8.4A  REVIEW OF CORRELATION TECHNIQUES

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The problem of correlation analysis in MST radar is to determine the scattered power, Doppler frequency and correlation time for a noisy signal.  It is assumed that coherent detection has been employed, with two accurately balanced quadrature receiving channels.  It is further assumed that coherent integration has been performed with a window length significantly less than the correlation time of the signal.

The analysis problem may be looked at from the point of view either of Fourier analysis or of correlation analysis, and it must be emphasized that the two approaches, if used properly, give identical results.  Why, then, use correlation analysis at all?  The reason can be seen from the spectrum and correlation function shown in Figure 1.  In each case 1 min of data is represented, for example (with 1/8 sec coherent integration time) 480 pairs of data points (real and imaginary).  Ordinary discrete Fourier analysis requires about $2 \times 10^5$ floating-point multiplications, all involving transcendentals.

In fact, however, only the area P under the echo spectrum, its position $f_1$ and its width $f_2$ are required; these quantities have to be calculated from the spectrum by separate algorithms.

It can easily be shown that the quantities P, $f_1$ and $f_2$ can be determined from the first few values of the complex autocovariance functions, shown on the lower part of Figure 1.  This function can be calculated out to a number of lags approaching the length of the sample, but almost no additional information is contained in the part of the function beyond the first few lags.  In examining such a function, it is necessary to make an assumption; namely, that the curve for lags other than zero can be extrapolated back to zero to give the signal power P and the noise power N as shown; this is possible because the noise power is uncorrelated from one coherently integrated sample to the next, while the coherent integration time has, as indicated above, been chosen so as to make the correlation between one sample and the next very good.

The correlation time (or time to correlation equals .5) is estimated as shown after the noise power has been removed.  The spectral width $f_2$ is the reciprocal of the correlation time.  The Doppler frequency $f_1$ is found from the slope of the imaginary part of the correlation function at the origin, care being taken to eliminate the unwanted variance N.

In principle, only 3 points on the complex autocorrelation function need to be calculated, which would require about 6000 multiplications, many fewer than the number required for the spectral approach.  In fact, a larger number of lags (up to 12) is often calculated with the idea of improving the analysis.  As described by COUNTRYMAN and BOWHILL (1979), the values of the argument of the complex covariance for the various lags can be weighted to give a more accurate value.  A maximum likelihood analysis of the estimation problem is given in Appendix 1.  This method of analysis has not yet been applied to experimental data.

The calculations of autocorrelation functions, however, need not necessarily use multiplication at all.  BOWHILL (1955) described a method of

finding autocorrelations using the mean difference of separate samples rather than their mean product. Appendix 2 gives a description of the algorithms used. This technique has been successfully applied in a microcomputer operating system for the real-time processing of MST data (see paper 8.3-D).

There are other ways in which the correlation process can be speeded up. HAGEN and FARLEY (1973) describe 11 methods. Table 1 illustrates several of these algorithms, together with the efficiency in terms of use of the input data. It should be emphasized that the technique of Appendix 2 has an efficiency in excess of 95%. The question arises as to whether a hardware correlator is worthwhile for MST radar. It is my opinion that the use of coherent integration, which reduces the number of input data by about 2 orders of magnitude, makes a special hardware device unnecessary, particularly with the use of algorithms such as those described in this paper.
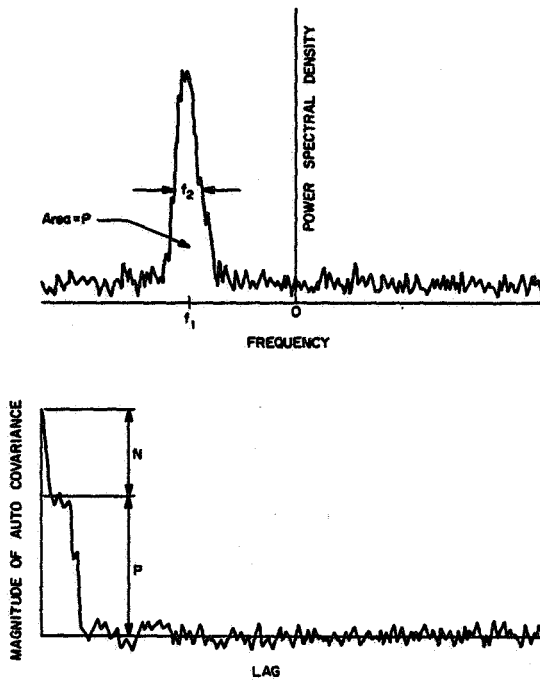


Figure 1. Power spectrum and autocovariance function corresponding to a coherently scattered signal plus noise.

Table 1

| Type of Correlation | Output | Efficiency (%) |
|---|---|---|
| Multibit | $\sigma^2 \rho$ | 100 |
| One-bit x multibit | $(2/\pi)^{1/2} \sigma \rho$ | 64 |
| One-bit | $(2/\pi) \sin^{-1} \rho$ | 41 |

**ACKNOWLEDGEMENTS**

**REFERENCES**

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Hagen, J. B. and D. T. Farley (1973), Digital-correlation techniques in radio science, _Radio Sci., 8_, 775-784.

APPENDIX 1.    Maximum-Likelihood Estimation of C-S Parameters.

Suppose that the real part of the complex autocorrelation function is given by

$$r(\tau) = r_0 \exp(-a\tau^2) \cos\omega\tau + \varepsilon_0(\tau)$$

where      $\varepsilon_0(\tau) = 1-r_0 \quad (\tau = 0)$
$$= 0 \qquad (\tau > 0)$$

and      $\tau = 0, 1, 2 \dots n.$

Now let the observed autocorrelation function have real part $R(\tau)$. For the imaginary part, $i(\tau) = r_0 \exp(-a\tau^2) \sin\omega\tau + \varepsilon_0(\tau)$ if the r and i channels have equal sensitivity.

Now let the noise n on the r and i channels have a Gaussian distribution $\exp(-bn^2)$ then the likelihood of a given set of $I(\tau)$ and $R(\tau)$ is

$$L = \exp[-b \sum_0^n \{R(\tau) - r(\tau)\}^2 - b \sum_0^n \{I(\tau) - i(\tau)\}]$$

and      $$-\frac{1}{b}\ln L = \sum_0^n [R(\tau) - r(\tau)]^2 + \sum_0^n [I(\tau) - i(\tau)]^2$$

and maximum likelihood amounts to a least squares fit of $R(\tau)$ and $I(\tau)$ by $r(\tau)$ and $i(\tau)$.

Substituting for $r(\tau)$ and $i(\tau)$, and neglecting $\tau = 0$,

$$-\frac{1}{b}\ln L = \sum_1^n [R(\tau) - r_0 \exp(-a\tau^2) \cos\omega\tau]^2 + \sum_1^n [I(\tau) - r_0 \exp(-a\tau^2) \sin\omega\tau]^2$$

$$= \sum_1^n [R^2(\tau) + I^2(\tau) + r_0^2 \exp(-2a\tau^2) - 2r_0^2\exp(-a\tau^2) (R(\tau) \cos\omega\tau + I(\tau) \sin\omega\tau)]$$

Differentiating with respect to $r_0$, a and $\omega$, and equating to 0, we get the following equations which must be satisfied simultaneously:

$$r_0 \sum_1^n \exp(-2a\tau^2) - \sum_1^n \exp(-a\tau^2) \, [R(\tau) \, \cos\omega\tau + I(\tau) \, \sin\omega\tau] = 0$$

$$r_0 \sum_1^n \tau^2 \exp(-2a\tau^2) - \sum_1^n \tau^2 \exp(-a\tau^2) \, [R(\tau) \, \cos\omega\tau + I(\tau) \, \sin\omega\tau] = 0$$

$$\sum_1^n \exp(-a\tau^2) \, [R(\tau) \, \sin\omega\tau - I(\tau) \, \cos\omega\tau] = 0$$

or $\qquad \omega = \dfrac{\sum I(\tau) \, \exp(-a\tau^2)}{\sum R(\tau)\tau \, \exp(-a\tau^2)} \qquad$ for $\omega\tau \ll 1$.

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## APPENDIX 2. Rapid Pseudocorrelation Technique.

Consider two voltages $R(t)$ and $I(t)$, nominally the real and imaginary parts of the phasor of a radar return signal. The problem is to determine the center frequency and power, and correlation time, of an embedded signal of frequency $\omega$.

### REPRESENTATION OF R(t) AND I(t)

The signal of frequency $\omega$ can be represented by a phasor $S(\tau)$ at that frequency, giving $S_R(t) \cos\omega t$ and $S_I(t) \cos(\omega t + \phi)$ in the voltages $R(t)$ and $I(t)$, respectively. $S_R(t)$ and $S_I(t)$ are random variables, such that $S_R(t)/S_I(t) = $ constant. The phase shift $\phi$ is nominally $\pi/2$. A noise voltage $n(t)$ will also appear in $R(t)$ and $I(t)$, which is supposed to be completely uncorrelated from one pulse to the next.

We therefore have

$$R(t) = S_R(t) \cos\omega t + n_R(t)$$

$$I(t) = S_I(t) \cos(\omega t + \phi) + n_I(t)$$

### MEAN SQUARE DIFFERENCE DEFINITIONS

Let $\qquad RO^2 = \langle R^2(t) \rangle$

$\qquad\qquad IO^2 = \langle I^2(t) \rangle$

$\qquad RR^2(\tau) = \langle [R(t) - R(t + \tau)]^2 \rangle$

$\qquad II^2(\tau) = \langle [I(t) - I(t + \tau)]^2 \rangle$

$\qquad RI^2(\tau) = \langle [R(t) - I(t + \tau)]^2 \rangle$

$\qquad IR^2(\tau) = \langle [I(t) - R(t + \tau)]^2 \rangle$

Now all these quantities may be related to the mean absolute values of the squared quantities by the relation:

$$\langle X^2(t) \rangle = k[\langle |X(t)| \rangle]^2$$

where k is a constant for a given waveform.

EVALUATION OF DIFFERENCES

From the expression for R(t),

$$\langle R^2(t) \rangle = \frac{1}{2} \langle S_R^2(t) \rangle + \langle n_R^2(t) \rangle$$

since $S_R(t)$ and $n_R(t)$ are independent random variables. Further, we define

$$S_R^2 = \langle S_R^2(t) \rangle, \quad N_R^2 = \langle n_R^2(t) \rangle,$$

$$RO^2 = \frac{1}{2} S_R^2 + N_R^2$$

$$IO^2 = \frac{1}{2} S_R^2 + N_R^2$$

$$RR^2(\tau) = \langle [S_R(t) \cos\omega t + n_R(t) - S_R(t + \tau) \cos(\omega t + \omega\tau) - n_R(t + \tau)]^2 \rangle$$

$$= \frac{1}{2} S_R^2 + N_R^2 + \frac{1}{2} S_R^2 + N_R^2 - 2\langle S_R(t) S_R(t + \tau) \cos\omega\tau \cos(\omega t + \omega\tau) \rangle$$

$$= S_R^2 + 2N_R^2 - \langle S_R(t) S_R(\tau) \rangle \cos\omega\tau$$

and defining

$$\rho(\tau) = \langle S_R(t) S_R(t + \tau) \rangle / S_R^2$$

$$RR^2(\tau) = S_R^2[1 - \rho(\tau) \cos\omega\tau] + 2N_R^2$$

$$II^2(\tau) = S_I^2[1 - \rho(\tau) \cos\omega\tau] + 2N_I^2$$

Similarly,

$$RI^2(\tau) = \langle [S_R(t) \cos\omega t + n_R(t) - S_I(t + \tau) \cos(\omega t + \omega\tau + \phi) - n_I(t + \tau)]^2 \rangle$$

$$= \frac{1}{2} S_R^2 + N_R^2 + \frac{1}{2} S_I^2 + N_I^2$$

$$+ 2 \langle S_R(t) S_I(t + \tau) \cos\omega t \cos(\omega t + \omega\tau + \phi) \rangle$$

$$+ 2 \langle n_R(t) n_I(t + \tau) \rangle$$

and the latter term is always zero. So

$$RI^2(\tau) = \frac{1}{2} S_R^2 + \frac{1}{2} S_I^2 + N_R^2 + N_I^2$$

$$+ 2 \langle S_R(t) S_R(t + \tau) \frac{1}{2} \cos(\omega t + \phi) \rangle$$

$$= \frac{1}{2} S_R^2 + \frac{1}{2} S_I^2 + N_R^2 + N_I^2 + S_R S_I \rho(\tau) \cos(+\omega\tau + \phi)$$

Similarly,

$$IR^2(\tau) = \frac{1}{2} S_R^2 + \frac{1}{2} S_I^2 + N_R^2 + N_I^2 + S_R S_I \rho(\tau) \cos(-\omega\tau + \phi)$$

POWER AND FREQUENCY CALCULATIONS

From the above relations,

$$2RO^2 - RR^2 = S_R^2 \, \rho(\tau) \, \cos\omega\tau$$

$$2IO^2 - II^2 = S_I^2 \, \rho(\tau) \, \cos\omega\tau$$

$$RI^2 - RO^2 - IO^2 = S_R S_I \, \rho(\tau) \, \cos(\omega\tau + \phi)$$

$$IR^2 - RO^2 - IO^2 = S_R S_I \, \rho(\tau) \, \cos(-\omega\tau + \phi)$$

and

$$[(2RO^2 - RR^2)(2IO^2 - II^2)]^{1/2} = S_R S_I \rho(\tau) \, \cos\omega\tau$$

$$IR^2 - RI^2 = 2 \, S_R S_I \rho(\tau) \, \sin\omega\tau \cdot \sin\phi$$

$$RI^2 + IR^2 - 2(RO^2 + IO^2) = 2 \, S_R S_I \rho(\tau) \, \cos\omega\tau \cdot \cos\phi$$

Normally, $\phi$ is set to $\pi/2$, so the first two equations become

$$[(2RO^2 - RR^2)(2IO^2 - II^2)]^{1/2} = S_R S_I \rho(\tau) \, \cos\omega\tau$$

$$IR^2 - RI^2 = 2 \, S_R S_I \rho(\tau) \, \sin\omega\tau$$

and $S_R S_I$ and $\omega\tau$ may be found trigonometrically. $S_R S_I$ may be adopted as the measure of scattered power. The third equation may be used as a check upon the accuracy with which the phase-quadrature channels have been adjusted.