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ACOUSTIC PRESSURES EMANATING FROM A TURBOMACHINE STAGE

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Abstract

A knowledge of the acoustic energy emission of each blade row of a turbomachine is useful for estimating the overall noise level of the machine and for determining its discrete frequency noise content. Because of the close spacing between the rotor and stator of a compressor stage, the strong aerodynamic interactions between them have to be included in obtaining the resultant flow field. This paper outlines a three-dimensional theory for determining the discrete-frequency noise content of an axial compressor consisting of a rotor and a stator each with a finite number of blades. The lifting surface theory and the linearized equation of an ideal, nonsteady compressible fluid motion are used for thin blades of arbitrary cross section. The combined pressure field at a point of the fluid is constructed by linear addition of the rotor and stator solutions together with an interference factor obtained by matching them for net zero vorticity behind the stage. The rotor solution is obtained as a Fourier-Bessel series; the stator solution is expressed as a Fourier-Laguerre series. The coefficients of the series and the interaction factor are determined as the eigenvector of a set of algebraic equations in matrix form whose elements comprise the sum-integrals of the rotor and stator eigenfunctions. The resultant pressure field of the stage is the sum of the individual perturbation pressures in the presence of the interaction effects, expressions for which are given herein.

1. Introduction

When the rotor is rotating in a uniform and steady incoming free stream, a point behind the rotor experiences periodicity in the pressure and velocity characteristics, which are propagated as acoustic perturbations. The relative motion between the stator blades and the rotor efflux introduces periodicity in the inflow to the stator. Since the rotor efflux consists of both the helical vortex sheets and the viscous wakes of the rotor blades, the aerodynamic forces and moments on the stator blades and the outlet velocities and pressure of the stator are affected by the flow periodicity. Under subsonic flow conditions, periodicity in the stator flow field also affects the rotor. Reflection, transmission of the acoustic perturbation from the blade surface, and interaction with the duct resonance characteristics also have a strong effect on the resulting acoustic field at a point.

Kemp and Sears[1,2] studied the unsteady flow field of two rows of a two-dimensional thin airfoil cascade in incompressible flow although they did not dwell on the aeroacoustic aspects. The aeroacoustic field of two-dimensional single cascades was studied by several workers (Lane and Friedman[3], Carta[4], Carta and Fanti[5], Hetherington[6], Parker[7-9], Tyler and Sofrin[10,11], Smith[12], Whitehead[13], Morfey[14-16], Mani[17,18], and Mani and Horvay[19]). Kaji and Okazaki[20-22] studied the aerodynamic interaction between the two rows of infinite airfoil cascades and observed the discrete-frequency circumferential modes and their duct resonance characteristics. Slutsky[23] solved the discrete-frequency noise generation due to rotor forcing excitation and rotor-stator interaction by using a Kemp-Sears mechanism with bound vortices on the rotor and the associated three-dimensional velocity potential; he obtained the spin wave characteristics. Namba[24,25] proposed the lifting surface theory for a single blade row in a compressible three-dimensional flow to calculate the unsteady blade forces and their acoustic power generation. He found that the sound power generated was decreased by increasing the radial nonuniformity of a sinusoidal external disturbance in the circumferential direction. Mariano[26] considered the effect of reflection of the sound field from a plane boundary for a uniform distribution of sound sources.

In an earlier paper[27], the author outlined a lifting surface theory for a turbomachine stage in a three-dimensional incompressible flow that uses a distribution of sources and vortices to represent the blades of the two rows: the rotor and the stator. Another paper[28] treats the acceleration potential approach to the rotor-stator combination for compressible flow for both the aerodynamic loading and the acoustic characteristics. The concept of a rotor-stator interaction factor is introduced to isolate their mutual interaction effects, and axially attenuated or wavelike solutions are constructed for the general solution by using the Birnbaum series expansion as modulated by the eigenfunctions of the rotor and stator. Thus the composite solution of the rotor and stator is expressed through Fourier-Bessel and Fourier-Laguerre series expansions that simultaneously satisfy the surface boundary conditions on both the blade rows and arbitrarily match to give zero net resultant vorticity in the wake of the stage. The procedure followed in this paper for two blade rows parallels closely that of Namba[25] for one blade row. An additional condition is required, however, to quantify the magnitude of the interference between the two blade rows by matching them optimally. In Section 2 the Green's function is written for the linearized differential equations governing the perturbation pressure field of a pulsating unit pressure monopole placed at a point on a blade in a double Fourier-Bessel and Fourier-Laguerre series for the rotor and stator, respectively. From the monopole solution the pulsating unit pressure dipole solution is obtained in Section 3 as the derivative normal to the blade corresponding to the surface pressure distribution. The chordwise pressure distribution on the blade is represented in Section 4 as a Birnbaum series modulated by the pressure eigenfunctions of the rotor and stator corresponding to the camber and thickness effects. The combined rotor-stator field is matched to give a zero net vorticity condition behind the stage (Section 5), and the coefficients of the Birnbaum expansion are determined by satisfying the flow tangency condition on the blade surface (Section 6) followed by a short discussion in Section 7 of the combined pressure field. Because of space limitations, only the highlights of this work are presented herein. No attempt has been made here to consider the reflection and absorption of waves or the natural duct resonance characteristics. Numerical results of the present theory will be reported separately.

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2. Eigensolutions of Pulsating Pressure Pole on Rotor and Stator

We shall consider a stage of an axial compressor (Fig. 1) with the rotor followed by a stator located centrally in an infinite annular duct with an outer diameter such that the clearance between the blade tip and the casing is uniform and negligibly small. We shall consider a compressible nonviscous fluid and assume a finite number of rigid blades Z_r and Z_s in the rotor and stator, respectively. Assuming that the compressor stage is lightly loaded so that the perturbation pressures are small compared with the free-stream static pressure p_∞, the linearized differential equations for the perturbation pressures p_r and p_s of the rotor and stator are given by

$$a_\infty^2\left(\frac{\partial^2 p_r}{\partial r^2} + \frac{1}{r}\frac{\partial p_r}{\partial r} + \frac{\partial^2 p_r}{r^2\partial\theta^2} + \frac{\partial^2 p_r}{\partial z^2}\right) = V_r^2\frac{\partial^2 p_r}{r^2\partial\theta^2} + W_r^2\frac{\partial^2 p_r}{\partial z^2} + \frac{\partial^2 p_r}{\partial t^2} + 2\left(W_r\frac{\partial^2 p_r}{\partial z\partial t} + \frac{V_r}{r}\frac{\partial^2 p_r}{\partial\theta\partial t} + \frac{V_r W_r}{r}\frac{\partial^2 p_r}{\partial\theta\partial z}\right) \tag{1a}$$

$$a_\infty^2\left(\frac{\partial^2 p_s}{\partial r^2} + \frac{1}{r}\frac{\partial p_s}{\partial r} + \frac{\partial^2 p_s}{r^2\partial\theta^2} + \frac{\partial^2 p_s}{\partial z^2}\right) = V_s^2\frac{\partial^2 p_s}{r^2\partial\theta^2} + W_s^2\frac{\partial^2 p_s}{\partial z^2} + \frac{\partial^2 p_s}{\partial t^2} + 2\left(W_s\frac{\partial^2 p_s}{\partial z\partial t} + \frac{V_s}{r}\frac{\partial^2 p_s}{\partial\theta\partial t} + \frac{V_s W_s}{r}\frac{\partial^2 p_s}{\partial\theta\partial z}\right) \tag{1b}$$

in cylindrical polar coordinates where $(0, V_r, W_r)$ and $(0, V_s, W_s)$ are assumed to be free-stream velocity components along the radial, peripheral, and axial directions for the rotor and stator, respectively, and a_∞ is the free-stream speed of sound. Because the stator is situated downstream of the rotor in a region of rapid change, its inflow components $(0, V_s, W_s)$ cannot be defined a priori. However, we assume that the stator inlet conditions correspond approximately to that obtained from the velocity diagram (Fig. 2) so that we can write the components of inflow to the rotor and stator as

$$\left.\begin{aligned}(0, V_r, W_r) &= (0, \Omega r, W_a)\\ (0, V_s, W_s) &= (0, \Omega r - W_a\tan\alpha_{2r}, W_a)\end{aligned}\right\} \tag{2}$$

where the axial velocity W_a is assumed to be constant throughout the stage.

Equations (1) can be nondimensionalized, with all lengths and time expressed in terms of the rotor tip diameter and characteristic time, and written as

$$\frac{\partial^2 p_r}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial p_r}{\partial r_1} + \left(1 - \bar{M}^2 r_1^2\right)\frac{\partial^2 p_r}{r_1^2\partial\theta^2} + \beta^2\frac{\partial^2 p_r}{\partial z_1^2} - M^2\frac{\partial^2 p_r}{\partial t_1^2} - 2\left(M^2\frac{\partial^2 p_r}{\partial z_1\partial t_1} + M\bar{M}\frac{\partial^2 p_r}{\partial\theta\partial t_1} + M\bar{M}\frac{\partial^2 p_r}{\partial\theta\partial z_1}\right) = -\frac{\delta(\vec{r} - \vec{\rho}_r)}{r}e^{i\omega_r t_1} \tag{3a}$$

$$\frac{\partial^2 p_s}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial p_s}{\partial r_1} + \left[1 - \left(M_2 - \bar{M}r_1\right)^2\right]\frac{\partial^2 p_s}{r_1^2\partial\theta^2} + \beta^2\frac{\partial^2 p_s}{\partial z_1^2} - M^2\frac{\partial^2 p_s}{\partial t_1^2} - 2\left\{M^2\frac{\partial^2 p_s}{\partial z_1\partial t_1} - \frac{M(M_2 - \bar{M}r_1)}{r_1}\frac{\partial^2 p_s}{\partial\theta\partial t_1} - \frac{M(M_2 - \bar{M}r_1)}{r_1}\frac{\partial^2 p_s}{\partial\theta_1\partial z_1}\right\} = -\frac{\delta(\vec{r}_1 - \vec{\rho}_s)}{r_1}e^{i\omega_s t_1} \tag{3b}$$

where

$$\left.\begin{aligned}&r_1 = r/r_{tr};\ z_1 = z/r_{tr};\ t_1 = t/t_0;\ \beta^2 = 1 - M^2;\\ &\beta_2^2 = 1 - M_2^2;\ t_0 = r_{tr}/W_a;\ V_r = \Omega r;\ M = W_a/a_\infty;\\ &M_2 = M\tan\alpha_{2r};\ \bar{M} = \Omega r_{tr}/a_\infty\end{aligned}\right\} \tag{4}$$

Equations (3) express the pressure perturbation field of a pulsating unit pressure pole situated on a rotor or stator blade at the vector point $\vec{\rho}_r$ or $\vec{\rho}_s$ with a pulsation frequency ω_r or ω_s, respectively. We may write the general solutions of Eqs. (3) as in Namba[25]

$$\hat{p}_r(\vec{r}_1, \vec{\rho}_r) = \frac{1}{4\pi^2}\sum_{k_r=-\infty}^{\infty}\int_{-\infty}^{\infty} P_r(r_1, \rho_r;\ k_r, a_r) \times \exp i\left[k_r(\theta - \varphi_r) + a_r(z_1 - \zeta_r)\right]da_r \tag{5a}$$

$$\hat{P}_s(\vec{r}_1,\vec{\rho}_s) = \frac{1}{4\pi^2}\sum_{k_s=-\infty}^{\infty}\int_{-\infty}^{\infty} P_s(r_1,\rho_s;\ k_s,a_s)$$

$$\times \exp i[k_s(\theta - \varphi_s) + a_s(z_1 - \zeta_s)]da_s \qquad (5b)$$

where a_r and a_s are the rotor and stator axial wave numbers; k_r and k_s ($k_r,k_s = 0,\pm 1,\pm 2,\ldots,\pm\infty$) are the pair of integer separation constants; and P_r and P_s are the unknown pressure functions.

We shall express the delta functions in Eqs. (3) in a combined Fourier series - Fourier integral form as

$$\delta(\theta - \varphi_r;\ z_1 - \zeta_r)$$

$$= \frac{1}{4\pi^2}\sum_{k_r=-\infty}^{\infty}\int_{-\infty}^{\infty} \exp i[k_r(\theta - \varphi_r) + a_r(z_1 - \zeta_r)]da_r \qquad (6a)$$

$$\delta(\theta - \varphi_s;\ z_1 - \zeta_s)$$

$$= \frac{1}{4\pi^2}\sum_{k_s=-\infty}^{\infty}\int_{-\infty}^{\infty} \exp i[k_s(\theta - \varphi_s) + a_s(z_1 - \zeta_s)]da_s \qquad (6b)$$

and write p_r and p_s as

$$\left.\begin{aligned} p_r(\vec{r}_1,\ \vec{\rho}_r,\ t_1) &= \hat{p}_r(\vec{r}_1 - \vec{\rho}_r)e^{i\omega_r t_1} \\ p_s(\vec{r}_1,\ \vec{\rho}_s,\ t_1) &= \hat{p}_s(\vec{r}_1 - \vec{\rho}_s)e^{i\omega_s t_1} \end{aligned}\right\} \qquad (7)$$

Combining Eqs. (3), (5), (6), and (7) and equating the corresponding terms on the two sides, we obtain the nonhomogeneous second-order ordinary differential equations for the radial pressure functions P_r and P_s, given by

$$\frac{d^2P_r}{dr_1^2} + \frac{1}{r_1}\frac{dP_r}{dr_1} + \left[\frac{\hat{M}^2}{\beta^2} - \beta^2\left(a_r - \frac{M\hat{M}}{\beta^2}\right)^2 - \frac{k_r^2}{r_1^2}\right]P_r = -\frac{\delta(r_1 - \rho_r)}{r_1} \qquad (8a)$$

$$\frac{d^2\tilde{P}_s}{dr_1^2} + \tilde{P}_s\left(-\frac{\nu^{(k_s)^2}}{4} - \frac{\chi^{(ks)^2}}{r_1} + \frac{\frac{1}{4} - \bar{\alpha}^{(k_s)^2}}{r_1^2}\right) = -\frac{\delta(r_1 - \rho_s)}{r_1^{1/2}} \qquad (8b)$$

where we have defined

$$P_s = \frac{\tilde{P}_s}{r_1^{1/2}} \qquad \lambda^{(k_r)^2} = \beta^2\left[\frac{\hat{M}^2}{\beta^4} - \left(a_r - \frac{M\hat{M}}{\beta^2}\right)^2\right]$$

$$\bar{\alpha}^{(k_s)} = \beta_2 k_s \qquad \nu^{(k_s)^2} = 4\beta^2[(a_s - MM_*)^2 - M_*^2]$$

$$\hat{M} = (M\omega_r + \bar{M}k_s) \qquad \chi^{(k_s)} = 2M_2k_s(\bar{M}k_s + M\omega_s + Ma_s)$$

$$M_* = \frac{M\omega_s + \bar{M}k_s}{\beta^2} \qquad (9)$$

An examination of Eqs. (3) or (8) shows that when $\chi \equiv 0$ as in the case of an incompressible fluid or when the exit rotor blade angle $\alpha_{2r} = 0$ in a compressible fluid, both the equations reduce to the Bessel type. But, for all other cases, χ cannot be neglected since k_s extends over an infinite range of values, and it is essential to consider χ in a formal solution of Eq. (8b). Thus Eq. (8a) is the nonhomogeneous Bessel differential equation and Eq. (8b) is the classical Whittaker nonhomogeneous differential equation, for which the complementary solutions can be written in the form

$$P_r(r_1,\rho_r) = A_*I(\lambda^{(k_r)}r_1) + B_*K(\lambda^{(k_r)}r_1) \qquad (10a)$$

$$P_s(r_1,\rho_s) = \left[C_*W_{\chi^{(k_s)},\bar{\alpha}}(\nu^{(k_s)}r_1) + D_*W_{\chi^{(k_s)},-\bar{\alpha}}(\nu^{(k_s)}r_1)\right]\Big/\nu^{(k_s)^2} \qquad (10b)$$

in which $I(x)$ and $K(x)$ represent the modified Bessel and Hankel functions, respectively, and $W_{\chi,\mu}(x)$ is the Whittaker function. To write the solution of the complete nonhomogeneous equations, it is convenient to express the solution of the rotor function P_r in an infinite expansion in terms of the normalized functions $\Phi_\ell^{(k_r)}$ for for Eq. (10a). The function $\Phi_\ell^{(k_r)}$ is defined in the appendix. The solution for the stator function P_s is more conveniently written in terms of the set of generalized Laguerre functions that satisfy certain orthogonality relations than directly in terms of the Whittaker function, to which they are related by the expressions

$$W_{\chi,\mu}(z) = \frac{\Gamma(1 + 2\mu)\ \Gamma(\chi - \mu + \frac{1}{2})}{\Gamma(\chi + \mu + \frac{1}{2})}$$

$$\times e^{z/2}z^{(\mu+1/2)}L^{(2\mu)}_{(\chi-\mu-1/2)}(z) \qquad (11)$$

Thus we shall obtain the solutions of the nonhomogeneous Eqs. (8) for ρ_r and ρ_s in the form

$$P_r^{(k_r)}(r_1,\rho_r) = \sum_{\ell=0}^{\infty} g_\ell^{(k_r)}(a_r)\Phi_\ell^{(k_r)}(r_1)\Phi_\ell^{(k_r)}(\rho_r) \tag{12a}$$

$$P_s^{(k_s)}(r_1,\rho_s) = \sum_{\ell=0}^{\infty} h_\ell^{(k_s)}(a_s)\Psi_\ell^{(k_s)}(r_1)$$

$$\times \Psi_\ell^{(k_s)}(\rho_s)(r_{2s}\rho_{2s})^{\bar{\alpha}+1/2} \exp\left(-\frac{r_{2s}+\rho_{2s}}{2}\right) \tag{12b}$$

where $g_\ell^{(k_r)}$ and $h_\ell^{(k_s)}$ are coefficients of the expansion to be determined. The eigenfunctions $\Phi_\ell^{(k_r)}$ and $\Psi_\ell^{(k_s)}$ are combinations of Bessel and Laguerre functions, respectively, that satisfy

$$\frac{d^2\Phi_\ell^{(k_r)}}{dr_1^2} + \frac{1}{r_1}\frac{d\Phi_\ell^{(k_r)}}{dr_1} + \left(\lambda_\ell^{(k_r)^2} - \frac{k_r^2}{r_1^2}\right)\Phi_\ell^{(k_r)} = 0 \tag{13a}$$

$$\frac{d^2\Psi_\ell^{(k_s)}}{dr_{2s}^2} + \left(\frac{2\bar{\alpha}+1}{r_2} - 1\right)\frac{d\Psi_\ell^{(k_s)}}{dr_{2s}} + \frac{\tilde{\ell}}{r_{2s}}\Psi_\ell^{(k_s)} = 0 \tag{13b}$$

in which the independent variables r_{2s} and ρ_{2s} are defined by

$$\left.\begin{aligned} r_{2s} &= \nu_\ell^{(k_s)} r_1 \\ \rho_{2s} &= \nu_\ell^{(k_s)} \rho_s \end{aligned}\right\} \tag{14}$$

For the Laguerre equation (Eq. 10a), eigensolutions exist if[31,32]

$$\tilde{\ell} = x_\ell^{(k_s)} + \frac{\nu_\ell^{(k_s)}}{h_\ell^{(k_s)}} - (\bar{\alpha} + \tfrac{1}{2}) \tag{15}$$

is an integer, and we shall put

$$\ell_1 = \tilde{\ell} + \bar{\alpha} + \frac{1}{2} \tag{16}$$

$\lambda_\ell^{(k_r)}$ and $x_\ell^{(k_s)}$ $(\ell = 0,1,2,\ldots,\infty)$ may be regarded as the radial eigenvalues of the rotor and stator since we can write

$$\frac{\nu_\ell^{(k_s)}}{4\beta^2 M_*^2} = \frac{1}{M^2}\left(\frac{x_\ell^{(k_s)}}{2M_* M_2 k_s} - 1\right)^2 - 1 \tag{17}$$

The stator eigenfunction $\Psi_\ell^{(k_s)}$ is defined in the appendix. The functions $\Phi_\ell^{(k_r)}$ and $\Psi_\ell^{(k_s)}$ also satisfy the orthogonality relations within the domain of r_1, for each blade row such that[28]

$$\int_{h_r}^{1} r_1\Phi_\ell^{(k_r)}(r_1)\Phi_m^{(k_r)}(r_1)dr_1 = \delta_{\ell m} \tag{18a}$$

$$\int_{r_1=R_{sr}h_s}^{R_{sr}} e^{-r_{2s}} r_{2s}^{\bar{\alpha}}\Psi_\ell^{(k_s)}(r_{2s})\Psi_m^{(k_s)}(r_{2s})dr_{2s} = \delta_{\ell m} \tag{18b}$$

The Dirac delta functions $\delta(r_1 - \rho_r)$ and $\delta(r_1 - \rho_s)$ can be expanded in terms of the corresponding eigenfunctions such that

$$\delta(r_1 - \rho_r) = \sqrt{r_1\rho_r}\sum_{\ell=0}^{\infty}\Phi_\ell^{(k_r)}(\rho_r)\Phi_\ell^{(k_r)}(r_1) \tag{19a}$$

$$\delta(r_1 - \rho_s) = \sum_{\ell=0}^{\infty}\exp\left(-\frac{r_{2s}+\rho_{2s}}{2}\right)\Psi_\ell^{(k_s)}(\rho_{2s})$$

$$\times \Psi_\ell^{(k_s)}(r_{2s}) \tag{19b}$$

The pressure functions P_r and P_s must satisfy the end conditions of zero radial flow at both the hub and the tip for which the radial pressure gradients must vanish at both these points and we have the relations

$$\left.\begin{aligned} &\frac{d}{dr_1}\Phi_\ell^{(k_r)} = 0 \qquad \text{at } r_1 = h_r,\ r_1 = 1 \\ &\frac{d}{dr_1}\Psi_\ell^{(k_r)} + \left(\frac{\bar{\alpha}}{r_{2s}} - \frac{1}{2}\right)\Psi_\ell^{(k_s)} = 0 \\ &\text{at } r_{2s} = \nu_\ell^{(k_s)}R_{sr}h_s;\ r_{2s} = \nu_\ell^{(k_s)}R_{sr} \end{aligned}\right\} \tag{20}$$

from which we obtain an infinite set of eigenvalues $\lambda_\ell^{(k_r)}$, $\nu_\ell^{(k_s)}$ $(\ell = 0,1,2,\ldots,\infty)$ for each k_r and k_s.

The perturbation pressure amplitudes $\hat{p}_r$ and $\hat{p}_s$ in Eqs. (5) involve summation over k_r and k_s and a contour integration over a_r and a_s with the contour so chosen as to yield bounded or exponentially decaying terms in the axial direction. Because of the cumbersome and lengthy nature of the calculations, only the final expressions for subsonic axial flow are given herein as

$$p_r(\vec{r}_1,\vec{\rho}_r,t_1) = \frac{1}{4\pi\beta}\sum_{\ell=0}^{\infty}\sum_{k_r=-\infty}^{\infty}\frac{\Phi_\ell^{(k_r)}(\rho_r)\Phi_\ell^{(k_r)}(r_1)}{\Lambda_\ell^{(k_r)}}$$

$$\times \exp\left\{i\left[k_r(\theta - \varphi_r) + \omega_r t_1 + \left(\frac{M\hat{M}}{\beta^2}\right) Z_r\right] - \Lambda_\ell^{(k_r)} \frac{|\hat{Z}_r|}{\beta}\right\}$$

$$p_s(\vec{r}_1,\vec{\rho}_s,t_1) = \frac{i\beta^2 M}{\pi}\sum_{\ell=0}^{\infty}\sum_{k_s=-\infty}^{\infty}\frac{M_*^2\mu}{(\beta^2 - \mu^2M^2)\Delta_1\ell_1}$$

$$\times \exp\left[-\frac{r_{2s} + \rho_{2s}}{2}\right](r_{2s}\rho_{2s})^{\bar{\alpha}+\frac{1}{2}}\ \Psi_\ell^{(k_s)}(\rho_{2s})\Psi_\ell^{(k_s)}(r_{2s})$$

$$\times \exp i[k_s(\theta - \varphi_s) + \omega_s t_1] \qquad (21)$$

$$\times (CF \cos M\Delta_1\hat{Z}_s + SF \sin M\Delta_1\hat{Z}_s)$$

$$\times \exp [i\, MM_*\hat{Z}_s + M(i\mathcal{R}e\Delta_2 - |\mathcal{I}m\Delta_2|)\ |\hat{Z}_s|]$$

where

$$\mu = \frac{M_2k_s}{\ell_1};\ \Delta = 1 + \frac{\mu}{(\beta^2 - M^2\mu^2)};\ \hat{Z}_s = (z_1 - \zeta_s);$$

$$\hat{Z}_r = (z_1 - \zeta_s);\ \Delta_2 = \Delta - 1;$$

$$\Delta_1 = \left[(1 - \mu^2) + \beta^4\ \frac{(1 + \mu^2)}{M^2\mu^2}\right] \times \frac{\mu}{(\beta^2 - M^2\mu^2)} \qquad (22)$$

3. Representation of the Lifting Surface

The lifting surface of the rotor and stator blades can be regarded as composed of a distribution of pulsating pressure dipoles with constant interpole phase angle factors σ_r and σ_s and with the dipole axes normal to the local blade surface. The position of the unit dipole on the blade has the coordinates $[\rho_r, \bar{\varphi}_r + \varphi_r + m_r(\sigma_r + \delta_r); z_r + \zeta_r]$; $[\rho_s, \bar{\varphi}_s + \varphi_s + m_s(\sigma_s + \delta_s); z_s + \zeta_s]$ on the $(m_r)^{th}$ rotor and $(m_s)^{th}$ stator blade, respectively, in which φ_r and φ_s are the corresponding offset angles of the first blade (Fig. 3). The dipole field can be obtained from the pole field by differentiating along the local unit normal to the surface. The field of the Z_r and Z_s unit pressure dipoles on the blades of the rotor and stator can be obtained by summing over m_r and m_s. If Δp_r and Δp_s be the net upward-acting pressure at the points $\vec{\rho}_r(\rho_r,\Psi_r,\zeta_r)$ and $\vec{\rho}_s(\rho_s,\Psi_s,\zeta_s)$, we can obtain the resulting pressure field of the rotor and stator each in isolation by integrating the dipole field of the intensity Δp_r or Δp_s and integrating over the entire surface of each blade with respect to ρ and ζ. For subsonic axial flow, we have the rotor and stator pulsating dipole pressure field given by

$$P_R(\vec{r}_1,t_1) = -\frac{Z_r}{4\pi\beta}\int_{h_r}^{1}\int_{Z_{r_1}}^{z_{r2}}\sum_{\ell=0}^{\infty}\sum_{k_r=-\infty}^{\infty}\Delta p_r\Phi_*(\vec{r}_1,\vec{\rho}_r,t_1;\ \ell,k_r)\, d\rho_r d\zeta_r \qquad (23a)$$

$$P_s(\vec{r}_1,t_1) = \frac{i\beta^2MZ_s}{4\pi}\int_{R_{sr}h_s}^{R_{sr}}\int_{z_{s1}}^{z_{s2}}\sum_{\ell=1}^{\infty}\sum_{k_s=-\infty}^{\infty}\Delta P_s\Psi_*(\vec{r}_1,\vec{\rho}_s,t_1;\ \ell,k_s)\, d\rho_s d\zeta_s \qquad (23b)$$

where the functions Φ_* and Ψ_* are given in the appendix.

4. Effect of Camber and Thickness

The pressure loading Δp_r and Δp_s on the blades in isolation is caused by the combined effects of blade thickness, camber, and incidence and varies over the blade surface. It can be expressed by combining it with the rotor and stator eigenfunctions $\Phi_\ell^{(k_r)}$ and $\Psi_\ell^{(k_s)}$ as

$$\Delta p_r = \sum_{\ell=0}^{\infty} H_1(\tilde{\omega}_r,\tilde{\rho}_r;\ k_r,\ell) = \sum_{\ell=0}^{\infty}\left[F_{1,\ell}^{(k_r)}(\tilde{\omega}_r) + F_{2,\ell}^{(k_r)}(\tilde{\omega}_r)\right]\Phi_\ell^{(k_r)}(\rho_r) \qquad (24a)$$

$$\Delta p_s = \sum_{\ell=0}^{\infty} H_2(\tilde{\omega}_s,\rho_s;\ k_s,\ell) = \sum_{\ell=0}^{\infty}\left[G_{1,\ell}^{(k_s)}(\tilde{\omega}_s) + G_{2,\ell}^{(k_s)}(\tilde{\omega}_s)\right]\Phi_\ell^{(k_s)}(\rho_s) \qquad (24b)$$

where (F_1,F_2) and (G_1,G_2) are the chordwise variations for the rotor and stator due to the thickness and camber, respectively, in terms of the chordwise coordinates $\tilde{\omega}_r$ and $\tilde{\omega}_s$ related to the locally orthogonal coordinates (y'_r,z_r) and (y'_s,z_s) defined by

$$y'_r = -C_R \cos \tilde{\omega}_r \quad -C_r \leq y'_s \leq +C_R \quad 0 \leq \tilde{\omega}_r \leq \pi \qquad (25a)$$

$$y'_s = +C_S \cos \tilde{\omega}_s \quad -C_s \leq y'_s \leq +C_S \quad 0 \leq \tilde{\omega}_s \leq \pi \qquad (25b)$$

The orthogonal coordinates (r,y'_r,z'_r) and (r,y'_s,z'_s) are related to the Cartesian coordinates (X, Y, Z) by rotation through the angles (α_r,α_s) and (ψ_r,ψ_s) described in Eq. (26)

$$\begin{pmatrix} r \\ y'_r \\ z'_r \end{pmatrix} = \begin{pmatrix} \cos\psi_r & \sin\psi_r & 0 \\ -\sin\psi_r \sin\alpha_r & \cos\psi_r \sin\alpha_r & \cos\alpha_r \\ \sin\psi_r \cos\alpha_r & -\cos\psi_r \cos\alpha_r & \sin\alpha_r \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{26a}$$

$$\begin{pmatrix} r \\ y'_s \\ z'_s \end{pmatrix} = \begin{pmatrix} \cos\psi_s & \sin\psi_s & 0 \\ -\sin\psi_s \sin\alpha_s & \cos\psi_s \sin\alpha_s & \cos\alpha_s \\ \sin\psi_s \cos\alpha_s & -\cos\psi_s \cos\alpha_s & \sin\alpha_s \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{26b}$$

The axial positions of the leading and trailing edges of the blades are given by

$$z_{r1} = z_{r0} - C_R \cos\alpha_r;\; z_{r2} = z_{r0} + C_R \cos\alpha_r \tag{27a}$$

$$z_{s1} = z_{s0} - C_R \cos\alpha_s;\; z_{s2} = z_{s0} + C_S \cos\alpha_s \tag{27b}$$

where (ψ_r, ψ_s) and (ζ_r, ζ_s) are related to ω_r by the equations

$$\left.\begin{aligned} \psi_r &= \bar{\varphi}_r + \frac{2\pi m_r}{Z_r} - \cos^{-1}\left(\frac{2C_R \sin\alpha_r \cos\tilde{\omega}_r}{\rho_r}\right) \\ \psi_s &= \bar{\varphi}_s + \frac{2\pi m_s}{Z_s} - \cos^{-1}\left(\frac{2C_S \sin\alpha_s \cos\tilde{\omega}_s}{\rho_s}\right) \\ \zeta_r &= Z_{r,0} + 2c_R \cos\alpha_r \cos\tilde{\omega}_r \\ \zeta_s &= Z_{s,0} + 2c_S \cos\alpha_s \cos\tilde{\omega}_s \end{aligned}\right\} \tag{28}$$

The offset angle $\bar{\varphi}_r$ of the first blade on the rotor can be written as $\bar{\varphi}_r = \Omega_0 t_1$, where $\Omega_0 = \Omega/t_0$; the offset angle $\bar{\varphi}_s$ of the first blade on the stator can be set equal to zero. The terms $(2\pi m_r/Z_r)$ and $(2\pi m_r/Z_s)$ represent the polar angles of the $(m_r)^{th}$ rotor and $(m_s)^{th}$ stator blades, respectively, at the midpoint of the blade chord. The last terms in ψ_r and ψ_s give the increment in the polar angles for points (y',z') away from the midchord point at any radius. We shall represent the functions (F_1,F_2) and (G_1,G_2) in an infinite trigonometric series expansion corresponding to the Glauert/Birnbaum series as

$$F_{1,\ell}^{(k_r)}(\tilde{\omega}_r) = A_0^{(k_r)} \cot\frac{\tilde{\omega}_r}{2} + \sum_{m=1}^{\infty} A_m^{(k_r)} \sin m\tilde{\omega}_r \tag{29a}$$

$$F_{2,\ell}^{(k_r)}(\tilde{\omega}_r) = B_0^{(k_r)}\left(\cot\frac{\tilde{\omega}_r}{2} - 2\sin\tilde{\omega}_r\right) + \sum_{m=2}^{\infty} B_m^{(k_r)} \sin m\tilde{\omega}_r \tag{29b}$$

$$G_{1,\ell}^{(k_s)}(\tilde{\omega}_s) = C_0^{(k_s)} \cot\frac{\tilde{\omega}_s}{2} + \sum_{m=1}^{\infty} C_m^{(k_s)} \sin m\tilde{\omega}_s \tag{29c}$$

$$G_{2,\ell}^{(k_s)}(\tilde{\omega}_s) = D_0^{(k_s)}\left(\cot\frac{\tilde{\omega}_s}{2} - 2\sin\tilde{\omega}_s\right) + \sum_{m=2}^{\infty} D_m^{(k_s)} \sin m\tilde{\omega}_s \tag{29d}$$

where the coefficients $A_m^{(k_r)}$, $B_m^{(k_r)}$, $C_m^{(k_s)}$, and $D_m^{(k_s)}$ can be determined by making the blade surfaces as stream surfaces. The series in Eqs. (29) satisfy the Kutta condition at the blade trailing edge and have the usual singularity at the leading edge. It can be observed from Eqs. (24) that thickness and camber effects are inseparable and are interwoven. Note that the thickness effect corresponds to the monopole solution and can be obtained directly.

5. Boundary Conditions

By defining the mean camberline and thickness shapes of the blades through the equations

$$\left.\begin{aligned} z'_{Cr} &= z'_{Cr}(y'_r);\; z'_{Cs} = z'_{Cs}(y'_s) \\ z'_{Tr} &= z'_{Tr}(y'_r);\; z'_{Ts} = z'_{Ts}(y'_s) \end{aligned}\right\} \tag{30}$$

The upper and lower surface blade profiles are given by

$$\left.\begin{aligned} z'_{Ur} &= z'_{Cr} + z'_{Tr};\; z'_{Lr} = z'_{Cr} - z'_{Tr} \\ z'_{Us} &= z'_{Cs} + z'_{Ts};\; z'_{Ls} = z'_{Cs} - z'_{Ts} \end{aligned}\right\} \tag{31}$$

If $\vec{U}_r$ and $\vec{U}_s$ be the resultants of the free-stream and induced velocity perturbations of the rotor-stator flow field, we can represent the surface flow tangency condition on the upper and lower blade surfaces by the equations

$$\tau_{U1} = \left(\frac{dz'_U}{dy'}\right)_r = \left(\frac{u_{Rz'}}{u_{Ry'}}\right)_{z'_r=0+};\; \tau_{U2} = \left(\frac{dz'_U}{dy'_s}\right) = \left(\frac{u_{Sz'}}{u_{Sy'}}\right)_{z'_s=0+} \tag{32a}$$

$$\tau_{L1} = \left(\frac{dz'_L}{dy'}\right)_r = \left(\frac{u_{Rz'}}{u_{Ry'}}\right)_{z'_r=0-};\; \tau_{L2} = \left(\frac{dz'_L}{dy'}\right)_s = \left(\frac{u_{Sz'}}{u_{Sy'}}\right)_{z'_s=0-} \tag{32b}$$

In keeping with the thin airfoil theory postulates, the boundary conditions are satisfied on the blade chord.

To obtain the induced velocity perturbations of the rotor and stator, it is convenient to introduce a local intrinsic helical orthogonal coordinate system (Fig. 4) for each based on the corresponding undisturbed streamline with the helix angles θ_{h1} and θ_{h2} defined by

$$\tan \theta_{h1} = \frac{R_*}{r_1}$$

$$\tan \theta_{h2} = \frac{R_*}{(r_1 - M_2/\bar{M})} \tag{33}$$

The helical intrinsic coordinates $(\rho_r, \tau_1, s_1,)$ and (ρ_s, τ_2, s_2) of the rotor and stator are related to the cylindrical coordinates by the matrix transformations of differential lengths as

$$\begin{pmatrix} dr \\ d\theta \\ dZ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\theta_{h1} & -\cos\theta_{h1} \\ 0 & \cos\theta_{h1} & \sin\theta_{h1} \end{pmatrix} \begin{pmatrix} dR \\ d\sigma_1 \\ dt_1 \end{pmatrix}$$

$$\begin{pmatrix} dr \\ d\theta \\ dz \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\theta_{h2} & -\cos\theta_{h2} \\ 0 & \cos\theta_{h2} & \sin\theta_{h2} \end{pmatrix} \begin{pmatrix} dR \\ d\sigma_2 \\ d\tau_2 \end{pmatrix} \tag{34}$$

The equations of motion in the intrinsic coordinates can be written as

$$\left. \begin{aligned} \left(\frac{\partial v_{r1}}{\partial s_1}, \frac{\partial v_{\tau 1}}{\partial s_1}, \frac{\partial v_{\sigma 1}}{\partial s_1}\right) &= -\frac{1}{rM_r}\left(\frac{\partial P_r}{\partial r_1}, \frac{\partial P_r}{\partial \tau_1}, \frac{\partial P_r}{\partial s_1}\right) \\ \left(\frac{\partial v_{r2}}{\partial s_2}, \frac{\partial v_{\tau 2}}{\partial s_2}, \frac{\partial v_{\sigma 2}}{\partial s_2}\right) &= -\frac{1}{rM_s}\left(\frac{\partial P_s}{\partial r_2}, \frac{\partial P_s}{\partial \tau_2}, \frac{\partial P_s}{\partial s_2}\right) \end{aligned} \right\} \tag{35}$$

where M_r and M_s are the resultant free-stream Mach numbers of the rotor and stator, defined by

$$\left. \begin{aligned} M_r^2 &= M^2\left(1 + \frac{r_1^2}{R_*^2}\right) \\ M_s^2 &= M^2\left[1 + \left(\frac{r_1}{R_*} - \tan\alpha_{2r}\right)^2\right] \end{aligned} \right\} \tag{36}$$

The velocity perturbations $(v_{r1}, v_{\sigma 1}, v_{\tau 1})$ and $(v_{r2}, v_{\sigma 2}, v_{\tau 2})$ of the rotor and the stator separately can be obtained by integrating Eqs. (35) along a helical streamline by putting $\theta_{h1} = (\theta - z_1/R_*)$ and integrating over z_1. We thus obtain the resultant free-stream velocity $\vec{U}_r, \vec{U}_s$ from the equations

$$\begin{Vmatrix} U_{R1} \\ U_{Ry'} \\ U_{Rz'} \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \sin\alpha_r & \cos\alpha_r \\ 0 & -\cos\alpha_r & \sin\alpha_r \end{Vmatrix} \begin{Vmatrix} v_r \\ v_\theta + r_1/R_* \\ v_z + 1 \end{Vmatrix} \tag{37a}$$

$$\begin{Vmatrix} U_{S1} \\ U_{Sy'} \\ U_{Sz'} \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \sin\alpha_s & \cos\alpha_s \\ 0 & -\cos\alpha_s & \sin\alpha_s \end{Vmatrix} \begin{Vmatrix} v_r \\ v_\theta - \tan\alpha_2 + \frac{r_1}{R_*} \\ v_z + 1 \end{Vmatrix} \tag{37b}$$

where the total perturbation velocity of the rotor-stator combination (v_r, v_θ, v_z) is expressed as

$$\begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix} = \begin{pmatrix} v_{rr} \\ v_{r\theta} \\ v_{rz} \end{pmatrix} + \begin{pmatrix} v_{sr} \\ v_{s\theta} \\ v_{sz} \end{pmatrix} + \begin{pmatrix} v_{ir} \\ v_{i\theta} \\ v_{iz} \end{pmatrix} \tag{38}$$

The first two vectors on the right of Eq. (38) are the perturbation velocities of the rotor and stator, each in isolation, and the last vector is the interference velocity due to the simultaneous presence in the compressor stage of the two rows in close proximity. We shall assume that the interference velocity vector is a linear combination of the perturbation velocities due to the rotor and stator, so that Eq. (38) can be written

$$\begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix} = (1 + \varepsilon)\begin{pmatrix} v_{rr} + v_{sr} \\ v_{r\theta} + v_{s\theta} \\ v_{rz} + v_{sz} \end{pmatrix} \tag{39}$$

This is the simplest assumption that can be made for the interference effect and permits us to estimate the magnitude of the interaction. By defining the parameters

$$\left. \begin{aligned} &\theta_1 = \theta_{h1} - \alpha_r;\ \theta_3 = \theta_{h1} - \alpha_s;\ \varepsilon_1 = 1/(1 + \varepsilon); \\ &\theta_2 = \theta_{h2} - \alpha_r;\ \theta_4 = \theta_{h2} - \alpha_s \end{aligned} \right\} \tag{40}$$

and combining Eqs. (32) and (40), we can obtain the perturbations $v_{\sigma 1}$, $v_{\tau 1}$, $v_{\sigma 2}$, and $v_{\tau 2}$ in the form

$$\begin{pmatrix} v_{\sigma 1} \\ v_{\sigma 2} \\ v_{\tau 1} \\ v_{\tau 2} \end{pmatrix} [AA] = \varepsilon_1 [BB] \tag{41}$$

where the matrices [AA] and [BB] are given by

$$AA = \begin{Vmatrix} -(\sin\theta_1 + \tau_{U1}\cos\theta_1) & -(\sin\theta_2 + \tau_{U1}\cos\theta_2) & (\cos\theta_1 - \tau_{U1}\sin\theta_1) & (\cos\theta_2 - \tau_{U1}\sin\theta_2) \\ -(\sin\theta_1 + \tau_{L1}\cos\theta_1) & -(\sin\theta_2 + \tau_{L1}\cos\theta_2) & (\cos\theta_1 - \tau_{L1}\sin\theta_1) & (\cos\theta_2 - \tau_{L1}\sin\theta_2) \\ -(\sin\theta_3 + \tau_{U2}\cos\theta_3) & -(\sin\theta_4 + \tau_{U2}\cos\theta_4) & (\cos\theta_3 - \tau_{U2}\sin\theta_3) & (\cos\theta_4 - \tau_{U2}\sin\theta_4) \\ -(\sin\theta_3 + \tau_{L2}\cos\theta_3) & -(\sin\theta_4 + \tau_{L2}\cos\theta_4) & (\cos\theta_3 - \tau_{L2}\sin\theta_3) & (\cos\theta_4 - \tau_{L2}\sin\theta_4) \end{Vmatrix} \quad (42)$$

$$B = \left\{ \begin{array}{l} [(\tau_{U1}\sin\alpha_r + \cos\alpha_r)(r_1/R_*) + (\tau_{U1}\cos\alpha_r - \sin\alpha_r)]_{z_r'=0+} \\ [(\tau_{L1}\sin\alpha_r + \cos\alpha_r)(r_1/R_*) + (\tau_{L1}\cos\alpha_r - \sin\alpha_r)]_{z_r'=0-} \\ [(\tau_{U2}\sin\alpha_s + \cos\alpha_s)(r_1/R_* - \tan\alpha_2) + (\tau_{U2}\cos\alpha_s - \sin\alpha_s)]_{z_s'=0+} \\ [(\tau_{L2}\sin\alpha_s + \cos\alpha_s)(r_1/R_* - \tan\alpha_2) + (\tau_{L2}\cos\alpha_s - \sin\alpha_s)]_{z_s'=0-} \end{array} \right\} \quad (43)$$

The velocity perturbations $v_{\sigma 1}$, $v_{\sigma 2}$, $v_{\tau 1}$, and $v_{\tau 2}$ can be expressed in terms of the coefficients of the Glauert series expansions and written as the system of equations in matrix form

$$\begin{Vmatrix} f_{1,m} & g_{1,m} \\ f_{2,m} & g_{2,m} \\ f_{3,m} & g_{3,m} \\ f_{4,m} & g_{4,m} \end{Vmatrix} \begin{Vmatrix} \mathscr{A}_m^{(k)} \\ \mathscr{B}_m^{(k)} \end{Vmatrix} = \epsilon_1 \begin{Vmatrix} \mathscr{C}_1 \\ \mathscr{C}_2 \\ \mathscr{C}_3 \\ \mathscr{C}_4 \end{Vmatrix} \quad (44)$$

$$K = 0, \pm 1, \pm 2, \ldots, \pm\infty$$

The submatrices f, g, and $\mathscr{C}$ in Eq. (44) are functions of the position vector and are not given here because of space limitations but may be obtained from Ramachandra[28]. The coefficients $\mathscr{A}_m^{(k)}$ and $\mathscr{B}_m^{(k)}$ $(m = 0, 1, 2, \ldots, \infty)$ are defined in terms of $A_i^{(k)}$, $B_i^{(k)}$, $C_i^{(k)}$, and $D_i^{(k)}$ as

$$\mathscr{A}_0^{(k)} = A_0^{(k)} + B_0^{(k)}; \mathscr{A}_1^{(k)} = A_1^{(k)} - 2B_0^{(k)};$$

$$\mathscr{A}_m^{(k)} = A_m^{(k)} + B_m^{(k)}; \mathscr{B}_0^{(k)} = C_0^{(k)} + D_0^{(k)};$$

$$\mathscr{B}_1^{(k)} = C_1^{(k)} - 2D_0^{(k)}; \mathscr{B}_m^{(k)} = C_m^{(k)} + D_m^{(k)}$$

$$m = 2, 3, \ldots \quad (45)$$

6. Rotor-Stator Matching and Final Solution

With each of the rotor and stator blades is associated a bound vortex of strength Γ_r, Γ_s producing the local lift $L = \rho V \Gamma$ per unit span according to the Kutta-Joukowski theorem. The strengths Γ_r and Γ_s of the bound vortex can be obtained from a chordwise integration of Δp_r and Δp_s as a function of r_1 and z_1 as

$$\Gamma_r \rho_\infty W_a \left(1 + \frac{r_1^2}{R_*^2}\right)^{1/2} = \int_{\hat{Z}_{r1}}^{\hat{Z}_{r2}} \Delta P_r d\hat{Z}_r \quad (46a)$$

$$\Gamma_s \rho_\infty W_a \left[1 + \left(\frac{r_1}{R_*} - \tan\alpha_2\right)^2\right]^{1/2} = \int_{\hat{Z}_{r1}}^{\hat{Z}_{r2}} \Delta P_s d\hat{Z}_s$$

(46b)

We assume that the rotor and stator are matched at any operating condition when the net vorticity behind the stator is zero. If Γ_{rh} and Γ_{sh} are bound vortex strength of the rotor and stator at the blade hub, we express the matching condition by the relation

$$Z_r \Gamma_{rh} + Z_s \Gamma_{sh} = 0 \quad (47)$$

By combining Eqs. (24), (29), (45), and (46), we can express the matching condition (Eq. (46)) as

$$\sum_{m=0}^{\infty} \left(f_{5,m} \mathscr{A}_m^{(k)} + g_{5,m} \mathscr{B}_m^{(k)}\right) = 0 \qquad k = 0, \pm 1, \pm 2, \ldots, \pm\infty$$

(48)

The reader is referred to Ramachandra[29] for the expressions of f_5 and g_5 in Eq. (48). A composite equation can be written in matrix form for the unknowns $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{E}_1$ by combining Eqs. (44) and (48) as

$$\mathscr{F}^{(k)} * \mathscr{X}^{(k)} = 0 \quad (49)$$

where $\mathscr{X}^{(k)}$ is the vector of unknowns

$$\mathscr{X}^{(k)} = \mathscr{A}_m^{(k)} \mathscr{B}_m^{(k)} \mathscr{E}_1{}^T \qquad m = 0,1,2,\ldots \tag{50}$$

and $\mathscr{F}^{(k)}$ is the matrix

$$\mathscr{F}^{(k)} = \begin{Vmatrix} f_{1,m} & g_{1,m} - \mathscr{C}_1 \\ f_{2,m} & g_{2,m} - \mathscr{C}_2 \\ f_{3,m} & g_{3,m} - \mathscr{C}_3 \\ f_{4,m} & g_{4,m} - \mathscr{C}_4 \\ f_{5,m} & g_{5,m} - 0 \end{Vmatrix} \tag{51}$$

$$m = 0,1,2,\ldots$$

The eigenvectors of the matrix $\mathscr{F}$ in Eq. (49) give the coefficients $\mathscr{A}^{(k)}$, $\mathscr{B}^{(k)}$, and $\mathscr{E}_1$ ($K = 0,\pm1,\pm2,\ldots,\pm\infty$) from which the pressure loading Δp_r and Δp_s on the blades is obtained. Equations (45) are insufficient to determine the coefficients A, B, C, and D of the Glauert series of Eq. (29) in order to separate the thickness and camber contributions.

7. Combined Pressure Field

It is now possible to express the pulsating pressure fields P_R and P_S of the rotor and stator from Eq. (23) inclusive of the interaction effects. The resultant pressure P at any point in the flow is the sum of the pressures P_R and P_S

$$P(\vec{r}_1,t_1;\, k_r,k_s,\ell;\, \omega_r,\omega_s) = P_R(\vec{r}_1,t_1;\, k_r,\ell;\, \omega_r) + P_S(\vec{r}_1,t_1;\, k_s,\ell;\, \omega_s) \tag{52}$$

Assuming that the free-stream values of the streamwise components of the perturbation velocities $v_{\sigma1,\infty} = v_{\sigma2,\infty} = 0$, we can write Eq. (52) in the form

$$(4\pi\beta/Z_r)P(\vec{r}_1,t;\, k_r,k_s,\ell;\, \omega_r,\omega_s) = \sum_{j,\ell,m=0}^{\infty} \mathscr{U}\mathscr{R}e^{i\omega_r t_1} - i\mathscr{V}\mathscr{S}(4\beta^3 MZ_s/Z_r)e^{i\omega_s t_1} \tag{53}$$

where

$$\mathscr{U} = \mathscr{U}_1\mathscr{U}_2;\quad \mathscr{V} = \mathscr{V}_1\mathscr{V}_2;\quad \mathscr{R} = \mathscr{R}_1\mathscr{R}_2\mathscr{R}_3;\quad \mathscr{S} = \mathscr{S}_1\mathscr{S}_2\mathscr{S}_3 \tag{54}$$

The terms $(\mathscr{R}_1,\mathscr{S}_1)$, $(\mathscr{R}_2,\mathscr{S}_2)$, and $(\mathscr{R}_3,\mathscr{S}_3)$ are, respectively, functions of r_1, θ, and z_1 alone; $\mathscr{U}_1$, $\mathscr{V}_1$, $\mathscr{U}_2$, and $\mathscr{V}_2$ are constants obtained in terms of the geometry of the stage, the number of blades, and the flow conditions. Again, because of space limitations, the reader is referred to Ramachandra[28] for detailed expressions of these quantities.

The acoustic modes and their characteristics, including their attenuation and resonance properties in ducts, can be obtained from Eq. (53). However, these discussions will be given in separate papers elsewhere.

8. Discussions and Conclusions

A lifting surface theory has been formulated for the two rotor-stator blade rows of an axial compressor stage in terms of the acceleration potential. Since the acceleration potential equals the ratio of the perturbation pressure and the mean flow density, the method is equivalent to the determination of the acoustic field of a rotor-stator combination in a duct of infinite length in terms of the flow Mach number, the geometry, and the oscillation frequency of a distribution of pulsating pressure dipoles through a lifting surface theory. The combined effects of blade thickness and camber have been included as well as that of incidence although it has not been possible to separate them. Curiously, it has been possible to separate their effects when the fluid is incompressible[27]. The perturbation pressures designated in terms of their mode numbers (k_r, k_s, and ℓ) form a superposition of an infinite number of acoustic modes consisting of the circumferential wave numbers k_r and k_s and the radial mode numbers ℓ. It has also been possible to isolate the rotor-stator interaction effects by matching the blade rows to give zero vorticity downstream of the stator as in an ideal combination. This assumption may appear restrictive in a multistage compressor. However, this can be relaxed and the same method can be used to determine the coefficient vector by using a finite prescribed wake vorticity condition in Eq. (47). From this general solution it is possible to obtain special cases of interest like that of a single or twin tandem actuator disk or single or twin two-dimensional cascade rows. This procedure could be extended to the study of counterrotating propellers, of course, by modifying the end conditions in Eq. (20) at the blade tip. This results in a different set of eigenvalues and the corresponding eigenfunctions.

APPENDIX

In this appendix expressions for the quantities mentioned in the paper are given without their detailed derivations. The functions I and K are, respectively, modified Bessel and Hankel functions of the first kind normalized over the range $h_r \le r_1 \le 1$. The radial eigenfunction $\Phi^{(k_r)}$ given in Eq. (A1) appears somewhat different from the form used by McCune[33] for the rotor. The radial eigenfunctions $\Phi^{(k_r)}$ and $\Psi^{(k_s)}$ of the rotor and stator are given by

$$\Phi^{(k_r)}(r_1) = \left[\frac{K^{(k_r-1)}(\lambda^{(k_r)}h_r) + K^{(k_r+1)}(\lambda^{(k_r)}h_r)}{I^{(k_r-1)}(\lambda^{(k_r)}h_r) + I^{(k_r+1)}(\lambda^{(k_r)}h_r)}\right] \times I^{(k_r)}(\lambda^{(k_r)}r_1) + K^{(k_r)}(\lambda^{(k_r)}r_1) \tag{A1}$$

$$\Psi^{(k_s)}(r_{2s}) = \hat{C}^{(k_s)} L^{(2\alpha)}_{\left(\chi^{(k_s)}-\bar{\alpha}-1/2\right)}(r_{2s}) + L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)}+\bar{\alpha}-1/2\right)}(r_{2s}) \tag{A2}$$

where

$$\hat{C}^{(k_s)} = \frac{\left(\frac{2\bar{\alpha}}{\nu_1} + \frac{1}{2}\right) L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} + \bar{\alpha} - 1/2\right)}(\nu_1) - \frac{\left(\chi^{(k_s)} + \bar{\alpha} - \frac{1}{2}\right)}{\nu_1} L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} - \bar{\alpha} - 3/2\right)}(\nu_1)}{\frac{1}{2} L_{\left(\chi^{(k_s)} - \alpha - 1/2\right)}(\nu_1) - \frac{\left(\chi^{(k_s)} + \bar{\alpha} - \frac{1}{2}\right)}{\nu_1} L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} - \bar{\alpha} - 3/2\right)}(\nu_1)} \quad \text{(A3)}$$

The eigenvalues $\lambda^{(k_r)}$ and $\chi^{(k_s)}$ are obtained as roots of the equations

$$\left[K^{(k_r-1)}(\lambda_1) + K^{(k_r+1)}(\lambda_1)\right]\left[I^{(k_r-1)}\left(\lambda^{(k_r)}\right) + I^{(k_r+1)}\left(\lambda^{(k_r)}\right)\right] - \left[K^{(k_r-1)}\left(\lambda^{(k_r)}\right) + K^{(k_r+1)}\left(\lambda^{(k_r)}\right)\right] \times \left[I^{(k_r-1)}(\lambda_1) + I^{(k_r+1)}(\lambda_1)\right] = 0 \quad \text{(A4)}$$

$$\left[\left(\frac{2\bar{\alpha}}{\nu_1} + \frac{1}{2}\right) L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} + \bar{\alpha} - 1/2\right)}(\nu_1) - \frac{\chi^{(k_s)} + \bar{\alpha} - \frac{1}{2}}{\nu^{(k_s)} R_{sr}} \times L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} - \bar{\alpha} - 3/2\right)}(\nu_1)\right] \times \left[\frac{1}{2} L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} - \bar{\alpha} - 1/2\right)} \times \left(\nu^{(k_s)} R_{sr}\right) - \frac{\chi^{(k_s)} + \bar{\alpha} - \frac{1}{2}}{\nu^{(k_s)} R_{sr}} L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} - \bar{\alpha} - 3/2\right)}\left(\nu^{(k_s)} R_{sr}\right)\right]$$

$$+ \left[\left(\frac{2\bar{\alpha}}{\nu^{(k_s)} R_{sr}} + \frac{1}{2}\right) L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} + \bar{\alpha} - 1/2\right)}\left(\nu^{(k_s)} R_{sr}\right) - \frac{\chi^{(k_s)} + \bar{\alpha} - \frac{1}{2}}{\nu^{(k_s)} R_{sr}} L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} - \bar{\alpha} - 1/2\right)}\left(\nu^{(k_s)} R_{sr}\right)\right]$$

$$\times \left[\frac{1}{2} L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} - \bar{\alpha} - 1/2\right)}(\nu_1) - \frac{\chi^{(k_s)} + \bar{\alpha} - \frac{1}{2}}{\nu_1} - \frac{\chi^{(k_s)} + \bar{\alpha} - \frac{1}{2}}{\nu_1} L^{(2\bar{\alpha})}_{\left(\chi^{(k_s)} - \bar{\alpha} - 3/2\right)}(\nu_1)\right] = 0 \quad \text{(A5)}$$

$$\Phi_*(\vec{r}_1, \vec{\rho}_r, t_1;\, k_r, \ell) = \frac{\Phi_\ell^{(k_r)}(\rho_r)\,\Phi_\ell^{(k_r)}(r_1)}{\Lambda_\ell^{(k_r)}} \times \left[\frac{ik_r}{\rho_r} + \tan \alpha_r \left\{\frac{\Lambda_\ell^{(k_r)}}{\beta} \operatorname{sgn} \hat{Z}_r - iM\hat{M}\hat{Z}_r\right\}\right] \times \exp\left[iM\hat{M}\hat{Z}_r - \frac{\Lambda_\ell^{(k_r)}}{\beta} \left|\hat{Z}_r\right|\right] \times \exp i[k_r(\theta - \bar{\varphi}_r - \varphi_r) + \omega_r t_1] \quad \text{(A6)}$$

$$\Psi_*(\vec{r}_1, \vec{\rho}_s, t_1;\, k_s, \ell) = \exp\left[-\frac{(r_{2s} + \rho_{2s})}{2}\right]\left(r_{2s}\rho_{2s}\right)^{\bar{\alpha}+1/2} \times \Psi_\ell^{(k_s)}(\rho_{2s})\,\Psi_\ell^{(k_s)}(r_{2s})$$

$$\times \left[\left[-\frac{ik_s}{\rho_s}(CF \cos M\Delta_1\hat{Z}_s + SF \sin M\Delta_1\hat{Z}_s) + M \tan \alpha_s \times \left\{\Delta_1 (SF \cos M\Delta_1\hat{Z}_s - CF \sin M\Delta_1\hat{Z}_s) + [iM_* (i\mathcal{R}e\Delta_2 - |\mathcal{I}m\Delta_2| \operatorname{sgn} \hat{Z}_s] (CF \cos M\Delta_1\hat{Z}_s + SF \sin M\Delta_1\hat{Z}_s)\right\}\right]\right] \exp [iMM_*\hat{Z}_s + M (i\mathcal{R}e\Delta_2 - |\mathcal{I}m\Delta_2|)\ \hat{Z}_s\ ] \exp i[k_s(\theta - \bar{\varphi}_s - \varphi_s)] \times \exp (i\omega_s t_1) \quad \text{(A7)}$$

where CF and SF are functions defined in Ramachandra[28] and will not be given here because of their lengthy nature.

References

1. Kemp, N. H. and Sears, W. R., "Aerodynamic Interference Between Moving Blade Rows," Journal of the Aeronautical Sciences, vol. 20, Sept. 1953, pp. 585-597.

2. Kemp, N. H. and Sears, W. R.," The Unsteady Forces Due to Viscous Wakes in Turbomachines," Journal of the Aeronautical Sciences, vol. 22, July 1955, pp. 478-483.

3. Lane, F. and Friedman, M., "Theoretical Investigation of Subsonic Oscillatory Blade-Row Aerodynamics," NACA TN 4136, 1958.

4. Carta, F. O., "Unsteady Aerodynamic Theory of a Staggered Cascade of Oscillating Airfoils in Compressible Flow," United Aircraft Corp. Res. Rep. UAC R-0582-19, 1957.

5. Carta, F. O. and Fanti, R., "Aerodynamic Interaction Effects of Cascaded Airfoils Oscillating in Two Dimensional Compressible Flow," United Aircraft Corp. Res. Rep. UAC R-0582-17, 1957.

6. Hetherington, R., "Compressor Noise Generated by Fluctuating Lift Resulting from Rotor-Stator Interaction," AIAA Journal, vol. 1, Feb. 1963, pp. 473-474.

7. Parker, R., "Resonance Effects in Wake Shedding from Parallel Plates: Some Experimental Observations," Journal of Sound and Vibration, vol. 4, July 1966, pp. 62-72.

8. Parker, R., "Resonance Effects in Wake Shedding from Parallel Plates: Calculation of Resonant Frequencies," Journal of Sound and Vibration, vol. 5, March 1967, pp. 330-343.

9. Parker, R. and Pryce, D. C., "Wave Excited Resonances in an Annular Cascade: An Experimental Investigation," Journal of Sound and Vibration, vol. 37, Nov. 1974, pp. 247-261.

10. Tyler, J. M. and Sofrin, T. G., "Axial Flow Compressor Noise Studies," SAE Preprint 345D, Society of Automotive Engineers, 1961.

11. Tyler, J. M. and Sofrin, T. G., "Axial Flow Compressor Noise Studies," SAE Transactions, vol. 70, 1962, p. 309.

12. Smith, S. N., "Discrete Frequency Sound Generation in Axial Flow Turbomachines," ARC R&M 3709, 1973.

13. Whitehead, D. S., "Vibration and Sound Generation in a Cascade of Flat Plates in Subsonic Flow," ARC R&M 3685, 1970.

14. Morfey, C. L., "Sound Generation in Subsonic Turbomachinery," Transactions of the ASME Journal of Basic Engineering, Ser. D, vol. 92, Sept. 1970., pp. 450-458.

15. Morfey, C. L., "A Review of Sound Generating Mechanisms in Aircraft Engine Fans and Compressors," Aerodynamic Noise, Proceedings of the Symposium, edited by H. S. Ribner, Univ. of Toronto Press, Toronto, 1969, pp. 299-329.

16. Morfey, C. L., "Rotating Pressure Patterns in Ducts: Their Generation and Transmission," Journal of Sound and Vibration, vol. 1, Jan. 1964, pp. 60-87.

17. Mani, R., "Compressibility Effects in the Kemp and Sears Problem," Fluid Mechanics, Acoustics and Design of Turbomachinery, Part II, edited by B. Lakshminarayana, W. R. Britsch, and W. S. Gearhart, NASA SP-304, 1974, pp. 513-536.

18. Mani, R., "Discrete Frequency Noise Generation from an Axial Flow Fan Blade Row," Transactions of the ASME, Journal of Basic Engineering, Ser. D, vol. 93, Mar. 1970, p. 37.

19. Mani, R. and Horvay, G., "Sound Transmission Through Blade Rows", Journal of Sound and Vibration, vol. 12, May 1970, pp. 59-83.

20. Kaji, S. and Okazaki, T., "Propagation of Sound Through a Blade Row. Part I. Analysis Based on the Semi-Actuator Disk Theory," Journal of Sound and Vibration, vol. 11, Mar. 1970, pp. 339-353.

21. Kaji, S. and Okazaki, T, "Propagation of Sound Waves Through a Blade Row. Part II. Analysis Based on the Acceleration Potential Method," Journal of Sound and Vibration, vol. 11, Mar. 1970, pp. 355-375.

22. Kaji, S. and Okazaki, T., "Generation of Sound by Rotor-Stator Interaction," Journal of Sound and Vibration, vol. 13, Nov. 1970, pp. 281-307.

23. Slutsky, S., "Discrete Noise Generation and Propagation by a Fan Engine," Aerodynamic Noise, Proceedings of the Symposium, edited by H. S. Ribner, Univ. of Toronto Press, Toronto, 1969, pp. 331-349.

24. Namba, M., "Three-Dimensional Analysis of Blade Force and Sound Generation for an Annular Cascade in Distorted Flows," Journal of Sound and Vibration, vol. 50, Feb. 1977, pp. 479-508.

25. Namba, M., "Lifting Surface Theory for a Rotating Subsonic or Transonic Blade Row," ARC R&M 3740, 1974.

26. Mariano, S., "Ground Effect of a Plane Uniform Sound Source Distribution," Journal of Sound and Vibration, vol. 11, Jan. 1970, pp. 145-148.

27. Ramachandra, S. M., "Lifting Theory for a Turbomachine Stage." NASA TM to be published.

28. Ramachandra, S. M., "The Combined Pressure Field of an Axial Compressor Stage." NASA TM to be published.

29. Flugge, S., ed., "Handbuch Der Physik, Vol. 1, Mathematische Methoden 1," (Encyclopedia of Physics, vol. 1, Mathematical Methods 1), Springer-Verlag, Berlin, 1956.

30. Magnus, W., Oberhettinger, F., and Soni, R. P., "Formulas and Theorems for the Special Functions of Mathematical Physics," Springer-Verlag, New York, 1966.

31. Szego, Gabor, "Orthogonal Polynomials," American Mathematical Society, New York, 1959.

32. Kotorynski, W. P., "On an Irregular Sturm-Liouville Problem in Boundary Layer Theory," SIAM Journal of Applied Mathematics, vol. 16, Nov. 1968, pp. 1132-1140.

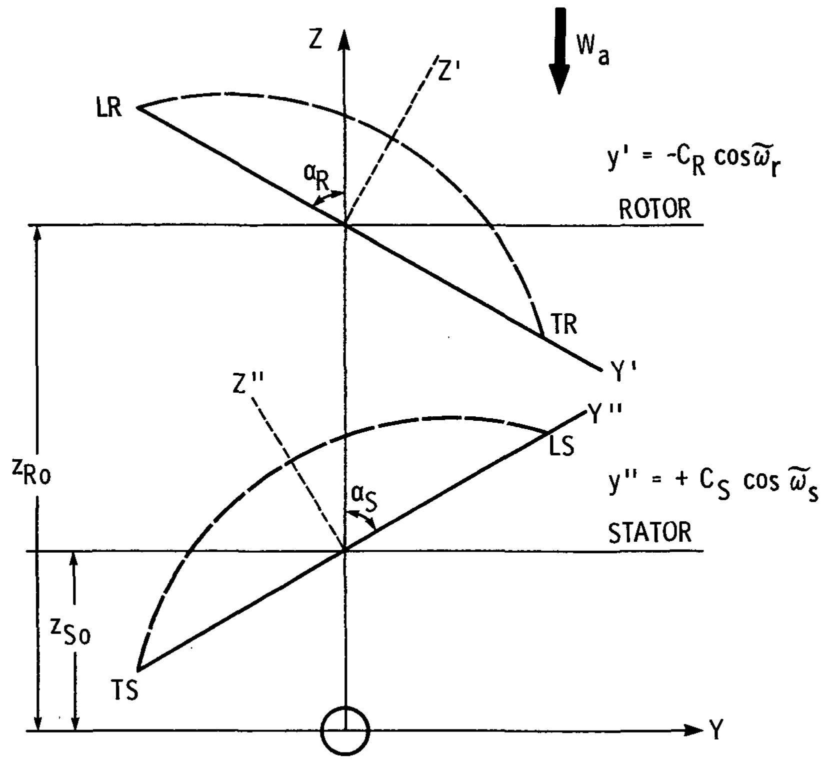

Fig. 1 Local coordinate system for blades.

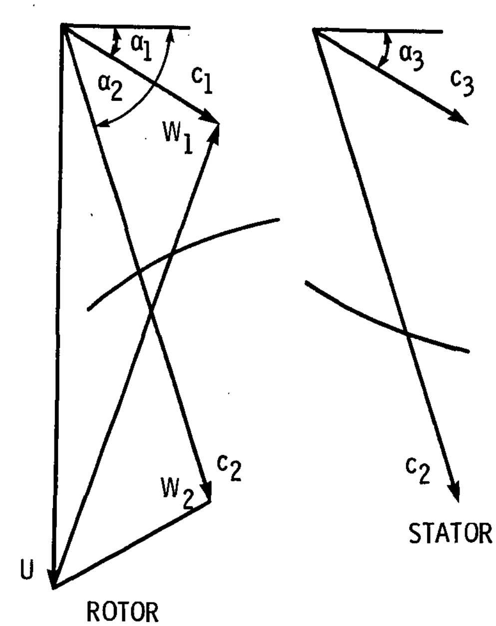

Fig. 2 Velocity diagram of stage.

$$\bar{\varphi}_r + \varphi_r + \frac{2\pi m_r}{z_r} = \psi_r$$

$$\bar{\varphi}_s + \varphi_s + \frac{2\pi m_s}{z_s} = \psi_s$$

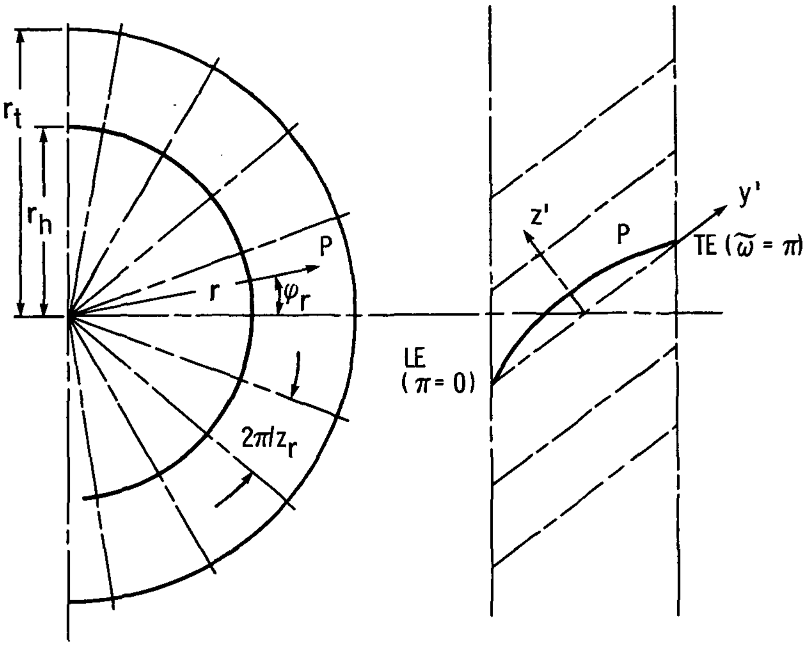

Fig. 3 Schematic diagram of casade.

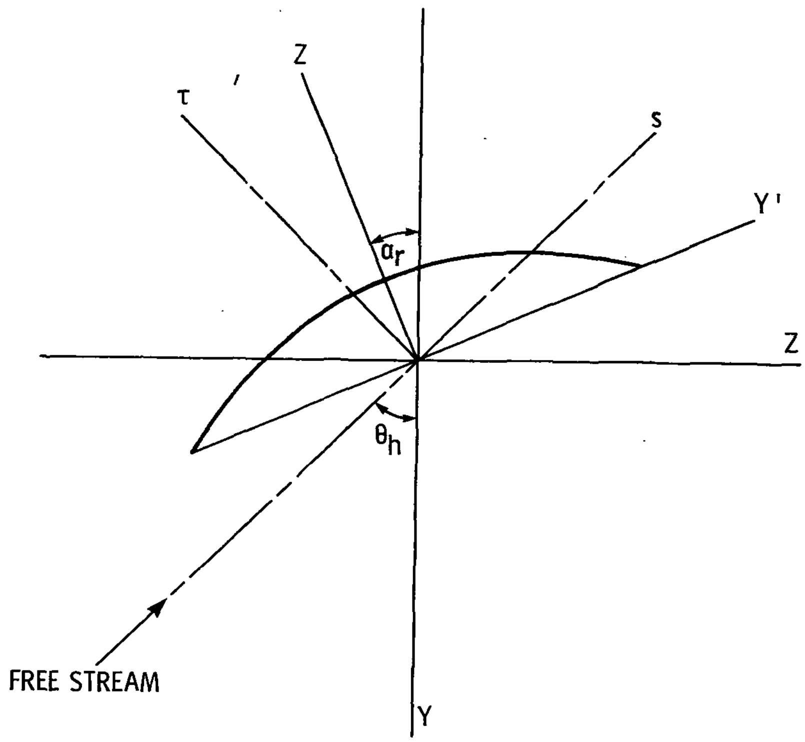

Fig. 4 Helical coordinate system for blade.

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15. Supplementary Notes

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16. Abstract

A knowledge of the acoustic energy emission of each blade row of a turbomachine is useful for estimating the overall noise level of the machine and for determining its discrete frequency noise content. Because of the close spacing between the rotor and stator of a compressor stage, the strong aerodynamic interactions between them have to be included in obtaining the resultant flow field. This paper outlines a three-dimensional theory for determining the discrete-frequency noise content of an axial compressor consisting of a rotor and a stator each with a finite number of blades. The lifting surface theory and the linearized equation of an ideal, nonsteady compressible fluid motion are used for thin blades of arbitrary cross section. The combined pressure field at a point of the fluid is constructed by linear addition of the rotor and stator solutions together with an interference factor obtained by matching them for net zero vorticity behind the stage. The rotor solution is obtained as a Fourier-Bessel series; the stator solution is expressed as a Fourier-Laguerre series. The coefficients of the series and the interaction factor are determined as the eigenvector of a set of algebraic equations in matrix form whose elements comprise the sum-integrals of the rotor and stator eigenfunctions. The resultant pressure field of the stage is the sum of the individual perturbation pressures in the presence of the interaction effects, expressions for which are given herein. The effects of wave absorption or reflection and the duct resonance modes have not been included in the present treatment.

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