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SPACE STRUCTURE VIBRATION MODES: HOW MANY EXIST? WHICH ONES ARE IMPORTANT?

by

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Abstract

This Note is the written version of invited remarks made at the "Workshop on Applications of Distributed System Theory to the Control of Large Space Structures," held at NASA's Jet Propulsion Laboratory in Pasadena, California, July 14-16, 1982, and appeared originally in the Proceedings of that Workshop. It attempts to shed some light on the two issues raised in the title, namely, How many vibration modes does a real structure have? and Which of these modes are important? Being a workshop organized and attended largely by persons who perceive the world as an assortment of continua, the surprise-free answers to these two questions are, respectively, "An infinite number" and "The first several modes." However heretical it may have seemed to such an audience, the author argues that the "Absurd Subspace" (all but the first billion modes) is not a strength of continuum modeling, but, in fact, a weakness. Partial differential equations are not real structures, only mathematical models. This Note also explains (a) that the PDE model and the finite element model are, in fact, the same model, the latter being a numerical method for dealing with the former, (b) that modes may be selected on dynamical grounds other than frequency alone, and (c) that long slender rods are useful as primitive cases but dangerous to extrapolate from.

Contents

SPACE STRUCTURE VIBRATION MODES: HOW MANY EXIST? WHICH ONES ARE IMPORTANT?

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AUTHOR'S PREFACE

To set the context of this paper, one or two prefatory remarks may be helpful.

Last summer, at the Third "Blacksburg" Conference on this subject, I was surprised to hear several speakers refer to the "fact" that "real" structures have "an infinite number of modes." These remarks were usually accompanied by the strong implication that any (mathematical) model of a structure that did not possess this essential characteristic was quite suspect, and that such models would therefore be difficult for sophisticated persons to tolerate. In fairness to the structural analysis community, I should hasten to add that this Infinite Modes Assertion was made chiefly by speakers who, whatever else their achievements, were not distinguished as structural analysts. If pressed to guess, I would suppose their backgrounds to be in controls and applied mathematics.

In any case, repeated references to the Infinite Modes Assertion at Blacksburg III prompted my recollection of a similar occasion just six years earlier where, at what some call the Zeroth Blacksburg Conference (organized by Prof. Peter Likins at UCLA), the kickoff panel session was titled "Primitive Methods." Not wishing to offend the members of that panel, Prof. Likins explained that in choosing this session title he was not implying that the panel members were themselves primitive. Instead, he said, he was using the word "primitive" in a narrow technical sense, to refer to methods based on "first principles." In essence, this meant the use of partial differential equations.

In spite of Prof. Likins' disclaimer, however, there remained the notion that if one's capability to analyse the dynamics of flexible space structures did not extend beyond PDE's, one was rather handicapped. That notion seemed sensible in 1975, and it seems even more sensible today. Unfortunately, this notion tends in practice to be inconsistent with the Infinite Modes Assertion (for reasons to be reviewed in this paper).

To return to Blacksburg III, I had the temerity during an end-of-conference panel session to question not only the importance of the Infinite Modes Assertion, but the Assertion itself. I would like to thank Dr. G. Rodriguez of JPL, who was present on that occasion, for the opportunity to expand on this theme at this workshop.

HOW MANY VIBRATION MODES DOES A REAL STRUCTURE HAVE?

A 'vibration mode' refers to a motion that is physically possible in the absence of any external influence, and in which the elastic displacements $\underline{u}(\underline{r},t)$ at position $\underline{r}$ and time t all move in unison: all displacements pass through zero simultaneously, and they all attain their maxima simultaneously. The concept of a 'vibration mode' is, in fact, a mathematical concept and can be stated most precisely and succinctly in mathematical form: if a distribution of elastic displacements of the form

$$\underline{u}(\underline{r},t) = \underline{\phi}(\underline{r})\eta(t) \tag{1}$$

is autonomously possible, $\underline{\phi}(\underline{r})$ is called the 'mode shape' and $\eta(t)$ shows the time dependence shared by the elastic displacements at all points in the structure. It is plain from (1) that the idea of 'mode shape' is a special case of the more general mathematical idea of 'separation of variables'.

Realization vs. Idealization

Much of the following argument rests on the important distinction between a 'real' (i.e., physical) structure and someone's mathematical model of that real structure. This distinction is, of course, essential on a philosophical level: whether dealing with high-energy particle physics, black holes, or flexible space structures, one is wise to discriminate between a symbolic representation of reality and reality itself. However, one hardly needs to evoke the Scientific Method to justify the distinction between the real structure and its mathematical representation. First, there is an almost unlimited quantity of experimental data on the dynamics of real structures; virtually none of this data agrees exactly with 'theory'. Second, if one returns to the fundamental assumptions that underlie 'theory', it is apparent that a large number of idealizations are made. These assumptions and idealizations are normally reasonable and defensible, but collectively they do constitute a well-documented case for distinguishing between the structure itself and its mathematical model.

Take, for example, what is arguably the simplest structure of all--the long, slender, uniform, cantilevered rod. This 'structure' is shown in Fig. 1a. (Its cousin, the 'two-rod satellite', accompanies it in Fig. 1b.) As is well known, the PDE and associated end conditions for the lateral displacements of the rod are

$$\begin{aligned} EIu'''' + \rho\ddot{u} &= f(x,t) \\ u(0,t) = u'(0,t) = u''(\ell,t) &= u'''(\ell,t) = 0 \end{aligned} \tag{2}$$

(A table of symbols is appended.)

(a) Long, Slender, Uniform Cantilevered Rod

(b) Simple Flexible Satellite

Fig. 1: The 'Simplest' Cases

Yet the following idealizations must be made to arrive at the Euler-Bernoulli equation (above) for this 'structure': (a) material continuum, (b) perfectly elastic material, (c) stress proportional to strain, (d) infinitesimally small deflections, (e) perfectly cantilevered root, (f) negligible rotational inertia, (g) negligible shear deflections. This list is undoubtedly incomplete but amply long enough already to demonstrate that properties of the PDE (2) will not likely be exactly the same as the corresponding properties of *actual* long slender uniform cantilevered rods. Experimental evidence tends to support this expectation; the model (2) is reasonable for many purposes if used intelligently, but (2) is not in any sense an *exact* representation of reality.

The Infinite Modes Assertion

There is no doubt that the PDE (2) has modes of the form (1), and that it has an infinite number of such modes. The question at issue is whether *real* rods also possess these properties. To state that a real structure has an infinite number of modes is, on reflection, to state an absurdity. How can a structure have more modes than it has molecules, or, for that matter, than there are molecules in the known universe? What does a frequency of $\omega = 10^{100}$ Hz mean? Does it mean, among other things, that particles in the structure move faster than the speed of light?

At this point the reader may retort, "Wait a minute. Let's not be extreme. When someone asserts that a structure has an infinite number of modes, all he really means is that the structure has a very large (but finite) number of modes." Not so, in the author's experience. The Infinite Modes Assertion is often made at technical meetings to an audience that includes individuals who are familiar with structural models that contain thousands of degrees of freedom (and therefore thousands of modes). To make the Assertion to such an audience clearly means that thousands of modes is not enough (in the Assertor's opinion); nothing less than infinity will do.

Yet it is clear that the Assertion is wrong, on the grounds of physical impossibility.

"All right," the reader may persist, "the Assertion is indeed made (in its strong form) and it is indeed wrong, but it is, after all, only a harmless misunderstanding". Again not so, in the author's opinion. Million-dollar R & D contract proposals on the dynamics and control of large space structures are currently under technical adjudication. If the adjudicators fall prey to a corollary of the Assertion--namely, that any methodology that does not use PDE's is faulty--they will tend to favor proposals that promise an infinite number of modes. In most cases, this viewpoint would be unwise and unjust.

How Many Modes Are There?

If a physical structure does not have an infinite number of modes, how many vibration modes *does* it have? The most precise (but not very helpful) answer is: "none". As an approximation, the mathematical concept of a 'mode' is still very useful, however. This is especially true for the lower modes. On the other hand, as one goes higher and higher in mode number (past the 100th mode, say, or the 1000th) the mathematical idea of a 'mode' tends to become increasingly inappropriate until, somewhere well this side of infinity, it is wholly inappropriate.

To emphasize this idea, we introduce the following definition in connection with mode shapes as a set of basis functions:

<u>Definition</u>: The *absurd subspace* associated with a PDE idealization of a structure is the subspace spanned by all but the first billion modes.

All PDE structural models have an absurd subspace. This absurd subspace is a flaw in these models but not an important one (unless glorified by the Assertion).

It is a curious paradox that the greatest advantage of modal analysis--the analyst can expand the general motion of a complex structure approximately in terms of a few important submotions--is lost if an infinite number of modes is insisted upon.

THE FINITE ELEMENT METHOD

When one analyses structures *in general*, one is not bothered by the necessity of generating numerical information. For example, it may suffice to say that the small deflection $\underline{u}(\underline{r},t)$ is related to the excitation $\underline{f}(\underline{r},t)$ via an appropriate operator $\underset{\sim}{K}$ that is,

$$\underset{\sim}{K}\underline{u} + \sigma\ddot{\underline{u}} = \underline{f}(\underline{r},t) \tag{3}$$

where σ is the mass density. $\underset{\sim}{K}$ is a symmetric, 3 x 3, partial differential stiffness operator. Assuming that rigid displacements are prevented (as in Fig. 2), $\underset{\sim}{K}$ is positive definite. The mode shapes for Eq. (3) satisfy

$$\underset{\sim}{K}\underline{\phi}_\alpha(\underline{r}) = \omega_\alpha^2\sigma\underline{\phi}_\alpha(\underline{r}) \tag{4}$$

and the orthonormality conditions are

$$\int_E \underline{\phi}_\alpha(\underline{r})\underline{\phi}_\beta(\underline{r})dm = \delta_{\alpha\beta} \tag{5}$$

where $dm = \sigma(\underline{r})dV$. For a system that deserves to be called a 'structure', there will be an infinite number of eigenfunctions (mode shapes). However, as we have seen above, the real structure that Eq. (3) represents does not share this 'infinite-modes' characteristic.

The modal coefficients of momentum and angular momentum (about O) are defined as follows:

$$\underline{p}_\alpha = \int_E \underline{\phi}_\alpha dm \ ; \ \underline{h}_\alpha = \int_E \underline{r}^\times\underline{\phi}_\alpha dm \tag{6}$$

It can be shown (Ref. 1) that the modal

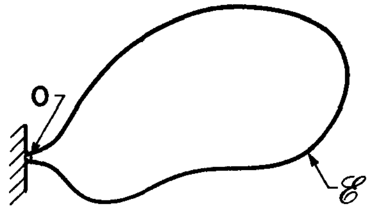

Fig. 2: General Elastic Structure

identities in the first column of Table 1 are satisfied by these coefficients. These modal identities and results like Eqs. (3), (4) and (5) for the generic structure of Fig. 2 are powerful in that they apply to all structures that satisfy the general assumptions that underlie Eq. (3).

The 'Mathematical Solution' Swindle

Operations like the integration $\int_E (\ )dm$ in Eq. (5), or the Σ_1^∞ in Table 1 can be performed with the stroke of a pen. Engineers dealing with *specific* space structures require numerical data, not just elegant theoretical results.

The classical method for dealing with PDE's like Eq. (3) is to expand the solution in terms of a series of functions that are defined, named, examined, cataloged, and expounded upon. Usually these functions are not especially easy to calculate. Even worse is to define the solution of Eq. (3) in terms of a difficult integral. This "solution" (as the mathematicians call it) is in practical terms often just another mathematically equivalent way of stating the problem. The Knotkwit function, whose origins are traced in Appendix A, furnishes an example of the different meanings that may be attached to the word 'solution' by a mathematician and an engineer.

Even the functions sin, cos, sinh, cosh that make up the well-known solution for the vibration modes of the simple rod in Fig. 1a require some numerical sophistication to calculate efficiently. For most structures of practical interest, 'closed-form' solutions are not available and, even if they were, they would not likely be much help in numerical calculations.

The Ritz Method Revisited

Frustrated by their difficulties in formulating PDE's for complex structures, and their further difficulties in extracting numerical information from these PDE's once they have them, structural analysts began to chop up complicated structures (on paper) into small elements. Each of these elements could be analysed and numerical data of the required accuracy extracted relatively easily. Initially this approach rested for its justification on physical understanding, but applied mathematicians (e.g., Ref. 2) have since shown that, if properly used, this finite element method model (FEM model) is, in fact, an ingenious implementation of the much older method of Ritz. A FEM model therefore enjoys the same theoretical foundations as the Ritz method. In particular, the conditions for convergence are known. This convergence is to the so-called 'exact' solution, i.e., to the elusive solution of the PDE model that has the same modeling assumptions as the FEM.

This property of convergence is a highly desirable one and can often be used to advantage--in connection with the identities of Table 1, for example. But in our celebration of this convergence to the 'exact' solution we should not overlook the fact that the 'exact' solution is 'exact' only for the PDE model. It is not 'exact' at all for the actual structure because the PDE model is not exact for the actual structure.

This raises the following question: How can an 'error' of (say) 1% matter, when the 'error' is with respect to an equation that is itself only valid to within (say) 10%? Yet it is this sort of error, no matter how small (and it can be made as small as desired by using sufficient finite elements), that seems to be the chief concern of the Infinite Mode Assertors. They do not trust the FEM model because it fails to predict the 'absurd subspace' (see earlier definition). In the author's opinion, however, this 'failure' is trivial and should, if anything, be counted as a point in the FEM model's favor because the absurd subspace doesn't exist physically anyway.

Unification

To this point in the discussion the FEM model and the PDE model have been treated as though they were competing alternatives. They are in an important sense *the same model*. The FEM model should be viewed as a numerical treatment of a corresponding PDE model. The finite element method must surely be one of the most spectacular success stories in the history of engineering analysis. FEM models circumvent the formulational and computational difficulties of their PDE counterpart models, while at the same time providing a numerical approximation to the latter that can be made arbitrarily accurate. If enough modeling elements are used, the error due to a finite number of coordinates can always be restricted to an 'absurd subspace'. The strength of the FEM model is that one can do numerical calculations for complicated structures; the weakness of the FEM model is that it can never be better than the associated PDE model to which it converges.

USES AND ABUSES OF LONG SLENDER RODS

A long, slender uniform cantilevered rod appears in Fig. 1 and its PDE model is given by Eq. (2). The attraction of this 'structure' is its simplicity and this makes it ideal as a learning tool. It provides a simple example for students being introduced to structural dynamics. For much the same reasons it is often cited to help in explaining new ideas to colleagues. Moreover, many satellites have rod-like appendages; in such cases the closed-form characteristics of cantilevered rods (summarized in Appendix B) have direct practical utility.

Nevertheless, because of its seductive simplicity, the slender rod structure tends to be focused upon rather more often than its limited range of application would warrant. In fact, the Infinite Modes Assertion is often a symptom of slender-rod overemphasis. If all the structures in the world were long slender rods, there certainly would be no need for the finite element method, at least not for structures. Slender rod enthusiasts often seem to imply that FEM models are really only undignified 'engineering approximations'. If such an enthusiast also wishes to ignore the crucial distinction between a physical structure and its PDE model, he has the right mind-set for accepting the Infinite Modes Assertion.

Modal Convergence

As a prelude to addressing the question 'Which modes are important?' we shall ourselves also use the long slender rod as a convenient starting point. Then, in the next section, a more realistic (and complicated) structure will be discussed. The notation and results in Appendix B will be taken for granted here.

The modal identities of Table 1 can be used as indicators of the error introduced into a structural model by modal truncation (i.e., error with respect to the 'exact' PDE representation, which is, as we have said repeatedly, not to be trusted too far itself). The modal parameters p_α and h_α are shown for the first few modes in Fig. 3. It is evident that they decrease nonotonically with mode number and that h_α decreases with α faster than p_α. These observations can be made also from Fig. 4, where the model error indices

$$\varepsilon_1(N) = 1 - 2 \sum_{\alpha=1}^{N} \lambda_\alpha^{-2} \kappa_\alpha^2 \tag{7}$$

$$\varepsilon_2(N) = 1 - 12 \sum_{\alpha=1}^{N} \lambda_\alpha^{-4} \tag{8}$$

have been introduced, corresponding respectively to the p_α and the h_α. With no modes, $\varepsilon_1(0) = \varepsilon_2(0) = 1$. For all the *theoretically* infinite number of modes, $\varepsilon_1(\infty) = \varepsilon_2(\infty) = 0$.

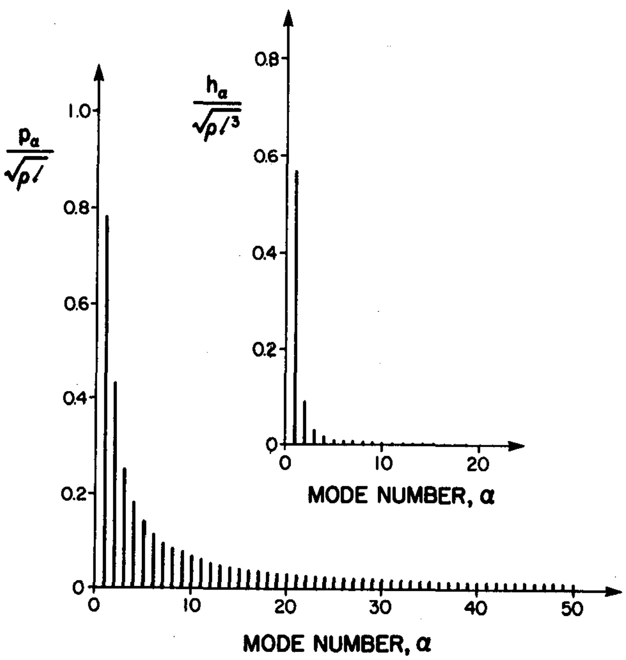

Fig. 3: Momentum Coefficients for Slender Rod

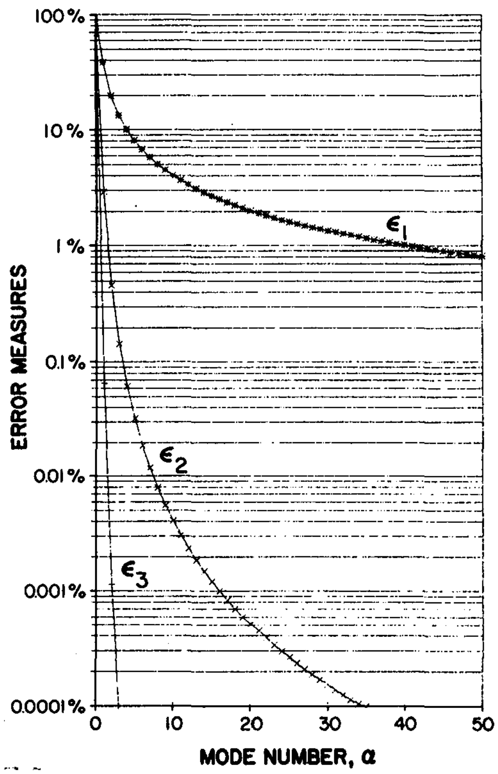

Fig. 4: Measures of Model Error

Also shown in Fig. 4 is the third measure of error,

$$\varepsilon_3(N) = 1 - \frac{1680}{11} \sum_{\alpha=1}^{N} \lambda_\alpha^{-8} \tag{9}$$

(see last entry in Table 1). This error indicator takes both momentum coefficients and frequencies into account and is thus a more plausible measure of model error than ε_1 or ε_2. The index ε_3 recognizes that, other things being equal ('other things' in this case being p_α and h_α), the low-frequency modes are more important than the high-frequency modes. If one wished to have a maximum of 1% model error, for example, as measured by ε_3, only the 1st mode should be retained and the rest deleted.

LARGE DEPLOYABLE SPACE REFLECTOR

Long, slender, uniform, cantilevered rods can be carried only so far. They are useful in teaching certain basic lessons, but some of these lessons are not true for more general structures. Therefore we now consider a typical space structure of current interest--a large deployable space reflector. Shown in Fig. 5 is the wrap-rib antenna reflector developed by the Lockheed Missiles and Space Corporation (Ref. 3). A FEM model has been developed for this reflector by the Jet Propulsion Laboratory (Ref. 4) and a typical mode shape, taken from Ref. 4, is shown in Fig. 6.

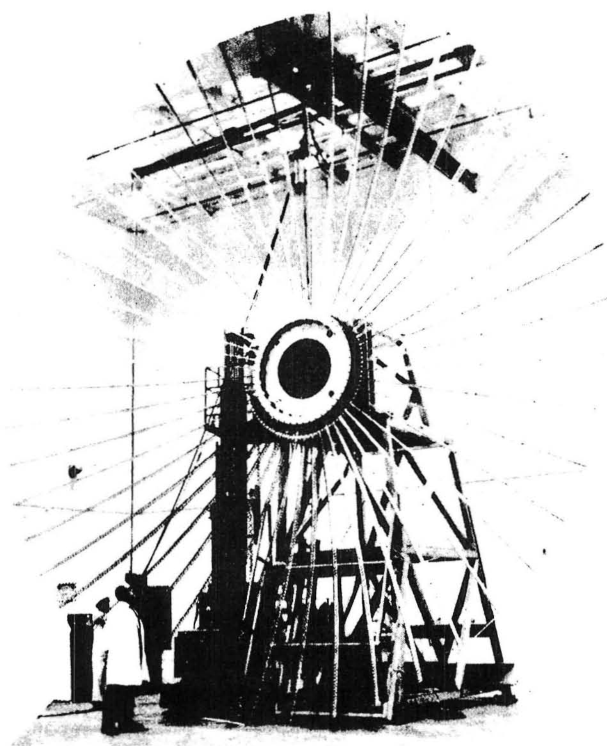

Fig. 5: Lockheed Wrap-Rib Reflector Used on ATS 6

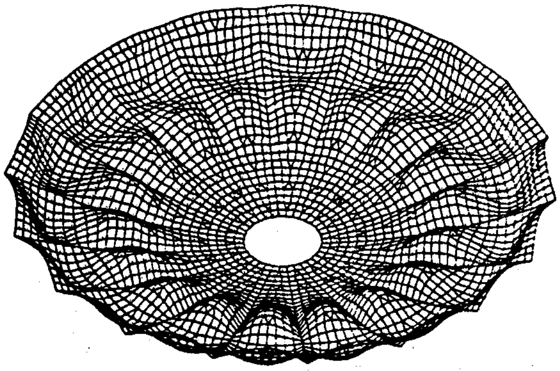

Fig. 6: Typical Wrap-Rib Mode Shape (from Ref. 4)

This model has several complexities that a simple rod does not have. The first is that a PDE model is very difficult and does not seem to have even been attempted. This leads to the use of a FEM model. The second complexity is three-dimensionality. For example, the model momentum coefficients $\underline{p}_\alpha$ and the modal angular momentum coefficients $\underline{h}_\alpha$ are no longer scalars, but are 3×1.

A Criterion for Mode Selection

A more subtle distinction between the wrap-rib reflector and the slender rod is that simple *modal truncation* becomes generalized to a process of *mode selection*. A glance back at Fig. 3 shows that for a slender rod the p_α and h_α decrease monotonically with α. In other words, whether we order the importance of the modes according to increasing frequency, or according to decreasing p_α, or according to decreasing h_α, the order of the modes is unchanged. This lesson, learned well for slender rods, must be unlearned for more complex structures. The question of which modes to keep is not simply a question of 'keeping the first N' and dropping the rest. There are several ideas available (Refs. 5,6)for mode selection, and the ones that rely solely on the structural dynamics are those that depend on ω_α, $\underline{p}_\alpha$, and $\underline{h}_\alpha$.

We can, for example, take the first three modal identities in Table 1. These three matrix identities correspond to 18 (independent) scalar identities. To create a single scalar indicator of how well these 18 identities are being satisfied, it is observed that they may be written as

$$\sum_{\alpha=1}^{\infty} \underline{M}_\alpha = \underline{M}_\infty \tag{10}$$

where the definitions

$$\underline{M}_\alpha = \begin{bmatrix} \underline{p}_\alpha \underline{p}_\alpha^T & \underline{p}_\alpha \underline{h}_\alpha^T \\ \underline{h}_\alpha \underline{p}_\alpha^T & \underline{h}_\alpha \underline{h}_\alpha^T \end{bmatrix}; \qquad \underline{M}_\infty = \begin{bmatrix} m\underline{1} & -\underline{c}^X \\ \underline{c}^X & \underline{J} \end{bmatrix} \tag{11}$$

have been introduced. Then the following scalar quantity is a measure of how well these identities are satisfied after the first N modes:

$$\varepsilon_M(N) = \rho[\underline{1} - \underline{M}_\infty^{-\frac{1}{2}}(\sum_{\alpha=1}^{N} \underline{M}_\alpha)\underline{M}_\infty^{-\frac{1}{2}}] \tag{12}$$

where $\rho[\cdot]$ stands for the spectral norm of $[\cdot]$. Note that $\underline{1}$ is here the 6 x 6 unit matrix, while in Eq. (11) $\underline{1}$ refers to the 3 x 3 unit matrix. (In other words, $\underline{1}$ always stands for a unit matrix of compatible size.)

The reasoning behind Eq. (12) is as follows: the Σ_1^N sum is normalized based on Eq. (10) in such a manner that symmetry is retained. The resulting matrix is compared to the ideal sum, $\underline{1}$. The cumulative sum in Eq. (12) is non-decreasing since $\underline{M}_\alpha$ is positive semi-definite. The matrix difference in Eq. (12) must be positive definite for finite N. Thus its eigenvalues will be six real numbers between 0 and 1. The greatest of these six numbers is defined to be the error, $\varepsilon_M(N)$.

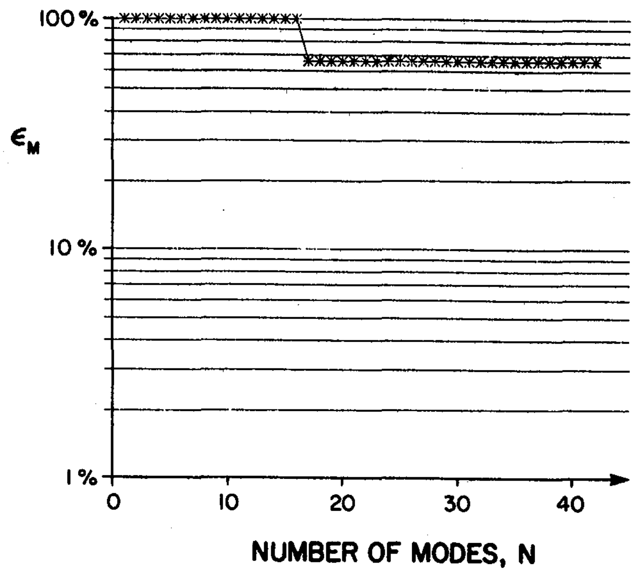

Fig. 7: Reduction of Model Error by First 42 Modes Using only Inertial Quantities in Error Measure, i.e., Using Eq. (12)

The error $\varepsilon_M(N)$ is plotted in Fig. 7 for data typical of a wrap-rib reflector with 48 ribs and 44.4 m in diameter. Even after 42 modes, $\varepsilon_M(42) = 0.66$. This slow convergence prompts the following comments.

(a) In the model used, some of the higher-wave-number modes have already been deleted. However, it is not expected that they would contribute materially to ε_M. (This is, in fact, why they were deleted.)

(b) Just because the $\varepsilon_M(N)$ vs. N curve is 'flat' does not mean that intermediate modes are not making a positive contribution. This behavior just means that they are not contributing to reducing the maximum eigenvalue of the matrix in Eq. (12).

(c) A more detailed examination of the six eigenvalues of the matrix in Eq. (12) discloses that it is the $\Sigma_1^\infty \underline{p}_\alpha \underline{p}_\alpha^T = m\underline{1}$ identity that is slow to converge. This is in accordance with the slow convergence of p_α for the slender rod in Fig. 3. The so-called 'breathing' modes for the wrap-rib reflector are few and far between; yet it is these modes that must produce convergence in the (3,3) element of $\Sigma_1^\infty \underline{p}_\alpha \underline{p}_\alpha^T = m\underline{1}$.

A Better Criterion for Mode Selection

Obviously the error criterion (12) is excessively harsh. It is counter-intuitive that a 42-mode model can have a 66% error. A goodly part of the problem is that the criterion (12) does not take the frequencies ω_α account. One of the messages in this paper is that frequency is not the only parameter of importance in modal selection. However, it would be extreme in the opposite direction to exclude the ω_α entirely, as Eq. (12) does. We therefore consider instead the last three modal identities in Table 1. These identities may be combined into the single 6 x 6 identity.

$$\sum_{\alpha=1}^{\infty} \underline{\Xi}_\alpha = \underline{\Xi}_\infty \tag{13}$$

where the definitions

$$\underline{\Xi}_\alpha = \omega_\alpha^{-2} \underline{M}_\alpha \tag{14}$$

$$\underline{\Xi}_\infty = \int_E \int_E \begin{bmatrix} \underline{1} \\ \underline{r}^X \end{bmatrix} \underline{F}(\underline{r},\underline{\xi}) \begin{bmatrix} \underline{1} \\ \underline{\xi}^X \end{bmatrix}^T dm_{\underline{r}} dm_{\underline{\xi}} \tag{15}$$

have been used.

The modal identity (13) suggests the following model error indicator:

$$\varepsilon_\Xi(N) = \rho[\underline{1} - \underline{\Xi}_\infty^{-1/2}(\sum_{\alpha=1}^{N} \underline{\Xi}_\alpha)\underline{\Xi}_\infty^{-1/2}] \tag{16a}$$

This indicator is patterned after Eq. (12), and is plotted in Fig. 8. According to this indicator, if an error of only 2.5% were the most that could be tolerated in the model, the first 28 modes would have to be kept.

There is, however, a hidden premise in this last procedure, namely, *the premise that the modes must be selected in their natural order* (i.e., by increasing frequency). There is no basis for this premise or this procedure. Figure 3 shows that, for a slender rod, p_α and h_α decrease monotonically with α, as would $p_\alpha^2/\omega_\alpha^2$, $h_\alpha^2/\omega_\alpha^2$, etc. Thus, for a slender rod, all methods of ordering modes produce the same order--the 'natural' order. For more complex structures this is no longer true. The error indicator in Eq. (16a) can therefore be improved (i.e., fewer modes required for the same model accuracy) by taking the modes in the cumulative sum in a different order. Thus we replace Eq. (16a) by

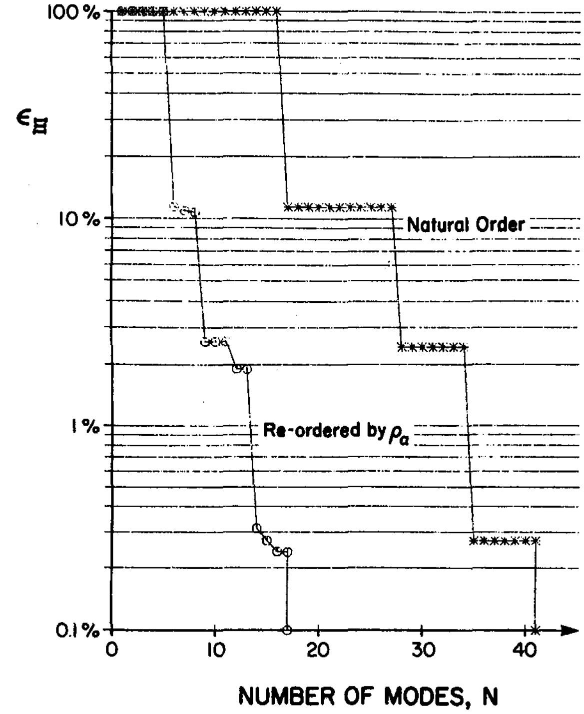

Fig. 8: Reduction of Model Error by First 42 Modes Using Eq. (16)

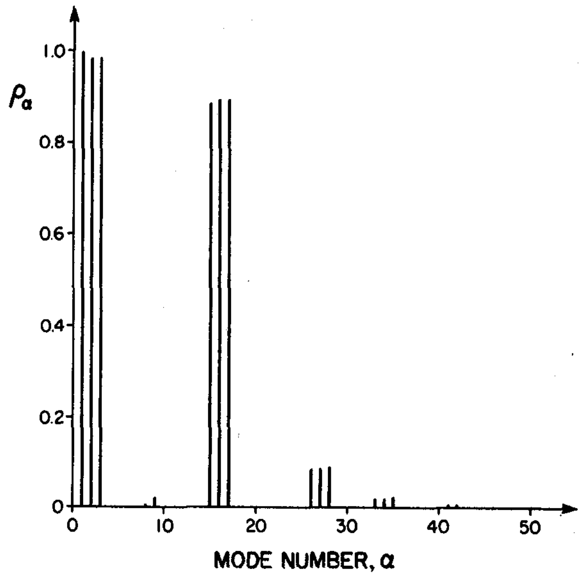

Fig. 9: Dynamical Significance of First 42 Modes as Measured by ρ_α

$$\epsilon_{\Xi}(N) = \rho[\underline{1} - \underline{\Xi}_{\infty}^{-\frac{1}{2}}(\sum_{\alpha=\alpha_1}^{\alpha_N} \underline{\Xi}_{\alpha})\underline{\Xi}_{\infty}^{-\frac{1}{2}}] \tag{16b}$$

where

$$\rho_{\alpha_1} > \rho_{\alpha_2} > \rho_{\alpha_3} > \ldots \tag{17}$$

and ρ_α is defined by

$$\rho_\alpha = \rho[\underline{\Xi}_{\infty}^{-\frac{1}{2}}\underline{\Xi}_{\alpha}\underline{\Xi}_{\infty}^{-\frac{1}{2}}] \tag{18}$$

(Note, however, that the spectral radius operator does not commute in addition; that is

$$\epsilon_{\Xi}(N) \neq 1 - \sum_{\alpha=\alpha_1}^{\alpha_N} \rho_\alpha$$

as might be assumed at first sight.)

As can be inferred from Fig. 9, ρ_α certainly does not decrease monotonically with α. This would suggest that the re-ordering of modes required by Eq. (16b) should be beneficial. The second plot in Fig. 8 shows that this is indeed the case. In fact, only 9 modes are now needed to give as low as 2.5% error--a saving of 19 modes (and a reduction in system order by 38 state variables) over the previous un-re-ordered scheme. Evidently mode selection can be, for complex structures, far superior to simple modal truncation.

CONCLUDING REMARKS

In summary, the main points discussed in this paper are the following:

(a) Neither a PDE model nor any other mathematical model of a structure is exact.

(b) For complicated structures, PDE models are very difficult to formulate and very difficult to extract numerical information from.

(c) Even when a PDE model does exist, the 'solution' in terms of 'known functions' may still require considerable effort to extract numerical information.

(d) Viewed as a Ritz method, a FEM model is not in competition with the corresponding PDE model; it is, instead, a very powerful numerical method for solving the PDE model.

(e) The idea of a 'mode' is, in essence, a mathematical one. It is highly unlikely that any real structure can vibrate *exactly* so that all its points move in unison; in other words, it is highly unlikely that any structure has any modes. As an approximation, however, the idea of a mode is an excellent one for many structures, especially for the 'lower modes'. The agreement between experiment and theory for the 'higher modes' tends to become weaker.

(f) In this approximate sense, most structures have a very large number of modes. It is elementary to show, however, that no real structure has an infinite number of modes. The Infinite Modes Assertion is false.

(g) The only utility of the Infinite Modes idea is within the purely mathematical domain. See, for example, the modal identities in Table 1.

(h) The long, slender, uniform cantilevered rod has a simplicity that is at once helpful and dangerous. It is a reasonable structure on which to explain a new idea, or to test a new idea, but the validation or generalization of the idea must be carried out on structures of more realistic complexity.

(i) Many 'error indices' can be defined as guidelines for structural modal order reduction. Simple modal truncation, although suggested by experience with slender rods, is naive. The proper process is mode selection, based on an appropriate error criterion.

(j) The error criterion in Eq. (12) is unnecessarily pessimistic because it ignores frequency information. It is as naive as a 'frequencies-only' criterion, at the opposite extreme.

(k) The error criterion in Eq. (16) is superior to Eq. (12), especially if the modes are selected according to the order specified by Eq. (17). This is illustrated for a wrap-rib antenna reflector in Fig. 8.

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Appendix A - The Origin of the Knotkwit Function

Some years ago, the eminent applied mathematician Professor Will Knotkwit encountered in his theoretical study of structures a certain PDE whose solution he could not express in closed form. Nor could he express the solution in terms of known functions. Eventually an important idea occurred to Prof. Knotkwit: he introduced a *new* function that was, by definition, the solution of his troublesome equation. He proceeded to write several papers on the interesting mathematical properties of the Knotkwit function (as it became known shortly before his retirement). Professor Knotkwit even lived to see his function referred to, by one of his former graduate students, as a 'known' function.

It is not likely that the Knotkwit function will ever be called an 'elementary' function. What is clear, however, is that any solution to a structural dynamics problem that can be written in terms of Knotkwit functions, or even that can be expressed as an integral whose integrand involves Knotkwit functions in a fairly simple manner, will be called a 'closed-form' solution.

Thus, ultimately, Professor Knotkwit achieved his 'closed-form' solution in terms of 'known' functions.

Appendix B - Long Slender Rod Modes

The well-known solution to Eq. (2) is

$$u(x,t) = \sum_{\alpha=1}^{\infty} \phi_\alpha(x)\eta_\alpha(t) \tag{B1}$$

where

$$\ddot{\eta}_\alpha + \omega_\alpha^2\eta_\alpha = \int_0^\ell \phi_\alpha(x)f(x,t)dx \tag{B2}$$

and

$$\phi_\alpha = (\rho\ell)^{-\frac{1}{2}}[(\cosh\lambda_\alpha\xi - \cos\lambda_\alpha\xi) - \kappa_\alpha(\sinh\lambda_\alpha\xi - \sin\lambda_\alpha\xi)] \tag{B3}$$

where

$$\lambda_\alpha = \frac{\rho\omega_\alpha^2\ell^4}{EI}\ ; \qquad \xi = \frac{x}{\ell}\ ; \qquad \kappa_\alpha = \frac{S_\alpha - s_\alpha}{C_\alpha + c_\alpha} \tag{B4}$$

with $s_\alpha = \sin\lambda_\alpha$, $c_\alpha = \cos\lambda_\alpha$, $S_\alpha = \sinh\lambda_\alpha$, $C_\alpha = \cosh\lambda_\alpha$.

The natural frequencies are calculated by numerical solution of the transcendental equation

$$c_\alpha C_\alpha + 1 = 0 \tag{B5}$$

The mode shapes of Eq. (B3) can readily be shown (directly from the differential equation) to satisfy the orthogonality conditions

$$\int_0^\ell \phi_\alpha(x)\phi_\beta(x)dx = 0 \qquad (\alpha \neq \beta) \tag{B6}$$

It is more onerous to show that Eq. (B3) satisfies the normality condition

$$\int_0^\ell \phi_\alpha^2 dm \equiv \rho\ell \int_0^1 \phi_\alpha^2 d\xi = 1 \tag{B7}$$

This latter fact is often omitted from textbook discussions.

In free vibration, the force and torque on the rod at O are (see Fig. 1a):

$$F(t) = \sum_{\alpha=1}^{\infty} p_\alpha \ddot{\eta}_\alpha \; ; \qquad G(t) = \sum_{\alpha=1}^{\infty} h_\alpha \ddot{\eta}_\alpha \tag{B8}$$

where p_α and h_α are the coefficients given (in general) by Eq. (6). For our present simple 'structure',

$$p_\alpha = \int \phi_\alpha dm \equiv \rho \ell \int_0^1 \phi_\alpha d\xi = 2(\rho\ell)^{\frac{1}{2}} \kappa_\alpha / \lambda_\alpha \tag{B9}$$

$$h_\alpha = \int x \phi_\alpha dm \equiv \rho \ell^2 \int_0^1 \xi \phi_\alpha d\xi = 2(\rho\ell^3)^{\frac{1}{2}} / \lambda_\alpha^2 \tag{10}$$

Therefore the modal identities of the first column in Table 1, which assume the special form shown in the second column for a slender rod, imply the identities shown in the third column in Table 1. Note that the sums involve an infinitude of transcendental numbers.

Appendix C - Table of Symbols

Roman

$\underline{c}$	first moment of inertia, $\int \underline{r} dm$
EI	flexural rigidity of a long slender rod
$f(x,t)$	force per unit length, at position x, at time t
$\underline{F}(\underline{r},\underline{\xi})$	deflection at position $\underline{r}$, due to unit force at position $\underline{\xi}$
$\underline{h}_\alpha$	modal angular momentum coefficient; see Eq. (6)
$\underline{J}$	(second) moment-of-inertia matrix
$\underset{\sim}{K}$	stiffness operator
ℓ	rod length
m	mass
N	number of modes retained
$\underline{p}_\alpha$	modal momentum coefficient; see Eq. (6)
$\underline{r}$	position vector
t	time
$\underline{u}$	small elastic displacement
x	distance along slender rod

Greek

α	modal index
$\delta_{\alpha\beta}$	1 if $\alpha = \beta$; otherwise 0
η_α	modal coordinate associated with mode α
κ_α	see Eq. (B4) in Appendix B
λ_α	see Eq. (B4) in Appendix B
ξ	x/ℓ for slender rod
$\underline{\xi}$	dummy position vector
ρ	mass per unit length for slender rod
σ	mass density function
$\underline{\phi}_\alpha$	mode shape for mode α
ω_α	natural frequency for mode α

Special Symbols

$\rho[\ ]$	spectral radius
$\underline{1}$	unit matrix (of appropr. size)
$(\ )'$	spatial derivative
$(\dot{\ })$	temporal derivative

TABLE I: SUMMARY OF MODAL IDENTITIES

	'MOST GENERAL' CASE (Linear Elastic Body)	'LEAST GENERAL' CASE (Long, Slender, Uniform Cantilever Beam)	TRANSCENDENTAL IMPLICATIONS
$\underline{p}_\alpha, \underline{h}_\alpha$	$\sum_{\alpha=1}^{\infty} \underline{p}_\alpha \underline{p}_\alpha^T = m\,\underline{1}$ $\sum_{\alpha=1}^{\infty} \underline{h}_\alpha \underline{p}_\alpha^T = \underline{c}^\times$ $\sum_{\alpha=1}^{\infty} \underline{h}_\alpha \underline{h}_\alpha^T = \underline{J}$	$\sum_{\alpha=1}^{\infty} p_\alpha^2 = \rho\ell$ $\sum_{\alpha=1}^{\infty} h_\alpha p_\alpha = \frac{\rho}{2}\ell^2$ $\sum_{\alpha=1}^{\infty} h_\alpha^2 = \frac{\rho}{3}\ell^3$	$\sum_{\alpha=1}^{\infty} \lambda_\alpha^{-2}\kappa_\alpha^2 = \frac{1}{4}$ $\sum_{\alpha=1}^{\infty} \lambda_\alpha^{-3}\kappa_\alpha = \frac{1}{8}$ $\sum_{\alpha=1}^{\infty} \lambda_\alpha^{-4} = \frac{1}{12}$
ω_α	$\sum_{\alpha=1}^{\infty} \omega_\alpha^{-2} = \text{trace} \int_E \underline{F}(\underline{r},\underline{r})\,dm$	$\sum_{\alpha=1}^{\infty} \omega_\alpha^{-2} = \frac{\rho}{12EI}\ell^4$	$\sum_{\alpha=1}^{\infty} \lambda_\alpha^{-4} = \frac{1}{12}$
$\omega_\alpha, \underline{p}_\alpha, \underline{h}_\alpha$	$\sum_{\alpha=1}^{\infty} \omega_\alpha^{-2} \underline{p}_\alpha \underline{p}_\alpha^T = \int_E \int_E \underline{F}(\underline{r},\underline{\xi})\,dm_{\underline{r}}\,dm_{\underline{\xi}}$ $\sum_{\alpha=1}^{\infty} \omega_\alpha^{-2} \underline{h}_\alpha \underline{p}_\alpha^T = \int_E \int_E \underline{r}^\times \underline{F}(\underline{r},\underline{\xi})\,dm_{\underline{r}}\,dm_{\underline{\xi}}$ $\sum_{\alpha=1}^{\infty} \omega_\alpha^{-2} \underline{h}_\alpha \underline{h}_\alpha^T = -\int_E \int_E \underline{r}^\times \underline{F}(\underline{r},\underline{\xi})\,\underline{\xi}^\times dm_{\underline{r}}\,dm_{\underline{\xi}}$	$\sum_{\alpha=1}^{\infty} \omega_\alpha^{-2} p_\alpha^2 = \frac{\rho^2}{20EI}\ell^5$ $\sum_{\alpha=1}^{\infty} \omega_\alpha^{-2} h_\alpha p_\alpha = \frac{13\rho^2}{360EI}\ell^6$ $\sum_{\alpha=1}^{\infty} \omega_\alpha^{-2} h_\alpha^2 = \frac{11\rho^2}{420EI}\ell^7$	$\sum_{\alpha=1}^{\infty} \lambda_\alpha^{-6}\kappa_\alpha^2 = \frac{1}{80}$ $\sum_{\alpha=1}^{\infty} \lambda_\alpha^{-7}\kappa_\alpha = \frac{13}{1440}$ $\sum_{\alpha=1}^{\infty} \lambda_\alpha^{-8} = \frac{11}{1680}$

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Hughes, P. C.

1. Vibration modes   2. Flexible spacecraft

I. Hughes, P. C.   II. UTIAS Technical Note No. 252

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