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CONJUGATE GRADIENT COUPLED WITH MULTIGRID FOR AN INDEFINITE PROBLEM

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Abstract

An iterative algorithm for the Helmholtz equation is presented. This scheme is based on the preconditioned conjugate gradient method for the normal equations. The preconditioning is one cycle of a multigrid method for the discrete Laplacian. The smoothing algorithm is red-black Gauss-Seidel and is constructed so it is a symmetric operator. The total number of iterations needed by the algorithm is independent of h. By varying the number of grids, the number of iterations depends only weakly on k when k^3h^2 is constant. Comparisons with a SSOR preconditioner are presented.

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Introduction

We consider the Helmholtz equation

$$\Delta u + k^2 n^2(x,y)u = f(x,y) \tag{1.1}$$

in a bounded two-dimensional region. In practice the region is unbounded in at least one direction. The unbounded region is then truncated with an artificial surface and a radiation boundary condition is imposed. In this paper we shall only consider Dirichlet and Neumann boundary conditions. Extensions to nonself-adjoint problems with local and nonlocal radiation boundary conditions verify the conclusions of this study. These results will be presented separately. The basic iterative method is presented in Bayliss, Goldstein, and Turkel [1]. There it is shown that a preconditioned conjugate gradient method is efficient for indefinite problem even when they are not self-adjoint.

A discretization of (1.1) produces a large linear system of equations to be solved

$$Au = b. \tag{1.2}$$

u is an approximation to the solution of (1.1) while b is determined by inhomogeneous data and boundary conditions. A is difficult to invert since it is not self-adjoint when radiation boundary conditions are used. In addition, the symmetric part of A is indefinite. Gaussian elimination is frequently used to solve (1.2). However, due to the large amount of storage needed, this method is limited to small k. As k increases, the solution becomes more oscillatory and more grid points are needed. In some simple

cases one requires k^3h^2 to be constant to achieve a fixed accuracy. In other cases the relationship of k to h can be worse than this [2].

To overcome this limitation an iterative method was proposed in [1]. The scheme is based on conjugate gradient for the normal equations. However, the normal equations have a condition number $O(h^{-4})$ for fixed k, and so the iterations will converge slowly. We therefore precondition the matrix A by a partial inverse to the discrete Laplacian. Hence, we solve

$$A'^* A' x' = A'^* b' \tag{1.3}$$

where

$$A' = Q^{-1} A Q^{-1*}, \qquad x' = Q^T x, \; b' = Q^{-1} b \tag{1.4}$$

and $M^{-1} = Q^{-1*} Q^{-1}$ is a partial inverse of A_0 where $A = A_0 + k^2 I$. Thus, A_0 is obtained by setting $k = 0$ in both the internal equations and the boundary conditions. We wish to choose M^{-1}, so that the condition number of $A'^* A'$ is substantially reduced and M^{-1} can be easily calculated.

In [1], Q^{-1} is constructed of one sweep of SOR. Hence, M^{-1} is point SSOR applied to the Laplacian. In [1] it was also suggested that a more efficient preconditioning might be obtained from one cycle of multigrid. Since the multigrid algorithm requires only $O(n)$ operations, it is expected that the preconditioned problem has a convergence rate independent of h. A red-black Gauss-Seidel relaxation scheme is used since it is an efficient method for the Laplacian [4]. Furthermore, it can lead to a symmetric preconditioner.

2. Iterative Algorithm

The preconditioned conjugate gradient algorithm is described in detail in [1]. For this algorithm we need to compute A times a vector and A^* times a vector. In addition we need to evaluate M^{-1} acting on a vector where M^{-1} represents a multigrid cycle. We first briefly review the basic multigrid algorithm [3].

Given

$$L\,U = f \quad \text{in} \quad \Omega, \qquad B\,U = g \quad \text{on} \quad \partial\Omega, \tag{2.1}$$

we approximate (2.1) on the finest grid by

$$L^m\,U^m = F^m \quad \text{in} \quad G^m, \qquad B^m\,U^m = g^m \quad \text{on} \quad \partial G^m. \tag{2.2}$$

The superscript m denotes an approximation on a grid with mesh h_m. We also use auxiliary grids $G^1,\cdots,G^{m-1}$ with $h_{i+1}/h_i = 1/2$. We denote by capital letters, the exact solution and by lower case letters, their approximation. I_k^{k-1} denotes the weighting operator from G^k to G^{k-1} and I_{k-1}^k is the interpolation operator from G^{k-1} to G^k.

The multigrid cycle begins with a "fine-to-coarse" process which starts with one relaxation sweep on the finest grid. We then have u^m which is an approximation to U^m. We then transfer the residual

$$R^m = F^m - L^m\,u^m \tag{2.3}$$

to the next coarser grid using

$$R^{m-1} = I_m^{m-1} R^m. \tag{2.4}$$

On the grid G^{m-1} we solve

$$L^{m-1} V^{m-1} = R^{m-1}, \tag{2.5}$$

where V^{m-1} is an approximation to the fine grid error

$$V^m = U^m - u^m. \tag{2.6}$$

This process is repeated to grids G^k until we get to the coarsest grid. The size of this coarsest grid is arbitrary and will be varied in the result section.

On the coarsest grid we use ℓ relaxation sweeps rather than the one relaxation sweep used on the finer grids. To return to the finer grid we begin by using ℓ relaxation sweeps on the coarsest grid. The "coarse-to-fine" process starts by calculating v^2_{new}, which is a new approximation to V^2, by

$$v^2_{new} = v^2_{old} + I_1^2 v^1. \tag{2.7}$$

We then do one relaxation sweep to improve v^2_{new}. This improved approximation is the correction to the next finer grid. This process is continued until we reach the finest grid, G^m. This process consists of one cycle of multigrid. When used as a iteration process, this cycle is repeated until convergence is reached. Since we use multigrid only as a preconditioner we shall only use one cycle of multigrid.

In the basic scheme (see e.g., [5]) Gauss-Seidel relaxation is used. I_k^{k-1} is injection from G^k to G^{k-1} while I_{k-1}^k is bilinear interpolation. For use as a preconditioning we require that the multigrid cycle be symmetric and positive definite, a characteristic which is not usually needed. Assuming a zero initial guess, the cycle is symmetric if all operations in the "fine-to-coarse" process are the adjoint of the operations in the "coarse-to-fine" process. Specifically,

(1) The fine-to-coarse relaxation must be the adjoint of the coarse-to-fine relaxation.

(2) The coarse-to-fine interpolation must be the adjoint of the fine-to-coarse residual weighting.

(3) A symmetric operation must be performed on the coarsest grid.

For the Laplace equation an efficient relaxation is Gauss-Seidel with red-black (RB) ordering [4]. It follows from property (1) that if RB is used when going to coarser grids, then BR must be used when returning to the finer grids. With this relaxation, full weighting in the fine-to-coarse residual transfer is preferable [5]. This agrees with property (2) for bilinear interpolation I_{k-1}^k. The cycle is given schematically by Figure 1.

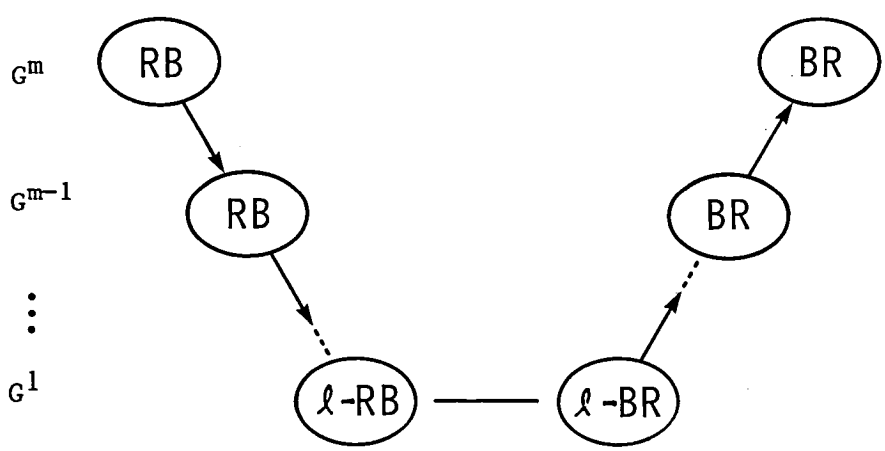

Figure 1

Dirichlet boundary conditions present no difficulties. For the Neumann boundary conditions we introduce fictitious points and use full weighting [5].

3. Numerical Results

The algorithm described above is valid for a general index of refraction $n(x,y)$. In this section we shall only consider constant coefficients. Consider

$$-\Delta u - k^2 u = 0 \qquad 0 \leq x,y \leq \pi \tag{3.1}$$

with boundary conditions

$$\begin{aligned} u_x(0,y) &= f(y) & u_x(\pi,y) &= g(y) \\ u_y(x,0) &= 0 & u(x,\pi) &= 0. \end{aligned} \tag{3.2}$$

The analytic solution is given by

$$u(x,y) = \cos\left(\frac{y}{2}\right)\left[\cos\left(k_x(x-\pi)\right) - 2\cos(k_x x)\right], \tag{3.3}$$

with $k^2 = k_x^2 + 1/4$. $f(y)$, $g(y)$ in (3.2) are given by those appropriate for (3.3). We use a uniform grid and (3.1) is approximated by

$$\frac{u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j}}{h^2} + k^2 u_{i,j} = 0. \tag{3.4}$$

In Table I we present results using SSOR as the M^{-1} preconditioner. We iterate until the residual is less than 10^{-6}. In Table II we present the results using one cycle of multigrid as the preconditioner. In this case we cycled down until we reached a 2×2 grid. We see that the number of cycles increases with finer mesh when SSOR preconditioning is used. Using multigrid, the number of iterations is independent of h, for fixed k. On the other hand, when h is fixed and k increases the number of iterations required by the multigrid preconditioner grows faster than that for the SSOR preconditioning. For a realistic problem, k and h are not independent. The exact relationship between k and h to achieve a given accuracy depends on the geometry of the region, boundary conditions, and distance to eigenvalues [2]. A typical case requires k^3h^2 to be constant for a fixed error tolerance. Hence, we wish to compare Tables I and II for this relationship.

We first consider the effect of choosing the coarsest mesh to be finer than the 2×2 grid. In Table III we give the number of iterations for a given number of levels. Here one-level corresponds to 2ℓ sweeps of Gauss-Seidel with red-black ordering without multigrid and level = 2 is a two-level algorithm. In all cases, 4 relaxation sweeps RB followed by 4 BR, are used on the coarsest grid. We see that as k increases, it pays to use fewer levels. As we approach the Laplace equation, $k = 0$, we prefer to use all the possible levels. In other tests we varied the number of relaxation sweeps on the coarsest grid. As k increases, one should use fewer sweeps on the coarsest grid. We see from Table III that for k^3h^2 fixed, the increase in iterations is a slowly increasing function of k. This growth is much slower than linear in k. However, when using a radiation condition, the growth seems to be closer to linear with k. Hence, this is much more efficient than SSOR.

The model problem considered is self-adjoint though indefinite. A. Bayliss (private communication) has done additional cases for a waveguide with a nonlocal radiation boundary condition which is no longer self-adjoint. Similar results were found which again show a slow growth with k when $k^3 h^2$ is fixed.

4. Conclusions

A preconditioned conjugate gradient algorithm is presented for solving the Helmholtz equation. The preconditioning is based on one cycle of multigrid for the Laplace equation. The number of levels used is a function of k and h. As k increases or coarser meshes are used, few levels should be used. In this case one should also use fewer sweeps on the coarsest grid. Conversely, as k decreases or h gets smaller, one should use more levels and more relaxation sweeps on the coarsest grid. When the optimal choice of levels is made the number of iterations is a slowly (less than linear) growing function of k when k^3h^2 is fixed.

Table I. SSOR Preconditioning

N	8	16	32	64
k				
0.69	27	52	108	240
1.39	31	61	125	272
2.77		69	145	333
4.16			166	369

Table II. Multigrid (to 2×2 grid) Preconditioning

N	8	16	32	64
k				
0.69	9	10	10	10
1.39	12	14	16	16
2.77		33	34	31
4.16			80	79

**Table III. Multigrid (to optimal, in computer time, coarsest grid) Preconditioning.
In parenthesis is given the number of grids used**

N	32	64	96
k			
0.69	10 (5)	10 (6)	11 (6)
1.39	12 (3)	12 (4)	15 (4)
2.77	15 (2)	17 (3)	19 (4)
4.16	22 (1)	24 (2)	21 (3)
5.56	21 (1)	22 (2)	29 (3)

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References

[1] A. Bayliss, C. I. Goldstein, E. Turkel, "An Iterative Method for the Helmholtz Equation," J. Comput. Phys., Vol. 48, 1983, pp. 443-457.

[2] A. Bayliss, C. I. Goldstein, E. Turkel, "On Accuracy Conditions for the Numerical Computation of Waves," to appear J. Comput. Phys., 1985.

[3] A. Brandt, "Multi-level Adaptive Solutions to Boundary Value Problems," Math. Comput., Vol. 31, 1977, pp. 333-390.

[4] A. Brandt, "Multigrid Solvers on Parallel Computers," Elliptic Problem Solvers (M. Schultz, ed.), Academic Press, 1981, pp. 39-84.

[5] A. Brandt, Guide to Multigrid Development, Springer-Verlag Lecture Notes in Mathematics, Vol. 960, 1981, pp. 280.

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