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Abstract

In this paper we show that the total variation diminishing (TVD) finite difference scheme which was analysed by Sweby [6] can be interpreted as a Lax-Wendroff scheme plus an upwind weighted artificial dissipation term. We then show that if we choose a particular flux limiter and remove the requirement for upwind weighting, we obtain an artificial dissipation term which is based on the theory of TVD schemes, which does not contain any problem dependent parameters and which can be added to existing MacCormack method codes. Finally, we conduct numerical experiments to examine the performance of this new method.

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1. Introduction

A major result of recent research into numerical methods for the solution of systems of hyperbolic conservation laws has been the development of second order accurate total variation diminishing (TVD) finite difference schemes. These schemes have a number of attractive properties including the fact that they resolve discontinuous solutions well, they do not exhibit spurious oscillations and, under certain circumstances, they can be proven to converge.

Unfortunately, the apparent complexity of these schemes has thus far discouraged their widespread adoption. Instead, most applications progams use standard methods such as the MacCormack variant of the Lax-Wendroff method and add additional artificial dissipation to damp the spurious wiggles that occur near discontinuities. These dissipation terms are usually chosen in an ad hoc fashion and usually contain problem-dependent parameters which must be fine tuned before the method will work.

In this paper we attempt to remedy this situation. In particular, we interpret the TVD finite difference scheme which was analysed by Sweby [6] as a Lax-Wendroff scheme plus an upwind weighted artificial dissipation term. We then attempt to simplify this artificial dissipation term by removing the requirement that it be upwind weighted. The result is a dissipation term which is based on the theory of TVD schemes and which does not contain any free parameters.

An outline of this paper is as follows. In section 2 we briefly review the theory of TVD finite difference schemes. Using this theory, we derive an artificial dissipation term for scalar hyperbolic equations in section 3 and we examine its performance in numerical experiments. These results are extended to systems of hyperbolic equations in section 4 and additional numerical experiments are performed. Section 5 contains some closing remarks.

2. Total Variation Diminishing Finite Difference Schemes

We consider the initial value problem for a scalar conservation law. That is

$$
(2.1)\qquad \begin{aligned} & u_t + f(u)_x = u_t - a(u)u_x = 0, \quad a(u) = \frac{df}{du}(u), \quad t > 0 \\ & u(x,0) = u_0(x), \quad -\infty < x < \infty \end{aligned}
$$

where $u_0(x)$ is assumed to have bounded total variation. A weak solution to this problem has the following monotonicity properties.

(1) No new extrema in x may be created.

(2) The value of a local minimum is nondecreasing and the value of a local maximum in nonincreasing.

The total variation of the solution to (2.1) at time t is defined by the formula

$$
(2.2)\qquad TV\big(u(x,t)\big) = \sup \sum_k \left| u(x_{k+1},t) - u(x_k,t) \right|,
$$

where the supremum is taken over all partitions of the real line.

It follows from this monotonicity property that the total variation in x of $u(x,t)$ does not increase in t. That is

$$
(2.3)\qquad TV\big(u(t_2)\big) \leq TV\big(u(t_1)\big), \qquad \text{for all } t_2 \geq t_1.
$$

Much recent research has been devoted to the construction of finite difference schemes that satisfy a discrete version of equation (2.2). We briefly describe this work below.

Consider explicit finite difference schemes in conservation form which approximate (2.1) and which we denote by

$$U^{n+1} = L \cdot U^n. \tag{2.4}$$

A scheme is called <u>total variation diminishing</u> if

$$TV\left(U^{n+1}\right) = TV\left(L \cdot U^n\right) \leq TV\left(U^n\right). \tag{2.5}$$

In addition, a scheme is called <u>monotonicity preserving</u> if the finite difference operator L is monotonicity preserving; that is, U^n a monotone mesh function implies that $L \cdot U^n$ is also a monotone mesh function. The following result, presented without proof is due to Harten [3].

Theorem 2.1 (Harten). <u>A total variation diminishing scheme is monotonicity preserving.</u>

This says that TVD schemes will not produce spurious oscillations. This is their chief attraction.

Another reason why TVD schemes are attractive is that it is very useful to have a bound on the total variation of the solution when proving convergence of nonlinear difference schemes [cf. 4, 5]. Equation (2.5) provides such a bound but convergence proofs are beyond the scope of this paper.

The scheme (2.4) can be written in the form

$$U_k^{n+1} = U_k^n - C_{k-1/2}\,\Delta U_{k-1/2}^n + D_{k+1/2}\,\Delta U_{k+1/2}^n \tag{2.6}$$

where

$$\Delta U^n_{k+1/2} = U^n_{k+1} - U^n_k \tag{2.7}$$

and $C_{k-1/2}$ and $D_{k+1/2}$ are functions of U^n. Harten [3] proves the following Lemma which provides a sufficient condition for the scheme (2.6) to be total variation diminishing.

Lemma 2.2 (Harten). <u>If the coefficients</u> C <u>and</u> D <u>of equation (2.6) satisfy the inequalities</u>

$$\begin{aligned} &0 \le C_{k+1/2} \\ &0 \le D_{k+1/2} \\ &0 \le C_{k+1/2} + D_{k+1/2} \le 1 \end{aligned} \tag{2.8}$$

<u>then the scheme (2.6) is total variation diminishing.</u>

Later, we use this Lemma to prove that our scheme is TVD.

3. Scalar Equations

For ease of presentation, we first consider the scalar linear equation

$$u_t + au_x = 0, \quad a = \text{const.} > 0. \tag{3.1}$$

Sweby [6] considers the solution to this problem using a scheme of the form

(3.2) $$U_k^{n+1} = U_k^n - \nu\{1 + \tfrac{1}{2}(1-\nu)[\phi(r_k^+)/r_k^+ - \phi(r_{k-1}^+)]\}\Delta U_{k-\frac{1}{2}}^n$$

where

(3.3) $$\nu = \frac{a\Delta t}{\Delta t}, \quad r_k^+ = \frac{\Delta U_{k-\frac{1}{2}}^n}{\Delta U_{k+\frac{1}{2}}^n}$$

and $\phi(r_k)$ is the flux limiter. This is a scheme of the form (2.6) with

(3.4) $$C_{k-\frac{1}{2}} = \nu\{1 + \tfrac{1}{2}(1-\nu)[\phi(r_k^+)/r_k^+ - \phi(r_{k-1}^+)]\}$$

$$D_{k+\frac{1}{2}} = 0 \quad .$$

A sufficient condition for the scheme (3.2) to be TVD is that $\nu \leq 1$ and

(3.5) $$|\phi(r_k^+)/r_k^+ - \phi(r_{k-1}^+)| \leq 2.$$

Sweby also specifies that $\phi(r) > 0$ and that $\phi(r) = 0$ for $r \leq 0$. Under these additional restrictions the bound (3.5) becomes

(3.6) $$0 \leq \phi(r)/r, \quad \phi(r) \leq 2.$$

If $\phi(r) = 1$, scheme (3.2) reduces to the centered difference Lax-Wendroff method. If $\phi(r) = r$, scheme (3.2) reduces to the second order upwind Warming and Beam [8] method.

The region defined by (3.6) is shown in Figure 1 along with the limiters corresponding to the Lax-Wendroff and Warming-Beam methods. Since these schemes are known to produce spurious wiggles in solutions with steep gradients, it is not surprising that these schemes are not uniformly within the TVD region.

Since the Lax-Wendroff method does not require one to determine an upwind direction and since many production computer codes are based on the Lax-Wendroff method or its variants, we wish to examine the possibility of adding terms to these codes to obtain a TVD scheme.

If we add a term of the form

$$K^{+}_{k+1/2}(r^{+}_{k})\Delta U^{n}_{k+1/2} - K^{+}_{k-1/2}(r^{+}_{k-1})\Delta U^{n}_{k-1/2} \tag{3.7}$$

to the Lax-Wendroff scheme

$$U^{n+1}_{k} = U^{n}_{k} - \frac{\nu}{2}\left(\Delta U^{n}_{k+1/2} + \Delta U^{n}_{k-1/2}\right) + \frac{\nu^{2}}{2}\left(\Delta U^{n}_{k+1/2} - \Delta U^{n}_{k-1/2}\right) \tag{3.8}$$

and rearrange this to the form (3.2) we get

$$U^{n+1}_{k} = U^{n}_{k} - \left[\nu\{1 + 1/2(1-\nu)[1/r^{+}_{k} - 1]\} - \left\{\frac{K^{+}_{k+1/2}}{r^{+}_{k}} - K^{+}_{k-1/2}\right\}\right]\Delta U^{n}_{k-1/2}\,. \tag{3.9}$$

That is

$$\begin{aligned} C_{k-1/2} &= \nu\{1 + 1/2(1-\nu)[1/r^{+}_{k} - 1]\} - \left\{\frac{K^{+}_{k+1/2}}{r^{+}_{k}} - K^{+}_{k-1/2}\right\} \\ D_{k+1/2} &= 0. \end{aligned} \tag{3.10}$$

A comparison of (3.9) with (3.2) shows that we can obtain Sweby's scheme if we choose

$$(3.11) \qquad K^{+}_{k+1/2} = \frac{\nu}{2}(1-\nu)[1 - \phi(r^{+}_{k})].$$

Next we consider the equation

$$(3.12) \qquad u_t + au_x = 0, \qquad a = \text{const.} < 0.$$

For this problem Sweby's scheme takes the form

$$(3.13) \qquad U^{n+1}_{k} = U^{n}_{k} + \nu\{-1 + 1/2(1+\nu)[\phi(r^{-}_{k+1}) - \frac{\phi(r^{-}_{k})}{r^{-}_{k}}]\}\Delta U^{n}_{k+1/2}$$

where

$$r^{-}_{k} = \frac{\Delta U^{n}_{k+1/2}}{\Delta U^{n}_{k-1/2}}, \qquad \nu = \frac{a\Delta t}{\Delta x} < 0$$

and our modified Lax-Wendroff scheme takes the form

$$(3.14) \quad U^{n+1}_{k} = U^{n}_{k} + \nu[\{-1 + 1/2(1+\nu)[1 - \frac{1}{r^{-}_{k}}]\} + \{K^{-}_{k+1/2} - \frac{K^{-}_{k-1/2}}{r^{-}_{k}}\}]\Delta U^{n}_{k+1/2}.$$

This reduces to Sweby's scheme if we choose

$$(3.15) \qquad K^{-}_{k+1/2} = \frac{\nu(1+\nu)}{2}[\phi(r^{-}_{k+1}) - 1].$$

A comparison of equations (3.11) and (3.15) shows how the sign of the coefficient a changes the method. In particular, we see that as a changes sign, the definition of r changes and the form of the term involving the Courant number ν changes. We can combine these two cases into a Lax-Wendroff method with an upstream weighted artificial dissipation term as follows. Put the scheme into the form

$$
\begin{aligned}
U_k^{n+1} &= U_k^n - \frac{\nu}{2}\left(U_{k+1}^n - U_{k-1}^n\right) + \frac{\nu^2}{2}\left(U_{k+1}^n - 2U_k^n + U_{k-1}^n\right) \\
&+ \left[K_{k+1/2}^+(r_k^+) + K_{k+1/2}^-(r_{k+1}^-)\right]\left(U_{k+1}^n - U_k^n\right) \\
&- \left[K_{k-1/2}^+(r_{k-1}^+) + K_{k-1/2}^-(r_k^-)\right]\left(U_k^n - U_{k-1}^n\right)
\end{aligned} \tag{3.16}
$$

where

$$
K_{k+1/2}^+ = \begin{cases} \frac{\nu}{2}(1-\nu)\left[1 - \phi(r_k^+)\right] & , \quad \text{if} \quad a > 0 \\ 0 & , \quad \text{if} \quad a \le 0 \end{cases}
$$

(3.17)

$$
K_{k+1/2}^- = \begin{cases} 0 & , \quad \text{if} \quad a \ge 0 \\ \frac{\nu}{2}(1+\nu)\left[\phi(r_{k+1}^-) - 1\right] & , \quad \text{if} \quad a < 0. \end{cases}
$$

This method still requires that we know which direction is upwind. For hyperbolic systems it is this requirement that makes upwind difference schemes complicated. Therefore we attempt to construct a method without upstream weighting by rewriting equation (3.17) as follows

$$K^{+}_{k+1/2} = \frac{|\nu|}{2}(1-|\nu|)[1 - \phi(r^{+}_{k})] \tag{3.18}$$

$$K^{-}_{k+1/2} = \frac{|\nu|}{2}(1-|\nu|)[1 - \phi(r^{-}_{k+1})].$$

It is obvious that the terms involving the K's are dissipative. The question that needs to be adressed is how much more dissipative (3.18) is than (3.17). To obtain some idea about this we compute the solution to the equation

$$u_t + u_x = 0 \tag{3.19}$$

with square wave initial data. We use the MacCormack version of the Lax-Wendroff method and the limiter defined by

$$\phi(r) = \begin{cases} \min(2r,1), & \text{if } r > 0 \\ 0 \quad , & \text{if } r \leq 0. \end{cases} \tag{3.20}$$

Figure 2a shows the result of this computation after 100 steps at a Courant number, $\nu = .9$ using the MacCormack method without additional dissipation. Note that the solution exhibits severe oscillations in those regions where r^{+} is small as predicted by Figure 1. Figure 2b is the result of the same computation using the MacCormack scheme with the upwind dissipation (3.17) and the flux limiter (3.20). Notice that the spurious oscillations have been removed. Finally, Figure 2c shows the result of the same computation using the MacCormack scheme with the simplified dissipation (3.18) and the flux limiter (3.20). These results are almost indistinguishable from those of Figure 2b. In addition we can prove the following:

Theorem 3.1. <u>The method (3.16), (3.18) with flux limiter (3.20) is TVD under the restriction that the Courant number</u> $|\nu| < 1$.

<u>Proof.</u> The proof is a direct appliction of Lemma 2.2. That is, we show that if the scheme is put into the form

$$U_k^{n+1} = U_k^n - C_{k-1/2}\,\Delta U_{k-1/2}^n + D_{k+1/2}\,\Delta U_{k+1/2}^n \tag{3.21}$$

then

$$C_{k+1/2} \geq 0 \tag{3.22}$$

$$D_{k+1/2} \geq 0 \tag{3.23}$$

$$0 \leq C_{k+1/2} + D_{k+1/2} < 1. \tag{3.24}$$

Here we prove the result for $\nu > 0$. The computations for $\nu < 0$ are similar.

Rearrange (3.16) as follows

$$\begin{aligned} U_k^{n+1} = U_k^n &- \Big[\nu\{1 + 1/2\,(1-\nu)(\frac{1}{r_k^+} - 1)\} \\ &- \{\frac{K_{k+1/2}^+}{r_k^+} - K_{k-1/2}^+ - K_{k-1/2}^-\}\Big]\Delta U_{k-1/2}^n + K_{k+1/2}^-\,\Delta U_{k+1/2}^n . \end{aligned} \tag{3.25}$$

That is

(3.26) $$C_{k+1/2} = \nu\{1 + 1/2\,(1-\nu)(\frac{1}{r^+_{k+1}} \quad -1)\} - \{\frac{K^+_{k+3/2}}{r^+_{k+1}} - K^+_{K+1/2} - K^-_{k+1/2}\}$$

(3.27) $$D_{k+1/2} = K^-_{k+1/2}\,,$$

$D_{K+1/2} > 0$ since $K^-_{k+1/2} > 0$.

To show that $C_{k+1/2} > 0$, substitute (3.18) into (3.26) and rearrange terms to obtain

(3.28) $$C_{K+1/2} = \nu + \frac{\nu(1-\nu)}{2}\,[\frac{\phi(r^+_{k+1})}{r^+_{k+1}} - \phi(r^+_k) + 1 - \phi(r^-_{k+1})].$$

Note that (3.20) implies

(3.29) $$0 < \frac{\phi(r)}{r} < 2$$

(3.30) $$0 < \phi(r) < 1$$

so

$$C_{k+1/2} \geq \nu - \frac{\nu(1-\nu)}{2} = \frac{\nu}{2} + \frac{\nu^2}{2} > 0.$$

Since $D_{k+1/2} > 0$ and $C_{k+1/2} > 0$, $C_{k+1/2} + D_{k+1/2} > 0$. To prove that

$$C_{k+1/2} + D_{k+1/2} < 1,$$

substitute (3.18) into (3.27) and add to (3.28). The result is

(3.31) $$C_{k+1/2} + D_{k+1/2} = \nu + \frac{\nu(1-\nu)}{2}\,[\frac{\phi(r^+_{k+1})}{r^+_{k+1}} - \phi(r^+_k) + 2 - 2\phi(r^-_{k+1})].$$

(3.31) is ≤ 1, if the term in the square brackets is ≤ 2. To show this, we note that

$$r_{k+1}^{-} = \frac{1}{r_{k+1}^{+}} \tag{3.32}$$

and we consider four cases.

Case 1. $r_{k+1}^{+} < 0 \Rightarrow \frac{1}{r_{k+1}^{+}} < 0 \Rightarrow \phi(r_{k+1}^{+}) = \phi(\frac{1}{r_{k+1}^{+}}) = 0.$

In this case the term in square brackets becomes

$$2 - \phi(r_k^{+}) \leq 2.$$

Case 2. $2r_{k+1}^{+} < 1 \Rightarrow r_{k+1}^{+} < 1/2,\ \frac{1}{r_{k+1}^{+}} > 2$, so

$$\phi(r_{k+1}^{+}) = 2r_{k+1}^{+},\quad \phi(\frac{1}{r_{k+1}^{+}}) = 1$$

and the term in square brackets is

$$2 - \phi(r_k^{+}) \leq 2.$$

Case 3. $1/2 < r_{k+1}^{+},\ \frac{1}{r_{k+1}^{+}} < 2 \Rightarrow \phi(r_{k+1}^{+}) = \phi(\frac{1}{r_{k+1}^{+}}) = 1.$

Then the term in square brackets becomes

$$\frac{1}{r_{k+1}^{+}} - \phi(r_k^{+}) \leq 2 - \phi(r_k^{+}) \leq 2.$$

Case 4. $\quad 2 \le r^+_{k+1} \Rightarrow \dfrac{2}{r^+_{k+1}} < 1 \Rightarrow \phi(r^+_{k+1}) = 1,\ \phi\left(\dfrac{1}{r^+_{k+1}}\right) = \dfrac{2}{r^+_{k+1}}$.

Then the term in square brackets becomes

$$\frac{1}{r^+_{k+1}} - \phi(r^+_k) + 2 - \frac{4}{r^+_{k+1}} = 2 - \frac{3}{r^+_{k+1}} - \phi(r^+_k) \le 2.$$

This completes the proof.

It is trivial to extend these results to scalar nonlinear problems. We simply define a local wave speed by

$$a_{k+1/2} = \begin{cases} \dfrac{\Delta f_{k+1/2}}{\Delta U_{k+1/2}}, & \text{if } \Delta U_{k+1/2} \neq 0 \\[2ex] \dfrac{df(U_k)}{dU}, & \text{if } \Delta U_{k+1/2} = 0 \end{cases} \tag{3.33}$$

and apply the schemes (3.16), (3.17) or (3.16), (3.18) as before. The wave speed definition (3.33) makes the resulting scheme conservative.

Figures 3a,3b and 3c show computed solutions of the inviscid Burgers' equation with square wave initial data and periodic boundary conditions. These results were obtained using the MacCormack scheme, the second order upwind scheme (3.16), (3.17) and the simplified scheme (3.16), (3.18), respectively. The MacCormack results exhibit severe oscillations in the vicinity of the shock and an entropy violating expansion shock. The upwind scheme eliminates the oscillations but not the expansion shock while the simplified scheme eliminates all but a small "entropy glitch" in the expansion region. There is no difference between the upwind method and the simplified method in their ability to resolve the shock.

The Burgers' equation results were highly dependent on the choice of initial conditions. Indeed, for certain initial conditions, the simplified method also computed solutions containing expansion shocks. To our knowledge, the only way to avoid expansion shocks in all cases is to explicitly add extra dissipation to the method when a sonic point occurs in an expansion region. Although we cannot guarantee that the simplified scheme will do this automatically, our computations indicate that this scheme is more robust than the unmodified upwind scheme.

4. Hyperbolic Systems

In this section we extend the results of the previous section to hyperbolic systems. To that end, we consider first the linear, constant coefficient system

$$u_t + Au_x = 0, \quad A = \text{const.} \tag{4.1}$$

where u is an m vector and A is an $m \times m$ matrix.

If the system (4.1) is hyperbolic, the matrix A has real eigenvalues and a complete set of linearly independent right eigenvectors. If we let P denote the matrix whose columns are the right eigenvectors of A, then

$$P^{-1}AP = \Lambda \tag{4.2}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \lambda_m \end{bmatrix} \tag{4.3}$$

and λ_k are the eigenvalues of A.

If we define a new set of dependent variables by the formula

$$v = P^{-1}u \tag{4.4}$$

and multiply (4.1) by P^{-1} we obtain

$$(P^{-1}u)_t + P^{-1}AP^{-1}(u)_x = 0 \tag{4.5}$$

or

$$v_t + \Lambda v_x = 0. \tag{4.6}$$

This is an uncoupled set of scalar equations. We solve (4.6) using (3.16). That is

$$\begin{aligned} v_k^{n+1} &= v_k^n - \frac{\nu}{2}\left(v_{k+1}^n - v_{k-1}^n\right) + \frac{\nu^2}{2}\left(v_{k+1}^n - 2v_k^n + v_{k-1}^n\right) \\ &+ \left[K_{k+1/2}^{+}(r_k^{+}) + K_{k+1/2}^{-}(r_{k+1}^{-})\right]\left(v_{k+1}^n - v_k^n\right) \\ &- \left[K_{k-1/2}^{+}(r_{k-1}^{+}) + K_{k-1/2}^{-}(r_k^{-})\right]\left(v_k^n - v_{k-1}^n\right) \end{aligned} \tag{4.7}$$

where $\nu = \Lambda\Delta t/\Delta x$ and $K^{\pm}$ and $r^{\pm}$ are defined below.

Multiply (4.7) by P to obtain an equation in terms of the original dependent variables. The result is

$$U_k^{n+1} = U_k^n - A\frac{\Delta t}{2\Delta x}\left(U_{k+1}^n - U_{k-1}^n\right) + A^2\frac{\Delta t}{2\Delta x}\left(U_{k+1}^n - 2U_k^n + U_{k-1}^n\right)$$

$$+ P[K_{k+1/2}^{+}(r_k^{+}) + K_{k+1/2}^{-}(r_{k+1}^{-})]P^{-1}(U_{k+1}^n - U_k^n) \tag{4.8}$$

$$- P[K_{k-1/2}^{+}(r_{k-1}^{+}) + K_{k-1/2}^{-}(r_k^{-})]P^{-1}(U_k^n - U_{k-1}^n).$$

The first three terms on the right of (4.8) comprise the well-known Lax-Wendroff scheme and need no further discussion. The last two terms require that we know the matrices P and P^{-1} and that we know which direction is upwind. In the following we construct a simplified version of this scheme which removes these requirements.

We remove the requirement that P and P^{-1} be known by approximating the diagonal matrices $K^{\pm}$ by scalar matrices. That is, we let

$$K^{\pm}(r^{\pm}) \approx \overline{K}^{\pm}(r^{\pm})\ I \tag{4.9}$$

where the $\overline{K}^{\pm}(r^{\pm})$ are scalar functions of $r^{\pm}$.

We remove the necessity to determine upwind directions by choosing

$$\overline{K}^{\pm}(r^{\pm}) = .5\ C(\nu)\ [1 - \phi(r^{\pm})] \tag{4.10}$$

where the Courant number ν is defined as

$$\nu = \max_j |\lambda_j| \frac{\Delta t}{\Delta x} \tag{4.11}$$

and $C(\nu)$ is chosen as follows.

$$C(\nu) = \begin{cases} \nu(1-\nu), & \text{if} \quad \nu \leq .5 \\ .25, & \text{if} \quad \nu > .5 \ . \end{cases} \tag{4.12}$$

This definition of $C(\nu)$ is an upper bound to the Courant number-dependent coefficient in (3.18).

Thus far we have not defined r^+ and r^-. In light of the fact that we do not wish to compute P or P^{-1}, we have chosen the following definitions for r^+ and r^-.

$$r_k^+ = \frac{\left(\Delta U^n_{k-1/2} \, , \, \Delta U^n_{k+1/2}\right)}{\left(\Delta U^n_{k+1/2} \, , \, \Delta U^n_{k+1/2}\right)} \tag{4.13a}$$

$$r_k^- = \frac{\left(\Delta U^n_{k-1/2} \, , \, \Delta U^n_{k+1/2}\right)}{\left(\Delta U^n_{k-1/2} \, , \, \Delta U^n_{k-1/2}\right)} \tag{4.13b}$$

where $(\cdot,\cdot)$ denotes the usual inner product on R^m.

If P does not vary significantly over adjacent mesh intervals, these definitions can be interpreted as averages of the scalar definitions. These were the most simple definitions that we could construct. They worked so well in our numerical experiments that we saw no reason to investigate more sophisticated $r^\pm$ definitions. We note in passing that other definitions of $r^\pm$ have been proposed by Sweby [6] and Chakravarthy and Osher [2].

With these simplifications, our numerical method takes the form

$$U_k^{n+1} = U_k^n - A\frac{\Delta t}{2\Delta x}\left(U_{k+1}^n - U_{k-1}^n\right) + A^2\frac{\Delta t}{2\Delta x}\left(U_{k+1}^n - 2U_k^n + U_{k-1}^n\right)$$

$$(4.14) \qquad + \left[\overline{K}_{k+1/2}^{+}(r_k^{+}) + \overline{K}_{k+1/2}^{-}(r_{k+1}^{-})\right](U_{k+1}^n - U_k^n)$$

$$- \left[\overline{K}_{k-1/2}^{+}(r_{k-1}^{+}) + \overline{K}_{k-1/2}^{-}(r_k^{-})\right](U_k^n - U_{k-1}^n),$$

where $\overline{K}^{\pm}$ and $r^{\pm}$ are defined by equations (4.10) and (4.13) respectively. Note that the resulting scheme does not depend explicitly on the transformation (4.4). Therefore, we can use the scheme without modification on nonlinear problems where (4.5) is not true. For the computations which follow, we replace the Lax-Wendroff scheme (the first three terms on the right of (4.14)) by the conservative MacCormack scheme. These schemes are equivalent for linear problems.

As a first test, we demonstrate the performance of this method on the Riemann problem. That is, we solve the Euler equations

$$(4.15a) \qquad u_t + f(u)_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

where

$$(4.15b) \qquad u = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, \quad f(u) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}$$

$$(4.15c) \qquad p = (\gamma-1)(E - \tfrac{1}{2}\rho u^2)$$

and $\gamma = 1.4$ with initial conditions

$$u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases} \tag{4.15d}$$

In this case the initial conditions were

$$u_L = \begin{bmatrix} .445 \\ .311 \\ 8.928 \end{bmatrix}, \quad u_R = \begin{bmatrix} .5 \\ 0 \\ 1.4275 \end{bmatrix} \tag{4.16}$$

Figure 4 shows the solution after 100 time steps computed on 140 grid points at a Courant number of .95. These conditions are the same as those used by Harten [3] and Chakravarty and Osher [2] in their numerical experiments. The results shown in Figure 4 cannot be distinguished from those shown in the cited references with one exception. Harten was able to obtain a dramatic improvement in the resolution of the contact discontinuity by selectively adding artificial compression to his second order upwind scheme. We intend to study this technique in the future.

Next we demonstrate the performance of our method on a two dimensional problem. Figure 5 shows a comparison of the present method with the second order upwind scheme of van Leer [7]. To obtain these results, we solve the two dimensional Euler equations

$$u_t + f(u)_x + g(u)_y = 0 \tag{4.17a}$$

where

$$u = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \quad f(u) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{bmatrix}, \quad g(u) = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{bmatrix} \tag{4.17b}$$

(4.17c) $$p = (\gamma-1)\left[E - \tfrac{1}{2}\left(\rho u^2 + \rho v^2\right)\right]$$

and $\gamma = 1.4$ for the problem of the reflection of an oblique shock from a plane wall.

For the computations shown, we specify a uniform $M = 2.9$ flow at the left boundary. At the top boundary, we specify the conditions behind a shock that would turn the flow 11^o (cf. [1]). A flow tangency condition is specified at the wall and all variables are extrapolated at the right boundary. The computation is started with the upstream conditions specified everywhere except the top boundary. The results shown consist of a three dimensional plot of the converged density solution and a longitudinal section of this plot taken at $y = .5$. Once again it is difficult to distinguish between the results computed using the two methods.

Finally, we compare the present method to the second order upwind method of van Leer on a model transonic flow problem.

The two dimensional Euler equations (4.17) are solved for the flow over a 10% thick parabolic arc bump in a channel. The flow is assumed to be uniform initially and then the condition that the flow be tangent to the bump is applied at the wall in the manner of small disturbance theory. Nonreflecting boundary conditions are applied at both upstream and downstream boundaries. Figures 6a and 6b show the converged pressure distribution on the wall for a flow with inlet Mach number .675 using the van Leer method and the present method, respectively.

The flow consists of an expansion region over the forward part of the bump, $0 \leq x \leq .5$, followed by a shock near $x = .75$. This is the type of problem that the Van Leer method was designed to solve since the shock is steady and nearly aligned with the computing grid. Figure 6a shows that the van Leer scheme does an admirable job on this problem. In particular, the flow expands smoothly through the sonic point $(P \approx .717)$ and there is only one grid point within the shock. By comparison, Figure 6b shows that the flow computed using the present method is not quite as smooth in the expansion region and there are two points within the shock. Still, considering the fact that the current method is considerably easier to program and that it runs in 2/3 the time of the Van Leer scheme, the results shown are quite acceptable.

During the course of this work, we discovered that, for two dimensional problems, the flux limiter (3.20) imposes a severe Courant number stability restriction on the method beyond that of the two-dimensional MacCormack scheme. To prevent that we define a new flux limiter by the formula

$$\phi(r) = \min(2|r|, 1), \tag{4.18}$$

A simple application of the maximum principle shows that the artificial dissipation based on this limiter is stable for two dimensional problems under the same Courant number restriction as the two-dimensional MacCormack method. At this time we have not analysed the three dimensional case.

5. Concluding Remarks

In this paper we construct a simple artificial viscosity term for Lax-Wendroff type methods which is based on the theory of Total Variation Diminishing upwind finite difference schemes. This method has advantages over both conventional artificial viscosity schemes and the TVD upwind schemes. In particular, the method does not contain the problem dependent parameters of conventional artificial viscosity schemes and it does not require the complex logic of upwind schemes.

The numerical experiments that have been performed thus far have been very encouraging but more numerical experimentation is needed. We intend to carry this out in the future and also to apply the ideas of this paper to other numerical schemes.

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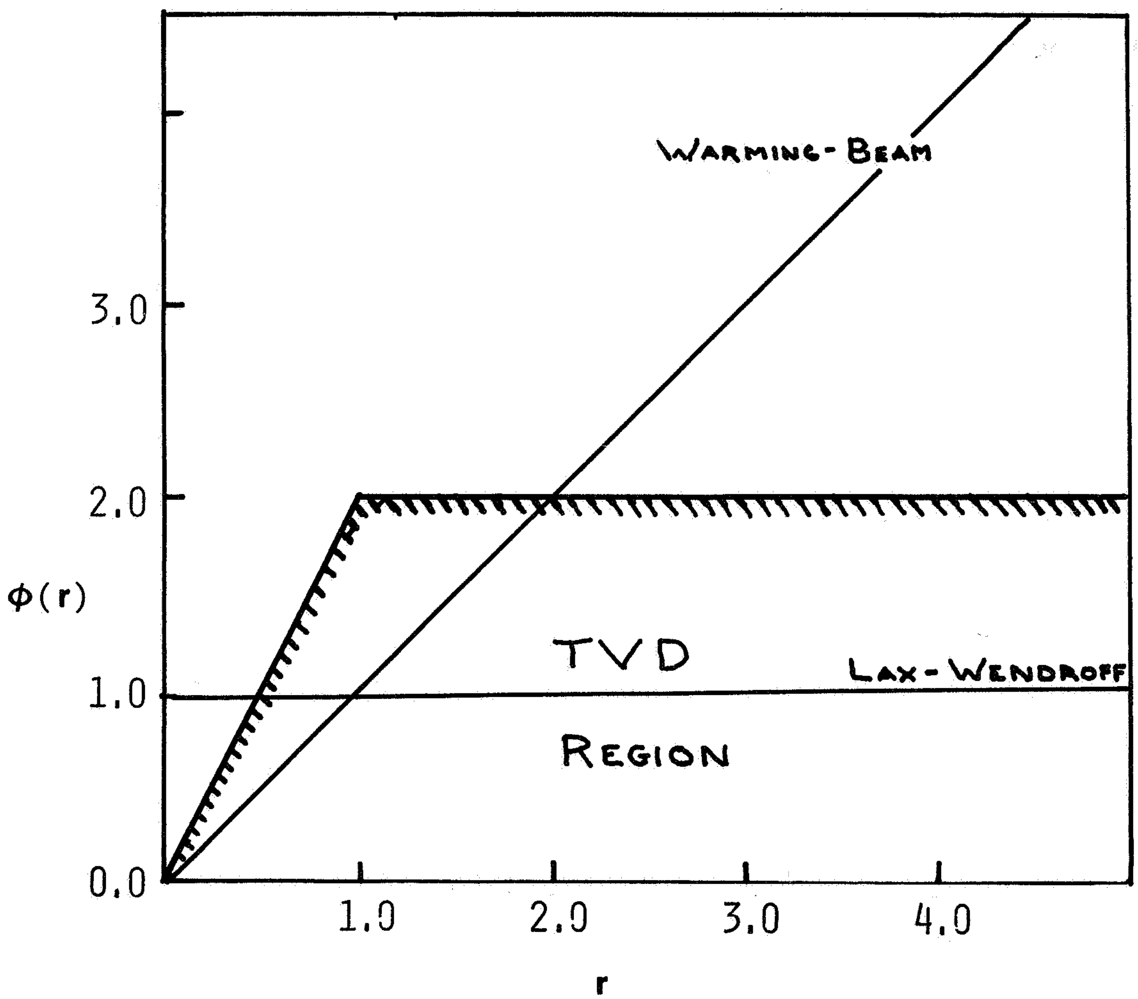

Figure 1. TVD Region

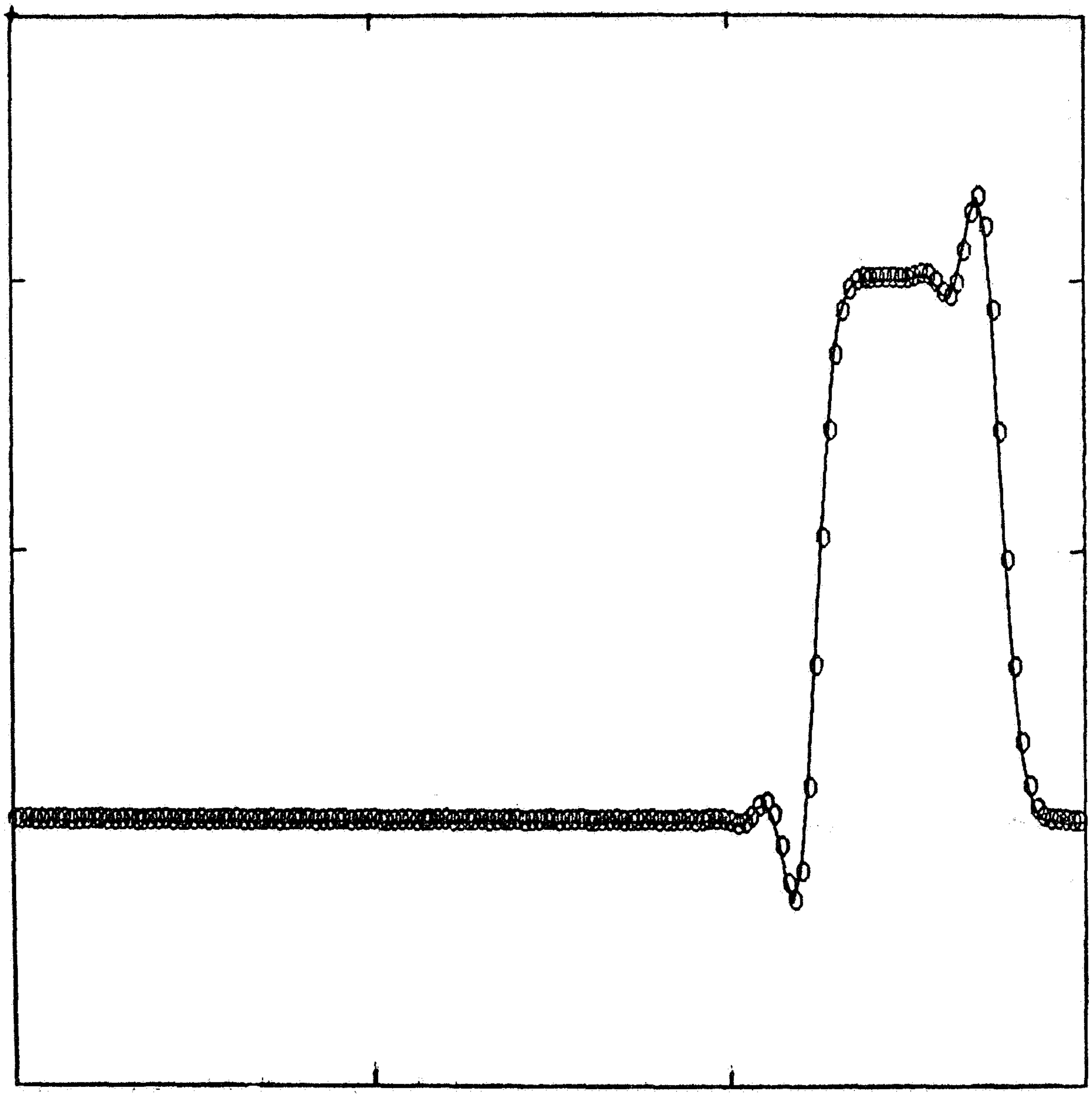

Figure 2a. Solution of (3.19) with square wave initial data after 100 steps at $\nu = .9$ using MacCormack method.

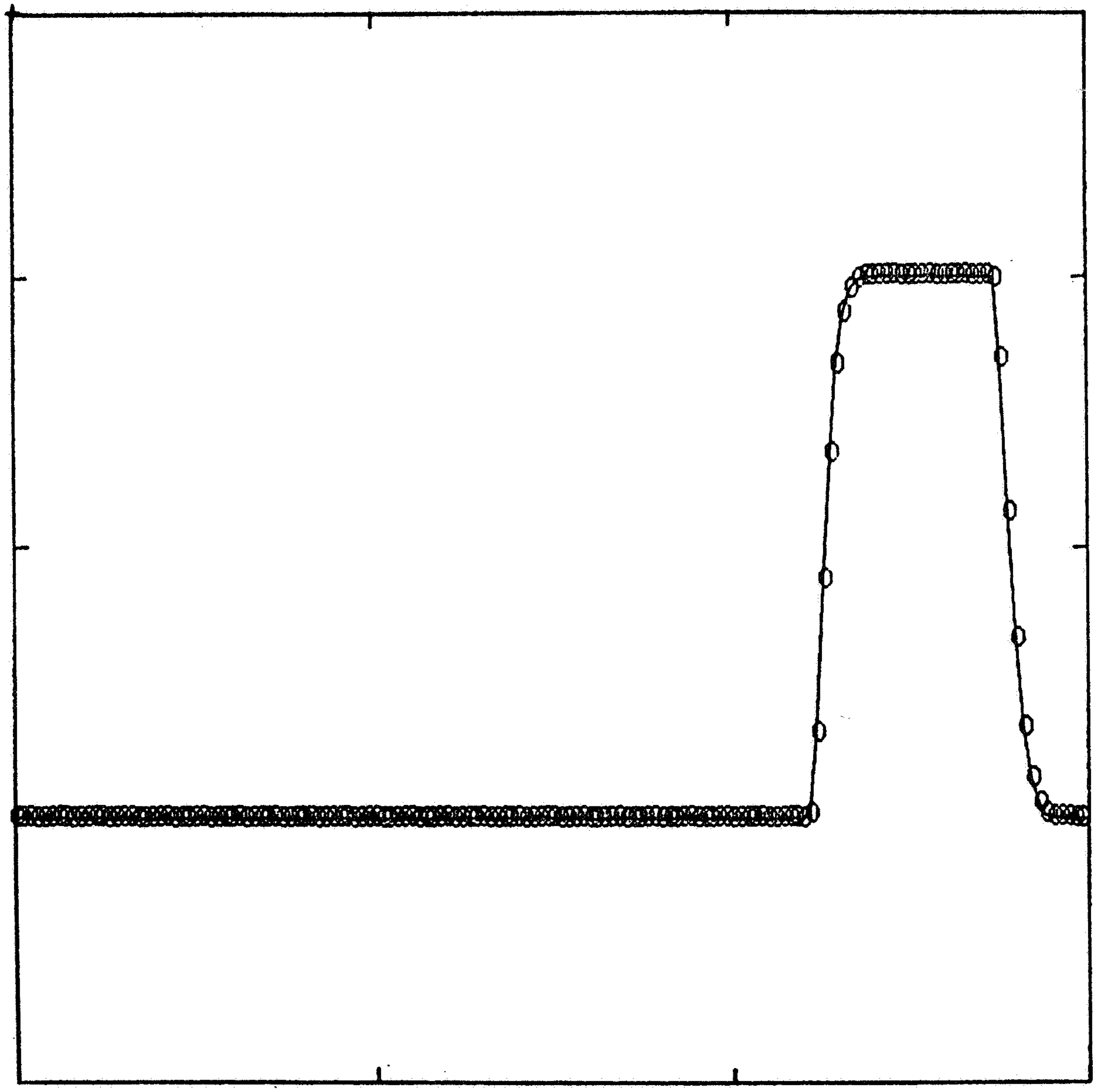

Figure 2b. Solution of (3.19) with square wave initial data after 100 steps at $\nu = .9$ using upwind scheme (3.16), (3.17).

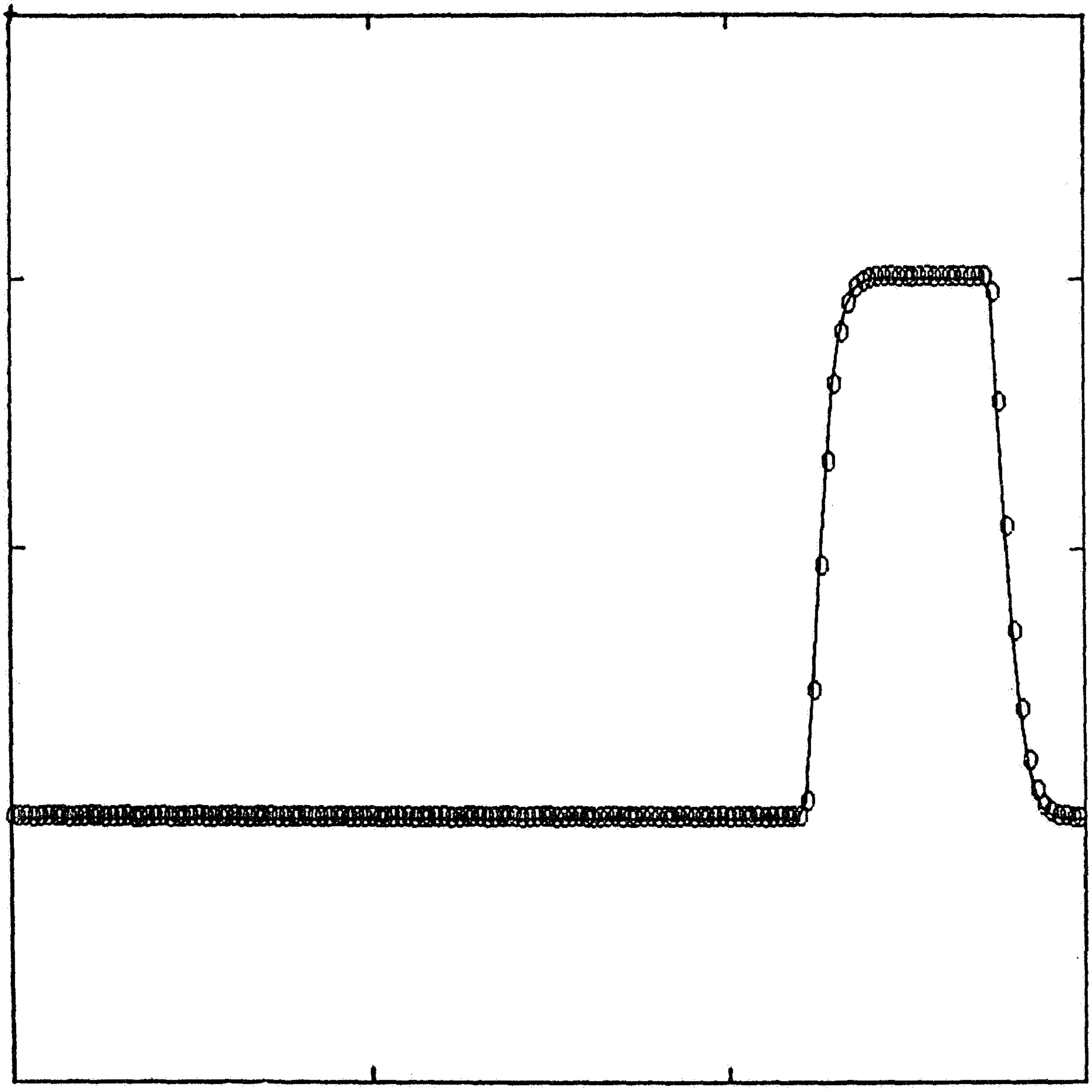

Figure 2c. Solution of (3.19) with square wave initial data after 100 time steps at $\nu = .9$ using TVD scheme (3.16), (3.18).

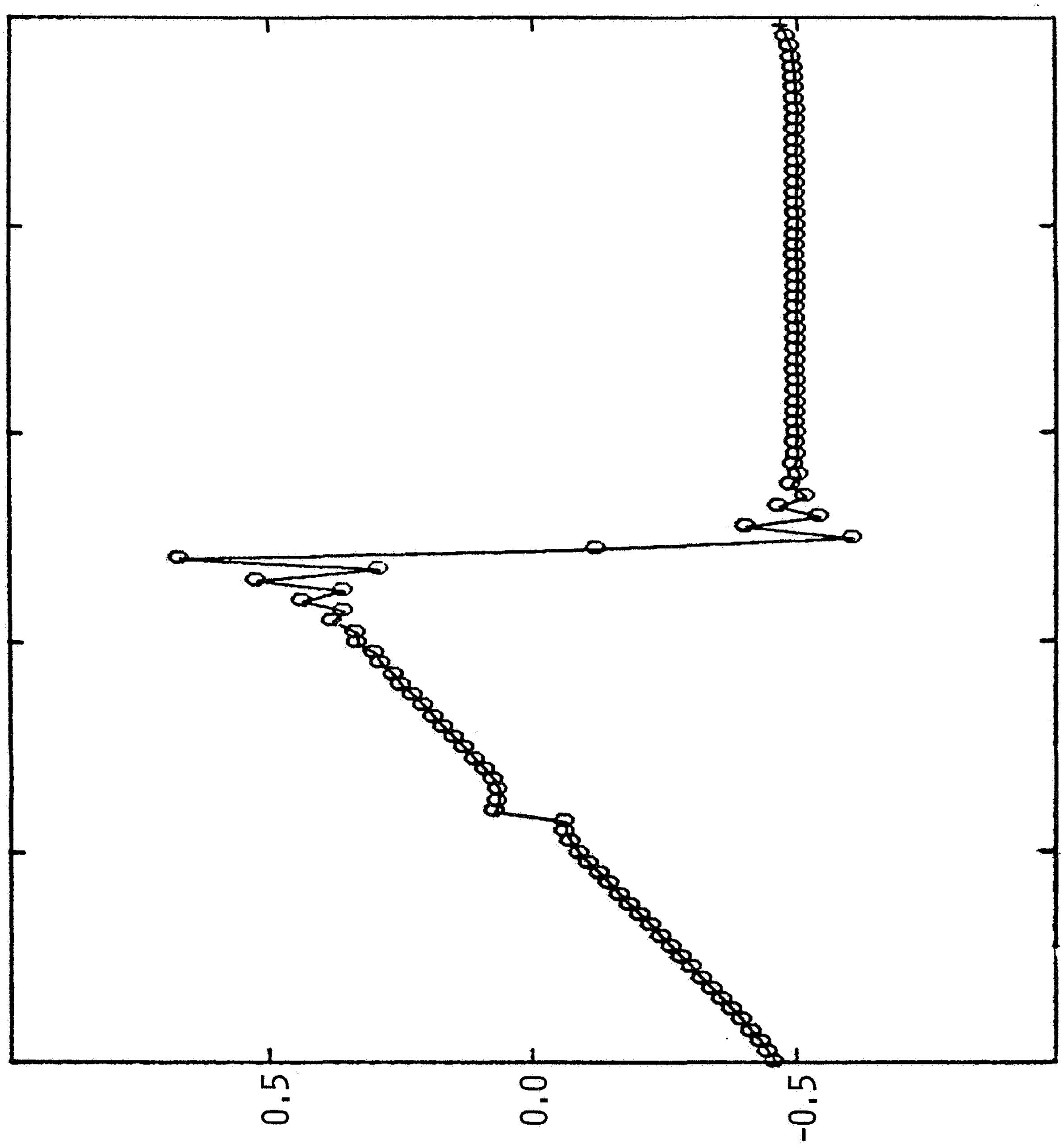

Figure 3a. Solution of Burgers' equation using MacCormack scheme.

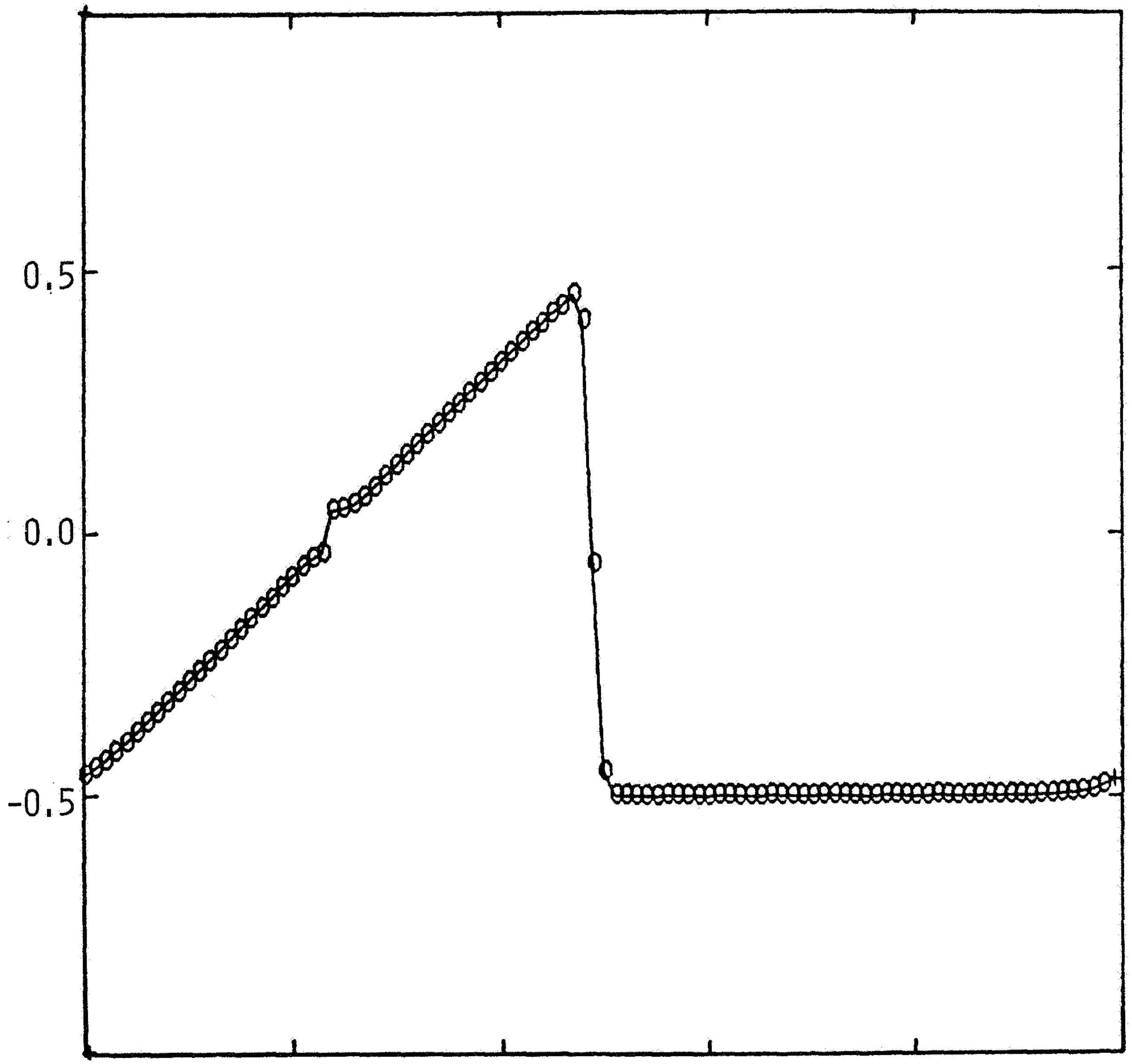

Figure 3b. Solution to Burgers' equation using upwind scheme. (3.16), (3.17)

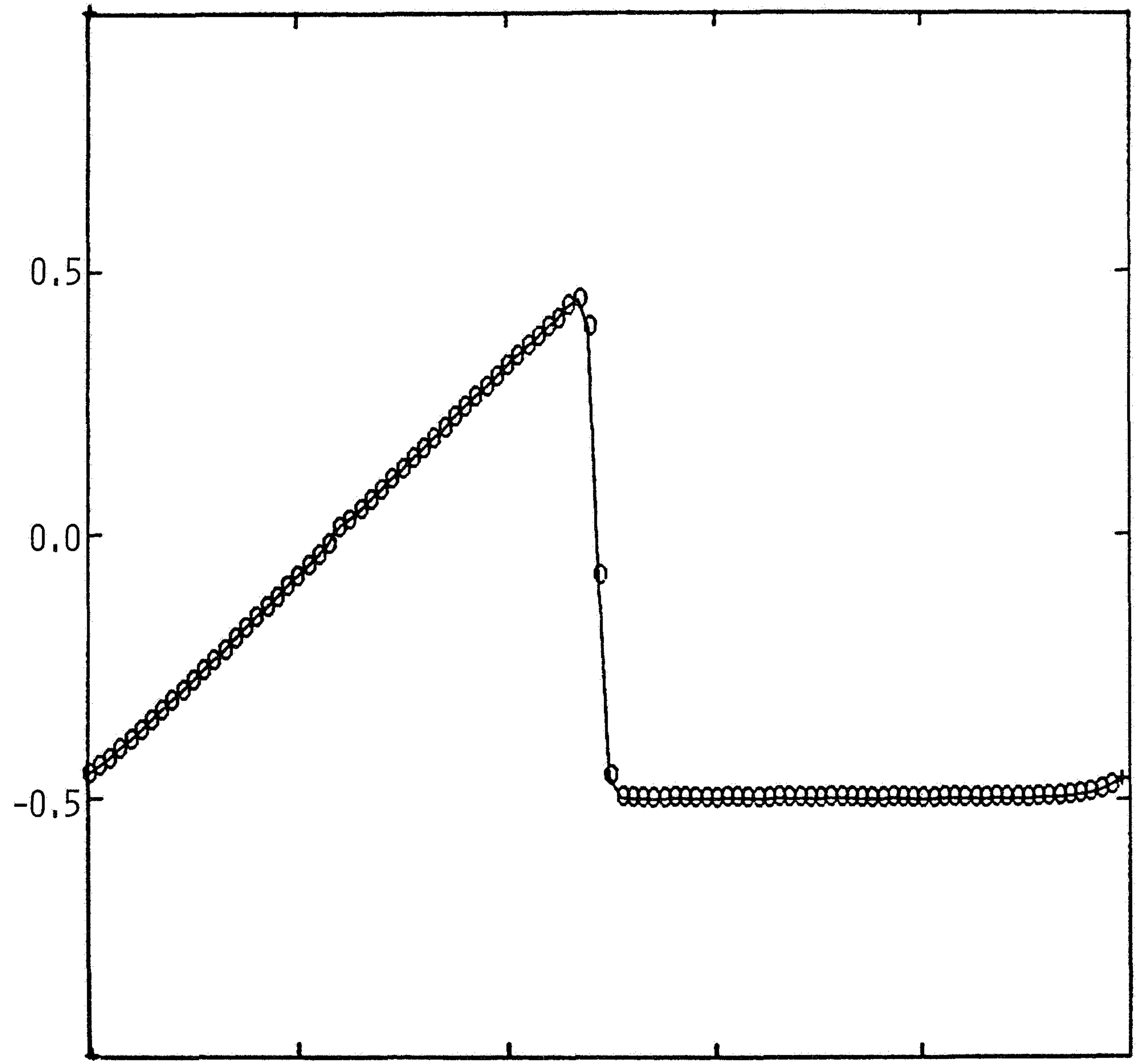

Figure 3c. Solution to Burgers' equation using TVD scheme (3.16), (3.18).

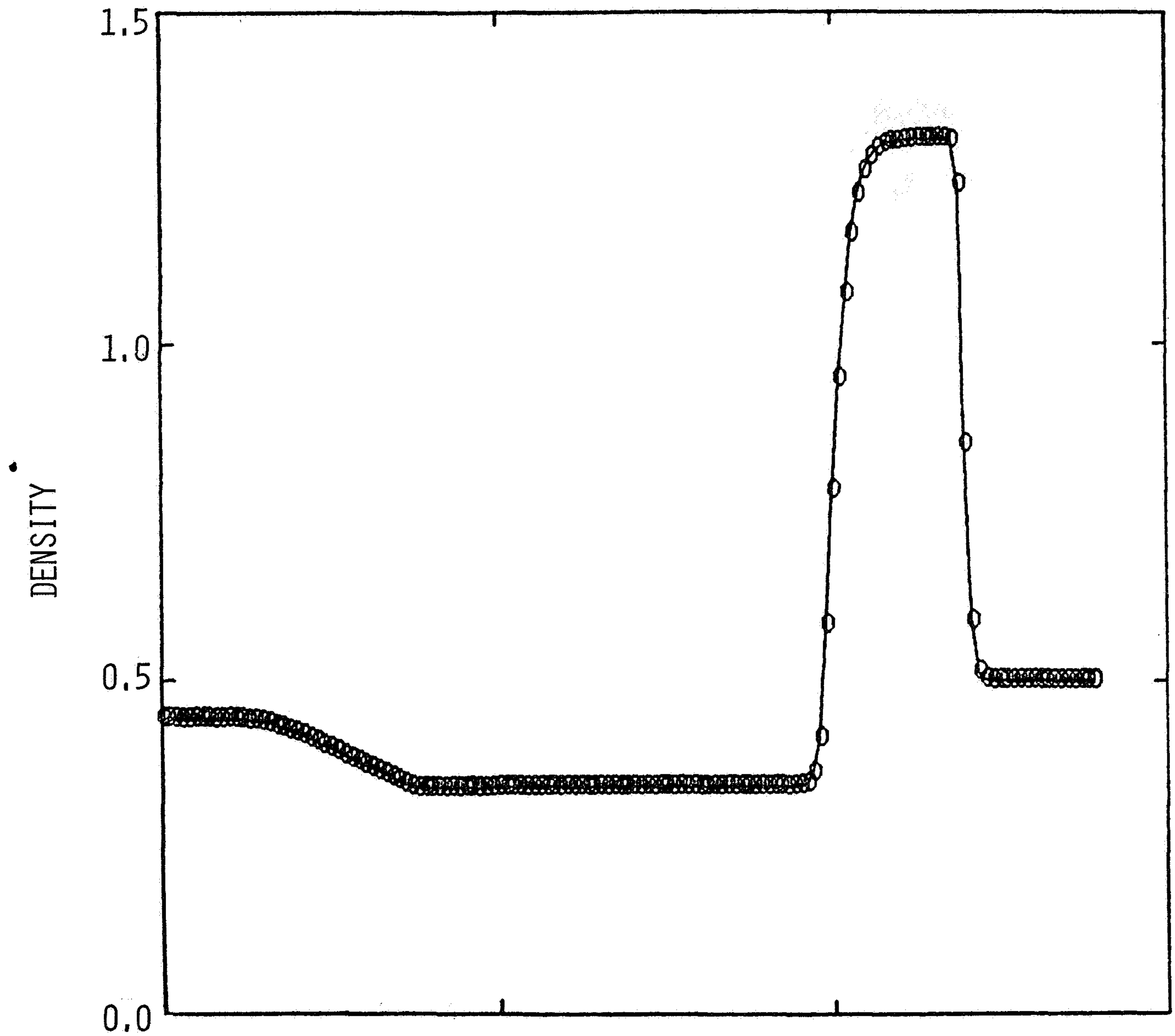

Figure 4. Solution to Riemann problem after 100 time steps at $\nu = .95$ using scheme (4.14), (4.10).

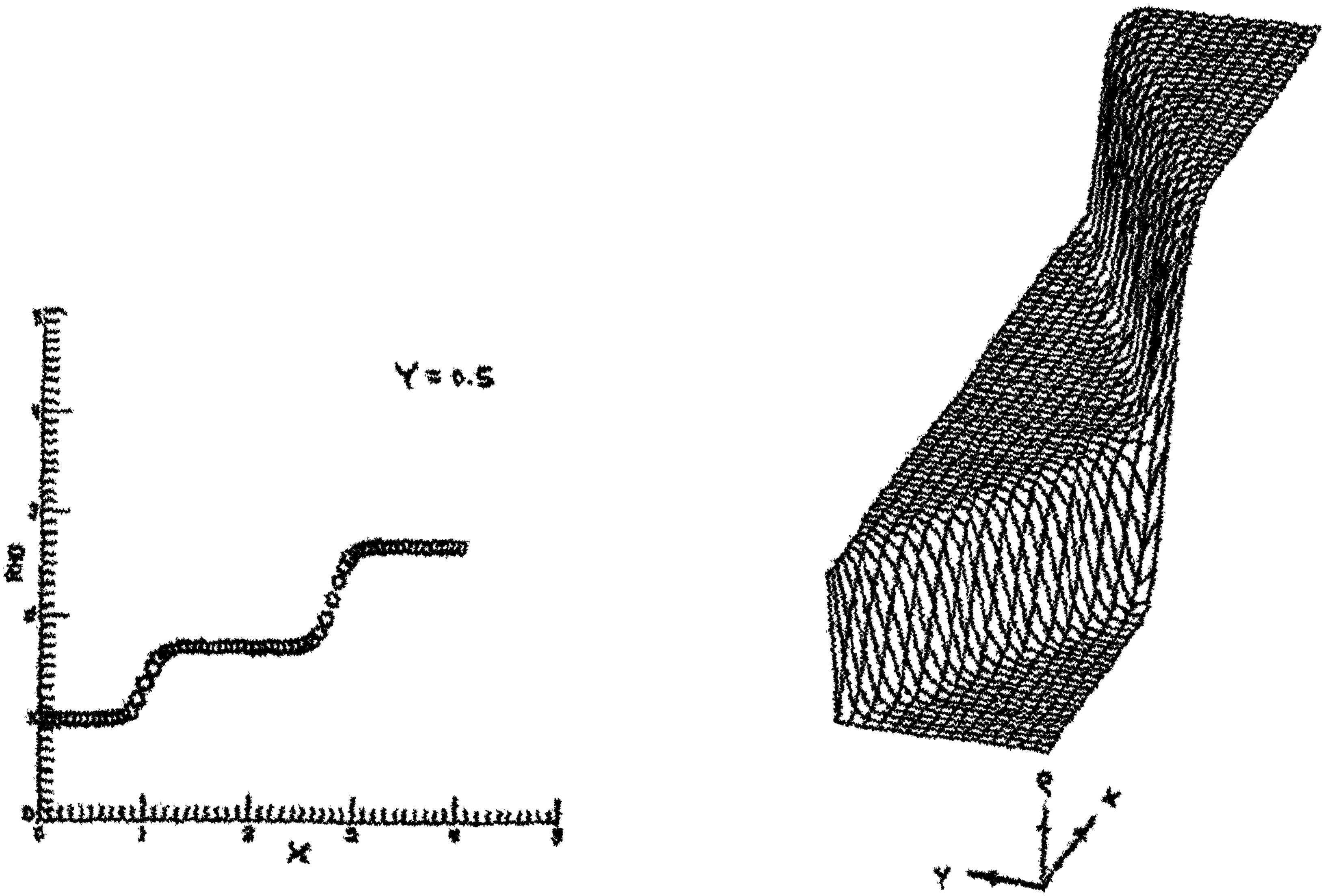

Figure 5a. Density profile for shock reflection problem using Van Leer method.

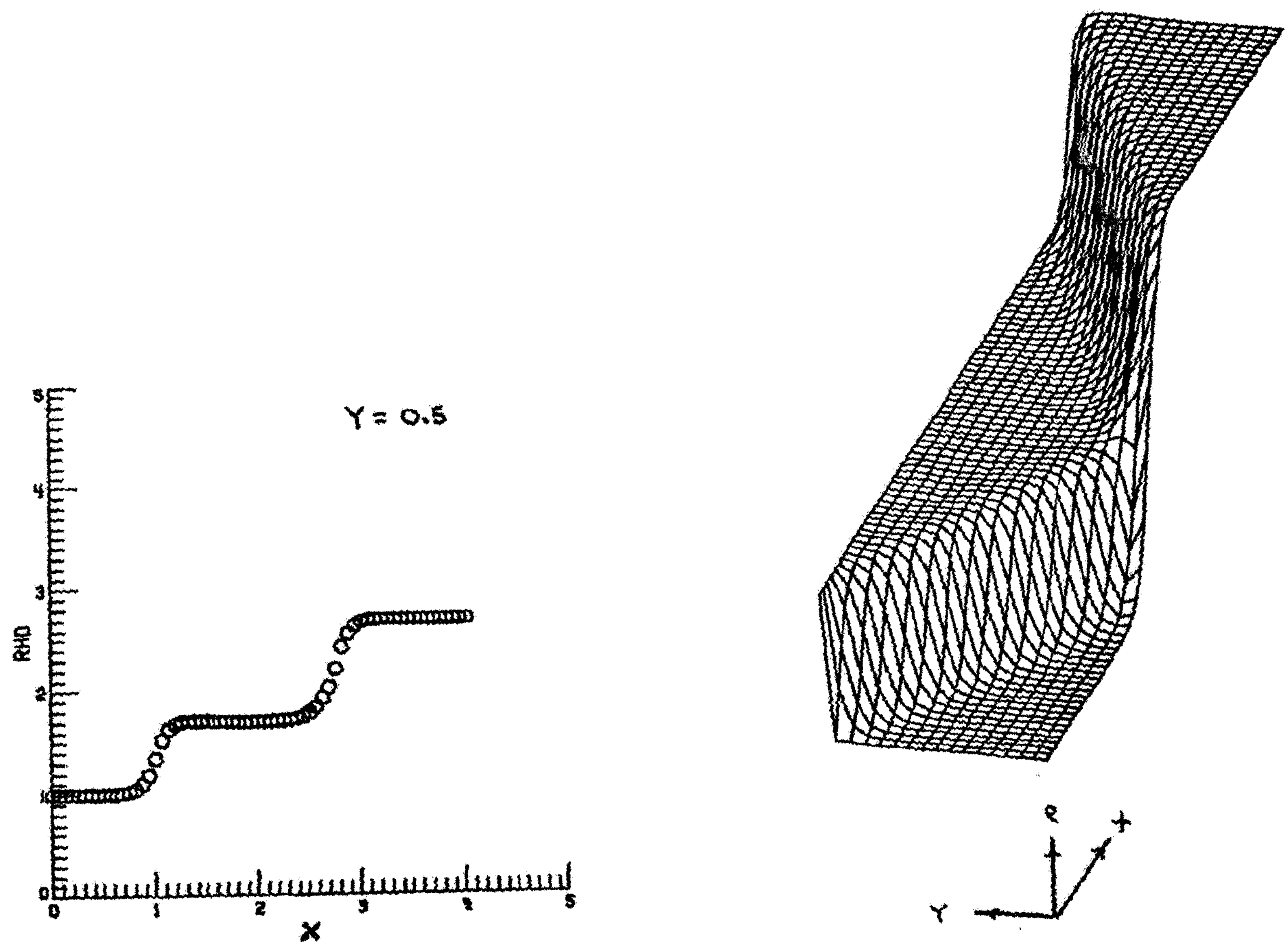

Figure 5b. Density profile for shock reflection problem using modified MacCormack scheme.

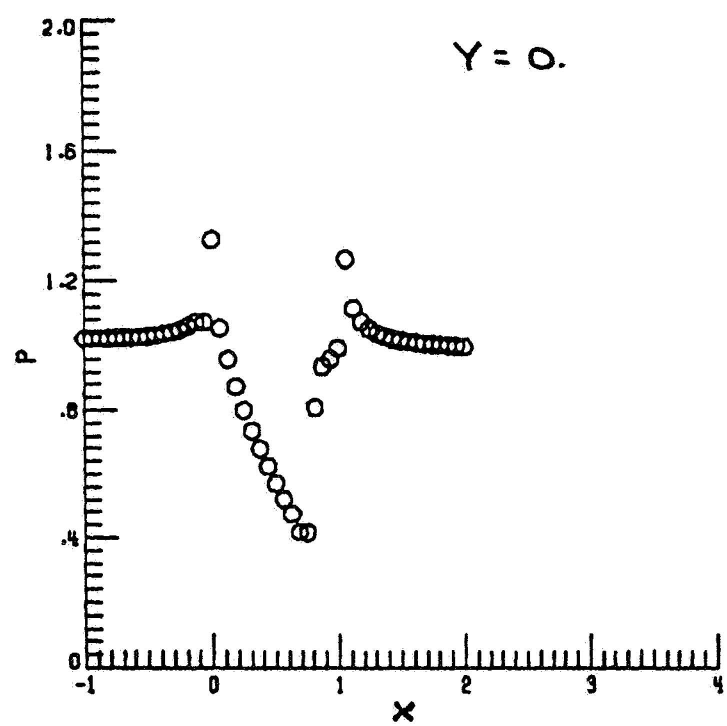

Figure 6a. Pressure distribution along wall for transonic problem using Van Leer scheme.

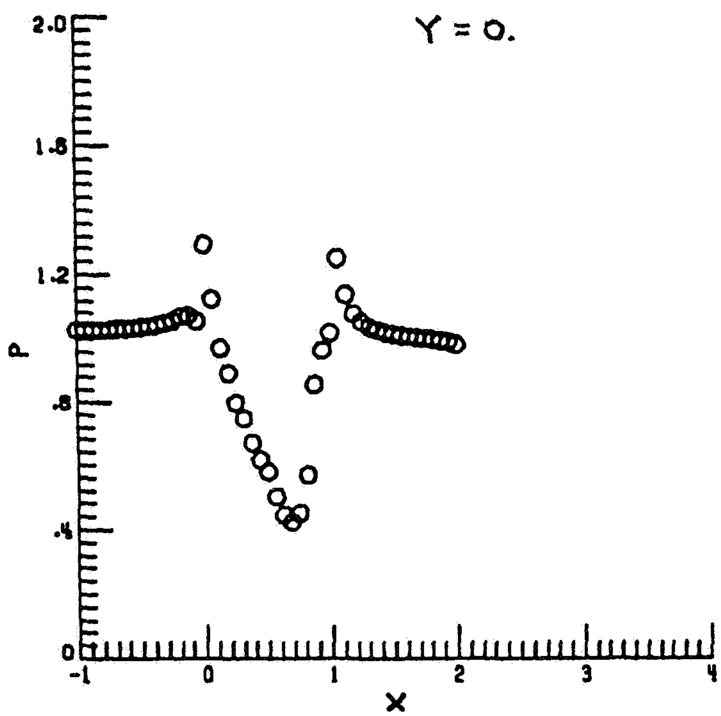

Figure 6b. Pressure distribution along wall for transonic problem using modified MacCormack scheme.

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16. Abstract

In this paper we show that the total variation diminishing (TVD) finite difference scheme which was analysed by Sweby [6] can be interpreted as a Lax-Wendroff scheme plus an upwind weighted artificial dissipation term. We then show that if we choose a particular flux limiter and remove the requirement for upwind weighting, we obtain an artificial dissipation term which is based on the theory of TVD schemes, which does not contain any problem dependent parameters and which can be added to existing MacCormack method codes. Finally, we conduct numerical experiments to examine the performance of this new method.

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