NASA Contractor Report 172380

ICASE REPORT NO. 84-21

NASA-CR-172380
19840021491

ICASE

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Contracts Nos. NAS1-17070, NAS1-17130

June 1984

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

ENTER:
9 1 1 RN/NASA-CR-172380
DISPLAY 09/6/1
84N29560*# ISSUE 19 PAGE 3082 CATEGORY 64 RPT#: NASA-CR-172380
ICASE-84-21 NAS 1.26:172380 CNT#: NAS1-17070 NAS1-17130 84/06/00 35
PAGES UNCLASSIFIED DOCUMENT
UTTL: Spectral methods for exterior elliptic problems TLSP: Final Report
AUTH: A/CANUTO, C.; B/HARIHARAN, S. I.; C/LUSTMAN, L. PAA: A/(CNR, Pavia, Italy); C/(Systems and Applied Sciences Corp., Hampton, Va.)
CORP: National Aeronautics and Space Administration. Langley Research Center, Hampton, Va. AVAIL.NTIS SAP: HC A03/MF A01
MAJS: /*BOUNDARIES/*ELLIPTIC DIFFERENTIAL EQUATIONS/*PROBLEM SOLVING/*SPECTRUM ANALYSIS
MINS: / COORDINATES/ DOMAINS/ FAST FOURIER TRANSFORMATIONS/ INFINITY/ ITERATION
ABA: M.A.C.

SPECTRAL METHODS FOR EXTERIOR ELLIPTIC PROBLEMS

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Abstract

This paper deals with spectral approximations for exterior elliptic problems in two dimensions. As in the conventional finite difference or finite element methods, it is found that the accuracy of the numerical solutions is limited by the order of the numerical farfield conditions. We introduce a spectral boundary treatment at infinity, which is compatible with the "infinite order" interior spectral scheme. Computational results are presented to demonstrate the spectral accuracy attainable. Although we deal with a simple Laplace problem throughout the paper, our analysis covers more complex and general cases.

Research for the first author was supported by NASA Contract No. NAS1-17070 and for the second author by NASA Contract No. NAS1-17130 while both were in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

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1. INTRODUCTION

In this paper we address some questions concerning applications of spectral methods to elliptic equations in exterior domains. Our goal is to investigate if one can obtain spectral accuracy in this class of problems. It is known through other methods that the accuracy of numerical solutions is governed by the order of the farfield conditions in addition to the order of the scheme itself. As an example for finite element methods see Bayliss et al. [BGT]. On the other hand boundary integral equations are very effective for exterior problems. They use appropriate Green's functions which automatically take care of the farfield behavior (see [HMC] and [GW]). However, they are limited by the need to use explicitly known kernels and thus cannot treat variable coefficient problems.

We believe spectral methods are useful alternative ways to address these difficulties. Even more than in other numerical procedures, the formulation of the farfield boundary conditions is an essential issue: a poor radiation condition may waste the high precision of spectral methods. A quite natural way of treating this problem within a spectral context is presented here, and forms the crucial part of this paper.

The basic problem to be treated here is as follows: let D be a simply connected bounded domain in $\mathbf{R}^2$, whose smooth boundary will be denoted by Γ and whose exterior region by $\Omega = \mathbf{R}^2 - \overline{D}$. We

want to solve the problem

$$
\begin{aligned}
Lu &= 0 \qquad \text{in } \Omega \\
u &= g \qquad \text{on } \Gamma \\
B_\infty(u) &\to 0 \qquad \text{as } \mathrm{r} = \sqrt{x^2+y^2} \to \infty,
\end{aligned}
\tag{1.1}
$$

where L is a second order uniformly elliptic differential operator in Ω , g is a smooth data on Γ and the last condition represents the behavior of the solution at infinity (a radiation condition).

As an example of a physical problem described by this formulation, one can think of D as a conductor in the field of a line source (located at $\underline{x} = \underline{x}_0$). Then L is the Laplacian operator Δ, u is the electrostatic potential, $g(\underline{x}) = -\log \mid \underline{x} - \underline{x}_0 \mid$ and $B_\infty(u) = u - \log \mathrm{r}$. Another example is given by the incompressible, irrotational flow around a body D: again $L = \Delta$, u is now the velocity potential, $g(\underline{x}) \equiv 0$ and $B_\infty(u) = u - U_0 x - \frac{\Gamma}{2\pi} \log \mathrm{r}$, where U_0 is the main stream velocity in the x-direction and Γ is the circulation. Although (1.1) is the simplest model one can consider, we will see that the treatment by spectral methods is quite general. Even for eigenvalue problems - such as the Helmholtz equation, which is not solved here - the numerical farfield algorithm remains applicable.

When the elliptic equation is discretized, the computational domain is obtained by placing an artificial boundary Γ_∞ surrounding the body at a finite distance. The radiation condition at infinity in (1.1) must be replaced by a boundary condition

$$
B(u) = 0 \quad \text{on} \quad \Gamma_\infty,
\tag{1.2}
$$

which makes the problem mathematically well-posed and mimics the radiation condition at infinity. Such farfield conditions may be of local (or differential) type (see for instance [BGT], [G], [KM]), or of global (or integro-differential) type (see, e.g., [ADK], [FM], [M], [MCM]). They are usually obtained by approximately similar conditions satisfied exactly by the physical solution on Γ_∞ . Hence they introduce an error on the numerical solution, which in principle should be comparable with the discretization error of the numerical scheme. The higher the order of the scheme, the higher the "order" of the farfield radiation condition. Since spectral methods produce discretization errors which decay faster than algebraically in the mesh-size, the farfield condition to be prescribed in conjunction with such methods must be particularly accurate to preserve the high precision of the interior scheme.

The radiation condition we present here meets optimally this stringent requirement, because its error is just the truncation error. Our farfield condition is somewhat analogous to the global boundary condition used in [MCM], but it takes most advantage from being implemented in a spectral context, both in terms of computational efficiency and accuracy. Numerical evidence shows that our treatment of the radiation condition produces overall spectral accuracy on the solution.

The plan of the paper is as follows. Section 2 is devoted to the discussion of the farfield conditions on the artificial boundary. In Section 3 we describe a pseudospectral algorithm for solving problem (1.1). Finally, Section 4 contains the results of some numerical tests.

2. AN INFINITE-ORDER RADIATION CONDITION

Let us assume that the problem we want to solve is the electrostatic potential problem, namely

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \Gamma \\ u - \log r &\text{ bounded} \qquad \text{as } r \to \infty. \end{aligned} \tag{2.1}$$

Although we are dealing with the Laplacian operator, the reader should keep in mind that the spectral procedure we are going to describe is particularly designed for those situations - such as the case of operators with variable coefficients approaching a constant state at infinity - where the integral equation method is not feasible.

The starting point for deriving several families of radiation conditions is the expansion of the exact solution into a convergent series of eigenfunctions, usually through separation of variables in a neighborhood of infinity, where the differential operator has constant coefficients. For (2.1) we have

$$u(r,\varphi) = \log r + \sum \frac{a_k}{r^{|k|}} e^{ik\varphi}, \tag{2.2}$$

(r,φ) being the polar coordinates in the plane. Note that the right-hand side satisfies the radiation condition at ∞. The coefficients a_k are unknown. The farfield conditions are obtained by eliminating these constants, or a finite number of them, on the artificial boundary Γ_∞.

Bayliss, Gunzburger and Turkel use in [BGT] differential operators in this process. Let us briefly recall their far-field conditions,

which we also implemented in a spectral context. The idea is to differentiate (2.2) m times in the r direction and eliminate the a_k's for $|\, k \,| \leq m$ through a linear combination of such derivatives. For each $m \geq 1$, this yields a differential operator

$$B_m = \prod_{j=1}^{m} \left(\frac{\partial}{\partial r} + \frac{2j-1}{r} \right) \tag{2.3}$$

which exactly anihilate the terms of order up to $2m$ in the series (2.2), i.e.,

$$B_m(u - \log r - a_0) = 0 \left(\frac{1}{r^{2m+1}} \right) , \; m = 1, 2, \ldots \tag{2.4}$$

with a_0 estimated by averaging u in the farfield. The approximate solution u^{ap} is required to satisfy the farfield condition

$$B_m(u^{ap} - \log r - a_0) = 0 \qquad \text{on } \Gamma_\infty \tag{2.5}$$

By comparing (2.5) with (2.4), it is seen that this method produces an "a priori" (i.e., independent of the numerical discretization) error on the approximate solution, due to having dropped the terms on the right-hand side of (2.4). This error decays algebraically with the distance of the artificial boundary. It can be made arbitrarily small by raising the order of the farfield operator, but this may lead to a cumbersome or inefficient numerical procedure.

The alternative approach which we follow in our construction of an infinitely accurate boundary operator consists of expressing <u>each</u> coefficient a_k as a functional of u, rather than eliminating a finite number of them via a differential operator. Observe that for any

$r > 0$, $a_k/r^{|k|}$ is the k-th Fourier coefficient of the periodic function $\varphi \mapsto u(r,\varphi)$, as (2.2) shows. Thus the following representation holds

$$\frac{a_k}{r^{|k|}} = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)e^{-ik\theta}d\theta = \hat{u}_k(r). \tag{2.6}$$

If we differentiate (2.2) with respect to r:

$$u_r(r,\varphi) = \frac{1}{r}\left[1 - \sum_k \mid k \mid \frac{a_k}{r^{|k|}}\, e^{ik\varphi}\right]$$

and use (2.6), we get an integro-differential relation satisfied by the exact solution of (2.1) on every circle of radius r. Precisely we have

$$u_r(r,\varphi) = \frac{1}{r}\left[1 - \frac{1}{2\pi}\int_0^{2\pi} \sum_k \mid k \mid e^{ik(\varphi-\theta)}u(r,\theta)d\theta\right]$$

or

$$u_r = \frac{1}{r} - \frac{1}{r}\; K * u, \tag{2.7}$$

where $*$ denotes the convolution of u with the singular kernel

$$K(\eta) = \frac{1}{\pi}\sum_{k=1}^{\infty} k\cos k\eta.$$

We impose the radiation condition (2.7) on the approximate solution over the artificial boundary Γ_∞. Clearly, the precision of this farfield condition, as well as the efficiency of its implementation, rely upon the accuracy and the easiness with which the singular integral in (2.7) is evaluated. Accuracy and efficiency are guaranteed if the artificial boundary is a circle, which is not at all a restriction, and if the approximate solution is a trigonometric polynomial on Γ_∞, which is the case if a spectral Fourier method is used in the angular direction.

Assume that $\Gamma_\infty = \{(r,\varphi) \mid, \mid r \mid = R_\infty\}$ and that the approximate solution, which we denote by u^N is represented on Γ_∞ by a trigonometric polynomial of degree N

$$u^N(R_\infty,\varphi) = \sum_{|k|\leq N} \hat{u}_k^N(R_\infty) e^{ik\varphi}$$

(with $\hat{u}_N^N(R_\infty) = \hat{u}_{-N}^N(R_\infty)$). Then one can easily check that

$$K * u^N = \sum_{|k|\leq N} \mid k \mid \hat{u}_k^N(R_\infty)\, e^{ik\varphi},$$

i.e., the integral operator produces a new polynomial of degree N, whose Fourier coefficients are obtained from those of $u^N(R_\infty,.)$ simply by multiplication by the modulus of the wavenumber. If u^N is known on Γ_∞ through its values $u^N(R_\infty,\varphi_j)$ at the equally spaced points $j\pi/N, j = 0,1,...,2N-1$, rather than through its Fourier coefficients, $K * u^N$ can be computed at the same modes exactly and efficiently, by Fourier transforming the values of u^N to get its coefficients, then multiplying by the modulus of the wavenumbers and finally Fourier transforming back to get the point values of $K * u^N$. Thus $K * u^N$ can be evaluated exactly in order $N\log_2 N$ operations using a Fast Fourier transform, which is available anyway as part of the spectral calculations machinary. Moreover, the global character of the convolution $K * u^N$, which would destroy the locality of a finite difference or finite element discretization method, is not a drawback in a spectral context. Therefore, this kind of boundary operator appears to be very natural for our purposes.

Thus the spectral solution is required to satisfy the radiation

condition

$$u_r^N = \frac{1}{R_\infty}\left[1 - K * u^N\right] \quad \text{on } \Gamma_\infty. \tag{2.8}$$

Unlike the family of farfield conditions proposed in [BGT], this is precisely the same boundary condition satisfied by the exact solution, i.e., except for the truncation error (a finite number of modes), no error is introduced on the numerical solution by the radiation operator on the artificial boundary.

3. THE CHEBYSHEV-FOURIER METHOD

In this Section, we describe a pseudospectral method for solving the exterior problem (2.1), which uses the global radiation condition previously introduced. This algorithm consists of three main steps:

i) mapping the physical domain onto the computational domain;

ii) building up the spectral collocation operator, which includes the radiation condition;

iii) solving the resulting system of discrete equations by various relaxation techniques.

Let us describe each step separately.

i) The Coordinate Transformation

It is not restrictive to put the origin of the coordinates within the domain D. We assume that the boundary Γ of D is represented

in polar coordinates by the equation $r = R(\varphi), 0 \leq \varphi \leq 2\pi$, where R is a smooth function. The exterior problem (2.1) will be discretized in the anular region $\hat{\Omega}$ between Γ and the artificial boundary Γ_∞, which is a circle of radius R_∞ :

$$\hat{\Omega} = \{(r\cos\varphi, r\sin\varphi)\,\varepsilon \mathrm{R}^2 \mid 0 \leq \varphi < 2\pi, R(\varphi) < r < R_\infty\}.$$

This domain is mapped smoothly onto the computational domain, the rectangle

$$\tilde{\Omega} = \{(s,\theta)\,\varepsilon\,\mathbf{R}^2) \mid -1 \ < s < 1, \ 0 < \theta < 2\pi\}$$

in such a way that the image of Γ is the side $s \equiv -1$. Let us remark at the outset that we do not attempt to obtain any special properties in the map $(r,\varphi) \rightarrow (s,\theta)$, except, of course, boundary fitting. In particular, the map is not conformal, as this would produce an immediate solution.

The coordinate change has the form

$$\begin{aligned} \varphi &= \theta \\ r &= r(s,\theta) \end{aligned} \tag{3.1}$$

with $r(-1,\theta) = R(\theta)$ and $r(1,\theta) \equiv R_\infty$. $0 \leq \theta \leq 2\pi$. The function $r(s,\theta)$ may be chosen to be an exponential stretching in the s-direction, precisely

$$r(s,\theta) = R(\theta)\ exp\ (\alpha(\theta)(s+1)) \tag{3.2}$$

with $\alpha(\theta) = \frac{1}{2}\log(R_\infty/R(\theta))$. At least two reasons suggest this choice. First, the exact solution of (2.1) satisfies for r large enough

$$u \simeq \log \mathrm{r} = \log\left(R(\theta)e^{\alpha(\theta)(s+1)}\right) \simeq \alpha(\theta)s.$$

Accordingly, u at infinity is a power of s, which is correctly approximated by Chebyshev polynomials in s. The next reason is the exponential stretching avoids the unnecessary crowding near the artificial boundary of the Chebyshev points involved in the colloation process (see [G] for similar considerations).

Numerically, the exponential stretching produces more accurate results than other stretching functions of polynomial type (see [B]). However, for some classes of problems algebraic stretching is the best (see [B], [GO]). The choice is dependent upon the problem.

ii) The Pseudospectral Operator.

When written in the (s,θ) coordinates, the Laplace equation takes the form

$$Lu \equiv au_{ss} + bu_{s\theta} + cu_{\theta\theta} + du_s = 0, \tag{3.3}$$

for some variable coefficients a,b,c,d satisfying the ellipticity conditions $a > 0, c > 0$ and $ac - b^2 > 0$ in $\tilde{\Omega}$. The solution u is periodic in θ, satisfies a Dirichlet condition at $s = -1$ and the radiation condition (2.7) at $s = 1$. Therefore we look for an approximate solution u^N which is a trigonometric polynomial of degree N in θ and an algebraic polynomial of degree M in s:

$$u^N(s,\theta) = \sum_{m=o}^{M} {\sum_{|k|\leq N}}^{*} \hat{u}^N_{mk}\, T_m(s)\, e^{ik\theta}.$$

In the sum the asterisk means that $\hat{u}^N_{mN} = \hat{u}^N_{m(-N)}$ for all m, and $T_m(s)$ is the m-th Chebyshev polynomial of the first kind. u^N

is equivalently determined by its values u_{ij}^N at the $(M+1)$x$2N$ Chebyshev-Fourier nodes

$$(s_i, \theta_j) = (\cos\frac{i\pi}{M}, \frac{j\pi}{N}), i = 0, ..., M \quad j = 0, ..., 2N-1.$$

While u_{Mj}^N for j=0,...,$2N-1$ are prescribed by the Dirichlet condition on Γ, the unknowns u_{ij}^N for i=0,...,M-1, j=0,...,$2N$-1 are computed by solving the set of collocation equations

$$\mathcal{L}_{sp}(u^N)(s_i, \theta_j) = 0 \tag{3.4}$$

The operator $\mathcal{L}_{sp}$ is pseudospectral approximating to the operator L defined in (3.3), which makes use of the radiation condition (2.8) - as well as of the Dirichlet condition on Γ - in the evaluation of the partial derivatives of u^N at the mesh points. Let us describe this process. The derivative $u_{\theta\theta}^N$ does not involve boundary conditions, hence it is computed by a standard procedure through the discrete Fourier transform. The derivative u_s^N is computed spectrally from all the values of u^N on the mesh. On the artificial boundary, what is obtained does not necessarily satisfy the radiation condition (2.8), which in the (s, θ) coordinates takes the form

$$u_s^N = \frac{\partial s}{\partial r}\left[1 - K * u^N\right] \tag{3.5}$$

Thus we modify the values of u_s^N on the artificial boundary according to (3.5). The term $K*u^N$ is evaluated in the way described in Section 2. We end up with a modified partial derivative of u^N, let us call it $\tilde{u}_s^N$ which we further differentiate spectrally to produce u_{ss}^N and $u_{s\theta}^N$.

In this procedure, the boundary conditions are imposed implicitly, namely during the evaluation of the spectral operator $\mathcal{L}_{sp}$. Thus

$\mathcal{L}_{sp}$ is an affine operator, since the boundary conditions are non-homogeneous. In view of the further discussion, it is convenient to write (3.4) in the equivalent form

$$(L_{sp}u^N - f)(s_i, \theta_j) = 0 \qquad i = 0, ..., M-1, \quad j = 0, ..., 2N-1 \tag{3.6}$$

where L_{sp} denotes the linear part of $\mathcal{L}_{sp}$ and the vector f takes into account the inhomogeneity in both the Dirichlet and the radiation boundary conditions.

We conclude this section with a few remarks on the implementation of the local boundary conditions (2.5). For m=1 we have a Robin-type condition, which can be implemented implicitly as described previously, by modifying the computed values of u_s^N on the artificial boundary to satisfy the radiation condition. For $m \geq 2$, B_m involves higher order radial operators. They may be computed spectrally, and (2.5) can be imposed explicitly, as a set of linear equations at the points on Γ_∞ (in this case the elliptic equation is collocated at the interior points only). However, the resulting algebraic system exhibits wild eigenvalues which does not compromise the possibility of using efficient iterative methods of solution. So, it is preferable to eliminate the higher order radial operators by means of the differential equation (3.3) on the boundary. This process, although rather complex, produces a boundary operator which is just first order in the radial direction; the associated boundary conditions can be imposed implicitly, as described above.

iii) Iterative Methods for the Discrete Problem.

The direct inversion of the algebraic system (3.6) is extremely impractical since the associated matrix, which we still call L_{sp}, is full and severely ill-conditioned. This matrix is not even computed, as it would require too much storage. Instead, the matrix-vector multiplication $L_{sp}v$ can be done efficiently through fast Fourier transforms, a process which needs just few storage matrices of size Mx2N for u and its partial derivatives and the coefficients of the operator. Therefore we always resort to iterative methods of solution, and specifically to those methods which update the solution by few matrix-vector multiplications of the type $L_{sp}v$.

Explicit time advancing techniques are a source of such methods, as iteration may be thought of as an evolution towards steady state of the solution of $u_t = L_{sp}u - f$. Convergence requires the stability of the scheme for the operator of the problem; moreover, the convergence history is heavily influenced by the conditioning properties of the operator and by the choice of the pseudo-time step.

Among these methods, we tested the DuFort-Frankel algorithm ([G L_1], [G L_2], [F]), in the form

(3.7)

$$\left\{ \frac{u_{ij}^{n+1} - u_{ij}^{n-1}}{2\Delta t} = (L_{sp}u - f)_{ij}^{n} - \sigma\left(\frac{a_{ij}}{(\Delta s_j)^2} + \frac{c_{ij}}{(\Delta\theta)^2}\right)(u_{ij}^{n+1} - 2u_{ij}^{n} + u_{ij}^{n-1}), \right\}$$

where a and c are the coefficients in (3.3) and $\Delta s_j = s_{j-1} - s_j$, $\Delta\theta = \pi/N$. This algorithm is known to be unconditionally stable for constant a, c, Δs and for Dirichlet or Neumann boundary conditions, once σ has been taken large enough. However, for variable coef-

ficients and radiation boundary conditions there are limitations on Δt, and the scheme (3.7) was found to converge very slowly to steady state. Even when the parameters $\Delta t, \sigma$ were fixed at optimal values (by trial and error, since there are no formulas for the general case), or dynamically redefined at each time step the convergence was still slow.

Due to the ill-conditioning of spectral operators, it is generally accepted that iterative methods for the solution of steady state problems have to be used together with preconditioning techniques. As first pointed out in [D] and [Mo], low order finite difference approximations to the boundary value problem at the collocation nodes provide a good way of preconditioning spectral methods. The matrices obtained by this process, denoted generically by L_{fd} are sparse and hence easily invertible. Moreover, the condition number of the matrix $L_{fd}^{-1} L_{sp}$ is bounded independently of the mesh size, or it grows slowly with it. These facts have been proved for some constant coefficient operators ([O],[HLAD]), and observed numerically for a wide class of variable coefficient operators, subject to Dirichlet or Neumann boundary conditions (see, e.g., [CQ],[HLAD],[PZH]).

Hereafter, we will describe a finite difference preconditioning matrix L_{fd} for the spectral operator with integral radiation condition introduced in this section. At the mesh point (i, j) the operator (3.3) is approximated by second order finite differences over the molecule $\{(i, j), (i \pm 1, j), (i, j \pm 1), (i + 1, j + 1), (i - 1, j - 1)\}$. This yields a 7-diagonal matrix. The points $(i + 1, j + 1)$ and $(i - 1, j - 1)$ are needed for consistency with the mixed term $u_{s\theta}$. A 5-diagonal matrix is obtained instead by approximating the reduced operator

$au_{ss} + cu_{\theta\theta}$ over the molecule $\{(i,j),(i \pm 1,j),(i,j \pm 1)\}$ The latter matrix can be inverted less expensively than the former one, and it provides a fairly good preconditioning as long as the body shape does not deviate too much from a circle. Otherwise, the term $u_{s\theta}$ in the differential operator is significant, and the 7-diagonal matrix has to be preferred.

In both cases, the outer points of the computational molecules centered on the artificial boundary are eliminated using the homogeneous form of the radiation condition (2.8). The exact evaluation of the integral term $K * u$ would produce a full $2N$x$2N$ block corresponding to the unknowns on the artificial boundary. Instead, the band structure is preserved if the global operator is approximated by a three-term formula,

$$(K * u)_{oj} \simeq k_+ u_{0,j+1} + k_0 u_{oj} + k_- u_{0,j-1}.$$

The coefficients $k_\pm$ and k_0 are determined in such a way that the approximation is exact for $u = 1$, $\cos\theta$ and $\sin\theta$ on the boundary.

At the points on the body surface, homogeneous Dirichlet conditions are implemented.

In this way we end up with a matrix M, which has 7 non-zero diagonals (5 if the five-point molecule is used), plus 2(M-1) non zero entries, due to the periodicity condition in the θ direction. An incomplete LU decomposition of this matrix can be built up following Meijerink and van der Vorst [MV]. Precisely, a lower and an upper triangle matrix $\mathcal{L}$ and $\mathcal{U}$ are introduced, respectively with the same (a priori) non-zero elements as the lower and the upper triangular

parts of M. $\mathcal{U}$ is normalized to be identically 1 on the main diagonal. These matrices are defined by the condition that the product $\mathcal{LU}$ equals M at the (a priori) non-zero elements in M (A different condition has also been implemented, essentially with equivalent results: $\mathcal{LU}$ and M must agree at the off-diagonal non-zero elements of M, and the row sums of $\mathcal{LU}$ and M must be equal, see [W]). Finally, we set $L_{fd} = \mathcal{LU}$. This matrix is a close approximation to M, while it allows the inversion of a linear system in $O(MN)$ operations.

The algebraic system (3.6) with this preconditioning has been solved by Richardson iterations

$$u^{n+1} = u^n + \alpha L_{fd}^{-1}(f - L_{sp}u^n) \quad , n \geq 0. \tag{3.8}$$

Alternatively, one can use Dufort-Frankel iterations

$$u^{n+1} = u^n + \beta L_{fd}^{-1}\Big(f - L_{sp}u^N\Big) - \gamma\Big(u^{n+1} - 2u^n + u^{n-1}\Big), \tag{3.9}$$

which corresponds to a second order method. The following discussion concerns (3.8), but the conclusions hold true for (3.9), too (see [CQ]).

The choice of the acceleration parameter α in (3.8) is quite delicate. Formulas for α involving the eigenvalues of $A = L_{fd}^{-1}L_{sp}$ (see for instance [O]) are impractical, since no theoretical information on the spectrum of A is known. Attempts to estimate dynamically the extreme eigenvalues of A (see [HLAD] for such an algorithm) have given poor results in our situation. Instead, a minimal residual strategy for choosing α at each iteration proves to be quite efficacious. Previously, α^n is determined in order to minimize the ℓ^2-norm of the

residual $r^{n+1} = f - L_{sp}u^{n+1}$, (see [WH]), i.e.,

$$\alpha^n = \frac{(r^n, L_{sp}p^n)}{(L_{sp}p^n, L_{sp}p^n)}, \tag{3.10}$$

where

$$p^n = L_{fd}^{-1}r^n. \tag{3.11}$$

With this choice, (3.8) is nothing but a descent method of the form

$$u^{n+1} = u^n + \alpha^n p^n, \tag{3.12}$$

where the descent direction p^n is determined according to (3.11). This method is quite appealing, since it produces fast convergence with modest work and storage requirements. Its drawback lies in the fact that it may break down if the symmetric part of the matrix $L_{sp}L_{fd}^{-1}$ is indefinite, as α^n may become zero. This occurrence has never been observed in the case of full Dirichlet boundary conditions, or constant coefficient elliptic operators. Unfortunately, the breakdown does happen for the problem under consideration, in cases where the geometry is far from being circular in shape. Then, a descent direction other than (3.11) must be used to continue the method.

We found it very effective to determine p^n according to the method Orthodir, proposed by Young and Jea ([YJ]). In its original formulation p^n is computed to be A^TA - orthogonal to all the previous descent directions (recall that $A = L_{fd}^{-1}L_{sp}$), i.e.,

$$p^n = Ap^{n-1} - \sum_{k=0}^{n-1} \beta_{nk}p^k, \tag{3.13}$$

with

$$\beta_{nk} = -\frac{(A^2 p^{n-1}, Ap^k)}{(Ap^k, Ap^k)}.$$

The method is guaranteed to converge even if the symmetric part of A is indefinite. We use a truncated version of the method, in which the sum in (3.13) starts from $k=n-2$ (for this version, however, convergence is not assured anymore). One step of Orthodir requires twice as many operations as one step of the original method (3.8). Thus, we shift to Orthodir whenever the α^n given by (3.10)-(3.11) becomes too small, say $\alpha^n \leq .001$. After one step of Orthodir, we continue with Richardson's iterations (3.8). No break-down of this algorithm was observed.

4. NUMERICAL RESULTS.

Evidence about the good performances of the method proposed here can come - at the moment - from numerical tests only. Indeed, the present status of the analysis of spectral methods is unable to handle such "complex" situations as multidimensional, variable coefficient equations with integrodifferential boundary conditions (see [GHV] for an overview of mathematical results about spectral methods). In particular, analysis of variable-coefficient operators are known for boundary value problems of Dirichlet type only (see the paper by Canuto and Quarteroni in [GHV]). Results for the implicit imposition of boundary conditions of Neumann or third type have not been established yet, except for the one-dimensional, constant coefficient case.

The hard point is to prove that the collocation scheme described in Section 3 is stable; namely, that the Chebyshev-Fourier solution W^N to the problem

$$
\begin{aligned}
\Delta W^N &= F, \quad \text{in } \hat{\Omega} \\
W^N &= G, \quad \text{on } \Gamma \\
\frac{\partial W^N}{\partial r} + \frac{1}{r} K * W^N &= H \quad \text{on } \Gamma_\infty,
\end{aligned}
\tag{4.1}
$$

satisfies an estimate of the form

$$
\| W^N \|_{\tilde{\Omega}} \leq C \left\{ \| F \|_{\hat{\Omega}} + \| G \|_{\Gamma} + \| H \|_{\Gamma\infty} \right\}, \tag{4.2}
$$

(with $C > 0$ independent of N), where the norms in (4.2) are norms of suitable Sobolev spaces. If we assume that (4.2) holds, then it is immediate to prove not only that the approximation is convergent,

but also that the error decays spectrally. Actually, if u is the solution of (2.1), set $\tilde{u} = I_M P_N u$, where P_N denotes the truncation of a trigonometric series in θ to the order N and I_M denotes the algebraic interpolation of degree M in the s-direction at the collocation points. It is well known that the error between u and $\tilde{u}$ decays spectrally. Since $P_N(K * u) = K * P_N u$, it is readily seen that $W^N = u^N - \tilde{u}$ is the collocation solution of the problem

$$
\begin{aligned}
&\Delta W^N = \Delta(u - \tilde{u}) \qquad \text{in } \hat{\Omega}, \\
&W^N = g - P_N g \qquad \text{on } \Gamma \\
&\frac{\partial W^N}{\partial r} + \frac{1}{r} K * W^N = P_N\left(\frac{\partial u}{\partial r}\right) - \frac{\partial \tilde{u}}{\partial r} \qquad \text{on } \Gamma_\infty.
\end{aligned} \tag{4.3}
$$

All the right-hand sides in (4.3) decay faster than algebraically in the Sobolev norms as $N, M \to \infty$, hence so does W^N, according to (4.2). We conclude that u^N approximates u with spectral accuracy. Note that if (4.2) holds, the convergence is guaranteed for any fixed value R_∞ of the radius of the artificial boundary.

Let us present now some numerical results.

In order to compare the behavior of our global radiation condition, the Bayliss-Gunzburger-Turkel radiation conditions of the first and second order were also implemented. The same pseudospectral algorithm described previously was used.

In order to appraise our numerical results we need exact solution of the problem for an arbitrarily shaped domain D. We simulated this situation by considering a point source at $\underline{x}_o = (x_o, y_o) \varepsilon$ D, whose exact solution is

$$
\log |\underline{x} - \underline{x}|, \quad \underline{x} \varepsilon \mathbf{R}^2 \setminus \{\underline{x}_o\}.
$$

We prescribed this value on Γ. Note that this satisfies (2.1) with $a_0 \equiv 0$. The numerical procedure begins only with this value on Γ. Moreoever, one can add to this above solution any harmonic function which decays at infinity and still satisfied by (2.1). Such solutions are used for comparison purposes. We considered three significant situations.

CASE 1: the body is a circle of radius 1, and the exact solution is $u(x,y) = \log\left[(x-.7)^2 + y^2\right]^{\frac{1}{2}}$.

CASE 2: the body surface is described by the function $\mathbf{R}(\varphi) = 1 + .4\cos 2\varphi$ (see Fig. 1), and the exact solution is

$$u(r,\varphi) = \log r + \frac{\cos 2\varphi}{r^2} + 3\frac{\cos 3\varphi}{r^3}.$$

CASE 3: the body surface is described by the function $R(\varphi) = 1 + .7\cos\varphi + .3\sin 2\varphi$ (see Fig. 2), while the exact solution is

$$u(x,y) = \log\left[(x-1)^2 + (y-1)^2\right]^{\frac{1}{2}}.$$

The algorithms were tested on a 9x8, a 17x16 and a 33x32 grid, and for different positions of the artificial boundary. If we define the relative distance of the artificial boundary to be $\rho = R_\infty / \max_{0\le\varphi<2\pi} R(\varphi)$, then the range of ρ was the interval [1.2,5.].

Table 1 provides some information about the behavior of the iterative algorithm used to solve the system (3.6). We report the relative residual (i.e., the l^2 - norm of the current residual divided by the l^2 - norm of the initial residual) at some selected iterations. The convergence histories are relative to the intermediate case $\rho = 3$,

and to the use of the integral radiation condition (2.8). These results can be considered typical, in the sense that the rate of convergence becomes just slightly worse where ρ approaches 1 (because of the increased stretching of the coordinates), and it is essentially the same if the local radiation conditions are imposed instead. It is observed that the preconditioning is sensitive to the geometry (i.e., to the coefficients of the differential operator). Moreover, the spectral radius of the iteration matrix depends on the mesh-size (although some "strange" behaviors appear, such as in Case 2 where the convergence on the 17x16 grid is faster than on the 9x8 grid, or in Case 3 where the histories for N=16 and N=32 are essentially similar). These facts are clearly related to the use of an incomplete LU decomposition in the inversion of the finite difference system. However, the preconditioning is globally effective. The reader should take into account that over 1000 iterations of the Du-Fort-Frankel algorithm (3.7) are needed to drive the residual below 10^{-10}, in the simplest geometry of the circle. Moreover, in all the cases tested, at most some 50 iterations of our algorithm were enough to produce 2 significant digits in the spectral solution.

Tables 2-4 contain the results of our tests, which allow to compare the performance of the first order (F), the second order (S) [BGT] radiation conditions, and the integral (I) radiation condition on the issue of the spectral accuracy. In each cell, the upper number is the relative error between the exact and the spectral solution in the l^2 - norm at the grid in the computational domain. This error may appear geometry-dependent, since the grid points are displaced in the physical domain by the stretching of coordinates. For this reason

we have included the relative error in the maximum norm, which is independent of the mapping of the domain (this is the lower number in the cell). The two errors behave qualitatively in the same way.

In general, it is seen that the integral radiation condition always produces the best results, and it is the sole to guarantee the spectral accuracy in all the cases.

This is evident in Case 1 (Table 2), where the I-condition produces spectral accuracy for any value of ρ. Thus the error just comes from the discretization scheme, while it is weakly sensitive to the position of Γ_∞. On the contrary, the F- and the S- conditions produce spectral accuracy only when Γ_∞ is far enough from the body. Otherwise, the truncation error introduced by the local radiation condition dominates the discretization error, and the global error increases as Γ_∞ is brought close to the body. This is particularly evident on the 33x32 grid. The reason is that in the expansion (2.2) of the exact solution, the terms with index $\mid k \mid$ larger than 2 (or 3) are treated incorrectly by the local radiation conditons, and correctly by our global condition.

This phenomenon is even clearer in Case 2, where the exact solution is chosen on purpose to exhibit the different behavior of the radiation conditions. The I-condition is the only one which allows the artificial boundary to be brought very close to the body, yielding the smallest errors. Note that on the 9x8 and the 17x16 grids, the error decreases as Γ_∞ approaches D even when the F- or the S- conditions are used: the error in this case is dominated by the discretization

error, which takes advantage by the refinement of the mesh. But on the 33x32 grid, the same behavior as in Case 1 occurs.

Finally, Case 3 is an example of a situation where an exact solution with a singularity relatively close to the boundary of the body is coupled with a fairly complex geometry. Again, the I-condition performs the best, producing spectral accuracy. It should be noticed, however, that now even the error obtained by the I-condition gets worse if the artificial boundary is too close to the body. Since the exact solution is qualitatively comparable with the exact solution of Case 1, the origin of this phenomenon should be found in the fact that the coordinate mapping produces larger variations in the coefficients of the elliptic operator, hence there is an overall loss of precision. This remark may suggest the use of a multidomain technique, whenever the geometry is too complex to be handled accurately by the one domain mapping discussed here.

Acknowledgements:

Authors would like to thank Professors C.E. Grosch and E. Turkel for their valuable discussions.

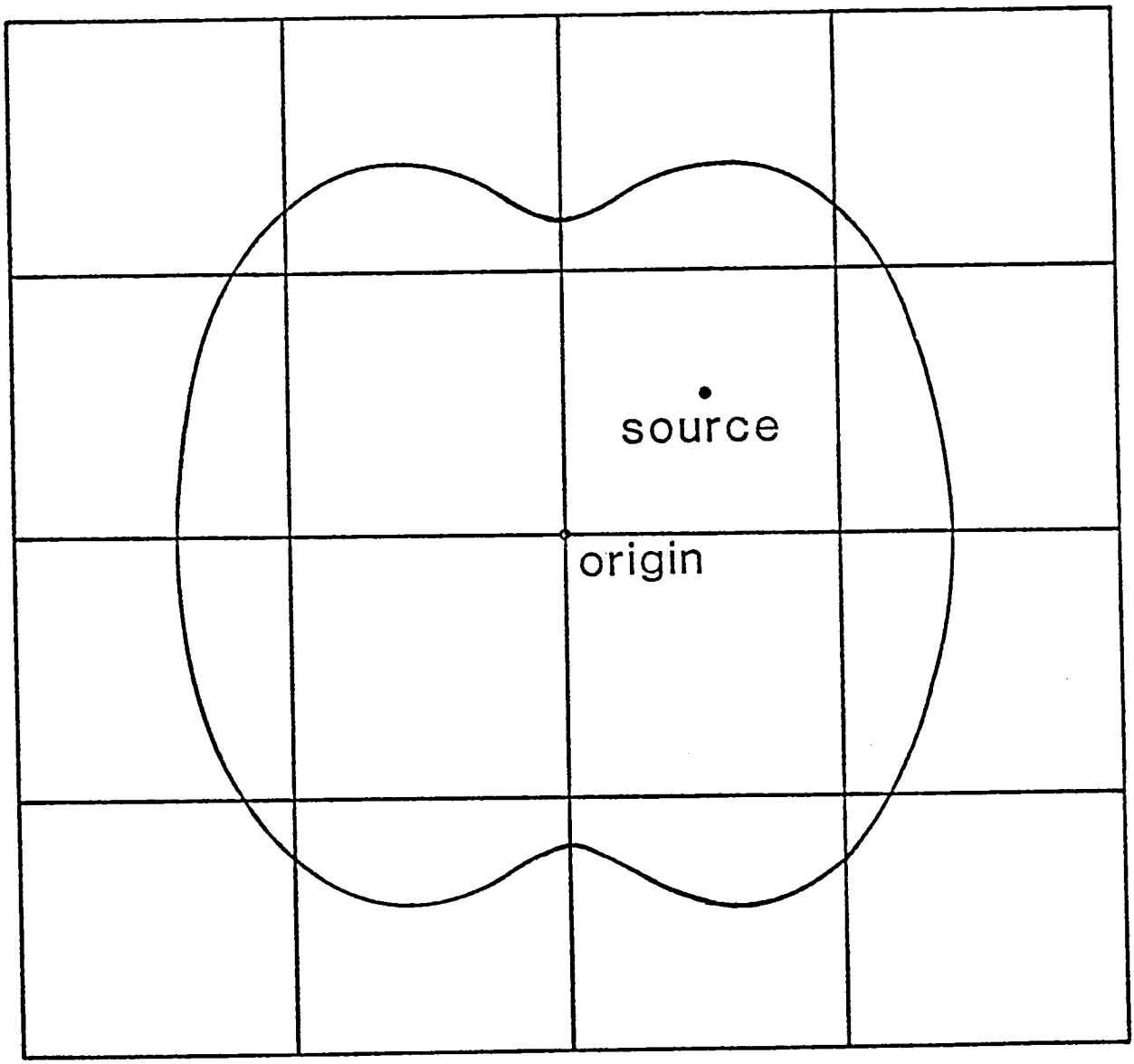

Figure 1.

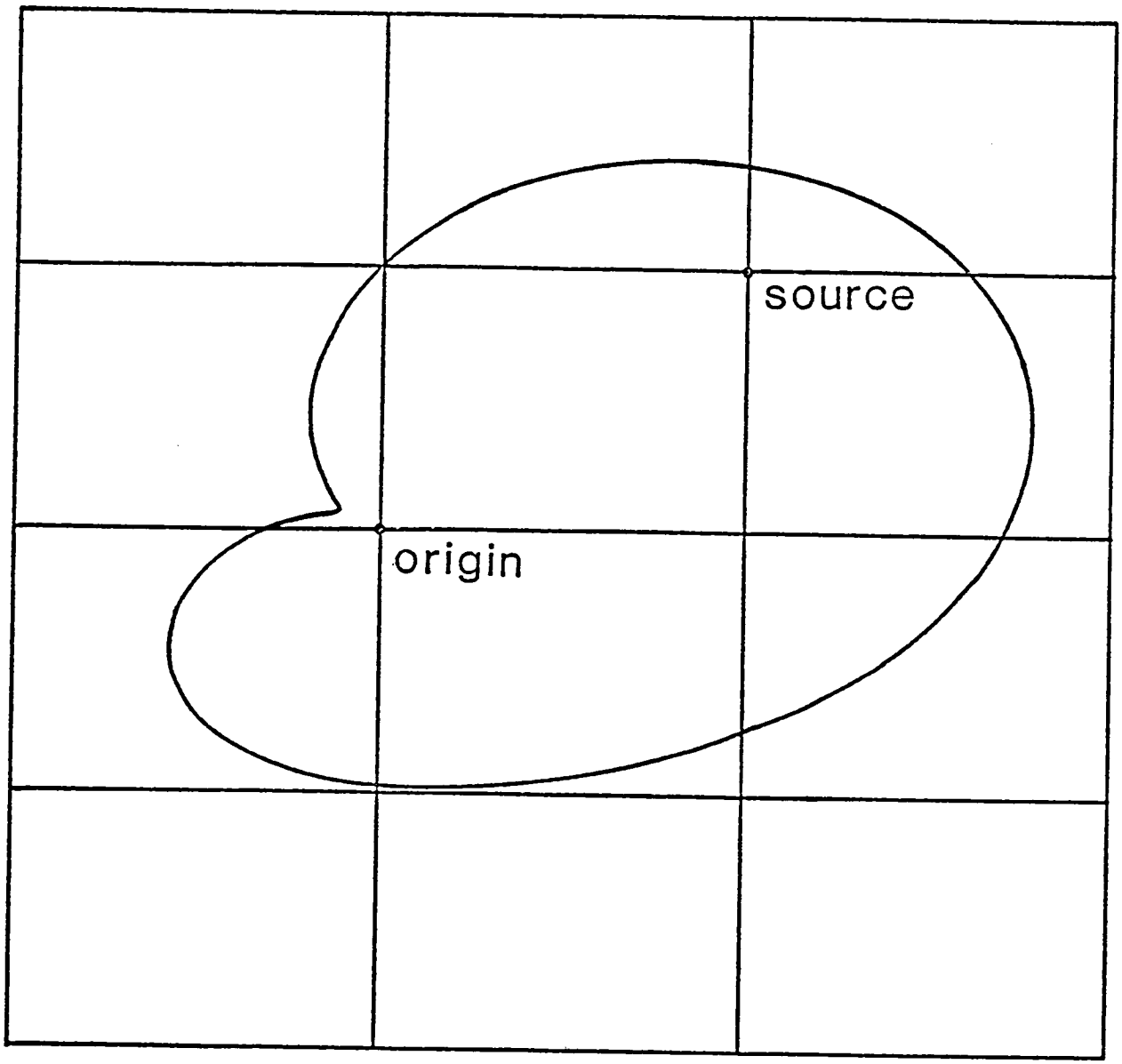

Figure 2.

ITER.	CASE 1			CASE 2			CASE 3		
	9X8	17X16	33X32	9X8	17X16	33X32	9X8	17X16	33X32
0	1.	1.	1.	1.	1.	1.	1.	1.	1.
50	.1E-9	.7E-6	.2E-5	.2E-6	.2E-7	.4E-5	.3E-6	.2E-3	.1E-3
100	.2E-21	.8E-12	.8E-10	.8E-11	.2E-13	.1E-7	.1E-8	.2E-4	.2E-4
150	*	.4E-18	.4E-15	.3E-15	.6E-17	.3E-9	.7E-10	.6E-5	.2E-5
200	*	*	.2E-20	.1E-19	.3E-21	.3E-10	.4E-11	.6E-6	.3E-6
250							.2E-13	.1E-6	.3E-7
300							.3E-14	.1E-7	.4E-8

TABLE 1: Convergence histories (* means: below 10^{-23})

ρ	9X8 grid			17X16 grid			33X32 grid		
	F	S	I	F	S	I	F	S	I
1.2	.68E-2 .67E-1	.46E-2 .36E-1	.25E-2 .24E-1	.20E-2 .33E-1	.56E-3 .61E-2	.18E-3 .30E-2	.94E-3 .12E-1	.21E-3 .28E-2	.37E-5 .49E-4
1.5	.80E-2 .65E-1	.70E-2 .48E-1	.38E-2 .33E-1	.18E-2 .13E-1	.57E-3 .43E-2	.27E-3 .46E-2	.83E-3 .81E-2	.10E-3 .69E-3	.49E-5 .80E-4
3.	.48E-2 .42E-1	.52E-2 .42E-1	.34E-2 .40E-1	.44E-3 .25E-2	.36E-3 .37E-2	.24E-3 .48E-2	.14E-3 .51E-3	.85E-5 .97E-4	.40E-5 .91E-4
5.	.34E-2 .37E-1	.38E-2 .38E-1	.27E-2 .36E-1	.25E-3 .16E-2	.26E-3 .16E-2	.18E-3 .15E-2	.34E-4 .52E-4	.33E-5 .57E-4	.30E-5 .61E-4

TABLE 2: Results for CASE 1

ρ	9X8 grid			17X16 grid			33X32 grid		
	F	S	I	F	S	I	F	S	I
1.2	.31E-1	.26E-1	.22E-1	.36E-2	.15E-2	.12E-2	.16E-2	.31E-3	.37E-5
	.38E-1	.38E-1	.37E-1	.86E-4	.47E-4	.30E-4	.49E-4	.74E-5	.18E-6
1.5	.35E-1	.36E-1	.28E-1	.26E-2	.18E-2	.14E-2	.95E-3	.18E-3	.46E-5
	.19E-1	.28E-1	.19E-1	.25E-3	.23E-3	.20E-3	.33E-4	.33E-5	.75E-6
3.	.54E-1	.60E-1	.42E-1	.38E-2	.37E-2	.24E-2	.15E-3	.25E-4	.10E-4
	.32E-1	.33E-1	.26E-1	.11E-2	.11E-2	.10E-2	.62E-5	.33E-5	.33E-5
5.	.59E-1	.66E-1	.47E-1	.41E-2	.46E-2	.31E-2	.44E-4	.16E-4	.14E-4
	.53E-1	.55E-1	.47E-1	.19E-2	.19E-2	.18E-2	.60E-5	.54E-5	.54E-5

TABLE 3: Results for CASE 2

ρ	9X8 grid			17X16 grid			33X32 grid		
	F	S	I	F	S	I	F	S	I
1.5	.12E-1	.12E-1	.68E-2	.14E-2	.51E-3	.39E-3	.56E-3	.63E-4	.11E-4
	.59E-1	.60E-1	.53E-1	.84E-2	.23E-2	.17E-2	.94E-3	.75E-3	.76E-4
2.	.77E-2	.79E-2	.55E-2	.63E-3	.44E-3	.25E-3	.25E-3	.20E-4	.65E-5
	.53E-1	.54E-1	.49E-1	.40E-2	.11E-2	.10E-2	.25E-3	.59E-4	.52E-4
3.	.57E-2	.60E-2	.52E-2	.89E-3	.10E-2	.46E-3	.85E-4	.11E-4	.47E-5
	.48E-1	.48E-1	.46E-1	.18E-2	.20E-2	.16E-2	.30E-4	.42E-4	.39E-4
5.	.61E-2	.67E-2	.67E-2	.13E-2	.15E-2	.74E-2	.26E-4	.15E-4	.12E-4
	.48E-1	.49E-1	.47E-1	.26E-2	.27E-2	.22E-2	.46E-4	.48E-4	.49E-4

TABLE 4: Results for CASE 3

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1. Report No. NASA CR-172380 ICASE Report No. 84-21	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Spectral Methods for Exterior Elliptic Problems		5. Report Date June 1984 6. Performing Organization Code
7. Author(s) C. Canuto, S. I. Hariharan, L. Lustman		8. Performing Organization Report No. 84-21
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering NASA Langley Research Center, Hampton, VA 23665		10. Work Unit No. 11. Contract or Grant No. NAS1-17070 NAS1-17130
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Contractor Report 14. Sponsoring Agency Code 505-31-83-01

15. Supplementary Notes

Langley Technical Monitor: Robert H. Tolson
Final Report

16. Abstract

This paper deals with spectral approximations for exterior elliptic problems in two dimensions. As in the conventional finite difference or finite element methods, it is found that the accuracy of the numerical solutions is limited by the order of the numerical farfield conditions. We introduce a spectral boundary treatment at infinity, which is compatible with the "infinite order" interior spectral scheme. Computational results are presented to demonstrate the spectral accuracy attainable. Although we deal with a simple Laplace problem throughout the paper, our analysis covers more complex and general cases.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
elliptic equations exterior problems spectral methods radiation conditions	64 Numerical Analysis Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	34	A03

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NASA-Langley, 1984