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ANALYSIS OF LATERAL AND TORSIONAL VIBRATION CHARACTERISTICS OF BEAMS AND SHAFTS WITH END LOCATED ROTATIONAL MASSES

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SUMMARY

Equations of motion are derived for free vibration of a uniform straight free-free beam with a rotational mass attached at each end. The mode-shapes are shown to be linear combinations of trigonometric and hyperbolic sine and cosine functions. By using boundary conditions, nonlinear algebraic equations are derived for both lateral and torsional vibrations. These solutions can then be used to compute natural frequencies and mode shapes.

A computer program was written to solve the nonlinear algebraic equations. This program uses the Newton-Raphson method to compute natural frequencies, mode shapes and node points for any number of modes, for any given parameters.

INTRODUCTION

The lateral and torsional vibration characteristics in the design of certain systems subject to external interference are not fully understood. Some common examples of such systems are the wing of an aircraft, the base of reciprocating engine, and the hull of a ship. Some of these systems can be effectively approximated by a single beam with varying end conditions. This paper will deal with the characteristics of free vibration of such beams.

Both the properties of natural frequency and location of node points provide the designer with those tools needed to either minimize or maximize the vibration of a system. Knowledge of the natural frequency of a system allows a designer to avoid resonance. If a concentrated force is located at a node point, then the induced vibration will be minimized. These characteristics are easily obtainable from tables of predetermined eigenvalues for simple beams such as cantilever, hinged-hinged, and a few others (refs. 1, 2, 3). However, the problem is more complicated for those configurations having something other than a hinged, clamped, or free end mass.

This paper describes the derivation of the differential equations for free vibration. The linearized equations satisfying the various boundary conditions, will then be obtained using the methods discussed in reference 1. The configuration to be considered is a free-free beam with rotating masses at both extremities. This configuration is considered to be the most general case, with all the classic cases able to be represented by the lower and upper limiting cases. The computer program FFBEAM which computes mode shapes, nodal points and natural frequencies of free vibration is discussed in the text of this paper. A listing of FFBEAM and sample input and output are also included in the appendices.

SYMBOLS

A	cross sectional area of beam or shaft
A_1, B_1, C_1, D_1	coefficients of mode shape equation
E	dimensionless position variable
EI	stiffness of beam
G	modulus of rigidity
I_p	polar moment of inertia
I_1	inertia of mass at $x = 0$
I_2	inertia of mass at $x = L$
L	length of beam or shaft
m_1	mass at $x = 0$ on beam
m_2	mass at $x = L$ on beam
M	moment
M_1	moment of $x = 0$ on beam
M_2	moment of $x = L$ on beam
$P(t)$	time solution of partial D.E.
$r(x), r(\varepsilon)$	lateral mode shape
t	time
T	torque
T_1	torque at $x = 0$ on beam
T_2	torque at $x = L$ on beam
$v(x,t)$	lateral displacement of beam
V	shear force
V_1	shear at $x = 0$ on beam
V_2	shear at $x = L$ on beam
X	location on axis of beam or shaft
$Z(\beta)$	eigenvalue matrix
α	phase angle (rad)
β	eigenvalue variable
ε	dimensionless position variable
$\theta(x)$	torsional mode shape
ρ	density of beam or shaft
$\phi(x,t)$	angular displacement of shaft
ω	rotational velocity (rad/sec)

DIFFERENTIAL EQUATION OF LATERAL FREE VIBRATION

The basic differential equation of lateral free vibration is best discussed using figure 1. Therein, a section of beam of length dx can be seen. Its longitudinal displacements and slopes are assumed to be negligible. Then, setting the transverse shear force acting on the element equal to the element's mass times acceleration, the following partial differential equation is obtained (ref. 1).

$$-\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{EI}{\rho A}\frac{\partial^4 v(x,t)}{\partial x^4} \tag{1}$$

where $v(x,t)$ is the lateral displacement.

This equation is solved by separation of variables. That is, $v(x,t)$ can be written as (ref. 1):

$$v(x,t) = p(t)\, r(x) \tag{2}$$

Then substituting the right half of equation (2) into the original D.E. and rearranging the variables.

$$-\frac{d^2 p(t)}{dt^2}\frac{1}{p(t)} = \frac{EI}{\rho A}\frac{1}{r(x)}\frac{d^4 r(x)}{dx^4} \tag{3}$$

But (3) can be true if, and only if, each side is identically equal to a constant. If this constant is $-\omega^2$, two ordinary homogeneous equations are obtained:

$$\frac{d^2 p(t)}{dt^2} + \omega^2 p(t) = 0 \tag{4}$$

and

$$\frac{d^4 r(x)}{dx^4} - \beta_1^4 r(x) = 0 \tag{5}$$

where

$$\beta_1^4 = \frac{\rho A}{EI}\omega^2 \tag{6}$$

The solutions for (4) and (5) are (ref. 1):

$$r(x) = A_1 \sin \beta x + B_1 \cos \beta x + C_1 \sinh \beta x + D_1 \cosh \beta x$$

and

$$p(t) = \cos(\omega t - \alpha) \tag{7}$$

where α represents the phase angle. But this solution is more convenient to use in its nondimensionalized form. Therefore, the new variable $\varepsilon = x/L$, where L is the length of the beam, is introduced. The solution of displacement after the substitution is found to be

$$r(\varepsilon) = A_1 \sin \beta_2 \varepsilon + B_1 \cos \beta_2 \varepsilon + C_1 \sinh \beta_2 \varepsilon + D_1 \cosh \beta_2 \varepsilon \tag{8}$$

where

$$\beta_2^4 = \frac{\rho A}{EI} \omega^2 L^4 \tag{9}$$

Boundary Conditions for Lateral Vibration

The configuration being considered is a beam with rotational masses at both ends. A sketch of the configuration is shown in figure 2. There the two free body diagrams show the anticipated reactions at the masses.

Since the original differential equation is of fourth order, four boundary conditions are needed. By using the equations for shear and bending moment (ref. 1):

$$V = \frac{EI}{L^3} \frac{d^3 v(\varepsilon,t)}{d\varepsilon^3} \tag{10}$$

$$M = \frac{EI}{L^2} \frac{d^2 v(\varepsilon,t)}{d\varepsilon^2} \tag{11}$$

one can obtain the four necessary linear equations.

I. Following reference 1 and referring to figure 2, the first boundary condition involves the shear V_1:

$$- V_1 = - \frac{EI}{L^3} \frac{d^3 v(\varepsilon,t)}{d\varepsilon^3} = m_1 \frac{d^2 v(\varepsilon,t)}{dt^2} \quad \text{at} \quad \varepsilon = 0 \tag{12}$$

By substituting the right-hand side of equation (2) into equation (12), the following is obtained:

$$- \frac{EI}{L^3} \frac{d^3 r(\varepsilon)}{d\varepsilon^3} p(t) = m_1 r(\varepsilon) \frac{d^2 p(t)}{dt^2} \tag{13}$$

Equation (4) can be rewritten as:

$$\frac{\dfrac{d^2 p(t)}{dt^2}}{p(t)} = - \omega^2 \tag{14}$$

Using equation (14) in equation (13), we have:

$$\omega^2 m_1 r(\varepsilon) - \frac{EI}{L^3} \frac{d^3 r(\varepsilon)}{d\varepsilon^3} = 0 \tag{15}$$

or

$$\frac{d^3 r(\varepsilon)}{d\varepsilon^3} = \frac{m_1}{\rho AL} \beta_2^4 r(\varepsilon) \quad \text{at} \quad \varepsilon = 0 \tag{16}$$

where β_2 is given by equation (9).

II. The derivation of the second boundary conditions employs the same procedure as that of the first boundary condition. The only difference is that they are applied to the mass m_2 and shear V_2. The resulting equation is:

$$\frac{d^3 r(\varepsilon)}{d\varepsilon^3} = \frac{-m_2}{\rho AL} \beta_2^4 r(\varepsilon) \quad \text{at} \quad \varepsilon = 1. \tag{17}$$

III. Following reference 1, the third boundary condition involves the moment M_1 (see figure 2):

$$M_1 = I_1 \ddot{\theta}(\varepsilon,t) = \frac{EI}{L^2} \frac{d^2 r(\varepsilon)}{d\varepsilon^2} p(t) \quad \text{at} \quad \varepsilon = 0 \tag{18}$$

By assuming small angular displacements, one may represent $\theta(\varepsilon,t)$ by the slope $\partial v(x,t)/\partial x$. Therefore, using equation (2), for small displacements, $\ddot{\theta}$ can be approximated by:

$$\frac{dr(x)}{dx} \cdot \ddot{p}(t)$$

thereby obtaining:

$$M_1 = \frac{I_1}{L} \frac{dr(\varepsilon)}{d\varepsilon} \frac{d^2 p(t)}{dt^2} = \frac{EI}{L^2} \frac{d^2 r(\varepsilon)}{d\varepsilon^2} \cdot p(t) \tag{19}$$

By using equation (14), equation (19) can be reduced:

$$\frac{d^2 r(\varepsilon)}{d\varepsilon^2} = - \frac{I_1 \; L \; \omega^2}{EI} \frac{dr(\varepsilon)}{d\varepsilon} \tag{20}$$

or

$$\frac{d^2 r(\varepsilon)}{d\varepsilon^2} = - \frac{-I_1}{\rho A L^3} \beta_2^4 \frac{dr(\varepsilon)}{d\varepsilon} \quad \text{at} \quad \varepsilon = 0 \tag{21}$$

IV. The fourth boundary condition uses the same method as the third. After employing the same steps as outlined above, the following is obtained:

$$\frac{d^2 r(\varepsilon)}{d\varepsilon^2} = \frac{I_2}{\rho A L^3} \beta_2^4 \frac{dr(\varepsilon)}{d\varepsilon} \quad \text{at} \quad \varepsilon = 1. \tag{22}$$

By substituting equation (8) and appropriate values of ε (i.e., $\varepsilon = 0$ at one boundary and $\varepsilon = 1$ at the other boundary) into equations (16), (17), (21) and (22), the following four linear equations are obtained:

$$- A_1 - \frac{m_1\beta_2}{\rho AL} B_1 + C_1 - \frac{m_1\beta_2}{\rho AL} D_1 = 0 \tag{23}$$

$$\left(\frac{m_2\beta_2}{\rho AL} \sin\beta_2 - \cos\beta_2\right) A_1 + \left(\frac{m_2\beta_2}{\rho AL} \cos\beta_2 + \sin\beta_2\right) B_1$$

$$+ \left(\frac{m_2\beta_2}{\rho AL} \sinh\beta_2 + \cosh\beta_2\right) C_1 + \left(\frac{m_2\beta_2}{\rho AL} \cosh\beta_2 + \sinh\beta_2\right) D_1 = 0 \tag{24}$$

$$\frac{I_1\beta_2^3}{\rho AL^3} A_1 - B_1 + \frac{I_1\beta_2^3}{\rho AL^3} C_1 + D_1 = 0 \tag{25}$$

$$\left(- \frac{I_2\beta_2^3}{\rho AL^3} \cos\beta_2 - \sin\beta_2\right) A_1 + \left(\frac{I_2\beta_2^3}{\rho AL^3} \sin\beta_2 - \cos\beta_2\right) B_1$$

$$+ \left(- \frac{I_2\beta_2^3}{\rho AL^3} \cosh\beta_2 + \sinh\beta_2\right) C_1 + \left(- \frac{I_2^3\beta_2}{\rho AL^3} \sinh\beta_2 + \cosh\beta_2\right) D_1 = 0 \tag{26}$$

Obtaining Nontrivial Solutions

Equations (23)-(26) can be compactly written in the vector-matrix form as follows:

$$\left[Z(\beta_2)\right] \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix} = 0 \tag{27}$$

where $Z(\beta)$ is the 4 x 4 coefficient matrix whose entries are functions of β. Non-zero solutions (A_1, B_1, C_1, D_1,) exist only when the determinant of $Z(\beta)$ is zero. Therefore, the first step in obtaining non-trivial solutions is to obtain the real solutions of the nonlinear equation:

$$\det \left[Z(\beta_2)\right] = 0 \tag{28}$$

where det [ ] denotes the determinant.

A solution β^* is substituted back in equation (27), and a degenerate system (usually of rank 3) of algebraic equations is obtained. Making $A_1 = 1$, the remaining three coefficients B_1, C_1 and D_1 can be uniquely determined for each β^*. The computer program MACSYMA was used to carry out these procedures. A listing and description of the steps can be found in Appendix C. The resulting determinant of $Z(\beta)$ obtained in MACSYMA is used in the program FFBEAM which is discussed below.

PROGRAM DISCUSSION

The computer program FFBEAM (see Appendix A for a listing) was written for the purpose of calculating non-trivial solutions using the method described in the preceding section. FFBEAM calculates the frequencies of vibration, mode shapes and the locations of nodes for any number of modes.

The program uses a simple root-finding subprogram (NONLIN) that employs the Newton-Raphson method to find the eigenvalues of the determinant. The size of the equation representing the determinant necessitated the separation of the equation into three separate quantities that are subsequently added together. FFBEAM uses the subprogram GELIM (for Gauss (-Seidel) elimination) to solve the degenerate system of equations (eq. 28) discussed in the previous section.

After computing the values of β, A_1, B_1, C_1 and D_1 from equations (27) and (28), the program uses simple substitution to find the frequencies. The subprogram NONLIN is again used to find the node points of the mode shape equation. A further explanation of the input as well as output is given below.

PROGRAM OPERATION

Input.- A total of eight input variables is needed to run FFBEAM. A consistent set of units is needed for the input (meters-kilograms-seconds or feet-pounds-seconds is recommended). The necessary input information includes two concentrated endpoint masses (M1 and M2), two endpoint inertias (I1 and I2), beam mass and length (MU and L, respectively), beam stiffness (EI), and the number of modes solutions are needed for (NMODE). Because of the small number of input variables the input can simply be written in the text of the program. A listing of sample input is given in Appendix B.

Output.- The output computed by FFBEAM includes the natural frequency of vibration, eigenvalue, mode shape equation and the location of node points for each mode. A listing of sample output is given in Appendix B. The first three pages of Appendix B include the frequencies, node positions, eigenvalues and mode shape equations. The final pages of Appendix B are the plots corresponding to each of the modes. These plots were found useful in identifying the relative motion of points of interest on the beam.

THE DIFFERENTIAL EQUATION OF TORSIONAL VIBRATION

The differential equation of torsional vibration will be derived next, using the method described in reference 1. This method utilizes a differential element of a shaft of length dx (as seen in fig. 4) and the dynamic equilibrium of this element.

It is known that the torque at any position of a circular shaft may be written as (ref. 1):

$$T = GI_p \frac{\partial \phi(x,t)}{\partial x} \tag{29}$$

Utilizing equation (29) and the differential element shown in figure 3, it follows that the net torque acting on the element at any time is (ref. 1):

$$\frac{\partial T}{\partial x}\, dx = GI_p \frac{\partial^2 \phi(x,t)}{\partial x^2}\, dx \tag{30}$$

This torque is opposed by an inertial torque:

$$- I_p \rho \frac{\partial^2 \phi(x,t)}{\partial t^2}\, dx \tag{31}$$

Hence, sum of these torques must be zero, and the differential equation governing torsional vibration of circular shafts is:

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = \frac{G}{\rho} \frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{32}$$

This equation can also be solved by separation of variables. The angular displacement of the shaft is written as:

$$\phi(x,t) = p(t)\ \theta(x) \tag{33}$$

Then, after substituting into equation (32) and rearranging, gives:

$$\frac{1}{p(t)} \frac{d^2 p(t)}{dt^2} = \frac{G}{\rho} \frac{1}{\theta(x)} \frac{d^2 \theta(x)}{dx^2} \tag{34}$$

The solutions for both $\theta(x)$ and $p(t)$ are obtained by setting both sides equal to $-\omega^2$:

$$p(t) = \cos(\omega t - \alpha) \tag{35}$$

$$\theta(x) = A_1 \sin \omega \sqrt{\frac{\rho}{G}}\ x + B_1 \cos \omega \sqrt{\frac{\rho}{G}}\ x \tag{36}$$

Using the dimensionless variable $\varepsilon = x/L$, equation (35) can be rewritten as:

$$\theta(\varepsilon) = A_1 \sin \beta_3\varepsilon + B_1 \cos \beta_3\varepsilon \tag{37}$$

where

$$\beta_3 = \omega L \sqrt{\frac{\rho}{G}} \tag{38}$$

Boundary Conditions of Torsional Vibrations

The configuration being considered, as seen in figure 4, is a shaft with a rotational mass at each end. The method used to obtain the two linear equations is outlined in reference 1. The only relationship needed to obtain the boundary conditions for this system is (ref. 1):

$$T = \frac{GI_p}{L} \frac{d\theta(\varepsilon)}{d\varepsilon} = - I \omega^2\theta(\varepsilon) \tag{39}$$

I. The first boundary condition involves the mass located at $\varepsilon = 0$ (that is, at the left end). Using the relationship in equation (39) one obtains:

$$T_1 = \frac{GI_p}{L} \frac{d\theta(\varepsilon)}{d\varepsilon} = - I_1 \omega^2\theta(\varepsilon) \quad \text{at} \quad \varepsilon = 0 \tag{40}$$

which can be rewritten as

$$\frac{d\theta(\varepsilon)}{d\varepsilon} = - \frac{I_1\beta_3^2\theta(\varepsilon)}{I_p L \rho} \quad \text{at} \quad \varepsilon = 0 \tag{41}$$

using $\beta_3 = \omega L \sqrt{\frac{\rho}{G}}$

II. The second boundary condition is obtained in a similar way with the result being:

$$\frac{d\theta(\varepsilon)}{d\varepsilon} = \frac{I_2\beta_3^2\theta(\varepsilon)}{I_p L \rho} \quad \text{at} \quad \varepsilon = 1 \tag{42}$$

After substituting equation (37) into equation (41) with $\varepsilon = 0$ and in (42) with $\varepsilon = 1$, the following two simultaneous equations are found:

$$\beta_3 \cdot A_1 + \frac{\beta_3^2 \cdot I_1}{\rho \, L \, I_p} B_1 = 0 \tag{43}$$

$$\left(\beta_3 \cos \beta_3 - \frac{I_2 \, \beta_3^2 \sin \beta_3}{\rho \, L \, I_p}\right) A_1 + \left(-\beta_3 \sin \beta_3 - \frac{I_2 \, \beta_3^2 \cos \beta_3}{\rho \, L \, I_p}\right) B_1 = 0 \tag{44}$$

Obtaining Nontrivial Solutions

The procedure for finding non-trivial solutions for A_1 and B_1 is identical to that described for the lateral case. It is straightforward to write the expression for the determinant of the 2×2 coefficient matrix in equations (43)and (44). After removing the common β_3^2 term (which corresponds to the rigid-body mode) from the expression for the determinant, and equating the expression to zero, the resulting transcendental equation can be solved by using any nonlinear root-finding algorithm.

CONCLUDING REMARKS

Equations of motion were derived for free vibrations of a uniform free-free beam with a rotational mass attached to each extremity. The simplified equations of motion considered in this paper were acquired by assuming the governing beam equations to be uncoupled. Using appropriate boundary conditions, nonlinear algebraic equations were obtained, the solutions of which yield the modal frequencies and the mode shapes for lateral as well as torsional vibrations. Computer programs were developed for computing the modal frequencies, mode shapes and node points. These programs can be used to generate modal data for any number of modes. While performing control systems studies based on structural models generated, using the computer program, care must be taken to normalize the mode-shapes by dividing them by the appropriate L_2-norm (i.e., square-root of the integral of mode-shape squared, integrated from 0 to L over the space variable). Mode slopes, which are necessary for control systems studies with torque actuators and attitude sensors, can be obtained simply by differentiating the corresponding normalized mode shapes with respect to the space variable.

Appendix A. Computer Program FFBEAM

```
      PROGRAM FFBEAM(INPUT,OUTPUT,TAPE5=OUTPUT)
      COMMON/VALUES/ I1,I2,L,M1,M2,MU,EI,A,B,C,D,CPD
      DIMENSION BET(10000),LAT(10000),BTA(1)
      REAL MU,L,M1,M2,I1,I2,LAT
C     PSEUDO INITIATES THE PLOTTING SUBROUTINE
      CALL PSEUDO
C     I AND J ARE INDICES OF LAT AND BET RESPECTIVELY
      I=1
      J=1
C     DLTX IS THE NUMBER ADDED TO BET(J) TO OBTAIN FIRST
C     GUESSES OF THE ROOTS.
      DLTX=.01
C**************************************************
C     THE FOLLOWING 8 VARIABLES ARE THE INPUT VARIABLES.
C
C        MU= mass OF THE BEAM ALONE
C        L=LENGTH OF THE BEAM
C        M1=MASS AT X=0. END OF BEAM
C        M2=MASS AT X=L END OF BEAM
C        I1=INERTIA OF M1
C        I2=INERTIA OF M2
C        EI=STIFFNESS OF THE BEAM
C        NMODE=THE NUMBER OF MODES SOLUTIONS ARE NEEDED FOR, STARTING
C              WITH #1 AND ENDING WITH #NMODE.
C**************************************************
      MU=177.
      L=100.
      M1=9.9E+55
      M2=250.
      I1=0.0
      I2=0.0
      EI=20000000.0
      NMODE=3
      WRITE(5,35) MU,L,M1,M2,I1,I2,EI
   35 FORMAT(/,52HTHESE ARE THE SOLUTIONS FOR LATERAL VIBRATION OF THE/
     140HBEAM WITH THE FOLLOWING CHARACTERISTICS./
     222HTHE MASS OF THE BEAM =F14.3,/
     323HTHE LENGTH OF THE BEAM=F14.3,/,
     426HTHE MASS LOCATED AT X=0. =F14.3,/
     525HTHE MASS LOCATED AT X=L =F14.3,/
     633HTHE INERTIA OF THE MASS AT X=0. =F14.3,/
     732HTHE INERTIA OF THE MASS AT X=L =F14.3,/
     827HTHE STIFFNESS OF THE BEAM =F14.3,/
     9/)
C      BET(1) IS THE STARTING POINT FOR FIRST GUESSES OF THE ROOTS
       BET(1)= 0.0
C      IN THIS LOOP ALL SOLUTIONS FOR THE MODES 1-NMODE ARE FOUND
      DO 10 K=1,NMODE
    5 X=(2*COS(BET(J))*COSH(BET(J))-2)*L**4*MU**4+((2*BET(J)*COS(BE
```

```
     1    T(J))*SINH(BET(J))-2*BET(J)*COSH(BET(J))*SIN(BET(J)))*L**4*M2+(
     2    2*BET(J)*COS(BET(J))*SINH(BET(J))-2*BET(J)*COSH(BET(J))*SIN(BET
     3    (J)))*L**4*M1+(I2*(-2*BET(J)**3*COS(BET(J))*SINH(BET(J))-2*BET(
     4    J)**3*COSH(BET(J))*SIN(BET(J)))+I1*(-2*BET(J)**3*COS(BET(J))*SI
     5    NH(BET(J))-2*BET(J)**3*COSH(BET(J))*SIN(BET(J))))*L**2)*MU**3
      Y=(((I2*(-2*BET(J)**4*COS(BET(J))*COSH(BET(J))-2*BET(J)**4)-4*I1*B
     7    ET(J)**4*COS(BET(J))*COSH(BET(J)))*L**2-4*BET(J)**2*SIN(BET(J))
     8    *SINH(BET(J))*L**4*M1)*M2+(I1*(-2*BET(J)**4*COS(BET(J))*COSH(BE
     9    T(J))-2*BET(J)**4)-4*I2*BET(J)**4*COS(BET(J))*COSH(BET(J)))*L**
     :    2*M1+4*I1*I2*BET(J)**6*SIN(BET(J))*SINH(BET(J)))*MU**2
      Z=(((I2*(2
     ;    *BET(J)**5*COSH(BET(J))*SIN(BET(J))-2*BET(J)**5*COS(BET(J))*SIN
     <    H(BET(J)))+I1*(2*BET(J)**5*COSH(BET(J))*SIN(BET(J))-2*BET(J)**5
     =    *COS(BET(J))*SINH(BET(J))))*L**2*M1+I1*I2*(2*BET(J)**7*COS(BET(
     >    J))*SINH(BET(J))+2*BET(J)**7*COSH(BET(J))*SIN(BET(J))))*M2+I1*I
     ?    2*(2*BET(J)**7*COS(BET(J))*SINH(BET(J))+2*BET(J)**7*COSH(BET(J)
     @    )*SIN(BET(J)))*M1)*MU+I1*I2*(2*BET(J)**8*COS(BET(J))*COSH(BET(J
     1    ))-2*BET(J)**8)*M1*M2
      LAT(I)=(X+Y+Z)/(L**4*MU**4)
C      THE FOLLOWING IF STATEMENT IS NECESSARY BECAUSE THERE IS ONLY
C      ONLY ONE VALUE OF LAT(I) AT I=1.
      IF(I.EQ.1) GO TO 15
C
C      THIS IF STATEMENT FINDS THE FIRST GUESS OF A ROOT BY FINDING
C      WHERE LAT(I) CHANGES FROM NEGATIVE TO POSITIVE.
      IF ((LAT(I)*LAT(I-1)).GT.0.0) GO TO 15
C
C      THIS STATEMENT CALLS THE ROUTINE THAT FINDS THE EXACT ROOT
C      AND CALCULATES THE POINTS FOR THE PLOT OF EACH MODE.
      WRITE(5,30) K
   30 FORMAT (//18HRESULTS FOR MODE #,I3,//)
      CALL NWTRPH(BET,J)
      GO TO 25
C
C      THIS STEP UPDATES BET(J) FOR THE NEXT CALCULATION OF LAT(I).
   15  BET(J+1)=BET(J)+DLTX
      I=I+1
      J=J+1
      GO TO 5
C
C      THIS STEP UPDATES BET(J) FOR THE NEXT CALCULATION OF LAT(I).
   25 BET(J+1)=BET(J)+DLTX
      I=I+1
      J=J+1
   10 CONTINUE
C
C      CALPLT TERMINATES THE PLOTTING ROUTINE.
      CALL CALPLT(0.,0.,999)
      STOP
```

```
      END

      SUBROUTINE NWTRPH(BET,J)
       COMMON/VALUES/ I1,I2,L,M1,M2,MU,EI,A,B,C,D,CRD
C
C      THESE EXTERNALS ARE NECESSARY FOR SUBROUTINE GELIM.
      EXTERNAL EVAL,JACOB
C**********************************************************************
C      BTA CONTAINS THE ROOT OF THE EQUATION,
C      AA IS A WORK ARRAY,
C      WK IS ALSO A WORK ARRAY
C      S IS THE MATRIX NECESSARY TO SOLVE FOR THE COEFFICIENTS
C           OF THE SHAPE EQUATION
C      T IS THE INPUT OUTPUT ARRAY FOR GELIM, THE INPUT CONTAINS THE
C           CONSTANTS OF THE ORIGINAL 3 EQUATIONS AND 3 UNKNOWNS.
C           THE OUTPUT CONTAINS THE COFFICIENTS OF THE SHAPE EQUATION.
C      IPIVOT IS A WORK ARRAY.
C      P IS THE VERTICAL AXIS OF THE SHAPE PLOT.
C      EPSLN IS THE HORIZONTAL AXIS AND IS EQUAL TO X/(LENGTH OF BEAM).
C      BET IS THE ARRAY FROM THE MAIN PROGRAM THAT CONTAINS A FIRST
C           GUESS FOR THE ROOT.
C*********************************************************************
      DIMENSION BTA(1),AA(1,1),WK(4),S(3,3),T(3,1),IPIVOT(3)
     1,P(2000),EPSLN(2000),BET(10000)
      REAL MU,M1,M2,I1,I2,L
      BTA(1)=BET(J)
C
C       NMAX,N,NSIG,INJAC AND ITMAX ARE ALL VARIABLES USED IN NONLIN
      NMAX=1
      N=1
      NSIG=14
      INJAC=1
      ITMAX=10000
C
C      NONLIN IS A SUBPROGRAM IN FTNMLIB THAT USES THE NEWTON-
C      RAPHSON METHOD TO FIND THE ROOTS OF LAT(I).
      CALL NONLIN(NMAX,N,BTA,NSIG,INJAC,JACOB,ITMAX,EVAL,WK,AA,IERR)
C
C      FREQ IS THE NATURAL FREQUENCY.
      FREQ=SQRT((EI*BTA(1)**4)/(MU/L*L**4))/(2*3.1415926)
      WRITE(5,10) FREQ
   10 FORMAT (18HNATURAL FREQUENCY=,F13.8,1X2HHZ,//)
C
C      NMAX,N,NRHS AND IFAC ARE ALL VARIABLES USED IN GELIM
      NMAX=3
      N=3
      NRHS=1
```

```
      IFAC=0
C
C      A IS THE NORMILIZATION COEFFICIENT NEEDED TO SOLVE
C      FOR THE MODE SHAPES
      A=1.
C
C     S(_,_) AND T(_,_) ARE THE NUMBERS IN THE COEFFICIENT
C     MATRIX TO BE SOLVED IN GELIM.
      S(1,1)=-M1*BTA(1)/MU
      S(1,2)=1.
      S(1,3)=-M1*BTA(1)/MU
      S(2,1)=-1.
      S(2,2)=I1*BTA(1)**3/(MU*L**2)
      S(2,3)=1.
      S(3,1)=(I2*BTA(1)**3/(MU*L**2)*SIN(BTA(1))-COS(BTA(1)))
      S(3,2)=(-I2*BTA(1)**3/(MU*L**2)*COSH(BTA(1))+SINH(BTA(1)))
      S(3,3)=(-I2*BTA(1)**3/(MU*L**2)*SINH(BTA(1))+COSH(BTA(1)))
      T(1,1)=A
      T(2,1)=-I1*BTA(1)**3/(MU*L**2)*A
      T(3,1)=(I2*BTA(1)**3/(MU*L**2)*COS(BTA(1))+SIN(BTA(1)))*A
      CALL GELIM(NMAX,N,S,NRHS,T,IPIVOT,IFAC,WK,IERR)
C
C      B,C AND D, ALONG WITH A ARE THE COFFICIENTS OF THE
C      MODE SHAPE EQUATION
      B=T(1,1)
      C=T(2,1)
      D=T(3,1)
      WRITE(5,15) A,B,C,D,BTA(1)
   15 FORMAT(34HTHE EQUATION OF THE MODE SHAPE IS://7HP(X/L)=F15.7,15H*S
     1IN(BETA*X/L)+F15.7,15H*COS(BETA*X/L)+/F15.7,16H*SINH(BETA*X/L)+F15
     2.7,15H*COSH(BETA*X/L)//11HWHERE BETA=F14.7//)
      JJ=1
C      DLTX IS THE INCREMENT ADDED TO EPSLN TO OBTAIN
C      POINTS FOR THE PLOT OF THE MODE SHAPE
      DLTX=.001
C
C      IEC,N,KX,KY,XMIN,XMAX,YMIN,YMAX,PCTPTS,NXMC,NYMC,
C      ISYM,SX,SY,XOFF AND YOFF ARE ALL VARIABLES NEEDED
C      IN INFOPLT.
      IEC=1
      N=1001
      KX=1
      KY=1
      XMIN=0.0
      XMAX=1.
      YMIN=-.1
      YMAX=.1
      PCTPTS=0.00
      NYMC=5
```

```
      NYMC=4
      ISYM=0
      SX=7.
      SY=5.
      XOFF=.75
      YOFF=.75
C
C      EPSLN(JJ) IS THE HORIZONTAL COMPONENT OF THE PLOT
C      WHERE AS R(K) IS THE VERTICAL.
      EPSLN(1)=0.
      DO 20 K=1,N
      R(K)=A*SIN(BTA(1)*EPSLN(JJ))+B*COS(BTA(1)*EPSLN(JJ))+C*SINH(BTA(1)
     1*EPSLN(JJ))+D*COSH(BTA(1)*EPSLN(JJ))
      IF (K.EQ.1) GO TO 5
C
C      THE FOLLOWING IF STATEMENT IS USED TO OBTAIN A
C      FIRST GUESS AS TO THE LOCATION OF THE NODES AND
C      SUBROUTINE NODE USES THE NEWTON RAPHSON METHOD TO
C      OBTAIN THE TRUE ROOTS.
      IF (R(K)*R(K-1).LE.0.) CALL NODE(EPSLN,JJ,BTA)
    5 EPSLN(JJ+1)=EPSLN(JJ)+DLTX
      JJ=JJ+1
   20 CONTINUE
      CALL INFOPLT(IEC,N,EPSLN(1),KX,R(1),KY,XMIN,XMAX,YMIN,YMAX,
     1PCTPTS,NXMC,5HEPSLN,NYMC,4HR(E),ISYM,SX,SY,XOFF,YOFF)
      RETURN
      END

      SUBROUTINE EVAL(BTA,XYZ)
      COMMON/VALUES/ I1,I2,L,M1,M2,MU,EI,A,B,C,D,CFD
      REAL I1,I2,L,M1,M2,MU
      DIMENSION BTA(1),XYZ(1)
      X=(2*COS(BTA(1))*COSH(BTA(1))-2)*L**4*MU**4+((2*BTA(1)*COS(BT
     1  A(1))*SINH(BTA(1))-2*BTA(1)*COSH(BTA(1))*SIN(BTA(1)))*L**4*M2+(
     2  2*BTA(1)*COS(BTA(1))*SINH(BTA(1))-2*BTA(1)*COSH(BTA(1))*SIN(BTA
     3  (1)))*L**4*M1+(I2*(-2*BTA(1)**3*COS(BTA(1))*SINH(BTA(1))-2*BTA(
     4  1)**3*COSH(BTA(1))*SIN(BTA(1)))+I1*(-2*BTA(1)**3*COS(BTA(1))*SI
     5  NH(BTA(1))-2*BTA(1)**3*COSH(BTA(1))*SIN(BTA(1))))*L**2)*MU**3
      Y=(((I2*(-2*BTA(1)**4*COS(BTA(1))*COSH(BTA(1))-2*BTA(1)**4)-4*I1*B
     7  TA(1)**4*COS(BTA(1))*COSH(BTA(1)))*L**2-4*BTA(1)**2*SIN(BTA(1))
     8  *SINH(BTA(1))*L**4*M1)*M2+(I1*(-2*BTA(1)**4*COS(BTA(1))*COSH(BT
     9  A(1))-2*BTA(1)**4)-4*I2*BTA(1)**4*COS(BTA(1))*COSH(BTA(1))))*L**
     :  2*M1+4*I1*I2*BTA(1)**6*SIN(BTA(1))*SINH(BTA(1)))*MU**2
      Z=(((I2*(2
     ;  *BTA(1)**5*COSH(BTA(1))*SIN(BTA(1))-2*BTA(1)**5*COS(BTA(1))*SIN
     <  H(BTA(1)))+I1*(2*BTA(1)**5*COSH(BTA(1))*SIN(BTA(1))-2*BTA(1)**5
     =  *COS(BTA(1))*SINH(BTA(1))))*L**2*M1+I1*I2*(2*BTA(1)**7*COS(BTA(
     >  1))*SINH(BTA(1))+2*BTA(1)**7*COSH(BTA(1))*SIN(BTA(1))))*M2+I1*I
```

```
     ?   2*(2*BTA(1)**7*COS(BTA(1))*SINH(BTA(1))+2*BTA(1)**7*COSH(BTA(1)
     @   )*SIN(BTA(1)))*MI)*MU+I1*I2*(2*BTA(I)**8*COS(BTA(I))*COSH(BTA(I
     1   ))-2*BTA(1)**8)*M1*M2
      XYZ(1)=(X+Y+Z)/(L**4*MU**4)
      RETURN
      END


       SUBROUTINE JACOR(BTA,DPRIME)
      COMMON/VALUES/ I1,I2,L,M1,M2,MU,EI,A,B,C,D,CRD
      REAL I1,I2,L,M1,M2,MU
       DIMENSION BTA(1),DPRIME(1,1)
         U =(2*COS(BTA(1))*SINH(BTA(I))-2*COSH(BTA(I))*SIN(BTA(1)))*
     1   L**4*MU**4+(((2*COS(BTA(1))-4*BTA(1)*SIN(BTA(1)))*SINH(BTA(1))-
     2   2*COSH(BTA(1))*SIN(BTA(1)))*L**4*M2+((2*COS(BTA(1))-4*BTA(1)*SI
     3   N(BTA(1)))*SINH(BTA(1))-2*COSH(BTA(1))*SIN(BTA(1)))*L**4*M1+((-
     4   6*I2-6*I1)*BTA(1)**2*COS(BTA(I))*SINH(BTA(I))+(-6*I2-6*I1)*BTA(
     5   1)**2*COSH(BTA(1))*SIN(BTA(1))+(-4*I2-4*I1)*BTA(1)**3*COS(BTA(1
     6   ))*COSH(BTA(1)))*L**2)*MU**3
      V = ((((-8*BTA(1)*SIN(BTA(1))-4*BTA(1)
     7   **2*COS(BTA(1)))*SINH(BTA(I))-4*BTA(I)**2*COSH(BTA(I))*SIN(BTA(
     8   1)))*L**4*M1+((-2*I2-4*I1)*BTA(1)**4*COS(BTA(1))*SINH(BTA(1))+(
     9   2*I2+4*I1)*BTA(1)**4*COSH(BTA(I))*SIN(BTA(1))+(-8*I2-16*I1)*BTA
     :   (1)**3*COS(BTA(1))*COSH(BTA(1))-8*I2*BTA(I)**3)*L**2)*M2+((-4*I
     ;   2-2*I1)*BTA(1)**4*COS(BTA(1))*SINH(BTA(1))+(4*I2+2*I1)*BTA(1)**
     <   4*COSH(BTA(1))*SIN(BTA(1))+(-16*I2-8*I1)*BTA(1)**3*COS(BTA(1))*
     =   COSH(BTA(1))-8*I1*BTA(1)**3)*L**2*M1+(24*I1*I2*BTA(1)**5*SIN(BT
     >   A(1))+4*I1*I2*BTA(1)**6*COS(BTA(1)))*SINH(BTA(1))+4*I1*I2*BTA(I
     ?   )**6*COSH(BTA(I))*SIN(BTA(I)))*MU**2
      W = (((((4*I2+4*I1)*BTA(1)**5*
     @   SIN(BTA(1))+(-10*I2-10*I1)*BTA(1)**4*COS(BTA(1)))*SINH(BTA(1))+
     1   (10*I2+10*I1)*BTA(1)**4*COSH(BTA(1))*SIN(BTA(1)))*L**2*M1+14*I1
     2   *I2*BTA(1)**6*COS(BTA(1))*SINH(BTA(1))+14*I1*I2*BTA(1)**6*COSH(
     3   BTA(1))*SIN(BTA(1))+4*I1*I2*BTA(1)**7*COS(BTA(1))*COSH(BTA(1)))
     4   *M2+(14*I1*I2*BTA(1)**6*COS(BTA(I))*SINH(BTA(1))+14*I1*I2*BTA(I
     5   )**6*COSH(BTA(1))*SIN(BTA(1))+4*I1*I2*BTA(1)**7*COS(BTA(1))*COS
     6   H(BTA(1)))*M1)*MU+(2*I1*I2*BTA(1)**8*COS(BTA(1))*SINH(BTA(1))-2
     7   *I1*I2*BTA(1)**8*COSH(BTA(1))*SIN(BTA(I))+16*I1*I2*BTA(1)**7*CO
     8   S(BTA(1))*COSH(BTA(1))-16*I1*I2*BTA(1)**7)*M1*M2
      DPRIME(1,1)=(U+V+W)/(L**4*MU**4)
      RETURN
      END


      SUBROUTINE NODE(EPSLN,JJ,BTA)
      COMMON/VALUES/I1,I2,L,M1,M2,MU,EI,A,B,C,D,CRD
C
C       ABLE AND BAKER ARE THE EXTERNALS THAT CONTAIN THE
C       MODE SHAPE EQUATION AND ITS DERIVATIVE WHICH ARE
```

```
C       USED BY NONLIN.
        EXTERNAL ABLE,BAKER
        DIMENSION BTA(1),EPSLN(2000),ANOD(1),WK(4),AA(1,1),CRD(1)
        REAL MU,M1,M2,L,I1,I2
        ANOD(1)=EPSLN(JJ)
C
C       NMAX,N,NSIG,INJAC AND ITMAX ARE ALL USED BY NONLIN.
        NMAX=1
        N=1
        NSIG=14
        INJAC=1
        ITMAX=10000
        CRD(1)=BTA(1)
        CALL NONLIN(NMAX,N,ANOD,NSIG,INJAC,BAKER,ITMAX,ABLE,WK,AA,IERR)
        X=ANOD(1)*L
        WRITE(5,10) X
   10   FORMAT(23HA NODE IS LOCATED AT X=F14.5)
        RETURN
        END

        SUBROUTINE ABLE(ANOD,RR)
        COMMON/VALUES/ I1,I2,L,M1,M2,MU,EI,A,B,C,D,CRD
        DIMENSION ANOD(1),RR(1),CRD(1)
        RR(1)=A*SIN(CRD(1)*ANOD(1))+B*COS(CRD(1)*ANOD(1))+C*SINH(CRD(1)*
       1ANOD(1))+D*COSH(CRD(1)*ANOD(1))
        RETURN
        END

        SUBROUTINE BAKER(ANOD,DRR)
        COMMON/VALUES/I1,I2,L,M1,M2,MU,EI,A,B,C,D,CRD
        DIMENSION ANOD(1),DRR(1,1),CRD(1)
        DRR(1,1)=A*CRD(1)*COS(CRD(1)*ANOD(1))-B*CRD(1)*SIN(CRD(1)*ANOD(1)
       1)+C*CRD(1)*COSH(CRD(1)*ANOD(1))+D*CRD(1)*SINH(CRD(1)*ANOD(1))
        RETURN
        END
```

Appendix B. Sample Input and Output for FFBEAM

EXAMPLE OF LATERAL VIBRATION PROGRAM

```
C*****************************************************
C      THE FOLLOWING 8 VARIABLES ARE THE INPUT VARIABLES.
C
C         MU= MASS  OF THE BEAM ALONE
C         L=LENGTH OF THE BEAM
C         M1=MASS AT X=0. END OF BEAM
C         M2=MASS AT X=L END OF BEAM
C         I1=INERTIA OF M1
C         I2=INERTIA OF M2
C         EI=STIFFNESS OF THE BEAM
C         NMODE=THE NUMBER OF MODES SOLUTIONS ARE NEEDED FOR, STARTING
C                 WITH #1 AND ENDING WITH #NMODE.
C*****************************************************
      MU=177.
      L=100.
      M1=80000.0
      M2=250.
      I1=9380141.7
      I2=250.
      EI=20000000.0
      NMODE=5
```

```
THESE ARE THE SOLUTIONS FOR LATERAL VIBRATION OF THE
BEAM WITH THE FOLLOWING CHARACTERISTICS.
THE MASS OF THE BEAM =        177.000
THE LENGTH OF THE BEAM=        100.000
THE MASS LOCATED AT X=0. =     80000.000
THE MASS LOCATED AT X=L =      250.000
THE INERTIA OF THE MASS AT X=0. =   9380141.700
THE INERTIA OF THE MASS AT X=L =        250.000
THE STIFFNESS OF THE BEAM =  20000000.000

RESULTS FOR MODE #  1

NATURAL FREQUENCY=    .00000000 HZ

THE EQUATION OF THE MODE SHAPE IS:

R(X/L)=      1.0000000*SIN(BETA*X/L)+      -.0000001*COS(BETA*X/L)+
```

```
       1.0000000*SINH(BETA*X/L)+        -.0000001*COSH(BETA*X/L)

 WHERE BETA=      -.0000000



 RESULTS FOR MODE #  2

 NATURAL FREQUENCY=     .08318431 HZ

 THE EQUATION OF THE MODE SHAPE IS:

 R(X/L)=       1.0000000*SIN(BETA*X/L)+      -1.3492285*COS(BETA*X/L)+
       -1.2622367*SINH(BETA*X/L)+       1.3452145*COSH(BETA*X/L)

 WHERE BETA=      1.2469426

 A NODE IS LOCATED AT X=        17.79221

 RESULTS FOR MODE #  3

 NATURAL FREQUENCY=     .85667199 HZ

 THE EQUATION OF THE MODE SHAPE IS:

 R(X/L)=       1.0000000*SIN(BETA*X/L)+      -1.0020106*COS(BETA*X/L)+
       -1.0058983*SINH(BETA*X/L)+       1.0009016*COSH(BETA*X/L)

 WHERE BETA=      4.0015941

 A NODE IS LOCATED AT X=          .91457
 A NODE IS LOCATED AT X=        96.44120

 RESULTS FOR MODE #  4

 NATURAL FREQUENCY=    2.68845526 HZ

 THE EQUATION OF THE MODE SHAPE IS:
```

```
R(X/L)=      1.0000000*SIN(BETA*X/L)+      -1.0018573*COS(BETA*X/L)+
     -1.0010610*SINH(BETA*X/L)+      1.0012328*COSH(BETA*X/L)

WHERE BETA=      7.0888723

A NODE IS LOCATED AT X=         .36140
A NODE IS LOCATED AT X=       55.64698
A NODE IS LOCATED AT X=       98.77667

RESULTS FOR MODE #  5

NATURAL FREQUENCY=   5.52148750 HZ

THE EQUATION OF THE MODE SHAPE IS:

R(X/L)=      1.0000000*SIN(BETA*X/L)+      -1.0007834*COS(BETA*X/L)+
     -1.0003601*SINH(BETA*X/L)+      1.0003477*COSH(BETA*X/L)

WHERE BETA=     10.1590663

A NODE IS LOCATED AT X=         .20791
A NODE IS LOCATED AT X=       38.79191
A NODE IS LOCATED AT X=       69.62796
A NODE IS LOCATED AT X=       99.45123
```

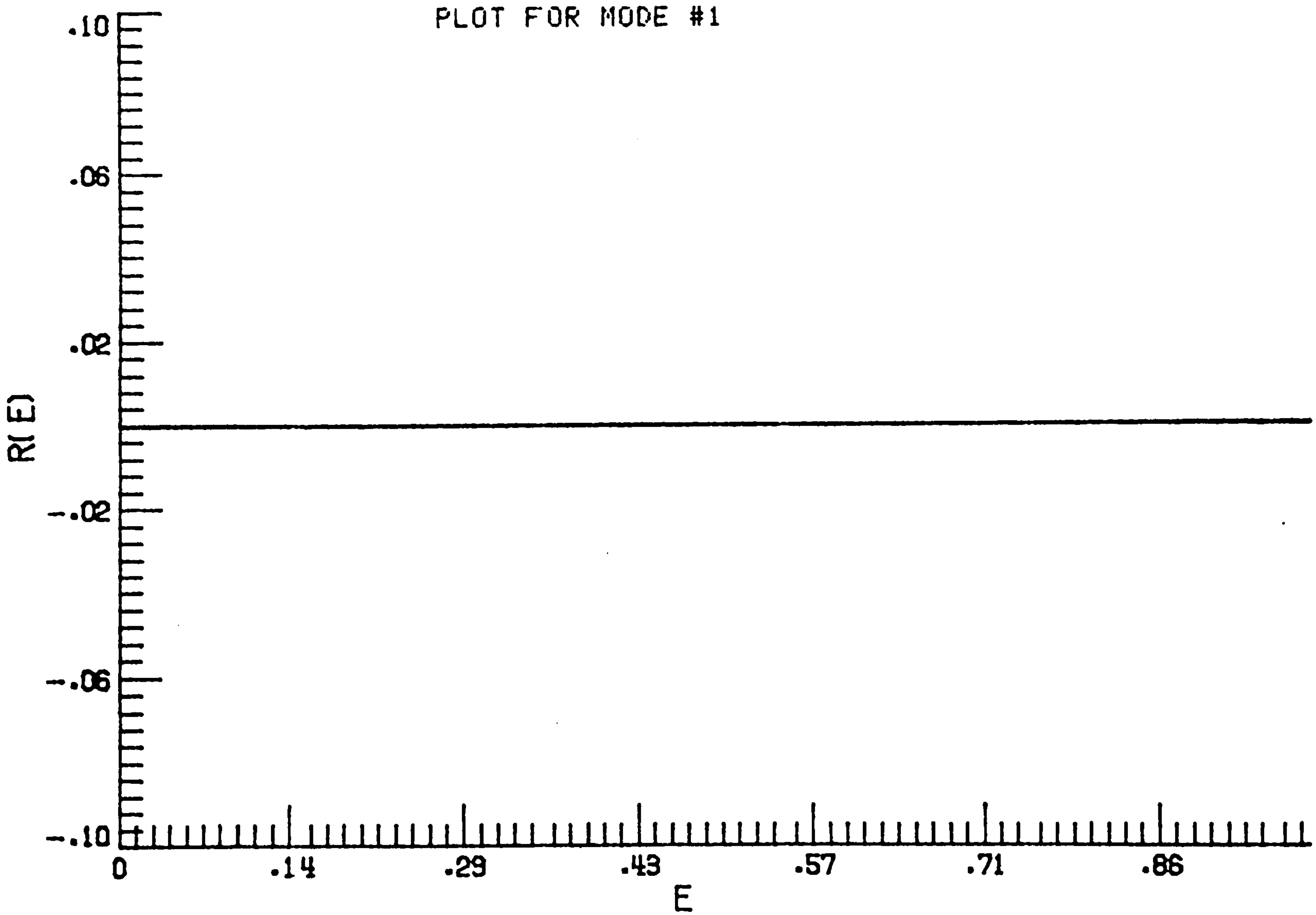

Figure 5.- Output from FFBEAM plotting displacement R(E) versus nondimensional position variable E for first mode.

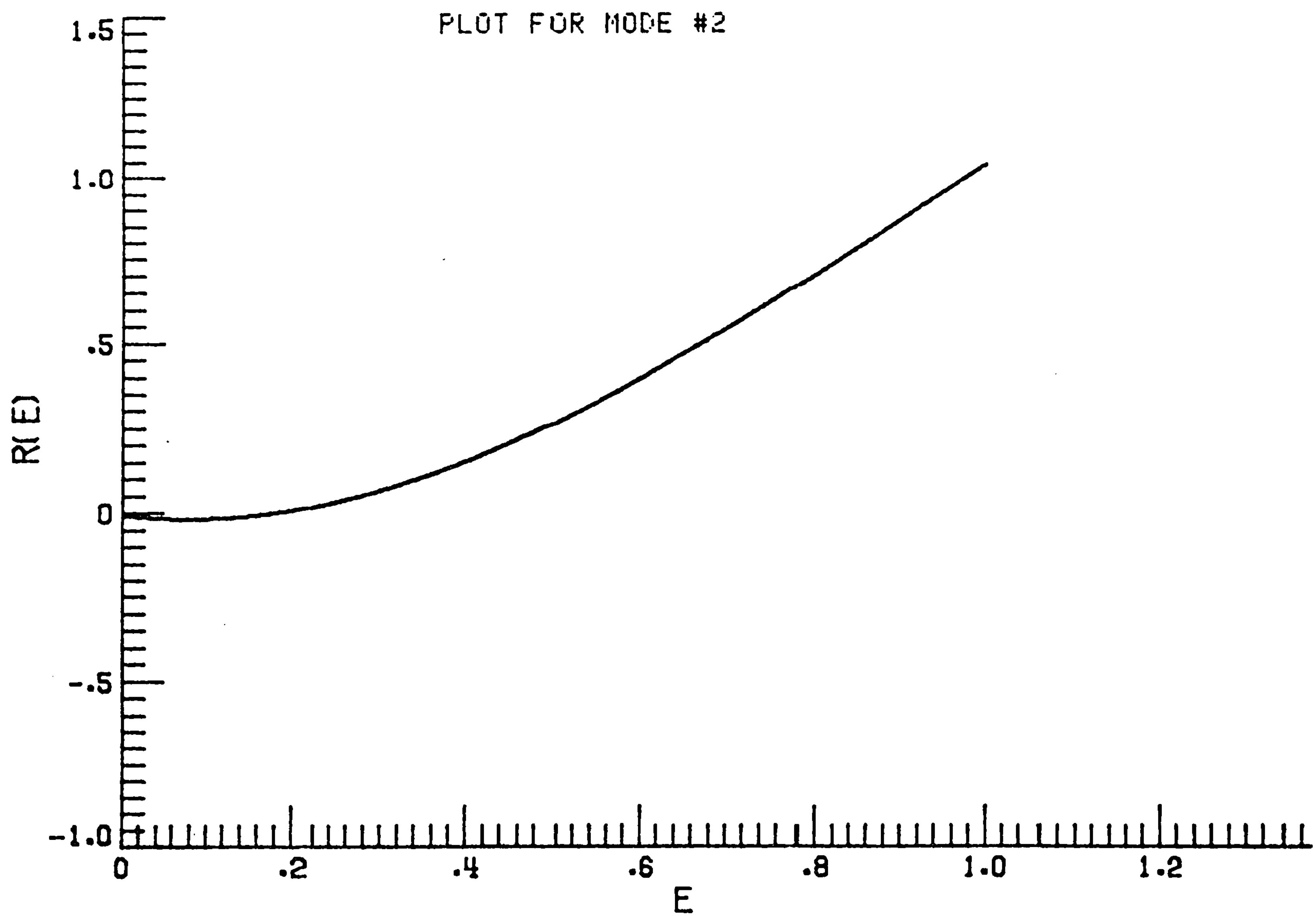

Figure 6.- Output from FFBEAM plotting displacement R(E) versus nondimensional position variable E for second mode.

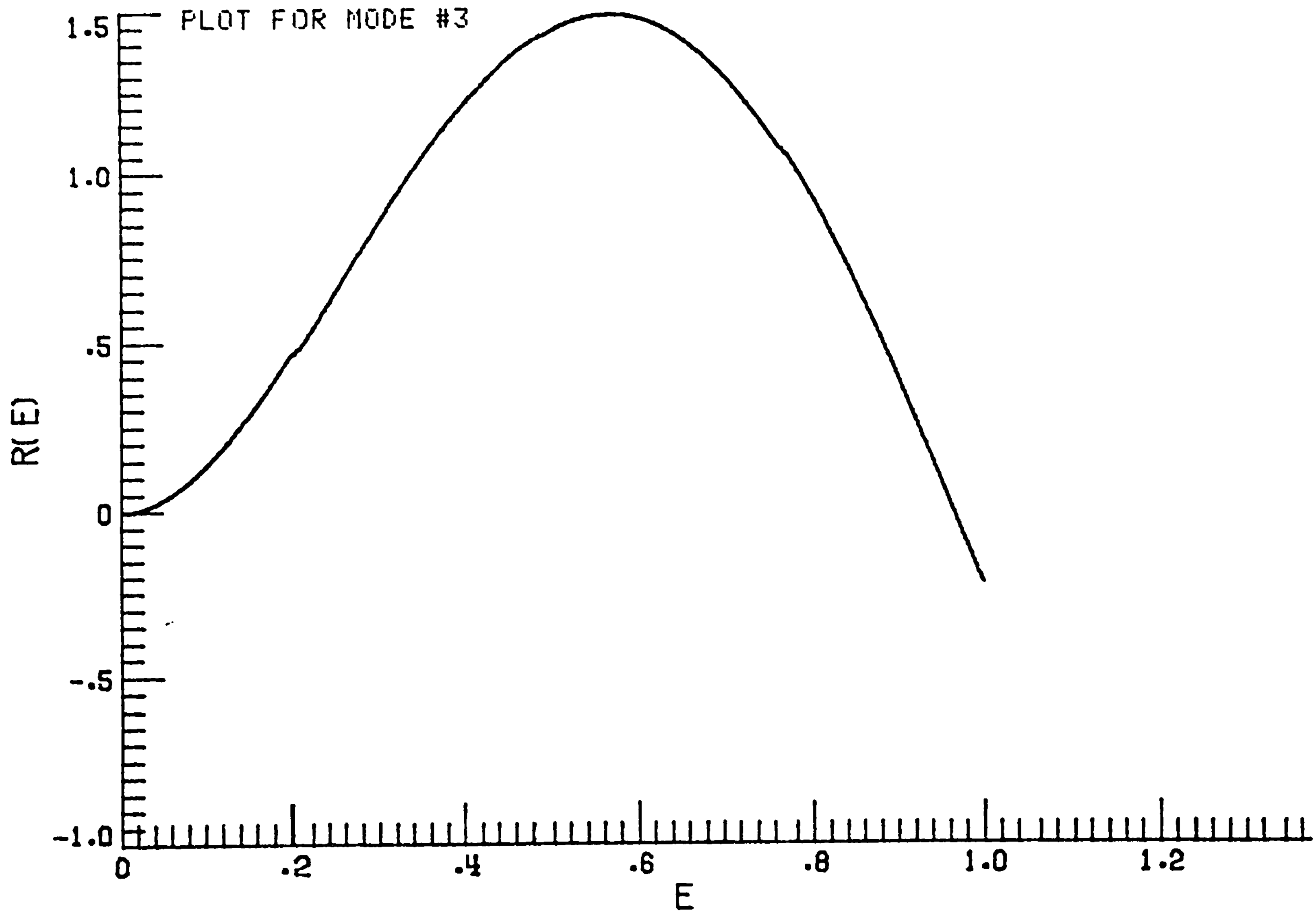

Figure 7.- Output from FFBEAM plotting displacement R(E) versus nondimensional position variable E for third mode.

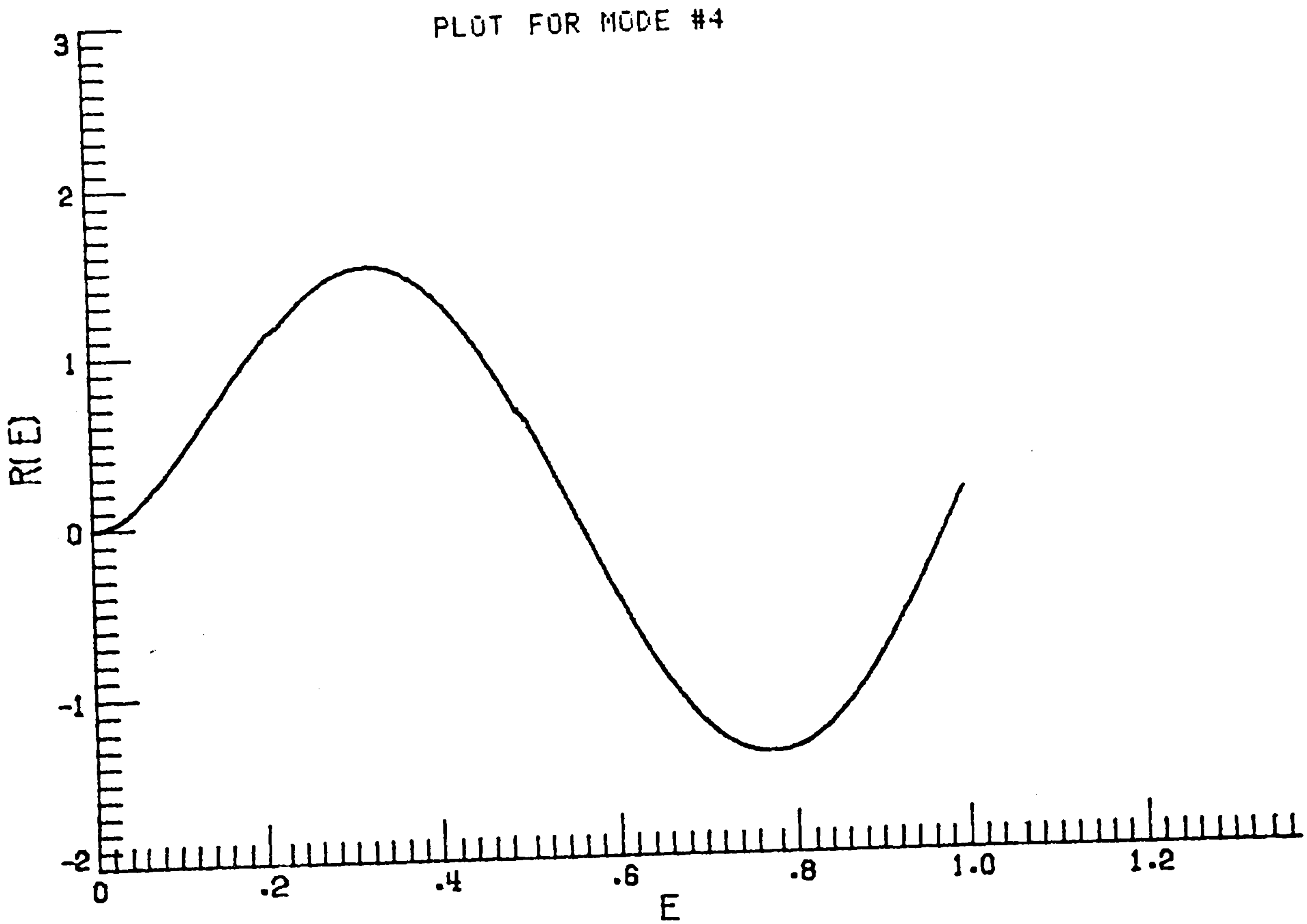

Figure 8.- Output from FFBEAM plotting displacement R(E) versus nondimensional position variable E for fourth mode.

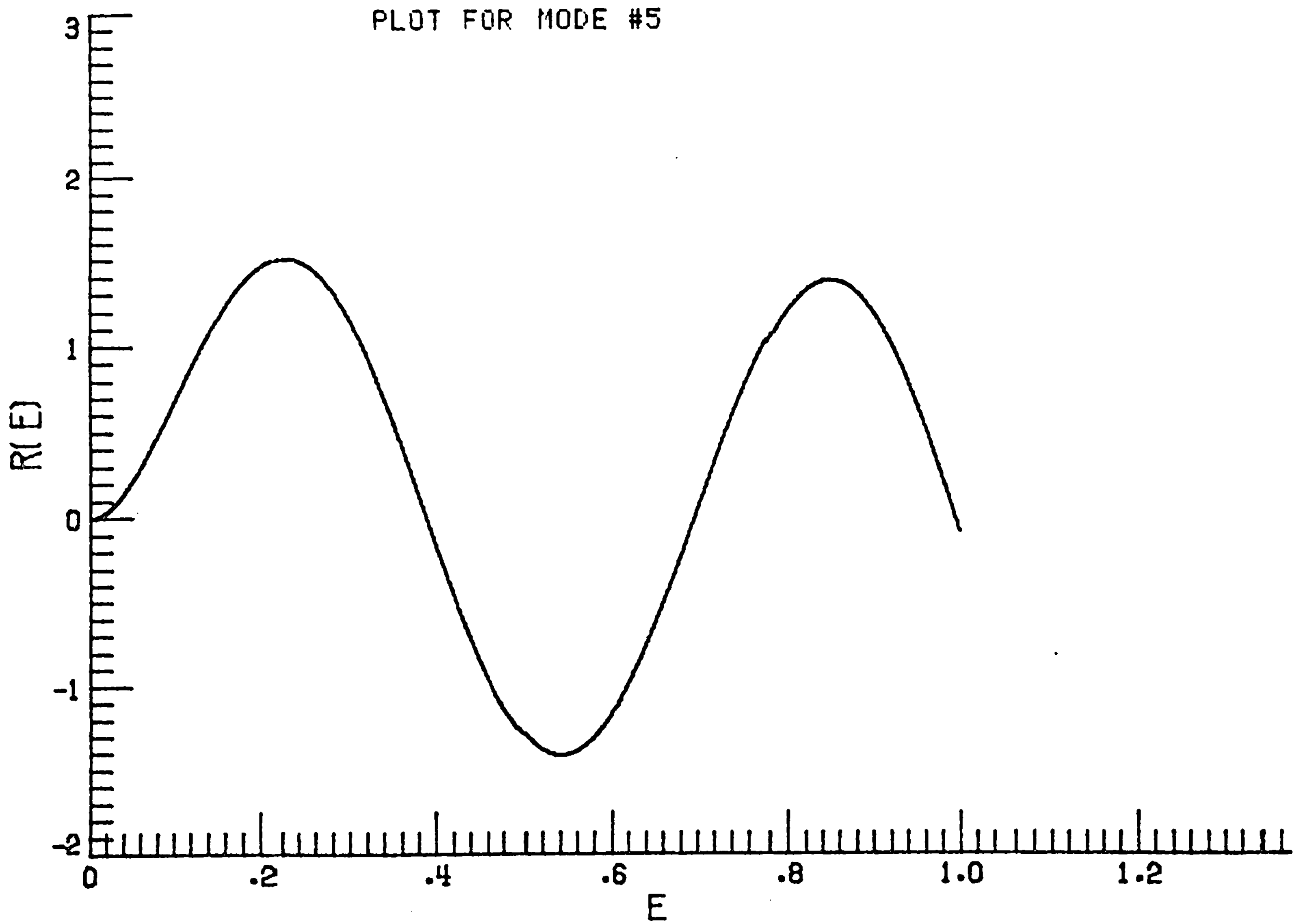

Figure 9.- Output from FFBEAM plotting displacement R(E) versus nondimensional position variable E for fifth mode.

APPENDIX C.

MACSYMA PROCEDURE

This appendix lists and explains the code used to derive the equations necessary to solve for the eigenvalues for free lateral vibration of the configurations described earlier. The following is a list of a batch file which can be run on MACSYMA to duplicate the derivation that was used for lateral vibrations:

```
 1.  A*SIN(BETA*E)+B*COS(BETA*E)+C*SINH(BETA*E)+D*COSH(BETA*E);
 2.  DIFF(D2,E,1);
 3.  DIFF(D2,E,2);
 4.  DIFF(D2,E,3);
 5.  D5-M1/MU*BETA**4*D2=0.;
 6.  SUBST(0.,E,D6);
 7.  D5+M2/MU*BETA**4*D2=0.;
 8.  SUBST(1.,E,D8)
 9.  D4+I1/(MU*L**2)*BETA**4*D3=0.;
10.  SUBST(0.,E,D10);
11.  D4-I2/(MU*L**2)*BETA**4*D3=0.;
12.  SUBST(1.,E,D12);
13.  COEFMATRIX([D7,D9,D11,D13],[A,B,C,D]);
14.  DETERMINANT(%);
15.  RATSIMP(%);
16.  DET=D16;
17.  DIFF(D16,BETA,1);
18.  RATSIMP(%);
19.  DPRIME=D19;
20.  FORTRAN(D17);
21.  FORTRAN(D20);
```

The following is an explanation of each line individually:

Line Number	Definition
1	This line simply writes equation (8).
2	This line differentiates equation (8) with respect to ε.
3	This line differentiates equation (8) twice with respect to ε.
4	This line differentiates equation (8) three times with respect to ε.
5	This line writes the first boundary condition: equation (16).
6	This line substitutes zero for ε to satisfy the boundary conditions.
7	This line writes the second boundary condition: equation (18).
8	This line substitutes one for ε to satisfy second boundary condition.
9	This line writes the third boundary condition: equation (22).
10	This line substitutes zero for ε to satisfy third boundary condition.
11	This line writes the fourth boundary condition: equation (23).
12	This line substitutes one for ε to satisfy fourth boundary condition.
13	This line generates the four by four matrix.
14	This line generates the determinant of the matrix.
15	This line simplifies the determinant.
16	This line writes the determinant in the form DET=....

Line Number	Definition
17	This line differentiates the determinant with respect to β. This is needed to use the Newton-Raphson method of finding roots.
18	This line simplifies the derivative.
19	This line writes the derivative in the form DPRIME=....
20	This line writes the determinant in FORTRAN code.
21	This lines writes the derivative in FORTRAN code.

This file can be run on MACSYMA by typing the command:

```
BATCH(DKROBE,LATRAL,DSK,LRC);
```

This must be done as the first command in a session: on line "Cl".

REFERENCES

1. Gorman, Daniel J.: Free Vibration Analysis of Beams and Shafts. John Wiley and Sons, Inc., 1975.

2. Hansen, H. M.; and Chenea, Paul F.: Mechanics of Vibration. John Wiley and Sons, Inc., 1952.

3. Thomson, William T.: Mechanical Vibrations. Second Edition; Prentice-Hall, Inc., 1953.

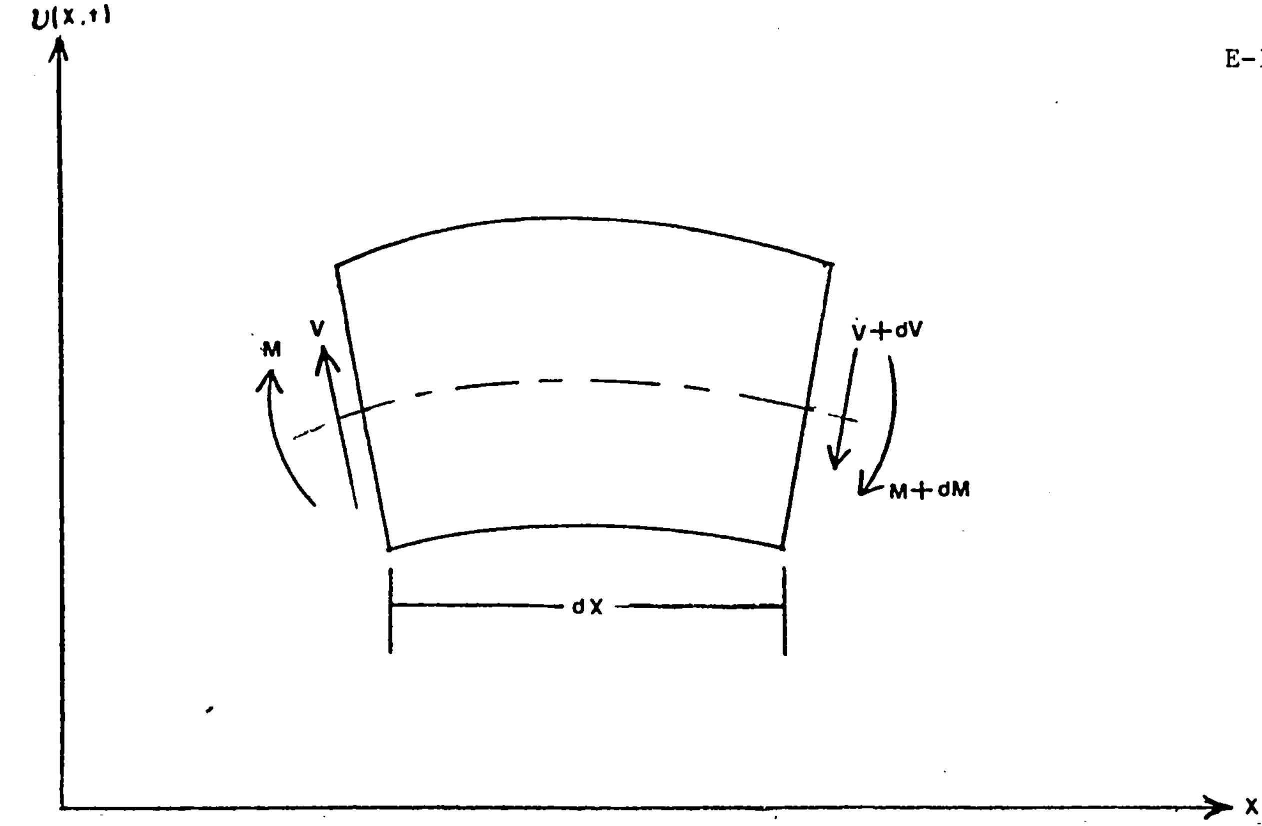

Figure 1.- A differential beam element showing shears and moments resulting from displacement V(x,t).

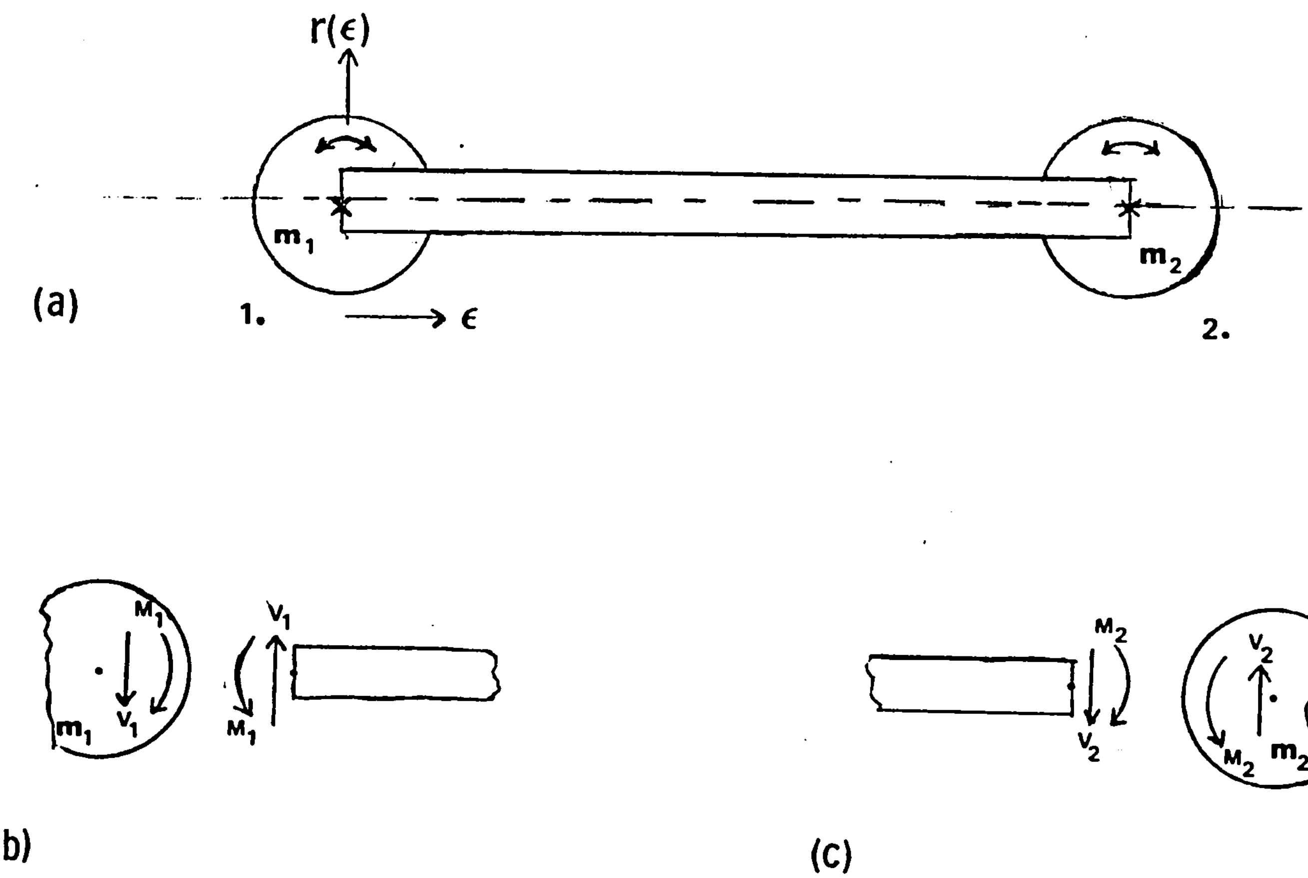

Figure 2.- (a) The beam configuration considered in this paper with rotating masses m_1 and m_2 at either end. Parts (b) and (c) show the force and moment reactions at the ends.

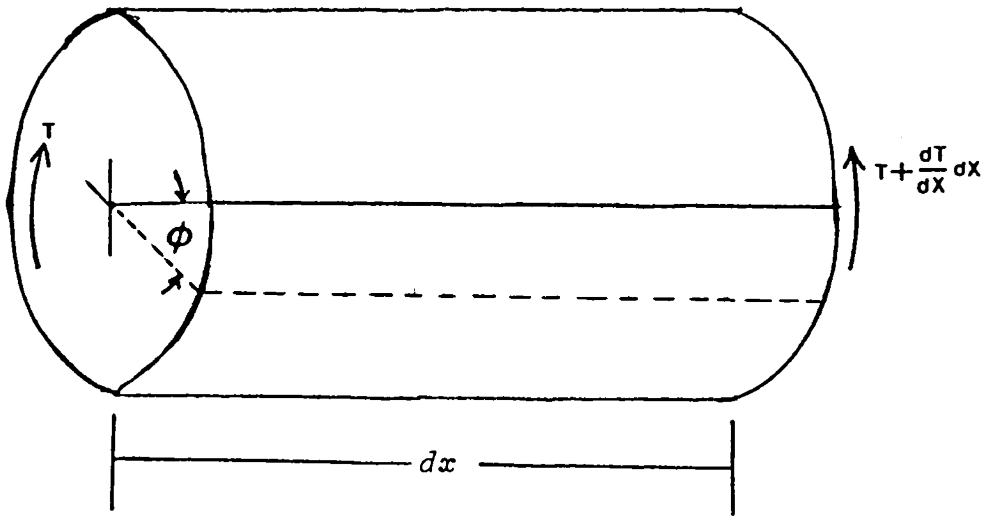

Figure 3.- A differential element showing torques resulting from angular displacement $\phi(x,t)$.

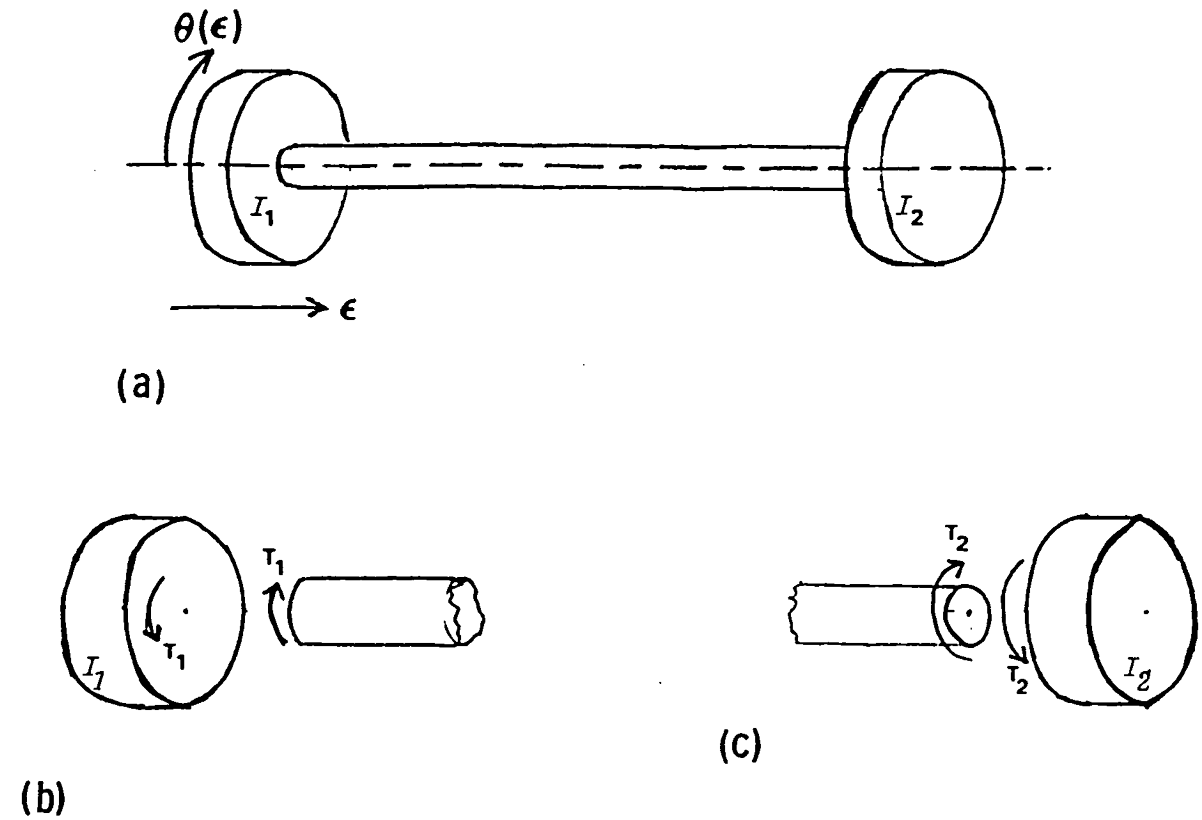

Figure 4.- (a) A shaft with rotational masses at either end. Parts (b) and (c) show the reaction torques at both ends.

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16. Abstract

Partial differential equations are derived for free lateral and torsional vibration of a uniform free-free beam with a rotational mass attached to each extremity. For appropriate boundary conditions, nonlinear algebraic equations are obtained using a symbolic manipulation computer program, the solutions of which enable the computation of the natural frequencies and mode-shapes. The mode-shapes are linear combinations of trigonometric and hyperbolic sine and cosine functions. A computer program is written for the numerical solution of the algebraic equations mentioned above, which can compute the natural frequencies, mode-shapes, and node points for any given set of parameters, for any given number of modes.

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