zeta_4 & \cos\phi_4 \cos\zeta_4 - \sin\phi_4 \sin\beta_4 \sin\zeta_4 & \sin\phi_4 \cos\beta_4 \\ \sin\phi_4 \sin\zeta_4 - \cos\phi_4 \sin\beta_4 \cos\zeta_4 & -\sin\phi_4 \cos\zeta_4 - \cos\phi_4 \sin\beta_4 \sin\zeta_4 & \cos\phi_4 \cos\beta_4 \end{bmatrix}$$

By equating the components of the top row to a, b, and c, respectively, expressions for the sines and cosines of ζ_4 and β_4 can be determined. These can then be substituted back into $[\mathcal{T}_4]$. By defining the twist variable in terms of ϕ_4, $[\mathcal{T}_4]$ is completely determined. An alternative approach for defining the twist variable is discussed later in this report.

Based on the rotation scheme discussed above, there are six possible sequences:

1. Flap-lag-twist $(\beta_1\text{-}\zeta_1\text{-}\phi_1)$
2. Flap-twist-lag $(\beta_2\text{-}\phi_2\text{-}\zeta_2)$
3. Twist-flap-lag $(\phi_3\text{-}\beta_3\text{-}\zeta_3)$
4. Lag-flap-twist $(\zeta_4\text{-}\beta_4\text{-}\phi_4)$
5. Lag-twist-flap $(\zeta_5\text{-}\phi_5\text{-}\beta_5)$
6. Twist-lag-flap $(\phi_6\text{-}\zeta_6\text{-}\beta_6)$

In terms of a, b, c, and the appropriate twist variable, the final form of $[\mathcal{T}]$ associated with each rotation sequence is listed in table 1.

Rotor blades are relatively stiff in torsion, and twist deformations are usually assumed to be small — of $O(\varepsilon^2)$ in size. Retaining all terms through $O(\varepsilon^2)$, there are two distinct forms for $[\mathcal{T}]$. The flap-lag form can be expressed as

$$[\mathcal{T}_i] \cong \begin{bmatrix} 1 - b^2/2 - c^2/2 & b & c \\ -b - c\phi_i & 1 - b^2/2 - \phi_i^2/2 & \phi_i - bc \\ -c + b\phi_i & -\phi_i & 1 - c^2/2 - \phi_i^2/2 \end{bmatrix} ; \qquad i = 1,2,3$$

and the lag-flap form as

$$[\mathcal{T}_j] \cong \begin{bmatrix} 1 - b^2/2 - c^2/2 & b & c \\ -b - c\phi_j & 1 - b^2/2 - \phi_j^2/2 & \phi_j \\ -c + b\phi_j & -\phi_j - bc & 1 - c^2/2 - \phi_j^2/2 \end{bmatrix} ; \qquad j = 4,5,6$$

For the linear case, in which terms smaller than $O(\varepsilon)$ are neglected, all six transformations reduce to the single form,

$$[\mathscr{T}_k] \cong \begin{bmatrix} 1 & b & c \\ -b & 1 & \phi_k \\ -c & -\phi_k & 1 \end{bmatrix} \; ; \qquad k = 1,2,3,4,5,6$$

Relationships between the six twist variables introduced above are listed in table 2. Included are relationships of each of the other five twist variables to ϕ_1. Manipulating these expressions, one can also find relationships between the five. The relationships of the lag-flap-twist variables, ϕ_5 and ϕ_6, to ϕ_4 are listed. Several others are added for use later. These expressions were derived by equating components of the different forms of $[\mathscr{T}]$. It is easily verified that by substituting the expression for one twist variable in terms of another, identical transformation matrices are produced.

Initial Twist

The pretwist, θ, is built-in about the blade's elastic axis before deformation. This pretwist can be a function of x but not of time. Figure 3 illustrates the undeformed geometry of the blade. The tangent to the elastic axis after deformation can still be described as

$$\mathbf{n}_{x3} = a\mathbf{n}_x + b\mathbf{n}_y + c\mathbf{n}_z$$

Relating the deformed geometry to that of the undeformed blade

$$\begin{Bmatrix} \mathbf{n}_{x3} \\ \mathbf{n}_{y3} \\ \mathbf{n}_{z3} \end{Bmatrix} = [T] \begin{Bmatrix} \mathbf{n}_\xi \\ \mathbf{n}_\eta \\ \mathbf{n}_\zeta \end{Bmatrix} = \begin{bmatrix} A & B & C \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{bmatrix} \begin{Bmatrix} \mathbf{n}_\xi \\ \mathbf{n}_\eta \\ \mathbf{n}_\zeta \end{Bmatrix}$$

where

$$A = a$$

$$B = b \cos \theta + c \sin \theta$$

$$C = -b \sin \theta + c \cos \theta$$

The lag-flap-twist ($\zeta_{\theta4}$-$\beta_{\theta4}$-$\phi_{\theta4}$) sequence is illustrated in figure 4. To evaluate $[T_4]$, substitute A for a, B for b, C for c and $\phi_{\theta4}$ for ϕ_4 in $[\mathscr{T}_4]$. In the same manner, the forms of [T] associated with the flap-lag-twist ($\beta_{\theta1}$-$\zeta_{\theta1}$-$\phi_{\theta1}$), flap-twist-lag ($\beta_{\theta2}$-$\phi_{\theta2}$-$\zeta_{\theta2}$), twist-flap-lag ($\phi_{\theta3}$-$\beta_{\theta3}$-$\zeta_{\theta3}$), lag-twist-flap ($\zeta_{\theta5}$-$\phi_{\theta5}$-$\beta_{\theta5}$), and twist-lag-flap ($\phi_{\theta6}$-$\zeta_{\theta6}$-$\beta_{\theta6}$) rotation sequences can be evaluated. The triad, $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$, can also be related to $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ by $[\mathscr{T}_\theta]$ where

$$\begin{Bmatrix} \mathbf{n}_{x3} \\ \mathbf{n}_{y3} \\ \mathbf{n}_{z3} \end{Bmatrix} = [\mathscr{T}_\theta] \begin{Bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{Bmatrix} = [T] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{Bmatrix}$$

Several forms of $[\mathscr{T}_\theta]$ are listed in table 3.

Again, when retaining terms to $O(\varepsilon^2)$, the six forms of $[\mathscr{T}_\theta]$ reduce to two. The flap-lag form is expressed as

$$
[\mathscr{T}_{\theta i}] \cong \left[\begin{array}{l|l|l}
1 - \dfrac{b^2}{2} - \dfrac{c^2}{2} & b & c \\
\hline
\begin{array}{l} -b\cos\theta - c\sin\theta \\ + \phi_{\theta i}(b\sin\theta - c\cos\theta) \end{array} & \begin{array}{l} -(\phi_{\theta i} + bc)\sin\theta \\ + \left[1 - b^2 + \dfrac{1}{2}(b\cos\theta + c\sin\theta)^2 \right. \\ \left. \quad - \dfrac{\phi_{\theta i}^2}{2}\right]\cos\theta \end{array} & \begin{array}{l} (\phi_{\theta i} - bc)\cos\theta \\ + \left[1 - c^2 + \dfrac{1}{2}(b\cos\theta + c\sin\theta)^2 \right. \\ \left. \quad - \dfrac{\phi_{\theta i}^2}{2}\right]\sin\theta \end{array} \\
\hline
\begin{array}{l} b\sin\theta - c\cos\theta \\ + \phi_{\theta i}(b\cos\theta + c\sin\theta) \end{array} & \begin{array}{l} -\phi_{\theta i}\cos\theta \\ - \left[1 - \dfrac{1}{2}(b\sin\theta - c\cos\theta)^2 \right. \\ \left. \quad - \dfrac{\phi_{\theta i}^2}{2}\right]\sin\theta \end{array} & \begin{array}{l} -\phi_{\theta i}\sin\theta \\ + \left[1 - \dfrac{1}{2}(b\sin\theta - c\cos\theta)^2 \right. \\ \left. \quad - \dfrac{\phi_{\theta i}^2}{2}\right]\cos\theta \end{array}
\end{array}\right]
$$

$$i = 1,2,3$$

and the lag-flap form as

$$
[\mathscr{T}_{\theta j}] \cong \left[\begin{array}{l|l|l}
1 - \dfrac{b^2}{2} - \dfrac{c^2}{2} & b & c \\
\hline
\begin{array}{l} -b\cos\theta - c\sin\theta \\ + \phi_{\theta j}(b\sin\theta - c\cos\theta) \end{array} & \begin{array}{l} -\phi_{\theta j}\sin\theta \\ + \left[1 - \dfrac{1}{2}(b\cos\theta + c\sin\theta)^2 \right. \\ \left. \quad - \dfrac{\phi_{\theta j}^2}{2}\right]\cos\theta \end{array} & \begin{array}{l} \phi_{\theta j}\cos\theta \\ + \left[1 - \dfrac{1}{2}(b\cos\theta + c\sin\theta)^2 \right. \\ \left. \quad - \dfrac{\phi_{\theta j}^2}{2}\right]\sin\theta \end{array} \\
\hline
\begin{array}{l} b\sin\theta - c\cos\theta \\ + \phi_{\theta j}(b\cos\theta + c\sin\theta) \end{array} & \begin{array}{l} -(\phi_{\theta j} + bc)\cos\theta \\ - \left[1 - b^2 + \dfrac{1}{2}(b\sin\theta - c\cos\theta)^2 \right. \\ \left. \quad - \dfrac{\phi_{\theta j}^2}{2}\right]\sin\theta \end{array} & \begin{array}{l} -(\phi_{\theta j} - bc)\sin\theta \\ + \left[1 - c^2 + \dfrac{1}{2}(b\sin\theta - c\cos\theta)^2 \right. \\ \left. \quad - \dfrac{\phi_{\theta j}^2}{2}\right]\cos\theta \end{array}
\end{array}\right]
$$

$$j = 4,5,6$$

Neglecting terms smaller than $O(\varepsilon)$, all six forms of $[\mathscr{T}_\theta]$ reduce to

$$[\mathscr{T}_{\theta k}] \cong \begin{bmatrix} 1 & b & c \\ -b\cos\theta - c\sin\theta & \cos\theta - \phi_{\theta k}\sin\theta & \sin\theta + \phi_{\theta k}\cos\theta \\ b\sin\theta - c\cos\theta & -\sin\theta - \phi_{\theta k}\cos\theta & \cos\theta - \phi_{\theta k}\sin\theta \end{bmatrix} ; \qquad k = 1,2,3,4,5,6$$

Note that no assumption has been made about the size of θ.

Relationships between the six twist variables introduced above can be determined by equating components of the different forms of [T]. Equivalently, the expressions derived for the case of no-pretwist (table 2) can be extended to this case by substituting A for a, B for b, C for c, and ϕ_θ for ϕ. Some revised expressions are listed in table 4. The relationships now include a θ-dependence.

Combined Twists

Let the transformation matrix, $[\mathscr{T}_\alpha]$, describe the difference in orientation between $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ and $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$. As before, the components of the top row are equated to a, b, and c, respectively. The pretwist is combined with the deformation twist in the rotation sequence used to evaluate the rest of $[\mathscr{T}_\alpha]$. Six rotation sequences are considered:

1. Flap-lag-twist ($\beta_{\alpha 1}$-$\zeta_{\alpha 1}$-α_1)
2. Flap-twist-lag ($\beta_{\alpha 2}$-α_2-$\zeta_{\alpha 2}$)
3. Twist-flap-lag (α_3-$\beta_{\alpha 3}$-$\zeta_{\alpha 3}$)
4. Lag-flap-twist ($\zeta_{\alpha 4}$-$\beta_{\alpha 4}$-α_4)
5. Lag-twist-flap ($\zeta_{\alpha 5}$-α_5-$\beta_{\alpha 5}$)
6. Twist-lag-flap (α_6-$\zeta_{\alpha 6}$-$\beta_{\alpha 6}$)

where

$$\alpha_i = \theta + \phi_{\alpha i} ; \qquad i = 1,2,3,4,5,6$$

A different form of $[\mathscr{T}_\alpha]$ is associated with each rotation sequence. The results for $[\mathscr{T}_i]$ given in table 1 can be used for $[\mathscr{T}_{\alpha i}]$ by substituting α_i for ϕ_i.

In table 5, expressions for $[\mathscr{T}_\alpha]$ are listed in which all terms greater than $O(\varepsilon^2)$ are neglected. Neglecting terms smaller than $O(\varepsilon)$ reduces the six forms of $[\mathscr{T}_\alpha]$ to

$$[\mathscr{T}_{\alpha i}] \doteq \begin{bmatrix} 1 & b & c \\ -b\cos\theta - c\sin\theta & \cos\theta - \phi_{\alpha i}\sin\theta & \sin\theta + \phi_{\alpha i}\cos\theta \\ b\sin\theta - c\cos\theta & -\sin\theta - \phi_{\alpha i}\cos\theta & \cos\theta - \phi_{\alpha i}\sin\theta \end{bmatrix} ; \qquad i = 1,2,3,4,5,6$$

Note that the linear expression above is identical to that derived for the case of initially applied pretwist.

By equating components of the different $[\mathscr{T}_\alpha]$, relationships can be determined between the different α-variables. Since α is not necessarily of $O(\varepsilon)$, the results to $O(\varepsilon^2)$ that are listed in table 2 cannot be extended to the present case. However, the other results in table 2 can be extended by replacing ϕ_i with α_i. Some modified expressions are listed in table 6. As before, the different forms of $[\mathscr{T}_\alpha]$ can be reduced to one by the appropriate change of the twist variable, α.

The trigonometric identity,

$$\tan(\phi_\alpha + \theta) = \frac{\tan\phi_\alpha + \tan\theta}{1 - \tan\phi_\alpha \tan\theta}$$

can be used to relate the ϕ_α-variables directly. Some expressions are listed in table 7. Although the relationships are largely θ-dependent, the relationship between $\phi_{\alpha1}$ and $\phi_{\alpha4}$ is identical to the zero pretwist case. Recall that $\phi_{\alpha1}$ is associated with the flap-lag-twist sequence and $\phi_{\alpha4}$ with the lag-flap-twist sequence. This result is a direct consequence of the matrix identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

Both $[\mathscr{T}_{\alpha1}]$ and $[\mathscr{T}_{\alpha4}]$ can be expressed as products of the θ-matrix and matrices independent of θ. Therefore, in equating $[\mathscr{T}_{\alpha1}]$ and $[\mathscr{T}_{\alpha4}]$, the expressions are independent of θ.

RELATING TWIST VARIABLES ASSOCIATED WITH THE INITIAL PRETWIST AND COMBINED PRETWIST TREATMENTS

Relationships between twist variables have been determined for the case in which the pretwist is applied initially and for the case in which the pretwist is combined with the deformation twist. It is possible, by equating components of the appropriate forms of the transformation matrix, to relate twist variables associated with one pretwist treatment to those associated with the other. Equivalently, results derived in previous sections can be used to determine the desired relationships.

For the twist-flap-lag and twist-lag-flap rotation sequences, lumping the pretwist with twist in the sequence does not alter the final form of the transformation matrix; that is, $[\mathscr{T}_{\alpha3}]$ and $[\mathscr{T}_{\alpha6}]$ are identical in form to $[\mathscr{T}_{\theta3}]$ and $[\mathscr{T}_{\theta6}]$, respectively. Therefore,

$$\phi_{\alpha3} = \phi_{\theta3}$$

$$\phi_{\alpha6} = \phi_{\theta6}$$

Consider one of the six forms of $[\mathscr{T}_\alpha]$ — say $[\mathscr{T}_{\alpha 1}]$. The components of $[\mathscr{T}_{\alpha 1}]$ are functions of a, b, c, and α_1. This form of the transformation matrix can be made identical to $[\mathscr{T}_{\alpha 3}]$ by substituting for α_1 in terms of α_3 where (from table 6)

$$\tan\alpha_1 = \frac{a \tan\alpha_3}{(1 - b^2) - bc \tan\alpha_3}$$

By definition,

$$\alpha_3 = \theta + \phi_{\alpha 3}$$

and, since $[\mathscr{T}_{\alpha 3}]$ is identical to $[\mathscr{T}_{\theta 3}]$,

$$\phi_{\alpha 3} = \phi_{\theta 3}$$

Substituting for α_3 and using trigonometric identities, $[\mathscr{T}_{\alpha 3}]$ can be expressed in terms of the sines and cosines of $\phi_{\alpha 3}$ and, therefore, of $\phi_{\theta 3}$. Then, using another change of variable based on the relationship (from table 4),

$$\tan\phi_{\theta 3} = \frac{[1 - (b \cos\theta + c \sin\theta)^2]\tan\phi_{\theta 1}}{a + (-b \sin\theta + c \cos\theta)(b \cos\theta + c \sin\theta)\tan_{\theta 1}}$$

what was originally $[\mathscr{T}_{\alpha 1}]$ can be made identical to $[\mathscr{T}_{\theta 1}]$.

Figure 5 illustrates the process with a flow chart. Although the procedure outlined above is indirect, it shows that, in general, any $[\mathscr{T}_\alpha]$ can be made identical to any other $[\mathscr{T}_\theta]$ by a series of previously derived changes of twist variable. The sequence of variable changes has been condensed for the two examples given in table 8.

ALTERNATIVE METHOD FOR EVALUATING THE TRANSFORMATION MATRIX

In general, a sequence of three Euler-like rotations is used to evaluate six unknown components of the transformation matrix. Suppose the rotation sequence was defined in terms of two sequential rotations instead of three. The overall rotation could be broken into a bending part and a twisting part. As before,

$$\mathbf{n}_{x3} = a\mathbf{n}_x + b\mathbf{n}_y + c\mathbf{n}_z$$

where a, b, and c are functions of the displacement at P and $a^2 + b^2 + c^2 = 1$. To characterize the bending rotation, an axis of rotation and an angle of rotation about this axis are required. The axis of rotation, $\mathbf{n}_R$, is to be perpendicular to both $\mathbf{n}_x$ and $\mathbf{n}_{x3}$ (fig. 6). To determine $\mathbf{n}_R$, then,

$$\mathbf{n}_R = \frac{\mathbf{n}_x \times \mathbf{n}_{x3}}{\|\mathbf{n}_x \times \mathbf{n}_{x3}\|}$$

$$= -\frac{c}{(b^2 + c^2)^{1/2}}\mathbf{n}_y + \frac{b}{(b^2 + c^2)^{1/2}}\mathbf{n}_z$$

$$= R_x\mathbf{n}_x + R_y\mathbf{n}_y + R_z\mathbf{n}_z$$

The sine and cosine of the angle of rotation, Φ, are

$$\cos\Phi = \mathbf{n}_x \cdot \mathbf{n}_{x3} = a$$

$$\sin\Phi = \|\mathbf{n}_x \times \mathbf{n}_{x3}\| = (b^2 + c^2)^{1/2}$$

Knowing the axis of rotation, the associated transformation matrix can be evaluated with the matrix equation (ref. 12, p. 35)

$$[B] = \cos\Phi\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + (1 - \cos\Phi)\begin{bmatrix}R_x^2 & R_xR_y & R_xR_z\\ R_yR_x & R_y^2 & R_yR_z\\ R_zR_x & R_zR_y & R_z^2\end{bmatrix} + \sin\Phi\begin{bmatrix}0 & R_z & -R_y\\ -R_z & 0 & R_x\\ R_y & -R_x & 0\end{bmatrix}$$

$$= \begin{bmatrix}a & b & c\\ -b & a + c^2/1 + a & -bc/1 + a\\ -c & -bc/1 + a & a + b^2/1 + a\end{bmatrix}$$

Consider the bend-twist sequence illustrated in figure 7(a) for the case of no pretwist. The transformation matrix between $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$ and $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ is

$$[\mathscr{T}_B] = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi_T & \sin\phi_T\\ 0 & -\sin\phi_T & \cos\phi_T\end{bmatrix}[B]$$

$$= \begin{bmatrix}a & b & c\\ -b\cos\phi_T - c\sin\phi_T & \left(a + \frac{c^2}{1+a}\right)\cos\phi_T - \frac{bc}{1+a}\sin\phi_T & \left(a + \frac{b^2}{1+a}\right)\sin\phi_T - \frac{bc}{1+a}\cos\phi_T\\ b\sin\phi_T - c\cos\phi_T & -\left(a + \frac{c^2}{1+a}\right)\sin\phi_T - \frac{bc}{1+a}\cos\phi_T & \left(a + \frac{b^2}{1+a}\right)\cos\phi_T + \frac{bc}{1+a}\sin\phi_T\end{bmatrix}$$

Now consider the twist-bend sequence shown in figure 7(b). For this case, the bending transformation matrix represents a rotation between $\mathbf{n}_{x\phi}$-$\mathbf{n}_{y\phi}$-$\mathbf{n}_{z\phi}$ and $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$. The rotation axis can be described in terms of components in the $\mathbf{n}_{x\phi}$-$\mathbf{n}_{y\phi}$-$\mathbf{n}_{z\phi}$ directions,

$$\mathbf{n}_R = \begin{bmatrix} 0 & \dfrac{b\ \sin\phi_T - c\ \cos\phi_T}{(b^2 + c^2)^{1/2}} & \dfrac{b\ \cos\phi_T + c\ \sin\phi_T}{(b^2 + c^2)^{1/2}} \end{bmatrix} \begin{Bmatrix} \mathbf{n}_{x\phi} \\ \mathbf{n}_{y\phi} \\ \mathbf{n}_{z\phi} \end{Bmatrix}$$

The expression for Φ is unchanged and the associated transformation matrix becomes

$$[B_\phi] = \begin{bmatrix} a & b\ \cos\phi_T + c\ \sin\phi_T & -b\ \sin\phi_T + c\ \cos\phi_T \\ -b\ \cos\phi_T - c\ \sin\phi_T & a + \dfrac{(b\ \sin\phi_T - c\ \cos\phi_T)^2}{1+a} & \dfrac{(b\ \cos\phi_T + c\ \sin\phi_T)(b\ \sin\phi_T - c\ \cos\phi_T)}{1+a} \\ b\ \sin\phi_T - c\ \cos\phi_T & \dfrac{(b\ \cos\phi_T + c\ \sin\phi_T)(b\ \sin\phi_T - c\ \cos\phi_T)}{1+a} & a + \dfrac{(b\ \cos\phi_T + c\ \sin\phi_T)^2}{1+a} \end{bmatrix}$$

Then, the transformation matrix between $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ and $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$ becomes

$$[\mathscr{T}_B] = [B_\phi] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_T & \sin\phi_T \\ 0 & -\sin\phi_T & \cos\phi_T \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \\ -b\ \cos\phi_T - c\ \sin\phi_T & \left(a + \dfrac{c^2}{1+a}\right)\cos\phi_T - \dfrac{bc}{1+a}\ \sin\phi_T & \left(a + \dfrac{b^2}{1+a}\right)\sin\phi_T - \dfrac{bc}{1+a}\ \cos\phi_T \\ b\ \sin\phi_T - c\ \cos\phi_T & -\left(a + \dfrac{c^2}{1+a}\right)\sin\phi_T - \dfrac{bc}{1+a}\ \cos\phi_T & \left(a + \dfrac{b^2}{1+a}\right)\cos\phi_T + \dfrac{bc}{1+a}\ \sin\phi_T \end{bmatrix}$$

It is apparent that there is only one ϕ_T and associated form for $[\mathscr{T}_B]$ for either the bend-twist or the twist-bend sequence.

Suppose the blade is pretwisted by an amount θ. For the pretwist-bend-twist sequence illustrated in figure 8, the bending transformation matrix represents a rotation between $\mathbf{n}_\xi$-$\mathbf{n}_\eta$-$\mathbf{n}_\zeta$ and $\mathbf{n}_{xB}$-$\mathbf{n}_{yB}$-$\mathbf{n}_{zB}$. The rotation axis is

$$\mathbf{n}_R = \begin{bmatrix} 0 & \dfrac{b\ \sin\theta - c\ \cos\theta}{(b^2 + c^2)^{1/2}} & \dfrac{b\ \cos\theta + c\ \sin\theta}{(b^2 + c^2)^{1/2}} \end{bmatrix} \begin{Bmatrix} \mathbf{n}_\xi \\ \mathbf{n}_\eta \\ \mathbf{n}_\zeta \end{Bmatrix}$$

The expression for Φ is again unaffected and $[B_\theta]$ is the same as $[B_\phi]$ with θ substituted for ϕ_T. Then the transformation matrix between $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ and $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$ is

$$[\mathscr{T}_{B\theta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{T\theta} & \sin\phi_{T\theta} \\ 0 & -\sin\phi_{T\theta} & \cos\phi_{T\theta} \end{bmatrix} [B_\theta] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \\ -b\cos\alpha - c\sin\alpha & \left(a + \frac{c^2}{1+a}\right)\cos\alpha - \frac{bc}{1+a}\sin\alpha & \left(a + \frac{b^2}{1+a}\right)\sin\alpha - \frac{bc}{1+a}\cos\alpha \\ b\sin\alpha - c\cos\alpha & -\left(a + \frac{c^2}{1+a}\right)\sin\alpha - \frac{bc}{1+a}\cos\alpha & \left(a + \frac{b^2}{1+a}\right)\cos\alpha + \frac{bc}{1+a}\sin\alpha \end{bmatrix}$$

where $\alpha = \theta + \phi_{T\theta}$. For the pretwist-twist-bend sequence, the transformation matrix is identical to that above. From the form of $[\mathscr{T}_{B\theta}]$, it is apparent that whether the transformation is determined by applying θ first or combined with the twist variable, the resulting expressions are the same. To $O(\varepsilon^2)$ in the sense already established,

$$[\mathscr{T}_{B\theta}] \cong \left[\begin{array}{ccc} 1 - \frac{b^2}{2} - \frac{c^2}{2} & b & c \\ \hline -b\cos\theta - c\sin\theta & \left(1 - \frac{b^2}{2} - \frac{\phi_{T\theta}^2}{2}\right)\cos\theta & \left(1 - \frac{c^2}{2} - \frac{\phi_{T\theta}^2}{2}\right)\sin\theta \\ + \phi_{T\theta}(b\sin\theta - c\cos\theta) & - \left(\phi_{T\theta} + \frac{bc}{2}\right)\sin\theta & + \left(\phi_{T\theta} - \frac{bc}{2}\right)\cos\theta \\ \hline b\sin\theta - c\cos\theta & -\left(1 - \frac{b^2}{2} - \frac{\phi_{T\theta}^2}{2}\right)\sin\theta & \left(1 - \frac{c^2}{2} - \frac{\phi_{T\theta}^2}{2}\right)\cos \\ + \phi_{T\theta}(b\cos\theta + c\sin\theta) & - \left(\phi_{T\theta} + \frac{bc}{2}\right)\cos\theta & - \left(\phi_{T\theta} - \frac{bc}{2}\right)\sin\theta \end{array}\right]$$

The matrix $[\mathscr{T}_{B\theta}]$ can, with the appropriate change of variable, be made identical to any of the previous forms of $[\mathscr{T}_\theta]$ or $[\mathscr{T}_\alpha]$. The relationship between $\phi_{\alpha 1}$ and $\phi_{T\theta}$ and between $\phi_{\alpha 4}$ and $\phi_{T\theta}$ are listed here:

$$\tan\phi_{\alpha 1} = \frac{(a + a^2 + c^2)\tan\phi_{T\theta} + bc}{-bc\tan\phi_{T\theta} + (a + a^2 + c^2)}$$

$$\tan\phi_{\alpha 4} = \frac{(a + a^2 + b^2)\tan\phi_{T\theta} - bc}{bc\tan\phi_{T\theta} + (a + a^2 + b^2)}$$

To $O(\varepsilon^2)$, these can be expressed as

$$\phi_{\alpha 1} \cong \phi_{T\theta} + \frac{bc}{2}$$

$$\phi_{\alpha 4} \cong \phi_{T\theta} - \frac{bc}{2}$$

The physical significance of $\phi_{T\theta}$ seems clear. It represents the total deformation twist — the bending, as well as the elastic contribution. Whether this twist variable has any real advantages over previously discussed twist variables remains to be demonstrated in further studies.

CURVATURE-DEFINED TWIST VARIABLES

Another approach for defining the twist variable is through the curvature. As before, the transformation matrix is derived in terms of a, b, c, and some deformation-twist variable. An expression for the torsional curvature is determined from the transformation matrix and equated to an elastic-twist rate. Solving for the deformation twist in terms of the elastic twist, the expression can be substituted back into the transformation matrix.

To develop the expressions for the curvature, recall the matrix relation

$$\begin{Bmatrix} \mathbf{n}_{x3} \\ \mathbf{n}_{y3} \\ \mathbf{n}_{z3} \end{Bmatrix} = \begin{bmatrix} a & b & c \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{Bmatrix}$$

and note that

$$\frac{d}{dx_3} \begin{Bmatrix} \mathbf{n}_{x3} \\ \mathbf{n}_{y3} \\ \mathbf{n}_{z3} \end{Bmatrix} = \frac{dx}{dx_3} \begin{bmatrix} a' & b' & c' \\ \ell_2' & m_2' & n_2' \\ \ell_3' & m_3' & n_3' \end{bmatrix} \begin{Bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{Bmatrix}$$

Using a kinetic-energy analogy (ref. 13, pp. 381-385), the curvature vector is defined as

$$\omega = \omega_{x3}\mathbf{n}_{x3} + \omega_{y3}\mathbf{n}_{y3} + \omega_{z3}\mathbf{n}_{z3}$$

such that

$$\frac{d\mathbf{n}_{x3}}{dx_3} = \omega \times \mathbf{n}_{x3} = \omega_{z3}\mathbf{n}_{y3} - \omega_{y3}\mathbf{n}_{z3}$$

$$\frac{d\mathbf{n}_{y3}}{dx_3} = \omega \times \mathbf{n}_{y3} = -\omega_{z3}\mathbf{n}_{x3} + \omega_{x3}\mathbf{n}_{z3}$$

$$\frac{d\mathbf{n}_{z3}}{dx_3} \quad \omega \times \mathbf{n}_{z3} = \omega_{y3}\mathbf{n}_{x3} - \omega_{x3}\mathbf{n}_{y3}$$

The components of ω can then be solved for

$$\omega_{x3} = \frac{d\mathbf{n}_{y3}}{dx_3} \cdot \mathbf{n}_{z3}$$

$$\omega_{y3} = \frac{-d\mathbf{n}_{x3}}{dx_3} \cdot \mathbf{n}_{z3}$$

$$\omega_{z3} = \frac{d\mathbf{n}_{x3}}{dx_3} \cdot \mathbf{n}_{y3}$$

and in terms of components in the $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ directions, one obtains

$$\omega_{x3} = \frac{dx}{dx_3} (\ell_2' \ell_3 + m_2' m_3 + n_2' n_3)$$

$$\omega_{y3} = - \frac{dx}{dx_3} (a' \ell_3 + b' m_3 + c' n_3)$$

$$\omega_{z3} = \frac{dx}{dx_3} (a' \ell_2 + b' m_2 + c' n_2)$$

These expressions can also be derived directly from the Serret-Frenet formulas for a space curve (ref. 14, pp. 501-503). The Serret-Frenet formulas refer to a three-dimensional space curve and the associated tangential ($\mathbf{n}_t$), normal ($\mathbf{n}_n$), and binormal ($\mathbf{n}_b$) vectors. The derivatives with respect to distance along the space curve, s, are

$$\frac{d\mathbf{n}_i}{ds} = \boldsymbol{\Omega} \times \mathbf{n}_i \; ; \qquad i = t,n,b$$

where Ω is the curvature vector,

$$\boldsymbol{\Omega} = \tau \mathbf{n}_t + \kappa \mathbf{n}_b$$

The unit vector, $\mathbf{n}_{x3}$ corresponds to $\mathbf{n}_t$, and $\mathbf{n}_{y3}$ and $\mathbf{n}_{z3}$ are in the same plane as $\mathbf{n}_n$ and $\mathbf{n}_b$. It is not difficult to relate the two triads in terms of a rotation about $\mathbf{n}_t$ and subsequently derive the expressions for ω. Despite statements to the contrary (ref. 7, p. 91), these relations do not rely on the coincidence of $\mathbf{n}_{y3}$ and $\mathbf{n}_{z3}$ with $\mathbf{n}_n$ and $\mathbf{n}_b$ — only on the coincidence of $\mathbf{n}_{x3}$ with $\mathbf{n}_t$.

For each twist variable and transformation matrix already discussed, there is an expression for $\boldsymbol{\omega}$. As in the case of the transformation matrix, one form of $\boldsymbol{\omega}$ can be made like another by appropriately changing the twist variable. The torsional curvature, ω_{x3}, can be expressed as a function of any one of the previously defined twist variables and its derivative, as well as a, b, c, θ, and their derivatives. Ideally, it would be possible to turn the expressions for ω_{x3} around, that is, to solve for each of the twist variables in terms of ω_{x3}, a, b, c, and θ. Then, if the resulting expressions were substituted back into the transformations, the different forms of the transformation matrix should reduce to a single form in terms of ω_{x3}, a, b, c, and θ.

For the case in which the blade has no built-in pretwist, let an elastic-twist variable, ϕ_e, be defined as

$$\frac{d}{dx_3}(\phi_e) = \omega_{x3}$$

In table 9, several relationships are listed between ϕ_e and the twist variables defined through different rotation sequences. Note that in integrating, the deformation twist and elastic twist are assumed to be zero at $x = 0$. For the flap-lag-twist and lag-flap-twist sequences, the twist variables, ϕ_1 and ϕ_4, respectively, can easily be solved for in terms of ϕ_e and an integral expression of a, b, and c. Like ϕ_1 and ϕ_4, the twist variable defined in the previous section, ϕ_T, can also be solved for in terms of ϕ_e. This is not the case for the other four Euler-like sequences. As an example, the relationship between ϕ_3 and ϕ_e is included.

Finally, when the blade has a built-in pretwist, θ, one way of defining the elastic-twist variable, ϕ_e, is

$$\frac{d}{dx_3}(\phi_e + \theta) = \omega_{x3}$$

In table 10, relationships are listed between ϕ_e and $\phi_{\theta 1}$, $\phi_{\alpha 1}$, $\phi_{\theta 4}$, $\phi_{\alpha 4}$, and $\phi_{T\theta}$. As in the zero-pretwist case, the deformation and elastic twist are assumed to be zero at $x = 0$.

COMPARISON OF RESULTS

Except for reference 10, few results have been published with which the relations derived in this paper can be compared. Results listed in reference 10 (pp. 16 and 22) are identical to those developed here relating $\phi_{\alpha 1}$ and $\phi_{\alpha 4}$, $\phi_{\alpha 1}$ and ϕ_e, and $\phi_{\alpha 4}$ and ϕ_e.

CONCLUSIONS

The deformation geometry of an elastic blade has been investigated. In evaluating the transformation matrix between the undeformed-blade triad $\mathbf{n}_x$-$\mathbf{n}_y$-$\mathbf{n}_z$ and the deformed-blade triad $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$, the direction of $\mathbf{n}_{x3}$ and the plane of $\mathbf{n}_{y3}$ and $\mathbf{n}_{z3}$ are specified by the deflections of the elastic axis. This is also true when there is built-in twist, regardless of whether it is considered to be applied before deformation or combined with the deformation twist. Despite different rotation sequences and pretwist treatments used to determine the transformation matrix, the use of a twist variable defined through the rotation sequence in the mathematical model allows each approach to orient $\mathbf{n}_{x3}$-$\mathbf{n}_{y3}$-$\mathbf{n}_{z3}$ identically.

An alternate approach to using an Euler-like sequence of rotations is discussed in which the change in orientation is broken into a bending and a twisting part. The results are independent of sequence and pretwist treatment. The twist variable defined with this approach represents the total twist deformation. Of the Euler-like sequences, the flap-lag-twist and the lag-flap-twist sequences appear to offer the simplest expressions for the transformation matrix and curvature vector. The alternate bending and twisting approach produces expressions for the transformation matrix and curvature vector of about equal simplicity. Whether the alternate approach offers any real advantages over the flap-lag-twist and lag-flap-twist sequences remains to be demonstrated.

Relationships between twist variables associated with the six different Euler-like sequences are determined in this paper for the zero-pretwist case, as well as for the case of initially applied pretwist and the case of combined pretwist and deformation twist. In addition, it is shown that twist variables associated with the combined twist case can be related to those for which the pretwist is applied before deformation (examples are included). Using these relationships, the different forms of the transformation matrix can be made identical by performing a change of twist variable.

The forms of the curvature vector associated with the different sequences and pretwist treatments can also be reduced to one form by performing a similar change of twist variable. Relations between the elastic twist and the twist variables associated with the flap-lag-twist sequence, lag-flap-twist sequence, and the alternate approach are listed.

By substituting expressions for one twist variable in terms of another, it is hoped that equations of motion developed using different twist treatments can be compared. Naturally, other differences in the derivation steps may also have to be dealt with in making any two sets of equations comparable.

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TABLE 1.- SIX FORMS OF THE TRANSFORMATION MATRIX FOR THE NO-PRETWIST CASE

Flap-lag-twist

$$[\mathcal{T}_1] = \begin{bmatrix} a & b & c \\ \dfrac{-(ab\cos\phi_1 + c\sin\phi_1)}{(1-b^2)^{1/2}} & (1-b^2)^{1/2}\cos\phi_1 & \dfrac{(a\sin\phi_1 - bc\cos\phi_1)}{(1-b^2)^{1/2}} \\ \dfrac{(ab\sin\phi_1 - c\cos\phi_1)}{(1-b^2)^{1/2}} & -(1-b^2)^{1/2}\sin\phi_1 & \dfrac{(a\cos\phi_1 + bc\sin\phi_1)}{(1-b^2)^{1/2}} \end{bmatrix}$$

Flap-twist-lag

$$[\mathcal{T}_2] = \begin{bmatrix} a & b & c \\ \dfrac{-[ab(\cos^2\phi_2 - b^2)^{1/2} + c\sin\phi_2]}{1-b^2} & (\cos^2\phi_2 - b^2)^{1/2} & \dfrac{[a\sin\phi_2 - bc(\cos^2\phi_2 - b^2)^{1/2}]}{1-b^2} \\ \dfrac{[ab\sin\phi_2 - c(\cos^2\phi_2 - b^2)^{1/2}]}{1-b^2} & -\sin\phi_2 & \dfrac{[a(\cos^2\phi_2 - b^2)^{1/2} + bc\sin\phi_2]}{1-b^2} \end{bmatrix}$$

Twist-flap-lag

$$[\mathcal{T}_3] = \begin{bmatrix} a & b & c \\ \dfrac{-a(b\cos\phi_3 + c\sin\phi_3)}{[1-(b\cos\phi_3 + c\sin\phi_3)^2]^{1/2}} & \dfrac{(1-b^2)\cos\phi_3 - bc\sin\phi_3}{[1-(b\cos\phi_3 + c\sin\phi_3)^2]^{1/2}} & \dfrac{(1-c^2)\sin\phi_3 - bc\cos\phi_3}{[1-(b\cos\phi_3 + c\sin\phi_3)^2]^{1/2}} \\ \dfrac{(b\sin\phi_3 - c\cos\phi_3)}{[1-(b\cos\phi_3 + c\sin\phi_3)^2]^{1/2}} & \dfrac{-a\sin\phi_3}{[1-(b\cos\phi_3 + c\sin\phi_3)^2]^{1/2}} & \dfrac{a\cos\phi_3}{[1-(b\cos\phi_3 + c\sin\phi_3)^2]^{1/2}} \end{bmatrix}$$

Lag-flap-twist

$$[\mathcal{T}_4] = \begin{bmatrix} a & b & c \\ \dfrac{-(b\cos\phi_4 + ac\sin\phi_4)}{(1-c^2)^{1/2}} & \dfrac{(a\cos\phi_4 - bc\sin\phi_4)}{(1-c^2)^{1/2}} & (1-c^2)^{1/2}\sin\phi_4 \\ \dfrac{(b\sin\phi_4 - ac\cos\phi_4)}{(1-c^2)^{1/2}} & \dfrac{-(a\sin\phi_4 + bc\cos\phi_4)}{(1-c^2)^{1/2}} & (1-c^2)^{1/2}\cos\phi_4 \end{bmatrix}$$

Lag-twist-flap

$$[\mathcal{T}_5] = \begin{bmatrix} a & b & c \\ \dfrac{-[b(\cos^2\phi_5 - c^2)^{1/2} + ac\sin\phi_5]}{1-c^2} & \dfrac{[a(\cos^2\phi_5 - c^2)^{1/2} - bc\sin\phi_5]}{1-c^2} & \sin\phi_5 \\ \dfrac{[b\sin\phi_5 - ac(\cos^2\phi_5 - c^2)^{1/2}]}{1-c^2} & \dfrac{-[a\sin\phi_5 + bc(\cos^2\phi_5 - c^2)^{1/2}]}{1-c^2} & (\cos^2\phi_5 - c^2)^{1/2} \end{bmatrix}$$

Twist-lag-flap

$$[\mathcal{T}_6] = \begin{bmatrix} a & b & c \\ \dfrac{-(b\cos\phi_6 + c\sin\phi_6)}{[1-(b\sin\phi_6 - c\cos\phi_6)^2]^{1/2}} & \dfrac{a\cos\phi_6}{[1-(b\sin\phi_6 - c\cos\phi_6)^2]^{1/2}} & \dfrac{a\sin\phi_6}{[1-(b\sin\phi_6 - c\cos\phi_6)^2]^{1/2}} \\ \dfrac{a(b\sin\phi_6 - c\cos\phi_6)}{[1-(b\sin\phi_6 - c\cos\phi_6)^2]^{1/2}} & \dfrac{-(1-b^2)\sin\phi_6 - bc\cos\phi_6}{[1-(b\sin\phi_6 - c\cos\phi_6)^2]^{1/2}} & \dfrac{(1-c^2)\cos\phi_6 + bc\sin\phi_6}{[1-(b\sin\phi_6 - c\cos\phi_6)^2]^{1/2}} \end{bmatrix}$$

TABLE 2.- RELATIONSHIPS BETWEEN TWIST VARIABLES FOR THE NO-PRETWIST CASE

	To $O(\varepsilon^2)$
$\tan\phi_2 = \dfrac{(1 - b^2)^{1/2} \tan\phi_1}{(1 + b^2 \tan^2\phi_1)^{1/2}}$	$\phi_2 \cong \phi_1$
$\tan\phi_3 = \dfrac{(1 - b^2)\tan\phi_1}{a + bc \tan\phi_1}$	$\phi_3 \cong \phi_1$
$\tan\phi_4 = \dfrac{a \tan\phi_1 - bc}{a + bc \tan\phi_1}$	$\phi_4 \cong \phi_1 - bc$
$\tan\phi_5 = \dfrac{a \tan\phi_1 - bc}{[(1 - b^2) - b^2c^2 + 2abc \tan\phi_1 + c^2 \tan^2\phi_1]^{1/2}}$	$\phi_5 \cong \phi_1 - bc$
$\tan\phi_6 = \dfrac{a \tan\phi_1 - bc}{(1 - b^2)}$	$\phi_6 \cong \phi_1 - bc$
$\tan\phi_5 = \dfrac{(1 - c^2)^{1/2} \tan\phi_4}{(1 + c^2 \tan^2\phi_4)^{1/2}}$	$\phi_5 \cong \phi_4$
$\tan\phi_6 = \dfrac{(1 - c^2)\tan\phi_4}{a - bc \tan\phi_4}$	$\phi_6 \cong \phi_4$
$\tan\phi_1 = \dfrac{a \tan\phi_3}{(1 - b^2) - bc \tan\phi_3}$	
$\tan\phi_4 = \dfrac{a \tan\phi_6}{(1 - c^2) + bc \tan\phi_6}$	
$\tan\phi_2 = \dfrac{a \tan\phi_3}{[(1 - b^2) - 2bc \tan\phi_3 + b^2 \tan\phi_3]^{1/2}}$	
$\tan\phi_3 = \dfrac{(1 - b^2)\tan\phi_2}{bc \tan\phi_2 + a[1 - b^2(1 + \tan^2\phi_2)]^{1/2}}$	
$\tan\phi_5 = \dfrac{a \tan\phi_6}{[(1 - c^2) + 2bc \tan\phi_6 + c^2 \tan^2\phi_6]^{1/2}}$	
$\tan\phi_6 = \dfrac{(1 - c^2)\tan\phi_5}{-bc \tan\phi_5 + a[1 - c^2(1 + \tan^2\phi_5)]^{1/2}}$	

TABLE 3.- FOUR FORMS OF THE TRANSFORMATION MATRIX FOR THE CASE OF INITIALLY APPLIED PRETWIST

Flap-lag-twist

$$[\mathscr{T}_{\theta 1}] = \begin{array}{c} \begin{array}{ccc} \quad a \quad & \quad b \quad & \quad c \quad \end{array} \\ \begin{bmatrix} \dfrac{[-a(b\cos\theta + c\sin\theta)\cos\phi_{\theta 1} + (b\sin\theta - c\cos\theta)\sin\phi_{\theta 1}]}{[1 - (b\cos\theta + c\sin\theta)^2]^{1/2}} & \dfrac{\{-a\sin\theta\sin\phi_{\theta 1} + [(1 - b^2)\cos\theta - bc\sin\theta]\cos\phi_{\theta 1}\}}{[1 - (b\cos\theta + c\sin\theta)^2]^{1/2}} & \dfrac{\{a\cos\theta\sin\phi_{\theta 1} + [(1 - c^2)\sin\theta - bc\cos\theta]\cos\phi_{\theta 1}\}}{[1 - (b\cos\theta + c\sin\theta)^2]^{1/2}} \\ \dfrac{[a(b\cos\theta + c\sin\theta)\sin\phi_{\theta 1} + (b\sin\theta - c\cos\theta)\cos\phi_{\theta 1}]}{[1 - (b\cos\theta + c\sin\theta)^2]^{1/2}} & \dfrac{\{-a\sin\theta\cos\phi_{\theta 1} - [(1 - b^2)\cos\theta - bc\sin\theta]\sin\phi_{\theta 1}\}}{[1 - (b\cos\theta + c\sin\theta)^2]^{1/2}} & \dfrac{\{a\cos\theta\cos\phi_{\theta 1} - [(1 - c^2)\sin\theta - bc\cos\theta]\cos\phi_{\theta 1}\}}{[1 - (b\cos\theta + c\sin\theta)^2]^{1/2}} \end{bmatrix} \end{array}$$

Twist-flap-lag

$$[\mathscr{T}_{\theta 3}] = \begin{array}{c} \begin{array}{ccc} \quad a \quad & \quad b \quad & \quad c \quad \end{array} \\ \begin{bmatrix} \dfrac{-a(b\cos\alpha_3 + c\sin\alpha_3)}{[1 - (b\cos\alpha_3 + c\sin\alpha_3)^2]^{1/2}} & \dfrac{(1 - b^2)\cos\alpha_3 - bc\sin\alpha_3}{[1 - (b\cos\alpha_3 + c\sin\alpha_3)^2]^{1/2}} & \dfrac{(1 - c^2)\sin\alpha_3 - bc\cos\alpha_3}{[1 - (b\cos\alpha_3 + c\sin\alpha_3)^2]^{1/2}} \\ \dfrac{(b\sin\alpha_3 - c\cos\alpha_3)}{[1 - (b\cos\alpha_3 + c\sin\alpha_3)^2]^{1/2}} & \dfrac{-a\sin\alpha_3}{[1 - (b\cos\alpha_3 + c\sin\alpha_3)^2]^{1/2}} & \dfrac{a\cos\alpha_3}{[1 - (b\cos\alpha_3 + c\sin\alpha_3)^2]^{1/2}} \end{bmatrix} \end{array}$$

Lag-flap-twist

$$[\mathscr{T}_{\theta 4}] = \begin{array}{c} \begin{array}{ccc} \quad a \quad & \quad b \quad & \quad c \quad \end{array} \\ \begin{bmatrix} \dfrac{[-(b\cos\theta + c\sin\theta)\cos\phi_{\theta 4} + a(b\sin\theta - c\cos\theta)\sin\phi_{\theta 4}]}{[1 - (b\sin\theta - c\cos\theta)^2]^{1/2}} & \dfrac{\{a\cos\theta\cos\phi_{\theta 4} - [(1 - b^2)\sin\theta + bc\cos\theta]\sin\phi_{\theta 4}\}}{[1 - (b\sin\theta - c\cos\theta)^2]^{1/2}} & \dfrac{\{a\sin\theta\cos\phi_{\theta 4} + [(1 - c^2)\cos\theta + bc\sin\theta]\sin\phi_{\theta 4}\}}{[1 - (b\sin\theta - c\cos\theta)^2]^{1/2}} \\ \dfrac{[(b\cos\theta + c\sin\theta)\sin\phi_{\theta 4} + a(b\sin\theta - c\cos\theta)\cos\phi_{\theta 4}]}{[1 - (b\sin\theta - c\cos\theta)^2]^{1/2}} & \dfrac{\{-a\cos\theta\sin\phi_{\theta 4} - [(1 - b^2)\sin\theta + bc\cos\theta]\cos\phi_{\theta 4}\}}{[1 - (b\sin\theta - c\cos\theta)^2]^{1/2}} & \dfrac{\{-a\sin\theta\sin\phi_{\theta 4} + [(1 - c^2)\cos\theta + bc\sin\theta]\cos\phi_{\theta 4}\}}{[1 - (b\sin\theta - c\cos\theta)^2]^{1/2}} \end{bmatrix} \end{array}$$

Twist-lag-flap

$$[\mathscr{T}_{\theta 6}] = \begin{array}{c} \begin{array}{ccc} \quad a \quad & \quad b \quad & \quad c \quad \end{array} \\ \begin{bmatrix} \dfrac{-(b\cos\alpha_6 + c\sin\alpha_6)}{[1 - (b\sin\alpha_6 - c\cos\alpha_6)^2]^{1/2}} & \dfrac{a\cos\alpha_6}{[1 - (b\sin\alpha_6 - c\cos\alpha_6)^2]^{1/2}} & \dfrac{a\sin\alpha_6}{[1 - (b\sin\alpha_6 - c\cos\alpha_6)^2]^{1/2}} \\ \dfrac{a(b\sin\alpha_6 - c\cos\alpha_6)}{[1 - (b\sin\alpha_6 - c\cos\alpha_6)^2]^{1/2}} & \dfrac{-(1 - b^2)\sin\alpha_6 - bc\cos\alpha_6}{[1 - (b\sin\alpha_6 - c\cos\alpha_6)^2]^{1/2}} & \dfrac{(1 - c^2)\cos\alpha_6 + bc\sin\alpha_6}{[1 - (b\sin\alpha_6 - c\cos\alpha_6)^2]^{1/2}} \end{bmatrix} \end{array}$$

TABLE 4.- RELATIONSHIPS BETWEEN TWIST VARIABLES FOR THE CASE OF INITIALLY APPLIED PRETWIST

	To $O(\varepsilon^2)$
$\tan\phi_{\theta 2} = \dfrac{(1 - B^2)^{1/2} \tan\phi_{\theta 1}}{(1 + B^2 \tan^2\phi_{\theta 1})^{1/2}}$	$\phi_{\theta 2} \cong \phi_{\theta 1}$
$\tan\phi_{\theta 3} = \dfrac{(1 - B^2)\tan\phi_{\theta 1}}{A + BC \tan\phi_{\theta 1}}$	$\phi_{\theta 3} \cong \phi_{\theta 1}$
$\tan\phi_{\theta 4} = \dfrac{A \tan\phi_{\theta 1} - BC}{A + BC \tan\phi_{\theta 1}}$	$\phi_{\theta 4} \cong \phi_{\theta 1} - BC$
$\tan\phi_{\theta 5} = \dfrac{A \tan\phi_{\theta 1} - BC}{[(1 - B^2) - B^2C^2 + 2ABC \tan\phi_{\theta 1} + C^2 \tan^2\phi_{\theta 1}]^{1/2}}$	$\phi_{\theta 5} \cong \phi_{\theta 1} - BC$
$\tan\phi_{\theta 6} = \dfrac{A \tan\phi_{\theta 1} - BC}{(1 - B^2)}$	$\phi_{\theta 6} \cong \phi_{\theta 1} - BC$
$\tan\phi_{\theta 5} = \dfrac{(1 - C^2)^{1/2} \tan\phi_{\theta 4}}{(1 + C^2 \tan^2\phi_{\theta 4})^{1/2}}$	$\phi_{\theta 5} \cong \phi_{\theta 4}$
$\tan\phi_{\theta 6} = \dfrac{(1 - C^2)\tan\phi_{\theta 4}}{A - BC \tan\phi_{\theta 4}}$	$\phi_{\theta 6} \cong \phi_{\theta 4}$
$\tan\phi_{\theta 3} = \dfrac{(1 - B^2)\tan\phi_{\theta 2}}{BC \tan\phi_{\theta 2} + A[1 - B^2(1 + \tan^2\phi_{\theta 2})]^{1/2}}$	
$\tan\phi_{\theta 6} = \dfrac{(1 - C^2)\tan\phi_{\theta 5}}{-BC \tan\phi_5 + A[1 - C^2(1 + \tan^2\phi_{\theta 5})]^{1/2}}$	

where

$$A = a$$

$$B = b \cos\theta + c \sin\theta$$

$$C = -b \sin\theta + c \cos\theta$$

TABLE 5.- SIX FORMS OF THE TRANSFORMATION MATRIX TO $O(\varepsilon^2)$ FOR THE COMBINED-TWISTS CASE

Flap-lag-twist

$$[\mathcal{T}_{\alpha 1}] \cong \begin{bmatrix} 1 - b^2/2 - c^2/2 & b & c \\ \begin{array}{l} -(b\cos\theta + c\sin\theta) \\ + \phi_{\alpha 1}(b\sin\theta - c\cos\theta) \end{array} & (1 - b^2/2 - \phi_{\alpha 1}^2/2)\cos\theta - \phi_{\alpha 1}\sin\theta & \begin{array}{l} (\phi_{\alpha 1} - bc)\cos\theta \\ + (1 - c^2/2 - \phi_{\alpha 1}^2/2)\sin\theta \end{array} \\ \begin{array}{l} (b\sin\theta - c\cos\theta) \\ + \phi_{\alpha 1}(b\cos\theta + c\sin\theta) \end{array} & -(1 - b^2/2 - \phi_{\alpha 1}^2/2)\sin\theta - \phi_{\alpha 1}\cos\theta & \begin{array}{l} -(\phi_{\alpha 1} - bc)\sin\theta \\ + (1 - c^2/2 - \phi_{\alpha 1}^2/2)\cos\theta \end{array} \end{bmatrix}$$

Flap-twist-lag

$$[\mathcal{T}_{\alpha 2}] \cong \begin{bmatrix} 1 - b^2/2 - c^2/2 & b & c \\ \begin{array}{l} -(b\cos\theta + c\sin\theta) \\ + \phi_{\alpha 2}(b\sin\theta - c\cos\theta) \end{array} & \begin{array}{l} (1 - b^2/2 - \phi_{\alpha 2}^2/2)\cos\theta \\ - (\phi_{\alpha 2} + b^2/2 \tan\theta)\sin\theta \end{array} & \begin{array}{l} (\phi_{\alpha 2} - bc)\cos\theta \\ + (1 - c^2/2 + b^2/2 - \phi_{\alpha 2}^2/2)\sin\theta \end{array} \\ \begin{array}{l} (b\sin\theta - c\cos\theta) \\ + \phi_{\alpha 2}(b\cos\theta + c\sin\theta) \end{array} & -(1 - \phi_{\alpha 2}^2/2)\sin\theta - \phi_{\alpha 2}\cos\theta & \begin{array}{l} -(\phi_{\alpha 2} - bc - b^2/2 \tan\theta)\sin\theta \\ + (1 - c^2/2 - \phi_{\alpha 2}^2/2)\cos\theta \end{array} \end{bmatrix}$$

Twist-flap-lag

$$[\mathcal{T}_{\alpha 3}] \cong \begin{bmatrix} 1 - b^2/2 - c^2/2 & b & c \\ \begin{array}{l} -(b\cos\theta + c\sin\theta) \\ + \phi_{\alpha 3}(b\sin\theta - c\cos\theta) \end{array} & \begin{array}{l} [1 - b^2 + 1/2(b\cos\theta + c\sin\theta)^2 - \phi_{\alpha 3}^2/2]\cos\theta \\ - (\phi_{\alpha 3} + bc)\sin\theta \end{array} & \begin{array}{l} (\phi_{\alpha 3} - bc)\cos\theta \\ + [1 - c^2 + 1/2(b\cos\theta + c\sin\theta)^2 - \phi_{\alpha 3}^2/2]\sin\theta \end{array} \\ \begin{array}{l} (b\sin\theta - c\cos\theta) \\ + \phi_{\alpha 3}(b\cos\theta + c\sin\theta) \end{array} & \begin{array}{l} -[1 - 1/2(b\sin\theta - c\cos\theta)^2 - \phi_{\alpha 3}^2/2]\sin\theta \\ - \phi_{\alpha 3}\cos\theta \end{array} & \begin{array}{l} -\phi_{\alpha 3}\sin\theta \\ + [1 - 1/2(b\sin\theta - c\cos\theta)^2 - \phi_{\alpha 3}^2/2]\cos\theta \end{array} \end{bmatrix}$$

Lag-flap-twist

$$[\mathcal{T}_{\alpha 4}] \cong \begin{bmatrix} 1 - b^2/2 - c^2/2 & b & c \\ \begin{array}{l} -(b\cos\theta + c\sin\theta) \\ + \phi_{\alpha 4}(b\sin\theta - c\cos\theta) \end{array} & \begin{array}{l} -(\phi_{\alpha 4} + bc)\sin\theta \\ + (1 - b^2/2 - \phi_{\alpha 4}^2/2)\cos\theta \end{array} & (1 - c^2/2 - \phi_{\alpha 4}^2/2)\sin\theta + \phi_{\alpha 4}\cos\theta \\ \begin{array}{l} (b\sin\theta - c\cos\theta) \\ + \phi_{\alpha 4}(b\cos\theta + c\sin\theta) \end{array} & \begin{array}{l} -(\phi_{\alpha 4} + bc)\cos\theta \\ - (1 - b^2/2 - \phi_{\alpha 4}^2/2)\sin\theta \end{array} & (1 - c^2/2 - \phi_{\alpha 4}^2/2)\cos\theta - \phi_{\alpha 4}\sin\theta \end{bmatrix}$$

Lag-twist-flap

$$[\mathcal{T}_{\alpha 5}] \cong \begin{bmatrix} 1 - b^2/2 - c^2/2 & b & c \\ \begin{array}{l} -(b\cos\theta + c\sin\theta) \\ + \phi_{\alpha 5}(b\sin\theta - c\cos\theta) \end{array} & \begin{array}{l} -(\phi_{\alpha 5} + bc + c^2/2 \tan\theta)\sin\theta \\ + (1 - b^2/2 - \phi_{\alpha 5}^2/2)\cos\theta \end{array} & (1 - \phi_{\alpha 5}^2/2)\sin\theta + \phi_{\alpha 5}\cos\theta \\ \begin{array}{l} (b\sin\theta - c\cos\theta) \\ + \phi_{\alpha 5}(b\cos\theta + c\sin\theta) \end{array} & \begin{array}{l} -(\phi_{\alpha 5} - bc)\cos\theta \\ - (1 - b^2/2 - c^2/2 - \phi_{\alpha 5}^2/2)\sin\theta \end{array} & \begin{array}{l} (1 - c^2/2 - \phi_{\alpha 5}^2/2)\cos\theta \\ - (\phi_{\alpha 5} + c^2/2 \tan\theta)\sin\theta \end{array} \end{bmatrix}$$

Twist-lag-flap

$$[\mathcal{T}_{\alpha 6}] \cong \begin{bmatrix} 1 - b^2/2 - c^2/2 & b & c \\ \begin{array}{l} -(b\cos\theta + c\sin\theta) \\ + \phi_{\alpha 6}(b\sin\theta - c\cos\theta) \end{array} & \begin{array}{l} -\phi_{\alpha 6}\sin\theta \\ + [1 - 1/2(b\cos\theta + c\sin\theta)^2 - \phi_{\alpha 6}^2/2]\cos\theta \end{array} & \begin{array}{l} [1 - 1/2(b\cos\theta + c\sin\theta)^2 - \phi_{\alpha 6}^2/2]\sin\theta \\ + \phi_{\alpha 6}\cos\theta \end{array} \\ \begin{array}{l} (b\sin\theta - c\cos\theta) \\ + \phi_{\alpha 6}(b\cos\theta + c\sin\theta) \end{array} & \begin{array}{l} -(\phi_{\alpha 6} + bc)\cos\theta \\ - [1 - b^2 + 1/2(b\sin\theta - c\cos\theta)^2 - \phi_{\alpha 6}^2/2]\sin\theta \end{array} & \begin{array}{l} [1 - c^2 + 1/2(b\sin\theta - c\cos\theta)^2 - \phi_{\alpha 6}^2/2]\cos\theta \\ - (\phi_{\alpha 6} - bc)\sin\theta \end{array} \end{bmatrix}$$

TABLE 6.- RELATIONSHIPS BETWEEN COMBINED-TWIST VARIABLES

$$\left\{\begin{aligned}
\tan\alpha_2 &= \frac{(1 - b^2)^{1/2} \tan\alpha_1}{(1 + b^2 \tan^2\alpha_1)^{1/2}} \\
\tan\alpha_3 &= \frac{(1 - b^2)\tan\alpha_1}{a + bc \tan\alpha_1} \\
\tan\alpha_4 &= \frac{a \tan\alpha_1 - bc}{a + bc \tan\alpha_1} \\
\tan\alpha_5 &= \frac{a \tan\alpha_1 - bc}{[(1 - b^2) - b^2c^2 + 2abc \tan\alpha_1 + c^2 \tan^2\alpha_1]^{1/2}} \\
\tan\alpha_6 &= \frac{a \tan\alpha_1 - bc}{(1 - b^2)}
\end{aligned}\right.$$

$$\tan\alpha_1 = \frac{a \tan\alpha_3}{(1 - b^2) - bc \tan\alpha_3}$$

$$\tan\alpha_2 = \frac{a \tan\alpha_3}{[(1 - b^2) - 2bc \tan\alpha_3 + b^2 \tan^2\alpha_3]^{1/2}}$$

$$\tan\alpha_4 = \frac{a \tan\alpha_6}{(1 - c^2) + bc \tan\alpha_6}$$

$$\tan\alpha_5 = \frac{a \tan\alpha_6}{[(1 - c^2) + 2bc \tan\alpha_6 + c^2 \tan^2\alpha_6]^{1/2}}$$

where

$$\alpha_1 = \theta + \phi_{1\theta}$$

$$\alpha_2 = \theta + \phi_{2\theta}$$

$$\alpha_3 = \theta + \phi_{3\theta}$$

$$\alpha_4 = \theta + \phi_{4\theta}$$

$$\alpha_5 = \theta + \phi_{5\theta}$$

$$\alpha_6 = \theta + \phi_{6\theta}$$

TABLE 7.- RELATIONSHIPS BETWEEN TWIST VARIABLES FOR THE COMBINED-TWISTS CASE

$$\tan\phi_{\alpha 4} = \frac{a \tan\phi_{\alpha 1} - bc}{a + bc \tan\phi_{\alpha 1}}$$

$$\phi_{\alpha 4} \approx \phi_{\alpha 1} - bc \quad \text{to} \quad O(\varepsilon^2)$$

$$\tan\phi_{\alpha 1} = \frac{[a + bc \tan\theta + (1 - b^2)\tan^2\theta]\tan\phi_{\alpha 3} - [(1 - b^2 - a)\tan\theta - bc \tan^2\theta]}{[(1 - b^2) - bc \tan\theta + a \tan^2\theta] - [bc + (1 - b^2 - a)\tan\theta]\tan\phi_{\alpha 3}}$$

$$\phi_{\alpha 1} \approx \phi_{\alpha 3} + \frac{1}{2}(b^2 - c^2)\sin\theta \cos\theta + bc \sin^2\theta \quad \text{to} \quad O(\varepsilon^2)$$

$$\tan\phi_{\alpha 2} = \frac{a(\tan\theta + \tan\phi_{\alpha 3}) - \tan\theta\{[(1 - b^2) - 2bc \tan\theta + b^2 \tan^2\theta] - 2[bc + (1 - 2b^2)\tan\theta - bc \tan^2\theta]\tan\phi_{\alpha 3} + [b^2 + 2bc \tan\theta + (1 - b^2)\tan^2\theta]\tan^2\phi_{\alpha 3}\}^{1/2}}{a \tan\theta(\tan\theta + \tan\phi_{\alpha 3}) + \{[(1 - b^2) - 2bc \tan\theta + b^2 \tan^2\theta] - 2[bc + (1 - 2b^2)\tan\theta - bc \tan^2\theta]\tan\phi_{\alpha 3} + [b^2 + 2bc \tan\theta + (1 - b^2)\tan^2\theta]\tan^2\phi_{\alpha 3}\}^{1/2}}$$

$$\phi_{\alpha 2} \approx \phi_{\alpha 3} + \frac{1}{2}(b^2 - c^2)\sin\theta \cos\theta + bc \sin^2\theta - \frac{b^2}{2}\tan\theta \quad \text{to} \quad O(\varepsilon^2)$$

$$\tan\phi_{\alpha 4} = \frac{[a - bc \tan\theta + (1 - c^2)\tan^2\theta]\tan\phi_{\alpha 6} - [(1 - c^2 - a)\tan\theta + bc \tan^2\theta]}{[(1 - c^2) + bc \tan\theta + a \tan^2\theta] - [-bc + (1 - c^2 - a)\tan\theta]\tan\phi_{\alpha 6}}$$

$$\phi_{\alpha 4} \approx \phi_{\alpha 6} - \frac{1}{2}(b^2 - c^2)\sin\theta \cos\theta - bc \sin^2\theta \quad \text{to} \quad O(\varepsilon^2)$$

TABLE 8.- RELATING TWIST VARIABLES ASSOCIATED WITH DIFFERENT PRETWIST TREATMENTS FOR THE FLAP-LAG-TWIST AND LAG-FLAP-TWIST ROTATION SEQUENCES

$$\tan\alpha_1 = \frac{a \tan\alpha_3}{(1 - b^2) - bc \tan\alpha_3}$$

$$\tan\phi_{\alpha 1} = \frac{[a + bc \tan\theta + (1 - b^2)\tan^2\theta]\tan\phi_{\alpha 3} - [(1 - b^2 - a)\tan\theta - bc \tan^2\theta]}{[(1 - b^2) - bc \tan\theta + a \tan^2\theta] - [bc + (1 - b^2 - a)\tan\theta]\tan\phi_{\alpha 3}}$$

$$\tan\phi_{\alpha 1} = \frac{[(1 - b^2) - bc \tan\theta + a \tan^2\theta]\tan\phi_{\theta 1} - [(1 - b^2 - a)\tan\theta - bc \tan^2\theta]}{[(1 - b^2) - bc \tan\theta + a \tan^2\theta] + [(1 - b^2 - a)\tan\theta - bc \tan^2\theta]\tan\phi_{\theta 1}}$$

$$\phi_{\alpha 1} \cong \phi_{\theta 1} + \frac{1}{2}(b^2 - c^2)\sin\theta \cos\theta + bc \sin^2\theta \quad \text{to} \quad 0(\varepsilon^2)$$

$$\tan\alpha_4 = \frac{a \tan\alpha_6}{(1 - c^2) + bc \tan\alpha_6}$$

$$\tan\phi_{\alpha 4} = \frac{[a - bc \tan\theta + (1 - c^2)\tan^2\theta]\tan\phi_{\alpha 6} - [(1 - c^2 - a)\tan\theta + bc \tan^2\theta]}{[(1 - c^2) + bc \tan\theta + a \tan^2\theta] - [-bc + (1 - c^2 - a)\tan\theta]\tan\phi_{\alpha 6}}$$

$$\tan\phi_{\alpha 4} = \frac{[(1 - c^2) + bc \tan\theta + a \tan^2\theta]\tan\phi_{\theta 4} - [(1 - c^2 - a)\tan\theta + bc \tan^2\theta]}{[(1 - c^2) + bc \tan\theta + a \tan^2\theta] + [(1 - c^2 - a)\tan\theta + bc \tan^2\theta]\tan\phi_{\theta 4}}$$

$$\phi_{\alpha 4} \cong \phi_{\theta 4} - \frac{1}{2}(b^2 - c^2)\sin\theta \cos\theta - bc \sin^2\theta \quad \text{to} \quad 0(\varepsilon^2)$$

TABLE 9.- RELATIONSHIPS BETWEEN THE ELASTIC TWIST AND SEVERAL DEFORMATION-TWIST VARIABLES FOR THE NO-PRETWIST CASE

$$\frac{dx}{dx_3}(\phi_e') = \frac{dx}{dx_3}\left[\phi_1' - \frac{b}{a}\left(c' + \frac{bc}{1 - b^2}\, b'\right)\right]$$

$$\phi_1 = \phi_e + \int_o^x \frac{b}{a}\left(c' + \frac{bc}{1 - b^2}\, b'\right) dx$$

$$\cong \phi_e + \int_o^x bc'\, dx \qquad \text{to} \quad 0(\varepsilon^2)$$

$$\frac{dx}{dx_3}(\phi_e') = \frac{dx}{dx_3}\left[\phi_4' + \frac{c}{a}\left(b' + \frac{bc}{1 - c^2}\, c'\right)\right]$$

$$\phi_4 = \phi_e - \int_o^x \frac{c}{a}\left(b' + \frac{bc}{1 - c^2}\, c'\right) dx$$

$$\cong \phi_e - \int_o^x b'c\, dx \qquad \text{to} \quad 0(\varepsilon^2)$$

$$\frac{dx}{dx_3}(\phi_e') = \frac{dx}{dx_3}\left[\frac{a^2\phi_3' + (b\cos\phi_3 + c\sin\phi_3)(b'\sin\phi_3 - c'\cos\phi_3) + (b\cos\phi_3 + c\sin\phi_3)^2(bc' - b'c)}{a[1 - (b\cos\phi_3 + c\sin\phi_3)^2]}\right]$$

$$\phi_3 \cong \phi_e + \int_o^x bc'\, dx \qquad \text{to} \quad 0(\varepsilon^2)$$

$$\frac{dx}{dx_3}(\phi_e') = \frac{dx}{dx_3}\left[\phi_T' + \frac{b'c - bc'}{1 + a}\right]$$

$$\phi_T = \phi_e + \int_o^x \left(\frac{b'c - bc'}{1 + a}\right) dx$$

$$\cong \phi_e + \int_o^x \left(\frac{b'c - b'c}{2}\right) dx \qquad \text{to} \quad 0(\varepsilon^2)$$

TABLE 10.- RELATIONSHIPS BETWEEN THE ELASTIC-TWIST AND SEVERAL DEFORMATION-TWIST VARIABLES FOR THE NONZERO-PRETWIST CASE

$$\frac{dx}{dx_3}\left(\phi_e' + \theta'\right) = \frac{dx}{dx_3}\left[\phi_{\theta 1}' + \frac{a^2\theta' + (b\cos\theta + c\sin\theta)(b'\sin\theta - c'\cos\theta) + (b\cos\theta + c\sin\theta)^2(c'b - b'c)}{a\left[1 - (b\cos\theta + c\sin\theta)^2\right]}\right]$$

$$\phi_{\theta 1} = \phi_e + \int_0^x \left\{\theta' - \left[\frac{a^2\theta' + (b\cos\theta + c\sin\theta)(b'\sin\theta - c'\cos\theta) + (b\cos\theta + c\sin\theta)^2(c'b - b'c)}{a\left[1 - (b\cos\theta + c\sin\theta)^2\right]}\right]\right\} dx$$

$$\cong \phi_e + \int_0^x \left[(b\cos\theta + c\sin\theta)(-b\sin\theta + c\cos\theta)' + \frac{1}{2}\left(b^2 + c^2\right)\theta'\right] dx \qquad \text{to} \quad O(\varepsilon^2)$$

$$\frac{dx}{dx_3}\left(\phi_e' + \theta'\right) = \frac{dx}{dx_3}\left[\phi_{\alpha 1}' + \theta' - \frac{b}{a}\left(c' + \frac{bc}{1 - b^2}\, b'\right)\right]$$

$$\phi_{\alpha 1} = \phi_e + \int_0^x \frac{b}{a}\left(c' + \frac{bc}{1 - b^2}\, b'\right) dx$$

$$\cong \phi_e + \int_0^x bc'\, dx \qquad \text{to} \quad O(\varepsilon^2)$$

$$\frac{dx}{dx_3}\left(\phi_e' + \theta'\right) = \frac{dx}{dx_3}\left[\phi_{\theta 4}' + \frac{a^2\theta' + (-b\sin\theta + c\cos\theta)(b'\cos\theta + c'\sin\theta) + (-b\sin\theta + c\cos\theta)^2(c'b - b'c)}{a\left[1 - (-b\sin\theta + c\cos\theta)^2\right]}\right]$$

$$\phi_{\theta 4} = \phi_e + \int_0^x \left\{\theta' - \left[\frac{a^2\theta' + (-b\sin\theta + c\cos\theta)(b'\cos\theta + c'\sin\theta) + (-b\sin\theta + c\cos\theta)^2(c'b - b'c)}{a\left[1 - (-b\sin\theta + c\cos\theta)^2\right]}\right]\right\} dx$$

$$\cong \phi_e - \int_0^x \left[(-b\sin\theta + c\cos\theta)(b\cos\theta + c\sin\theta)' - \frac{1}{2}\left(b^2 + c^2\right)\theta'\right] dx$$

$$\frac{dx}{dx_3}\left(\phi_e' + \theta'\right) = \frac{dx}{dx_3}\left[\phi_{\alpha 4}' + \theta' + \frac{c}{a}\left(b' + \frac{bc}{1 - c^2}\, c'\right)\right]$$

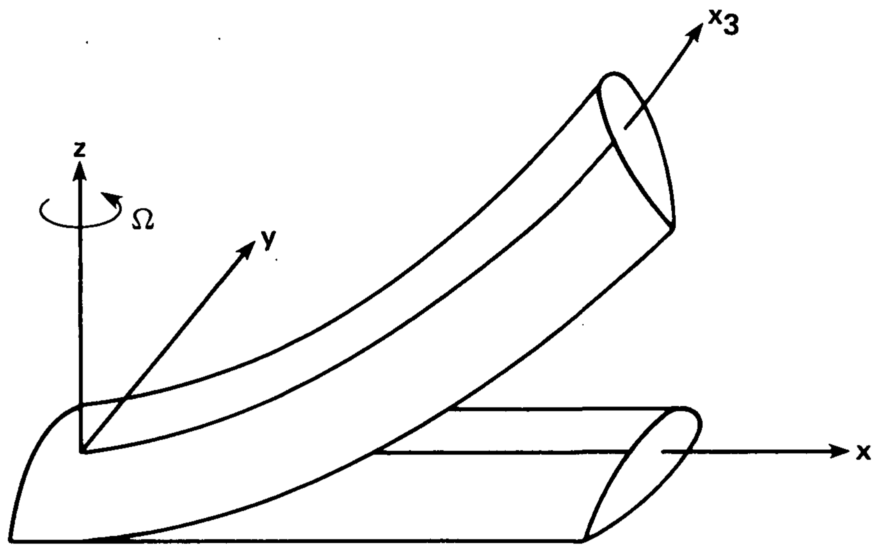
$$\phi_{\alpha 4} = \phi_e - \int_0^x \frac{c}{a}\left(b' + \frac{bc}{1 - c^2}\, c'\right) dx$$

$$\cong \phi_e - \int_0^x b'c\, dx \qquad \text{to} \quad O(\varepsilon^2)$$

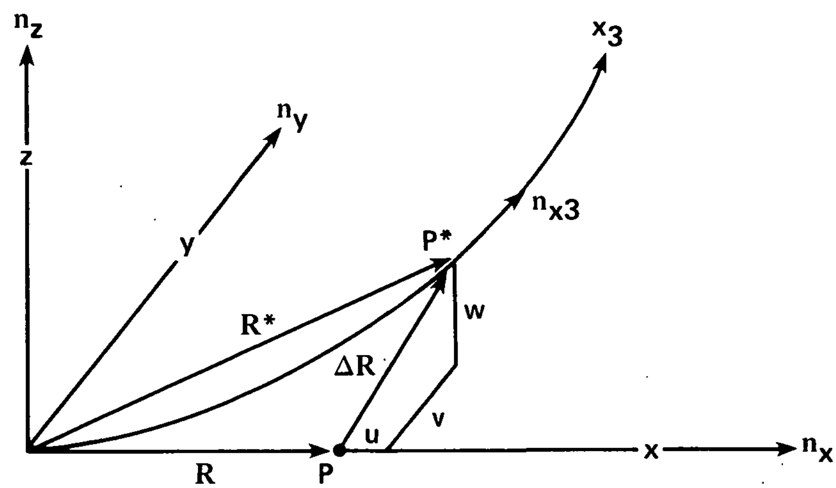
$$\frac{dx}{dx_3}\left(\phi_e' + \theta'\right) = \frac{dx}{dx_3}\left[\phi_{T\theta}' + \theta' + \frac{b'c - bc'}{1 + a}\right]$$

$$\phi_{T\theta} = \phi_e + \int_0^x \left(\frac{bc' - b'c}{1 + a}\right) dx$$

$$\cong \phi_e + \int_0^x \left(\frac{bc' - b'c}{2}\right) dx \qquad \text{to} \quad O(\varepsilon^2)$$



(a) Physical representation.



(b) Vector representation.

Figure 1.- Geometry of deformation.

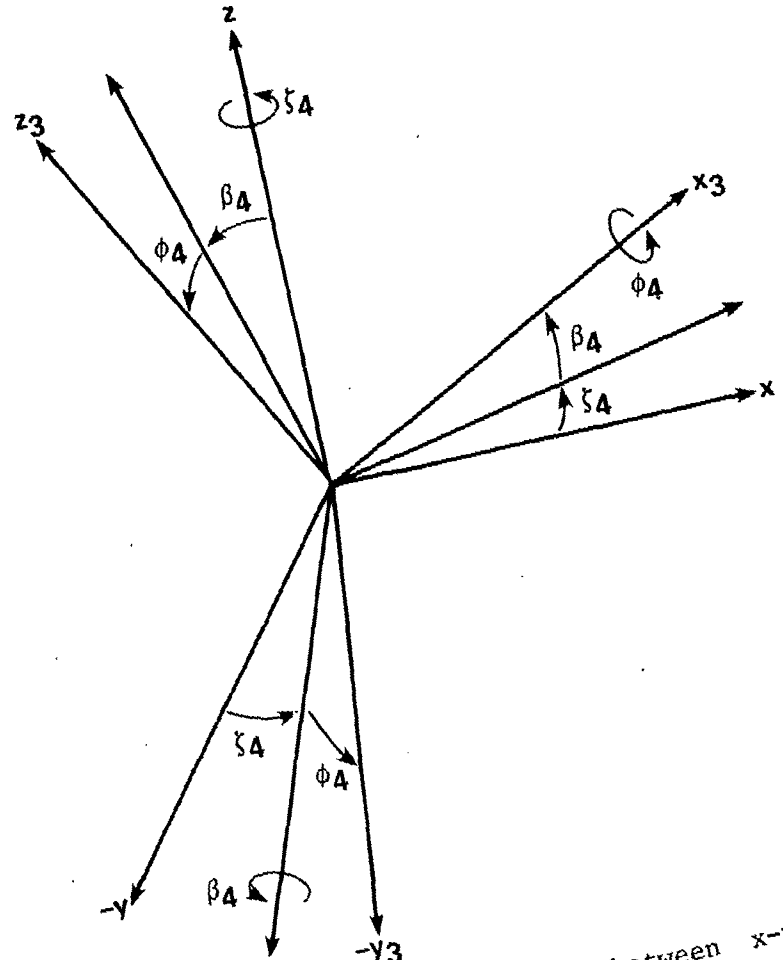


Figure 2.- Lag-flap-twist rotation sequence between x-y-z and x_3-y_3-z_3.

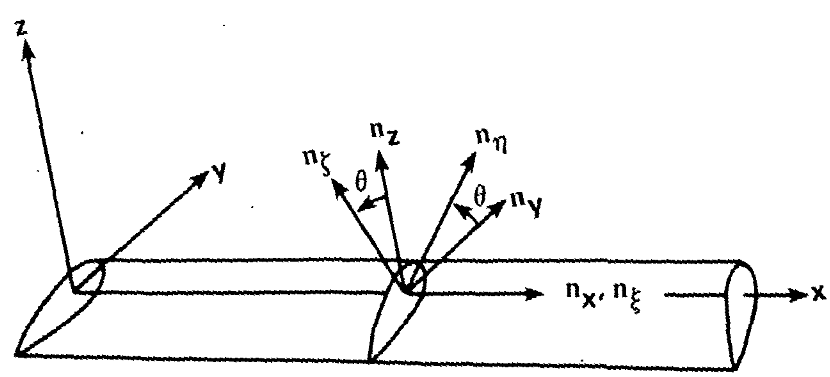


Figure 3.- Blade with built-in twist.

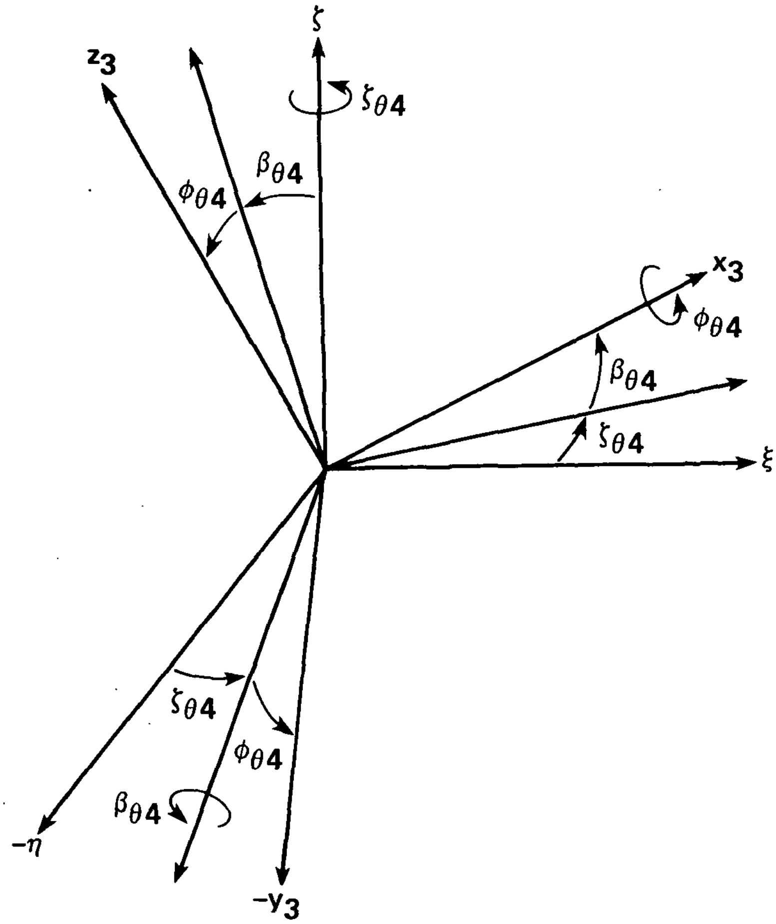


Figure 4.- Lag-flap-twist rotation sequence between ξ-η-ζ and x_3-y_3-z_3.

$$\left[\ \tau_{\alpha i}\ \right] \quad ; \ i = 1, 2, 3, 4, 5, 6$$

⬇

SUBSTITUTE FOR $\sin\alpha_i$, $\cos\alpha_i$ IN TERMS OF $\sin\alpha_j$, $\cos\alpha_j$ WHERE

$$\tan\alpha_i = f_i\,(a, b, c, \tan\alpha_j) \ ; \ \begin{array}{l} i = 1, 2, 3, 4, 5, 6 \\ j = 3, 6 \end{array}$$

$$\left[\ \tau_{\alpha j}\ \right]$$

⬇

SUBSTITUTE FOR $\sin\alpha_j$, $\cos\alpha_j$ WHERE

$$\alpha_j = \theta + \phi_{\alpha j} = \theta + \phi_{\theta j} \ ; \ j = 3, 6$$

$$\left[\ \tau_{\theta j}\ \right]$$

⬇

APPLY TRIGONOMETRIC IDENTITIES

$$\cos\alpha_j = \cos\theta\ \cos\phi_{\theta j} - \sin\theta\ \sin\phi_{\theta j} \ ; \ j = 3, 6$$
$$\sin\alpha_j = \cos\theta\ \sin\phi_{\theta j} + \sin\theta\ \cos\phi_{\theta j}$$

⬇

SUBSTITUTE FOR $\sin\phi_{\theta j}$, $\cos\phi_{\theta j}$ IN TERMS OF $\sin\phi_{\theta k}$, $\cos\phi_{\theta k}$ WHERE

$$\tan\phi_{\theta j} = f(a, b, c, \theta, \tan\phi_{\theta k}) \ ; \ \begin{array}{l} j = 3, 6 \\ k = 1, 2, 3, 4, 5, 6 \end{array}$$

$$\left[\ \tau_{\theta k}\ \right]$$

Figure 5.- A series of variable changes between transformation matrices associated with the combined twists and initial pretwist treatments.

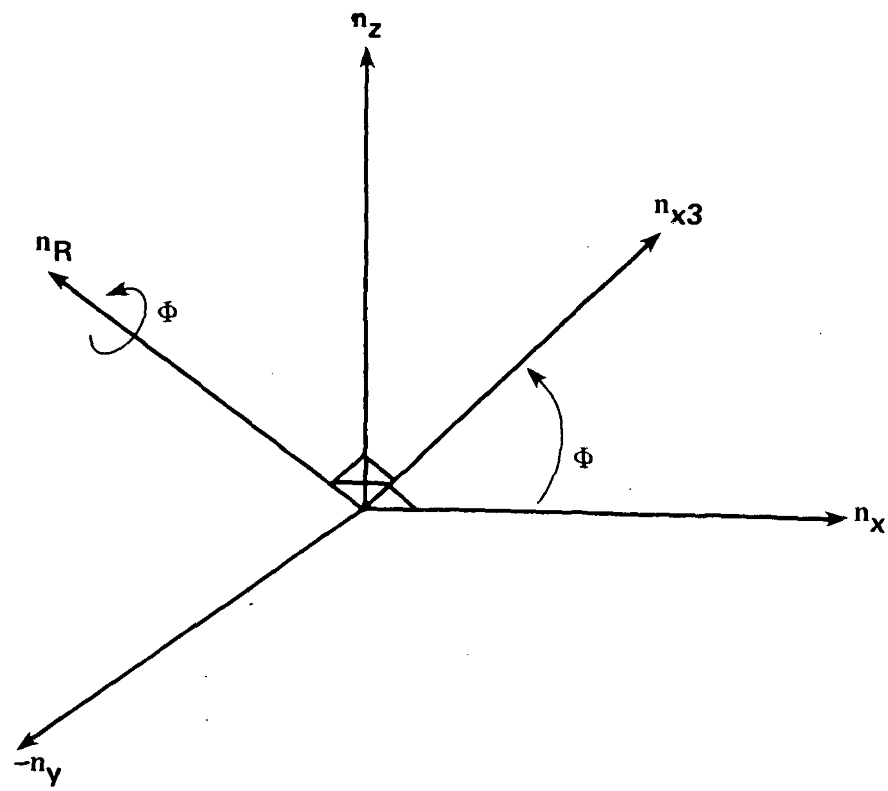
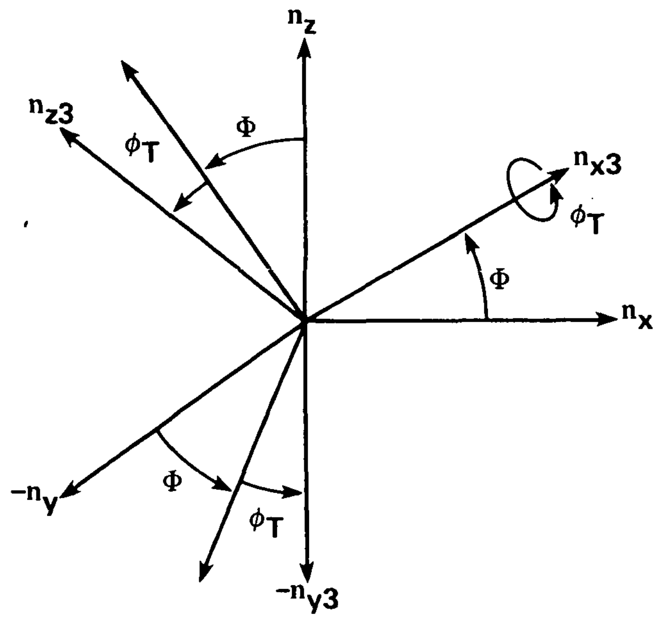
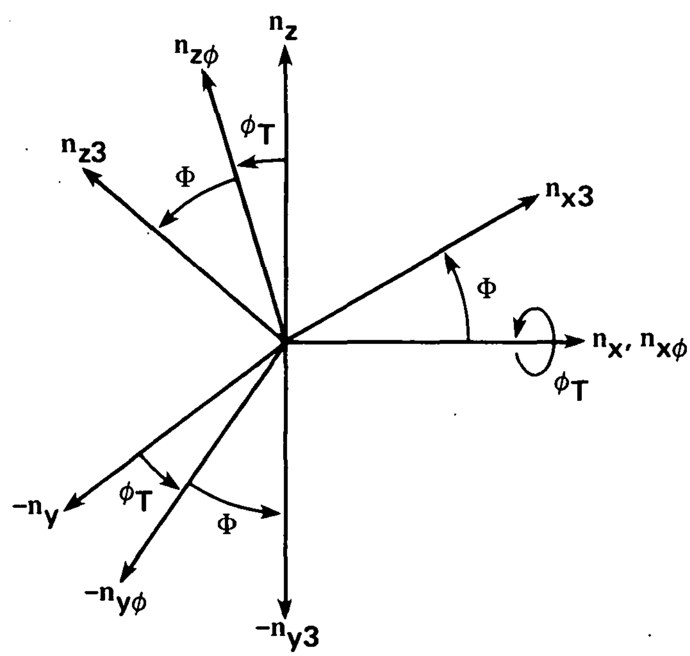


Figure 6.- A bending rotation.



(a) Bend-twist.



(b) Twist-bend.

Figure 7.- Bend-twist and twist-bend rotation sequences for the case of no pretwist.

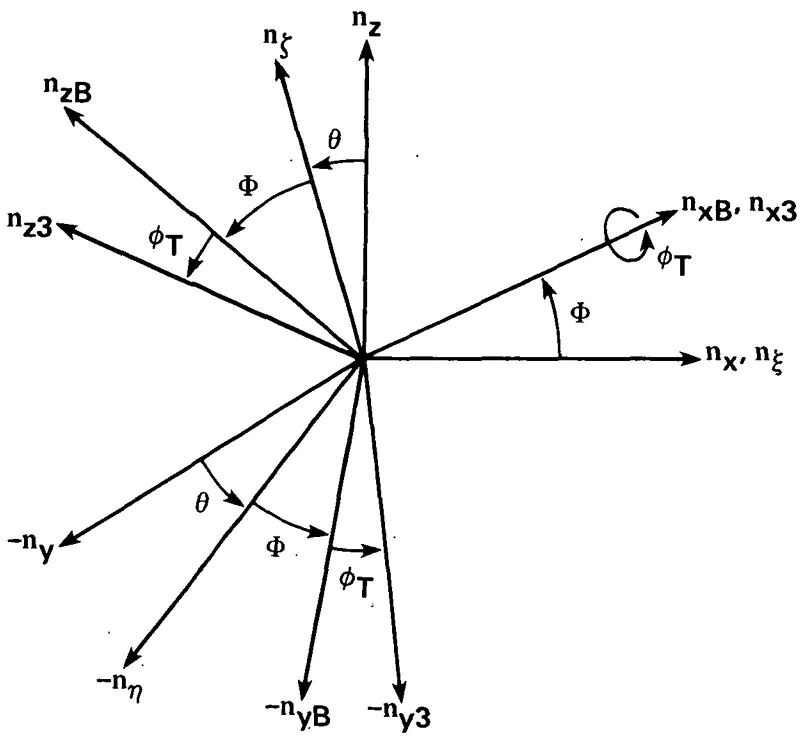


Figure 8.- Bend-twist rotation sequence with initially applied pretwist.

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15. Supplementary Notes

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16. Abstract

To adequately model elastic helicopter rotor blades experiencing moderately large deformations, a nonlinear analysis is necessary. This analysis must be based on an appropriate description of the blade's deformation geometry including elastic bending and twist. Built-in pretwist angles complicate the deformation process and its definition. In this study, relationships between the twist variables associated with different rotation sequences are listed as well as corresponding forms of the transformation matrix. Included are relationships between the twist variables associated with first, the pretwist applied initially, and second, the pretwist combined with the deformation twist. Many of the corresponding forms of the transformation matrix for the two cases are also listed. Moreover, it is shown that twist variables connected with the combined twist treatment can be related to those where the pretwist is applied initially. A method is outlined for determining these relationships and some results are given. Additionally, a procedure is demonstrated for evaluating the transformation matrix that eliminates the Euler-like sequence altogether. The resulting form of the transformation matrix is unaffected by rotation sequence or pretwist treatment.

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