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# A Selection Principle for Bénard-Type Convection

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1. Introduction. In a Bénard-type convection problem one seeks, e.g., to determine the stationary flows of an infinite layer of fluid lying between two rigid horizontal walls and heated uniformly from below. Such a problem possesses a unique, motionless conduction solution when the parameters of the problem lie within a certain range but, as the temperature difference across the layer increases beyond a certain value, other, convective, motions appear. These motions are often cellular in character in that their streamlines are confined to certain well-defined "cells" having, e.g., the shape of rolls or hexagons. The purpose of this paper is to formulate a "selection principle" that explains why hexagonal cells seem to be "preferred" for certain ranges of the parameters.

Bénard-type problems and their generalizations play an important role in fluid dynamics and have been investigated in recent years by a number of authors. Convection problems have been studied, e.g., by Schlüter, Lortz and Busse [19] and Fife and Joseph [4] using expansion methods, by Busse [1] using variational methods, by Kirchgässner [9,10] using the Lyapunov-Schmidt method, by Sattinger [17,18] and Golubitsky, Swift and Knobloch [5] using group-theoretic methods, and by Buzano and Golubitsky [2] using group-theoretic methods and singularity theory. The reader is referred to the above papers and to the book of Joseph [7] for a comprehensive introduction to Bénard-type problems.

An important aspect of the work of Busse [1] is that the "extremum principle" and the stability results there are independent of the number of critical wave vectors corresponding to a given critical wave number. In the same spirit an important aspect of this work is the formulation and verification of a selection principle in a setting that is independent of any fixed number of critical wave vectors. Although our study is restricted to functions doubly periodic in the horizontal plane, the (finite) number of critical wave vectors can be taken arbitrarily large by proper choice of the period rectangle. Moreover, in the case of the hexagonal lattice this choice can be made in such a way that the critical wave number and the "size" of the resulting hexagonal cells are kept fixed. Thus, whereas other methods offer a complete bifurcation analysis on the hexagonal lattice in the usual six-dimensional setting, the methods of this paper prove useful for a stability analysis on the hexagonal lattice in the general case of an arbitrarily large number of critical wave vectors (see also the discussion in Section 7).

To obtain a physical interpretation of the extremum principle in [1], Palm [15] derived in the time-dependent problem a minimum principle for a type of generalized dissipation, V, namely that, as time increases, V decreases and attains a minimum value on steady state solutions (see [15, p. 2414]). To treat the generalized Bénard problem studied here, we introduce an analogous sort of functional, $V$, called the generalized dissipation (see (3.23) in Section 3 below). It can be shown for time-dependent problems in a formal way as in [15] that the associated time-dependent $V$ decreases as time increases and assumes a minimum on steady state solutions. Since $V = 0$ for the motionless conduction solution and since $V$ initially increases in the steady state problem along a subcritical branch of convective solutions bifurcating from the conduction solution at the critical Rayleigh number, $R_c$, it is natural to conjecture that what we shall call a "selection principle" is related to the existence of a convective solution for which $V = 0$. Presumably, such a solution would correspond to a point on an "upper" branch because $V > 0$ on "lower" subcritical branches. Using such an interpretation, one could replace the formal "geometrical" condition for upper branches used in [1, p.633] by the exact analytical condition $V = 0$. This would be an important first step in a stability analysis since subcritical solutions lying on upper branches are the ones most likely to be stable.

The basic idea of the paper can now be stated as follows (see also the related but somewhat easier approach used in [12] to solve a class of variational problems arising in nonlinear shell theory--the parameter $\gamma$ in (2.3) plays the role of the "structure" parameter $\epsilon$ in [12]). Instead of solving only the Boussinesq-type equations given in (2.1) as is usually done, we solve the equations in (2.1) together with the constraint that, for fixed $\gamma$ near $\gamma = 0$, $V = 0$ is a local minimum of $V$. One anticipates here that the condition $V = 0$ will lead to a solution on an upper branch and that the minimization condition will lead to a stable solution. In this paper we show that such an approach does, in fact, yield stable, subcritical solutions of the generalized Bénard problem, when $\gamma$ is sufficiently small. Such solutions may even be considered as "large" solutions because they are both subcritical and stable whereas "small" subcritical solutions bifurcating from the conduction solution at $R_c$ are always unstable. In this sense our method may be regarded as a "selection principle" for obtaining "large", stable, subcritical solutions because the method selects certain solutions of equations (2.1) while excluding certain others. By "stability" here and throughout the remainder of the paper we mean "linearized stability" relative to some appropriate Hilbert space.

The outline of the paper is as follows. In Section 2 we give an operator-theoretic formulation of a certain type of generalized Bénard problem and in Sections 3 and 4 we reduce the given infinite-dimensional problem to one of solving a finite-dimensional system of equations, the so-called selection equations. The selection equations are derived by means of splitting techniques such as those used in the Lyapunov-Schmidt method in bifurcation theory but the equations obtained are not the usual bifurcation equations associated with the problem. The works of Kirchgässner [10] and Sattinger [17] play an important role in these preliminary sections. Sections 5 and 6 contain the main results of the paper. In Section 5 we solve the selection equations in a general setting by the use of variational methods and present a linearized stability analysis of the resultant stationary flows. In Section 6 we show for the hexagonal lattice that the classical hexagonal cellular solutions are generated from the absolute minimum of an appropriate selection functional and that such a minimization property is independent of the dimension of the basic underlying finite-dimensional problem. Thus, since the classical hexagonal cellular solutions are also stable, they are in some sense the preferred subcritical convection solutions.

2. Formulation of the problem. In this section we formulate a generalized Bénard problem for certain temperature-dependent fluids and introduce a Hilbert space setting for its study. The particular problem described below is chosen mainly for convenience. The methods of the paper apply also to a much wider class of convection problems (e.g., see [1]).

The generalized Bénard problem studied here is to determine the stationary flows of an infinite layer of fluid between two rigid, horizontal walls and heated uniformly from below. The fluid density, $\rho$, is assumed to be constant, say $\rho = \rho_0$, except in the gravity term where it is taken to be quadratic in the temperature, T, i.e.,

$$\rho = \rho_0[1 - a(T - T_0) - b(T - T_0)^2],$$

where $T_0$ is the average of the (constant) temperatures $T_2$ on the upper wall and $T_1$ on the lower wall. Under this assumption on $\rho$, one is led, after scaling the variables suitably, to the system of Boussinesq-type equations given in (2.1) below. The equations relate, at each point of the set

$$\Omega = \{\underset{\sim}{x} = (x,y,z): -\infty < x,y < \infty,\ -\tfrac{1}{2} < z < \tfrac{1}{2}\},$$

the fluid velocity vector, $\underset{\sim}{u} = (u_1,u_2,u_3)$, scalar pressure, p, and the scalar variable, $\theta$, measuring the change in temperature from its value for the pure conduction state (see, e.g., [9] where $\underset{\sim}{u}$, p, $\theta$ are related by a factor to those used here ):

(2.1) (a) $\quad -\Delta\underset{\sim}{u} - \lambda\hat{\underset{\sim}{k}}f_1(\theta) + \nabla p = -(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} + \hat{\underset{\sim}{k}}f_2(\theta)$

(b) $\quad -(Pr)^{-1}\Delta\theta - \lambda u_3 = -\underset{\sim}{u}\cdot\nabla\theta$

(c) $\quad \nabla\cdot\underset{\sim}{u} = 0$

(d) $\quad \underset{\sim}{u} = 0,\ \theta = 0 \quad \text{for} \quad z = \pm\frac{1}{2}\ .$

In (2.1), $\hat{k} = (0,0,1)$, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, and $\Delta$ is the Laplace operator; the Prandtl number, Pr, equals the ratio of kinematic viscosity, $\nu$, to thermal conductivity and is regarded as a fixed constant throughout the paper; the Grashof number $Gr = agd(T_1 - T_2)/\nu^2$ ($g$ = gravitational constant, $d$ = thickness of the unscaled layer); $\lambda = \sqrt{Gr}$ and

(2.2) (a) $f_1(\theta) = \theta(1 - 2\gamma z)$

(b) $f_2(\theta) = \gamma\theta^2$ ,

where $\gamma$ is a "structure" parameter given by

(2.3) $\gamma = b(T_1 - T_2)/a.$

The Rayleigh number, Ra, is related to $\lambda$ by $Ra = PrGr = Pr\lambda^2$.

We shall seek solutions having a doubly periodic cellular structure. Thus, given positive numbers $\alpha_1$ and $\alpha_2$ (to be specified below), we set

$$\Omega = \{\underset{\sim}{x} = (x,y,z): 0 < x < \frac{2\pi}{\alpha_1},\quad 0 < y < \frac{2\pi}{\alpha_2},\ -\frac{1}{2} < z < \frac{1}{2}\}.$$

We next introduce the (complex) Hilbert space, $H$, defined as the closure of the set $\{v = (u_1,u_2,u_3,\theta)$ : $v$ smooth, periodic in $x$ with period $\frac{2\pi}{\alpha_1}$, periodic in $y$ with period $\frac{2\pi}{\alpha_2}$, $v = 0$ in a neighborhood of $|z| = \frac{1}{2}$ and $\nabla \cdot v = 0\}$ in the norm $\|\cdot\|$ associated with the inner product

$$(v,w) = \int_\Omega [\sum_{j=1}^{3} \nabla v_j \cdot \nabla \bar{w}_j + \frac{1}{Pr} \nabla v_4 \cdot \nabla \bar{w}_4].$$

Here and throughout the paper a bar over a quantity denotes complex conjugation and the symbol $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, 0\right)$ when used with elements of $H$. Thus $\nabla \cdot v = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$ .

If we take the scalar product of (2.1a,b) with $w \in H$, use (2.1c,d) and integration by parts, then for $v = (u,\theta)$ we obtain

(2.4) $(v,w) - \lambda(L_\gamma v,w) = (F_\gamma(v),w).$

Here the linear operator $L_\gamma : H \to H$ and quadratic operator $F_\gamma : H \to H$ are given by

(2.5) $L_\gamma = L - \gamma M, \qquad F_\gamma = F + \gamma G$

and the operators L, M, F, G are defined by

(2.6) $$(Lv,w) = \int_\Omega [v_4\bar{w}_3 + v_3\bar{w}_4]$$

(2.7) $$(Mv,w) = \int_\Omega 2zv_4\bar{w}_3$$

(2.8) $$(F(v),w) = -\int_\Omega (v\cdot\nabla v)\cdot\bar{w}$$

(2.9) $$(G(v),w) = \int_\Omega (v_4)^2\bar{w}_3$$

for all v,w in $H$. Since in (2.4) w is an arbitrary element of $H$ we see that a smooth solution $v = (u,\theta)$ of (2.1) in $H$ satisfies the operator equation

(*) $$0 = v - \lambda L_\gamma v - F_\gamma(v), \quad v \in H, \quad \lambda \in \mathbb{R}^1.$$

In fact, one can apply standard regularity methods (e.g., see [11,13,14]) to show that problems (2.1) and (*) are equivalent.

In order to study solutions of (*) we shall require properties of the linearized version of (*) when $\gamma = 0$,

(2.10) $$0 = v - \lambda Lv, \quad v \in H, \quad \lambda \in \mathbb{R}^1.$$

The linear eigenvalue problem (2.10) is equivalent to the classical problem, for smooth $u, p, \theta$ periodic with periods $\frac{2\pi}{\alpha_1}$ in x and $\frac{2\pi}{\alpha_2}$ in y, obtained by omitting the nonlinear terms in (2.1). This linear problem is well studied (see, e.g., [3,6,10,11]). The eigenfunctions are complete in $H$ and are obtained from the relations $\underset{\sim}{k} = (k_1,k_2,0)$, $\sigma = (k_1^2 + k_2^2)^{1/2}$, $i = \sqrt{-1}$ and

(2.11) (a) $u_j = e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\phi_j(z)$, $j = 1,2,3$,

(b) $\theta = e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\phi_4(z)$,

(c) $p = e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\sigma^{-2}D^2\phi_3'$,

(d) $\phi_j = i\sigma^{-2}k_j\phi_3'$, $(j = 1,2)$ .

Here $D^2 = \frac{d^2}{dz^2} - \sigma^2$, a prime denotes $\frac{d}{dz}$, and $\phi_3$ and $\phi_4$ satisfy

(2.12) (a) $0 = D^4\phi_3 - \lambda\sigma^2\phi_4$,

(b) $0 = \frac{1}{Pr}D^2\phi_4 + \lambda\phi_3$,

(c) $\phi_3 = \phi_3' = \phi_4 = 0$ at $z = \pm\frac{1}{2}$ .

One can show (e.g., see [6]) for $\sigma > 0$ that the eigenvalue problem (2.12) has a countable sequence of positive, simple eigenvalues, $0 < \mu_1(\sigma) < \mu_2(\sigma) < \ldots$, depending continuously on $\sigma$. Moreover, $\mu_1(\sigma) \to \infty$ as $\sigma \to 0^+$ or $\sigma \to \infty$. Consequently, $\mu_1(\sigma)$ assumes an absolute minimum at some $\sigma_0 > 0$ depending only on the Prandtl number, Pr. We assume through the paper that $\sigma_0$ is unique so that $\mu_1(\sigma) > \mu_1(\sigma_0)$ if $\sigma \neq \sigma_0$. (This property is suggested by numerical calculations [3] and is usually assumed for Bénard-type problems.) For given integers $n_0$, $m_0$ we now choose $\alpha_1$, $\alpha_2$ such that

(2.13) $\sigma_0^2 = n_0^2\alpha_1^2 + m_0^2\alpha_2^2$ .

In Section 6 we consider some special cases, of the form $\alpha_1 = \sqrt{3}\alpha_2$, important for the study of both "classical" and "exotic" hexagonal-cellular solutions (see Remark 6.1).

Since the vectors $\underset{\sim}{k}$ in (2.11) are constrained by the requirement that $e^{ik\cdot x}$ have periods $2\pi/\alpha_1$ in x and $2\pi/\alpha_2$ in y, it follows that $n = k_1/\alpha_1$ and $m = k_2/\alpha_2$ are integers and $\underset{\sim}{k}$ must have the form $\underset{\sim}{k} = (n\alpha_1, m\alpha_2, 0)$. Thus, the only wave numbers, $\sigma$, corresponding to eigenfunctions having the required periods are those for which

(2.14) $\sigma^2 = n^2\alpha_1^2 + m^2\alpha_2^2$

for some integers m, n, i.e., such that the ellipse $\sigma^2 = x^2\alpha_1^2 + y^2\alpha_2^2$ passes through at least one lattice point $(n,m) \neq (0,0)$. (Note that there is no nontrivial solution of (2.12) if $\sigma = 0$.) There are countably many such wave numbers $0 < \sigma_1 < \sigma_2 < \ldots$, each of which corresponds to a finite, even number of lattice points $(\pm n, \pm m)$.

If $\sigma_p$ corresponds to the $2s_p$ lattice points

$$(n_{pj}, m_{pj}) \qquad -s_p \le j \le s_p, \quad j \neq 0,$$

where the indices are chosen so that $n_{p(-j)} = -n_{pj}$, $m_{p(-j)} = -m_{pj}$, then we set

(2.15) $$k_{pj} = (n_{pj}\alpha_1, m_{pj}\alpha_2, 0), \quad j = \pm 1, \pm 2, \dots, \pm s_p, \quad p = 1, 2, \dots,$$

and observe that

(2.16) $$k_{p(-j)} = -k_{pj}.$$

For each $\sigma = \sigma_p$ the reduced eigenvalue problem (2.12) has an infinite sequence of real, nontrivial solutions

(2.17) $$(\lambda, \phi_3, \phi_4) = (\mu_{pq}, \phi_3^{pq}, \phi_4^{pq}), \quad q = \pm 1, \pm 2, \dots .$$

Since $(-\lambda, \phi_3, -\phi_4)$ satisfies (2.12) whenever $(\lambda, \phi_3, \phi_4)$ satisfies (2.12), we may order the indices so that

(2.18) $$\mu_{p(-q)} = -\mu_{pq}, \quad \phi_3^{p(-q)} = \phi_3^{pq}, \quad \phi_4^{p(-q)} = -\phi_4^{pq}, \quad 0 < \mu_{p1} < \mu_{p2} < \dots .$$

The $\mu_{p1}$ are simple eigenvalues and the corresponding $\phi_3^{p1}, \phi_4^{p1}$ may be taken to be positive on $(-1/2, 1/2)$. Moreover, since $\sigma_0$ in (2.13) is equal to $\sigma_{p_0}$ in (2.14) for some unique, positive integer $p_0$,

(2.19) $$\mu_1 \equiv \mu_{p_0 1} = \min_{p=1,2,\dots} \mu_{p1}$$

is also a simple eigenvalue of (2.12) and, for $q \geq 1$, $\mu_{pq} > \mu_1$ if $p \neq p_0$.

One now sees from (2.11) that the full eigenvalue problem (2.10) has the solutions

(2.20) $$\lambda = \mu_{pq}, \quad v = \psi^{pqj}(x) = e^{ik_{pj}\cdot x}\, \phi^{pqj}(z), \quad j = \pm 1, \pm 2, \dots, \pm s_p,$$

for $p = 1, 2, 3, \dots$, $q = \pm 1, \pm 2, \pm 3, \dots$, where

(2.21) $$\phi^{pqj}(z) \equiv \left( \frac{i\alpha_1}{\sigma_p^2}\, n_{pj} \frac{d}{dz} \phi_3^{pq}, \; \frac{i\alpha_2}{\sigma_p^2}\, m_{pj} \frac{d}{dz} \phi_3^{pq}, \; \phi_3^{pq}, \phi_4^{pq} \right) .$$

Note that the $\phi^{pqj}$ depend on $j$ only in the first two components. According to (2.16), (2.20), (2.21) and the fact that $\phi_3^{pq},\phi_4^{pq}$ are real, we have

$$\phi^{pq(-j)} = \bar{\phi}^{pqj}. \tag{2.22}$$

It is shown in the Appendix that the eigenfunctions $\{\phi^{pqj}\}$ in (2.20) may be assumed orthonormal in $H$, after scaling with constants depending on $p$ and $q$ but not on $j$. Thus we suppose that

$$(\phi^{pqj},\phi^{rst}) = \delta_{pr}\delta_{qs}\delta_{jt} \tag{2.23}$$

where $\delta_{jt}$ is the usual Kronecker delta symbol.

The next lemma summarizes some of the properties just discussed. The compactness properties are essentially well known (e.g., see [11]), while (2.24) is easily derived from (2.7).

Lemma 2.1. (i) The operator $L:H \to H$ is bounded, linear, selfadjoint and compact. Its characteristic values and eigenfunctions are given by (2.20) and satisfy (2.22) and (2.23). The eigenfunctions are complete in $H$.

(ii) The operator $M:H \to H$ is bounded, linear and compact. Its adjoint, $M^*$, is characterized by

$$(M^*v,w) = 2\int_{\Omega} zv_3\bar{w}_4, \qquad v,w \in H. \tag{2.24}$$

3. The selection equations. We show next that the generalized Bénard problem can be reduced to a finite-dimensional one. This reduction is carried out by means of splitting methods using the "structure" parameter $\gamma$ in (2.8) as an "amplitude" parameter.

Since $\mu_1$ given by (2.19) is a simple eigenvalue of (2.12) and $\mu_1 \neq \mu_{pq}$ for $(p,q) \neq (p_0,1)$, it is also a characteristic value of $L$ of multiplicity $2N - 2s_{p_0}$. The associated nullspace $M$ of $I - \mu_1 L$ is spanned by

$$\phi^j = \psi^{p_0 1 j}, \quad j = \pm 1, \pm 2, \ldots, \pm N. \tag{3.1}$$

(When dealing with quantities on $M$ it will often be convenient to suppress the indices $p = p_0$, $q = 1$. Thus we write $k_{p_0 j} = k_j$, $\psi^{p_0 1 j} = \phi^j$, etc.) The orthogonal complement, $M^\perp$, of $M$ in $H$ is spanned by $\{\psi^{pqj} : (p,q) \neq (p_0,1)\}$.

We shall look for solutions of (*) having the form $v = \phi + \psi$ with $\phi$ in $M$ and $\psi$ in $M^\perp$. In order to study the way $L_\gamma$ and $F_\gamma$ act on $v$ it will be useful to introduce some related operators. Let $P: H \to M^\perp$ denote the orthogonal projection of $H$ onto $M^\perp$ and let $K: M^\perp \to M^\perp$ denote the inverse of the restriction of $I - \mu_1 L$ to $M^\perp$. In addition, we define bilinear operators $\Phi: H \times H \to H$ and $\Gamma: H \times H \to H$ by

$$(\Phi(u,v),w) = -\int_\Omega (u \cdot \nabla v) \cdot \bar{w}, \qquad u,v,w \in H \tag{3.2}$$

$$(\Gamma(u,v),w) = \int_\Omega u_4 v_4 \bar{w}_3, \qquad u,v,w \in H. \tag{3.3}$$

One sees easily from (2.8) and (2.9) that

$$F(v) = \Phi(v,v) \quad \text{and} \quad G(v) = \Gamma(v,v), \quad v \in H. \tag{3.4}$$

It will frequently be convenient to represent $v \in H$ by its Fourier series

$$v = \Sigma \xi_{pqj} \psi^{pqj}, \tag{3.5}$$

where the sum is extended over the set of integer triples $(p,q,j)$ with $1 \le |j| \le N$, $1 \le p < \infty$, $1 \le |q| < \infty$. When $v = \phi \in M$, (3.5) becomes

(3.6) $\quad \phi = \sum_{|j|=1}^{N} \beta_j \phi^j .$

Since $\phi^{-pqj} = \phi^{pq(-j)}$, it follows that $v$ in (3.5) (or $\phi$ in (3.6)) is real if and only if $\bar{\beta}_{pqj} = \beta_{pq(-j)}$ (or $\bar{\beta}_j = \beta_{-j}$).

The following lemma, proved in the Appendix, enables us in some situations to calculate with the operators introduced above. Here and throughout the paper $\delta(\xi)$ is zero whenever the (scalar or vector) parameter $\xi$ is not zero, and $\delta(0) = 1$.

Lemma 3.1. (i) If $v \in H$, then $Lv$, $Mv$ and $KPv$ can be obtained from (3.5) by formal calculation. E.g.,

(3.7) $$Lv = \Sigma \beta_{pqj} \mu_{pq}^{-1} \phi^{pqj} \quad , \; KPv = \Sigma_0 \beta_{pqj} \left( \frac{\mu_{pq}}{\mu_{pq} - \mu_1} \right) \phi^{pqj} \quad ,$$

where $\Sigma_0$ denotes summation over the same set of integer triples as $\Sigma$, except that $(p,q,j) \neq (p_0,1,j)$. In particular, $K$ is bounded, positive and selfadjoint on $M^{\perp}$.

(ii) $R(M) = M^{\perp}$, i.e., for $\phi \in M$, $(M\phi, \bar{\phi}^j) = 0$, $|j| = 1,2,\ldots,N$.

(iii) For $u,v,w \in H$, $(\Phi(u,v),\bar{w}) = -(\Phi(u,w),\bar{v})$ and $\Gamma(u,v) = \Gamma(v,u)$.

(iv) $\Phi: H \times M \to M^{\perp}$. In particular, $F: M \to M^{\perp}$ and $(\Phi(\phi^{pqr},\phi^j),\bar{\phi}^n) = 0$ for all $p,q,r,j,n$ with $1 \le p < \infty$, $1 \le |q| < \infty$ and $|r|,|j|,|n| = 1,2,\ldots,N$.

(v) If $\phi$ has the form (3.6), then there are real constants $b_0,b_1,b_2,b_3$ depending on $p_0$ but not on $n$, such that $b_3 > 0$,

(3.8) $$(MKM\phi,\bar{\phi}^n) = b_0\beta_{-n},$$

(3.9) $$(MKF(\phi),\bar{\phi}^n) = b_1 \sum_{|j|,|r|=1}^{N} \beta_j\beta_r\delta(k_j + k_r + k_n) \quad ,$$

(3.10) $$(\Phi(\phi,KM\phi),\bar{\phi}^n) = b_2 \sum_{|j|,|r|=1}^{N} \beta_j\beta_r\delta(k_j + k_r + k_n)$$

and

(3.11) $$(G(\phi),\bar{\phi}^n) = b_3 \sum_{|j|,|r|=1}^{N} \beta_j\beta_r\delta(k_j + k_r + k_n).$$

(vi) There are nonnegative constants $a_{p_0 jn}$ such that

$$(\Phi(\epsilon, KF(\phi)), \bar{\Phi}^n) = - \sum_{|j|=1}^{\infty} a_{p_0 jn} (2 - \delta_{jn}) \epsilon_j \epsilon_{-j} \epsilon_{-n} \quad (3.12)$$

The constant $a_{p_0 jn}$ depends only on $p_0$ and $p = |k_j + k_n|$ so that $a_{p_0 jn} = a_{p_0 nj} = a_{p_0 (-n)(-j)}$. The exceptional cases in which the constant $a_{p_0 jn}$ is zero are described in (A.24) and (A.37) of the Appendix. (See also Remark 5.1).

We shall need to relate the spectral analysis of the linear operator L to that of the linear operator $L_\gamma = L - \gamma M$. For small values of $\gamma$ it is well known (e.g., see [8]) that the characteristic values of $L_\gamma$ are perturbations of those of L. In fact, the characteristic values of $L_\gamma$ are determined by the problem obtained from (2.12) upon replacing (2.12a) with

$$0 = D^4 \phi_3 - \lambda \sigma^2 (1 - 2\gamma z) \phi_4 \ . \quad (3.13)$$

One finds, in particular, that the critical characteristic value, $\lambda_c = \lambda_c(\gamma)$, i.e., the characteristic value of $L_\gamma$ of least magnitude, is real and simple as an eigenvalue of the problem (2.12) with (2.12a) replaced by (3.13) and $\sigma$ set equal to $\sigma_0$. (The relationship between $\lambda_c$ and the critical Rayleigh number is the usual one described in [1;4;7].) The next lemma specifies the expansion in $\gamma$ of $\lambda_c$ and may be proved along the lines of the development for the nonlinear problem leading to equations (3.17) and (3.18).

Lemma 3.2. The critical characteristic value, $\lambda_c$, of $L_\gamma$ has the expansion

$$\lambda_c = \mu_1 - \gamma^2 \mu_1^3 b_0 + \Lambda_c(\gamma) \quad (3.14)$$

where $\mu_1$ is given by (2.19), $b_0$ is as in (3.8) and $\Lambda_c(\gamma)$ is real and satisfies $|\Lambda_c(\gamma)| = O(\gamma^3)$ as $\gamma \to 0$.

For small $\gamma$ we seek a solution of equation (*) in the form

$$v = \gamma(\varphi + \gamma\Psi), \qquad \lambda = \mu_1 - \gamma^2\mu_1(\mu_1^2 b_0 - \tau). \tag{3.15}$$

Here $\varphi \in M$, $\Psi \in M^{\perp}$ and $\tau \in \mathbb{R}^1$ are to be determined. If $\tau = \tau_0 + O(\gamma)$, then $\lambda - \lambda_c = \gamma^2\mu_1\tau_0 + O(\gamma^3)$ and a solution of the form (3.15), for small $\gamma$, is subcritical if $\tau_0 < 0$ or supercritical if $\tau_0 > 0$.

We substitute (3.15) in equation (*), use $P$ and $S = I - P$ to project onto $M^{\perp}$ and $M$, and use (ii) and (iv) of Lemma 2 to obtain the following equations on $M^{\perp}$ and $M$:

$$\begin{aligned}(3.16)\quad (a)\quad 0 = {} & (I - \mu_1 L)\Psi + \mu_1 M\varphi - F(\varphi) + \gamma P[\mu_1 M\Psi - \Phi(\varphi,\Psi) - \Phi(\Psi,\varphi) - G(\varphi)] \\ & + \gamma^2 P[-\mu_1(\mu_1^2 b_0 - \tau)M\varphi + \mu_1(\mu_1^2 b_0 - \tau)L\Psi - F(\Psi) - 2\Gamma(\varphi,\Psi)] \\ & + \gamma^3 P[-\mu_1(\mu_1^2 b_0 - \tau)M\Psi - G(\Psi)], \\ (b)\quad 0 = {} & (\mu_1^2 b_0 - \tau)\varphi + S[\mu_1 M\Psi - \Phi(\varphi,\Psi) - G(\varphi)] + \gamma S[F(\Psi) - 2\Gamma(\varphi,\Psi)] \\ & + \gamma^2 S[-\mu_1(\mu_1^2 b_0 - \tau)M\Psi - G(\Psi)].\end{aligned}$$

Since $K = (I - \mu_1 L)^{-1}$ is bounded on $M^{\perp}$, given $t_0 > 0$ there is a $\gamma_0 > 0$ such that if $(\varphi,\tau) \in M \times \mathbb{R}^1$ with $|\tau| + \|\varphi\| < t_0$ then one can solve (3.16a), by successive approximations, for $\Psi = \Psi(\varphi,\gamma)$ whenever $|\gamma| < \gamma_0$. In fact $\Psi$ satisfies

$$\Psi = -\mu_1 KM\varphi + KF(\varphi) + \gamma\Psi_1, \tag{3.17}$$

where $\Psi_1 = \Psi_1(\varphi,\tau,\gamma) \in M^{\perp}$ is bounded depending only on $t_0$. We next use (3.17) to eliminate $\Psi$ from (3.16b), taking (3.8) into account to get

$$0 = -\tau\varphi + S[\mu_1 MKF(\varphi) + \mu_1\Phi(\varphi,KM\varphi) - \Phi(\varphi,KF(\varphi)) - G(\varphi)] + R(\varphi,\tau,\gamma). \tag{3.18}$$

Here, for $|\gamma| < \gamma_0$ and $|\tau| + \|\varphi\| < t_0$, the remainder term

$$R(\varphi,\tau,\gamma) = \gamma S\{\mu_1 M\Psi_1 - \Phi(\varphi,\Psi_1) + F(\Psi) - 2\Gamma(\varphi,\Psi) + \gamma[-\mu_1(\mu_1^2 b_0 - \tau)M\Psi - G(\Psi)] \tag{3.19}$$

satisfies, for some $r_0 > 0$ depending only on $t_0$,

(3.20) $$\|R(\psi,\tau,\gamma)\| \leq \gamma r_0.$$

We take the inner product of $\psi^{-n}$ with equation (3.18), making use of the expansion (3.6) and various formulas in Lemma 3.1 to obtain

(3.21) $$0 = F_n(\beta,\tau,\gamma) = -\tau\beta_{-n} + b \sum_{|i|,|j|=1}^{N} \beta_i\beta_j\delta(k_i + k_j + k_n) + \sum_{|j|=1}^{N} a_{p_0 jn}(2-\delta_{jn})\beta_j\beta_{-j}\beta_{-n} + r_n(\beta,\tau,\gamma), \quad |n| = 1,2,\ldots,N .$$

Here $b = \mu_1(b_1 + b_2) - b_3$ and, according to (3.20),

(3.22) $$r_n(\beta,\tau,\gamma) = (R(\psi,\tau,\gamma),\psi^{-n}),$$

satisfies $|r_n(\beta,\tau,\gamma)| \leq \gamma r_0$.

For the reasons discussed in the introduction (see also the discussion in [12]) we must augment the system (3.21) by an equation, $V(\beta,\tau) = \varepsilon$, involving the so-called generalized dissipation $V$, where $\varepsilon$ is a real parameter and

(3.23) $$V(\beta,\tau) = -\frac{1}{2}\tau \sum_{|j|=1}^{N} \beta_j\beta_{-j} + \frac{1}{3}b \sum_{|i|,|j|,|m|=1}^{N} \beta_i\beta_j\beta_m\delta(k_i + k_j + k_m) + \frac{1}{4} \sum_{|i|,|j|=1}^{N} a_{p_0 ij}(2 - \delta_{ij})\beta_i\beta_{-i}\beta_j\beta_{-j} .$$

Thus, we consider the system of <u>selection equations</u>

(3.24) (a) $0 = F(\beta,\tau,\gamma)$,

(b) $\varepsilon = V(\beta,\tau)$, $\beta \in \mathbb{C}^{2N}$, $(\tau,\gamma,\varepsilon) \in \mathbb{R}^3$,

where $F = (F_n)_{|n|=1,\ldots,N}$ and $\beta = (\beta_{-N},\ldots,\beta_{-1},\beta_1,\ldots,\beta_N)$.

The functional $V$ is essentially the functional $E$ in [1, p.631] with $\varepsilon = \gamma$. In fact, setting $\varepsilon = \gamma$ in the analysis in [1], one obtains formally a number of expansions, equations, etc., that are closely related to various quantities used in the analysis here.

Perhaps one would hope to solve (3.24) by solving the equations, e.g., when $(\gamma,\varepsilon) = (0,0)$ and then using the implicit function theorem to extend such a solution to a small $(\gamma,\varepsilon)$ neighborhood of $(0,0)$. One anticipates, however, difficulty here in implementing the implicit-function theorem argument (e.g., see [17]) because the equations are invariant under translations of the $(x,y)$-plane. Consequently, the solutions will not be isolated and the relevant Jacobians will be zero. Thus, it is natural to seek solutions in a subspace of $H$, where one may hope that solutions <u>will</u> be isolated. This is conveniently done in the next section in terms of group representations as in [17].

4. The reduced selection equations. The basic subspace, $S_\pi$, used throughout the remainder of the paper is introduced in this section together with some technical lemmas regarding real solutions of equation (*).

Let r be the $2 \times 2$ matrix of a plane rotation or reflection and let $a = (a_1, a_2)$ be a translation vector. For $k = 3,4$ let $r_k$ denote the $k \times k$ matrix obtained from the identity by inserting r in place of the $2 \times 2$ identity matrix in the upper left-hand corner. Set $a_3 = (a_1, a_2, 0)$ and let $\sigma = \{r_3, a_3\}$ represent an arbitrary plane rigid motion of $x = (x,y,z)$ space that keeps z fixed: $\sigma x = r_3 x + a_3$. Then a representation, $\sigma \mapsto T_\sigma$, of this group, G, of rigid motions is defined by

$$(T_\sigma v)(x) = r_4 v(\sigma^{-1} x) \tag{4.1}$$

for smooth four-dimensional vector fields v defined for $x \in \mathbb{R}^3$.

When $\gamma = 0$ it is well known (e.g., see [11, 17]) that the Boussinesq equations in (2.1) are invariant under $T_\sigma$ for $\sigma \in G$. The next lemma shows that a corresponding invariance property holds for equation (*) when $\gamma = 0$ and that the invariance also extends to the case $\gamma \neq 0$. Such an invariance statement makes sense, of course, only for $\sigma$, v for which both v and $T_\sigma v$ lie in $H$.

Lemma 4.1. Let $\sigma \in G$ and suppose that u, v, $T_\sigma u$, $T_\sigma v$ all lie in $H$. Then each of the operators L, M, $\Phi$, $\Gamma$ is invariant in the sense that $L(T_\sigma v) = T_\sigma(Lv)$, $\Phi(T_\sigma u, T_\sigma v) = T_\sigma \Phi(u,v)$, etc. Consequently,

$$L_\gamma(T_\sigma v) = T_\sigma(L_\gamma v), \quad F_\gamma(T_\sigma v) = T_\sigma(F_\gamma(v)) \tag{4.2}$$

so that equation (*) is invariant under $T_\sigma$.

Proof. Each of the operators L, M, $\Phi$, $\Gamma$ is defined ((2.6),(2.7),(3.2), (3.3)) by an integral of the form $\int_\Omega A\bar{w}$, where A is a linear, $A(v)$, or bilinear, $A(u,v)$, term in the Boussinesq equations. If $A$ is invariant under $T_\sigma$, then it is easy to see that the corresponding operator is invariant under $T_\sigma$. E.g., for $\sigma$ as defined in (3.3) note that $A(u,v) = (0,0,u_4v_4,0)$. Since $(r_4)_{j3} = \delta_{j3}$, $A(u,v)$ is invariant under $T_\sigma$ because

$$(T_\sigma A(u,v))(x) = (0,0,u_4(\sigma^{-1}x)v_4(\sigma^{-1}x),0) = (0,0,(T_\sigma u)_4(x)(T_\sigma v)_4(x),0).$$

The invariance of $Au = (0,0,2zu_4,0)$, corresponding to the operator $M$, follows in a similar way; the invariance of the $A$'s corresponding to $L$ and is proved in [9].

Remark 4.1. Because of Lemma 4.1, we may study problem (*) on any of the closed linear subspaces, $S_\sigma = \{v \in H: T_\sigma v = v\}$, without the use of projections, by merely restricting the operators in (*) to $S_\sigma$. Under such a restriction the equation (*) is denoted $(*)_\sigma$ and retains its form; similarly the new selection equations, $(3.24)_\sigma$, are obtained from (3.24) merely by restricting the coefficients in a well-defined manner determined by $\sigma$. By the restriction to $S_\sigma$ we shall avoid the problem of zero Jacobians mentioned above. Of course a solution of $(*)_\sigma$ in $S_\sigma$ is also a solution of (*) in $H$. On the other hand, a stability proof in $S_\sigma$, although encouraging, is a weaker statement than one on $H$ but instability in $S_\sigma$ does imply instability in $H$.

Throughout the remainder of the paper we shall largely restrict our attention to $S_\pi$ and its subspaces, where $S_\theta \equiv S_\theta$ when $\theta$ denotes rotation by $\theta$ radians about the z-axis. Thus, $(x,y) \to (-x,-y)$ under $\pi$ and

$$(4.3) \qquad (T_\pi v)(x,y,z) = (-v_1,-v_2,v_3,v_4)(-x,-y,z).$$

It follows from (2.20)-(2.22) and (4.3) that

$$(4.4) \qquad T_\pi \psi^{pqj} = \psi^{pq(-j)} = \bar{\psi}^{pqj} .$$

Consequently, $v = \Sigma \beta_{pqj} \phi^{pqj}$ in (3.5) lies in $S_\pi$ if and only if each of the coefficients satisfies $\beta_{pqj} = \beta_{pq-j}$. In addition, it follows directly from (4.4) that $T_\pi$ satisfies

$$(4.5) \qquad (T_\pi u,v) = (u,T_\pi v), \qquad u,v \in H.$$

While $H$ is a complex Hilbert space, we are of course interested only in real solutions of (*). Since the basis elements satisfy (2.22), the coefficients in the expansion (3.5) satisfy $\bar{\beta}_{pqj} = \beta_{pq-j}$ whenever $v$ is real. In particular, for real $v$ in $S_\pi$ equation (4.5) implies further that the $\beta_{pqj}$ are real. Moreover, the following lemma shows that the operators in equation (*) are real operators.

Lemma 4.2. (i) The operators L, M, $\Phi$, $\Gamma$ are real in the sense that $\overline{Lv} = L\bar{v}$, $\overline{\Phi(u,v)} = \Phi(\bar{u},\bar{v})$, etc.

(ii) If $\psi \in M$ is real and $\gamma,\tau \in \mathbb{R}^1$ satisfy $|\gamma| < \gamma_0$, $|\tau| + \|\psi\| < t_0$, with $\gamma_0$ and $t_0$ such that (3.17)-(3.20) hold, then $\Psi = \Psi(\psi,\tau,\gamma)$ in (3.17) and $R = R(\psi,\tau,\gamma)$ in (3.19) are real.

Proof. Part (i) follows easily from the definitions, since the corresponding differential operators have real coefficients, e.g., $(\overline{\Gamma(u,v)},w) = \overline{(\Gamma(u,v),\bar{w})} = \int \bar{u}_4\bar{v}_4\bar{w}_3 = (\Gamma(\bar{u},\bar{v}),w)$. For part (ii), note that if $\gamma,\tau$ and $\psi$ are real then upon taking the complex conjugate of (3.16a) and using part (i) we see that $\bar{\Psi}$ is a solution of (3.16a) whenever $\Psi$ is a solution. But the successive-approximations solution of (3.16a) is unique in a small neighborhood of $-\omega_1 KM\psi + KF(\psi)$, which is real. Hence $\bar{\Psi} = \Psi$ is real and by (3.17) $\Psi_1$ is real. From (i) and (3.19), $R(\psi,\tau,\gamma)$ is real.

Since, according to Lemma 4.2, $\Psi(\psi,\tau,\gamma)$ is real whenever $\gamma,\tau$ and $\psi \in M$ are real, the problem of finding real solutions of (*) is reduced to that of finding, for sufficiently small $(\gamma,\varepsilon) \in \mathbb{R}^2$, solutions $(\beta,\tau)$ of the selection equations (3.24) with $\tau$ and $\psi = \sum_{|j|=1}^{N} \beta_j \psi^j$ real, i.e., with

(4.6) $$\tau \in \mathbb{R}^1, \ \beta_{-j} = \bar{\beta}_j, \quad j = 1,2,\ldots,N.$$

In the remainder of this section we consider problem $(*)_\pi$ obtained by restricting $(*)$ to $S_\pi$. The nullspace of $I - \mu_1 L$ restricted to $S_\pi$ is $M_\pi = M \cap S_\pi$. From (4.5) we see that if $\psi \in M_\pi$ then $\beta_{-j} = \beta_j$, $j = 1,\ldots,N$ and

(4.7) $$\psi = \sum_{|j|=1}^{N} \beta_j \psi^j = \sum_{j=1}^{N} \beta_j (\psi^j + \psi^{-j}) \quad .$$

Thus, $M_\pi$ is N-dimensional and we shall henceforth take the liberty of suppressing $\beta_{-1},\ldots,\beta_{-N}$ in the notation, i.e. we write $\beta = (\beta_1,\ldots,\beta_N)$ instead of $\beta = (\beta_N,\ldots,\beta_1,\beta_1,\ldots,\beta_N)$ and we regard $F_n$ and $V$ as functions of $(\beta,\tau,\gamma)$ in $\mathbb{C}^N \times \mathbb{R}^2$. Moreover, in the context of $S_\pi$ we have the following lemma (see also the related results in [17]).

Lemma 4.3. If $\psi \in M_\pi$ and $\gamma,\tau \in \mathbb{R}^1$ are sufficiently small then $F_n(\beta,\tau,\gamma) \equiv F_{-n}(\beta,\tau,\gamma)$, $n = 1,2,\ldots,N$. If, in addition, $\psi$ is real then $r_n = r_n(\beta,\tau,\gamma)$ in (3.21) is real, $n = 1,2,\ldots,N$.

Proof. Since $\beta_{-j} = \beta_j$ whenever $\psi = \sum_{|j|=1}^{N} \beta_j \psi^j$ belongs to $M_\pi$, and since $a_{p_0 nj} = a_{p_0(-n)(-j)}$ in (3.21), to show that $F_n = F_{-n}$ it suffices to show that $r_n = r_{-n}$ in (3.22). Using the fact that $T_\pi \psi = \psi$ for $\psi \in M_\pi$, one sees from the invariance of (3.16a) under $T_\pi$ and the uniqueness of $\Psi$ that $T_\pi \Psi = \Psi$ also holds. It follows that $T_\pi \Psi_1 = \Psi_1$ and $T_\pi \tilde{R} = \tilde{R}$, where $\Psi_1$ and $\tilde{R}$ are given by (3.17) and (3.19). Thus, one sees from (4.4) and (4.5) that $r_{-n} = (\tilde{R},\psi^n) = (\tilde{R},T_\pi\psi^{-n}) = (\tilde{R},\psi^{-n}) = r_n$. If, in addition, $\psi$ is real then (2.22) and Lemma 4.2 imply $\bar{r}_n = (\tilde{R},\psi^{-n}) = (\tilde{R},\psi^n) = r_{-n}$ so that $r_n = \bar{r}_n$.

Because of Lemma 4.3, the selection equations (3.24) in the setting of $S_\pi$ may be replaced by an equivalent system of $N+1$ equations in the $N$ (possibly complex) variables $\beta = (\beta_1,\ldots,\beta_N)$, the real variable $\tau$, and the real parameters $\gamma$ and $\varepsilon$:

$$\text{(4.8)}\quad \text{(a)}\quad 0 = F_n(\beta,\tau,\gamma) \equiv -\tau\beta_n + b\sum_{i,j=1}^{N} \Lambda_{ijn}\beta_i\beta_j + \sum_{i=1}^{N} A_{in}\beta_i^2\beta_n + r_n(\beta,\tau,\gamma),\quad n = 1,2,\ldots,N\ ,$$

$$\text{(b)}\quad 0 = V(\beta,\tau) \equiv -\tau\sum_{j=1}^{N}\beta_j^2 + \frac{2b}{3}\sum_{i,j,m=1}^{N}\Lambda_{ijm}\beta_i\beta_j\beta_m + \frac{1}{2}\sum_{i,j=1}^{N}A_{ij}\beta_i^2\beta_j^2\ ,$$

where $(\beta,\tau,\gamma,\varepsilon) \in \mathbb{C}^N \times \mathbb{R}^3$,

$$\text{(4.9)}\qquad \Lambda_{ijm} = \delta(k_i+k_j+k_m) + \delta(k_i+k_j-k_m) + \delta(k_i-k_j+k_m) + \delta(k_i-k_j-k_m),$$

$$\text{(4.10)}\qquad A_{ij} = a_{p_0 ij}(2-\delta_{ij}) + 2a_{p_0 i(-j)}\quad .$$

Moreover, since Lemma 4.3 shows also that $F = (F_1,\ldots,F_N)$ may be regarded as a mapping of a neighborhood of $(0,0,0)$ in $\mathbb{R}^N \times \mathbb{R}^2$ into $\mathbb{R}^N$, it is natural to seek solutions of the selection equations in (4.8) of the form $(\beta^*(\gamma,\varepsilon),\ \tau^*(\gamma,\varepsilon)) \in \mathbb{R}^{N+1}$ by use of the implicit function theorem near $\gamma = \varepsilon = 0$. If $(\beta^*,\tau^*) \in \mathbb{R}^{N+1}$ is such a solution of (4.8) near $\gamma = \varepsilon = 0$, then $(\beta_N^*,\ldots,\beta_1^*,\beta_1^*,\ldots,\beta_N^*,\tau^*)$ is a solution of (3.24) satisfying (4.6) with $\beta_{-j} = \beta_j$, i.e., a solution of $(3.24)_\pi$ satisfying (4.6). Thus, the above construction leads to real $\psi$ in $M_\pi$ and, hence, real solutions of $(*)_\pi$ in $S_\pi$. To actually carry out the above construction, we seek first the real solutions of the <u>reduced selection equations</u> obtained by setting $\gamma = \varepsilon = 0$ in (4.8):

$$\text{(4.11)}\quad \text{(a)}\quad 0 = F_n(\beta,\tau,0),\qquad n = 1,2,\ldots,N$$

$$\text{(b)}\quad 0 = V(\beta,\tau),\qquad (\beta,\tau) \in \mathbb{R}^{N+1}\ .$$

Remark 4.2. It is easy to check that $F_n(\beta,\tau,0) = \frac{1}{2}\frac{\partial V(\beta,\tau)}{\partial \beta_n}$, $n = 1,2,\ldots,N$ so that (4.11a) is a gradient system. Since (4.11a) is not the reduced bifurcation system associated with (*), this gradient structure is not identical to that used extensively in [1, 10, 17], although it is closely related. We note that the reduced system obtained from (3.24) by setting $\varepsilon = \gamma = 0$ has a similar structure, with $F_n(\beta,\tau,0) = \frac{\partial}{\partial \beta_n} V(\beta,\tau)$; the factor $\frac{1}{2}$ appears in the $S_3$ case because of the identification of $\beta_j$ and $\beta_{-j}$.

In developing a selection principle for stable subcritical hexagonal cells one needs to consider only the reduced selection equations in (4.11). Other choices of the reduced selection equations are also appropriate in convection problems, e.g., in the study of supercritical solutions and the exchange of stability between rolls and hexagonal cells, and will be considered in a subsequent paper.

5. Existence and stability of real solutions in $S_\psi$. In this section we solve the selection equations in a general setting by means of variational methods.

The following preliminary result yields real solutions of $(*)_\beta$ in $S_\psi$.

Theorem 5.1. Let $(\beta^*,\tau^*) \in \mathbb{R}^{N+1}$, with $\beta^* \neq 0$, satisfy the reduced selection equations (4.11) and suppose the Jacobian $\det\left[\frac{\partial F}{\partial \beta}\right]$ is not zero at $(\beta,\tau,\gamma,\varepsilon) = (\beta^*,\tau^*,0,0)$. Then there is a $\delta > 0$ such that for $(\gamma,\varepsilon) \in \mathbb{R}^2$ with $|\gamma| + |\varepsilon| < \delta$ the selection equations (4.8) have a solution $(\beta(\gamma,\varepsilon),\tau(\gamma,\varepsilon)) \in \mathbb{R}^{N+1}$ satisfying

(5.1) $$\lim_{(\gamma,\varepsilon)\to(0,0)} (\beta(\gamma,\varepsilon),\tau(\gamma,\varepsilon)) = (\beta^*,\tau^*).$$

Furthermore problem $(*)_\beta$ has a real solution of the form (3.15) with

(5.2) $$\phi = \sum_{j=1}^{N} \beta_j(\gamma,\varepsilon)(\psi^j + \psi^{-j}), \quad \tau = \tau(\gamma,\varepsilon),$$

and $\Psi$ obtained from $\phi,\tau$ by means of (3.17).

The result follows from the implicit-function theorem applied to $F,V$ near $(\beta,\tau,\gamma,\varepsilon) = (\beta^*,\tau^*,0,0)$, provided that $\det\frac{\partial(F,V)}{\partial(\beta,\tau)}$ is not zero when evaluated at $(\beta^*,\tau^*,0,0)$. But $\frac{\partial V}{\partial \beta} = 2F$ is zero at this point and $\frac{\partial V}{\partial \tau} = -|\beta^*|^2 \neq 0$. Thus,

$$\det\frac{\partial(F,V)}{\partial(\beta,\tau)}(\beta^*,\tau^*,0,0) = -|\beta^*|^2 \det\left[\frac{\partial F}{\partial \beta}\right](\beta^*,\tau^*,0,0) \neq 0$$

and the rest of the theorem follows easily.

To utilize Theorem 5.1 we seek solutions of the reduced selection equations with $\beta^* \neq 0$. We next show how this may be accomplished by exploiting the variational structure of the reduced problem (4.11).

Note that

(5.3) $$\tfrac{1}{2}V(\beta,\tau) = -\tfrac{1}{2}\tau|\beta|^2 + q(\beta) + c(\beta),$$

where

(5.4) (a) $$q(\beta) \equiv \frac{b}{3}\sum_{i,j,m=1}^{N} A_{ijm}\beta_i\beta_j\beta_m \quad ,$$

(b) $$c(\beta) \equiv \frac{1}{4}\sum_{i,j=1}^{N} A_{ij}\beta_i^2\beta_j^2 \quad .$$

In order to determine subcritical solutions of $(*)_\tau$ we shall impose the following hypotheses on $q$ and $c$:

$(H_q)$ $\quad q(\beta) \not\equiv 0$ on $\mathbb{R}^N$

$(H_c)$ $\quad c(\beta) > 0$ for all $\beta \neq 0$ in $\mathbb{R}^N$.

Remark 5.1. Hypothesis $(H_q)$ fails, in general, since $\delta(k_i + k_j + k_m)$ is zero unless the vectors $k_i$, $k_j$, $k_m$ form an equilateral triangle: $k_i + k_j + k_m = 0$. This latter condition is possible for hexagonal lattices, $\alpha_1 = \sqrt{3}\,\alpha$, $\alpha_2 = \alpha$, when $\alpha$ satisfies (2.13) for integers $n_0$, $m_0$ of the same parity. In such cases $(H_q)$ is satisfied if $b \neq 0$. Concerning $(H_c)$, the condition $A_{ij} > 0$ follows from (4.10) and the nonnegativity of the $a_{p_0ij}$ in (vi) of Lemma 3.1. So hypothesis $(H_c)$ is satisfied, e.g., if $a_{p_0ii} > 0$ for each $i = 1,2,\ldots,N$. The latter condition is fulfilled if at least one term in the sum defining $a_{p_0ii}$ is different from zero.

In the following discussion of the finite-dimensional problem (4.11), a prime denotes the gradient with respect to $\beta$. Thus

$$c'(\beta) = \left\{\frac{\partial c(\beta)}{\partial \beta_j}\right\}_{j=1}^{N} \quad , \qquad f''(\beta) = \left\{\frac{\partial^2 f(\beta)}{\partial\beta_i\,\partial\beta_j}\right\}_{i,j=1}^{N} \quad , \quad \text{etc.}$$

In view of Remark 4.2, the system (4.11) becomes

(5.5) (a) $$0 = -\tau\beta + q'(\beta) + c'(\beta) \ ,$$

(b) $$0 = -\frac{\tau}{2}|\beta|^2 + q(\beta) + c(\beta) \ .$$

We define selection functionals $f$ and $g$ by

$$(5.6)\quad (a)\quad f(\beta) = \begin{cases} \dfrac{q(\beta) + c(\beta)}{|\beta|^2}, & \text{if } \beta \neq 0 \\ 0, & \text{if } \beta = 0 \end{cases},$$

$$(b)\quad g(\beta) = q^2(\beta)/(4c(\beta)) .$$

Lemma 5.1. Let the functionals $q$ and $c$ be given by (5.4) and suppose $c$ satisfies $(H_c)$. Let $(\beta,\tau) \in \mathbf{R}^{N+1}$ with $\beta \neq 0$ and set $\hat{\beta} = \beta/|\beta|$. Then the following are equivalent.

(i) $(\beta,\tau)$ is a solution of (5.5),

(ii) $\beta$ is a critical point of $f$ with critical value $f(\beta) = \frac{\tau}{2}$,

(iii) $\hat{\beta}$ is a critical point of $g$ on $|\beta| = 1$ with critical value $g(\hat{\beta}) = -\frac{\tau}{2}$ and the magnitude of $\beta$ satisfies

$$(5.7)\qquad |\beta| = [-q(\hat{\beta})/2c(\hat{\beta})].$$

Proof. The critical points of $f(\beta)$ are determined by

$$(5.8)\qquad 0 = f'(\beta) = |\beta|^{-2}[-\tau\beta + q'(\beta) + c'(\beta)],$$

where

$$(5.9)\qquad \tau = 2[q(\beta) + c(\beta)]|\beta|^{-2} = 2f(\beta).$$

Since $\beta \neq 0$, equations (5.8), (5.9) are just (5.5). Thus (i) and (ii) are equivalent. The condition that $\hat{\beta}$ be a critical point of $g(\beta)$ on $|\beta| = 1$ with critical value $-\frac{\tau}{2}$ is

$$(5.10)\qquad -\tau\hat{\beta} = g'(\hat{\beta}) = q(\hat{\beta})[2c(\hat{\beta})]^{-2}[2c(\hat{\beta})q'(\hat{\beta}) - q(\hat{\beta})c'(\hat{\beta})] .$$

If we use the homogeneity of $q$, $q'$, $c$, $c'$, $g'$ and the Euler identities $\beta\cdot q'(\beta) = 3q(\beta)$, $\beta\cdot c'(\beta) = 4c(\beta)$, $\beta\cdot g'(\beta) = 2g(\beta)$, then from (5.10) we get

(5.11) $$-\tau = \hat{\beta}\cdot g'(\hat{\beta}) = 2g(\hat{\beta}),$$

(5.12) $$-\tau\beta = |\beta| g'(\hat{\beta}) = g'(\beta) = q(\beta)[2e(\beta)]^{-2}[2e(\beta)q'(\beta) - q(\beta)e'(\beta)].$$

If $(\beta,\tau)$ satisfies (5.5) with $\beta \neq 0$, then upon multiplying (5.5a) by $\beta$, using the Euler identities and subtracting twice (5.5b) we obtain

(5.13) $$0 = q(\beta) + 2e(\beta),$$

which implies (5.7). From $\beta \neq 0$, (5.13) and $(H_e)$ we have $q(\beta) = -2e(\beta) \neq 0$. For such $\beta$, equations (5.12) and (5.5a) are the same. Similarly, (5.5b) and (5.13) imply

$$\frac{\tau}{2}|\beta|^2 = [q(\beta) + e(\beta)]\left(-\frac{q(\beta)}{2e(\beta)}\right)^2 = -\frac{q^2(\beta)}{4e(\beta)} = -g(\beta) = -|\beta|^2 g(\hat{\beta}) \quad ,$$

so that (5.5b) and (5.11) are the same. Thus (i) implies (iii). Finally, let $\hat{\beta}, \tau$ and $|\beta|$ satisfy the conditions in (iii). Since $\beta \neq 0$ by assumption, (5.7) is equivalent to (5.13) so that again (5.5) is the same as (5.11), (5.12). Thus, (iii) implies (i).

It is clear from Lemma 5.1 that solutions $(\beta^*,\tau^*)$ of the reduced selection equations with $\beta^* \neq 0$ are obtained from those critical points $\hat{\beta}^*$ of $g(\hat{\beta})$ on $|\beta| = 1$ for which $g(\hat{\beta}^*) \neq 0$. Furthermore, it follows from $(H_e)$, (5.6a), (5.13) and (ii) of Lemma 5.1, that for such critical points

(5.14) $$\tau^* = 2f(\beta^*) = -2\,\frac{e(\beta^*)}{|\beta^*|^2} < 0.$$

Thus, on the basis of (3.15ff.), a solution of $(*)_\eta$ generated from $(\beta^*,\tau^*)$ will be subcritical, at least for small values of $\gamma$ and $\varepsilon$. According to Theorem 5.1, to extend such a solution of (5.5) to a solution of (4.8) we must show that $\det \frac{\partial F}{\partial \beta} \neq 0$ at $(\beta,\tau,\gamma,\varepsilon) = (\beta^*,\tau^*,0,0)$, i.e. $\det E \neq 0$, where $E$ is the symmetric matrix

$$(5.15) \qquad E = -\tau^* I + q''(\beta^*) + c''(\beta^*) \ .$$

Thus, $\det \frac{\partial F}{\partial \beta}$ is zero if and only if $E$ is singular. We have established the following result.

Theorem 5.2. Suppose $q$ and $c$ satisfy hypotheses $(H_q)$ and $(H_c)$. Let $(\beta^*, \tau^*)$, $\beta^* \neq 0$, be a solution of the reduced selection equations (5.5) such that the matrix $E$ in (5.15) is nonsingular. Then there exist $\varepsilon_1 > 0$ and $\gamma_1 > 0$ such that, when $|\gamma| < \gamma_1$ and $|\varepsilon| < \varepsilon_1$, equation $(*)_\pi$ has a real, subcritical solution $(v^*(\gamma,\varepsilon), \lambda^*(\gamma,\varepsilon))$ of the form (3.15) with $\tau = \tau(\gamma,\varepsilon) < 0$ and generalized dissipation $V = \varepsilon$. In fact,

$$(5.16) \quad (a) \qquad v^*(\gamma,\varepsilon) = \gamma \sum_{j=1}^{N} \beta_j^* (\psi^j + \psi^{-j}) + V(\gamma,\varepsilon),$$

$$(b) \qquad \lambda^*(\gamma,\varepsilon) = \mu_1 - \gamma^2 \mu_1 (\mu_1^2 b_0 - \tau^*) + \Lambda(\gamma,\varepsilon) = \lambda_c + \gamma^2 \mu_1 \tau^* + \Lambda(\gamma,\varepsilon),$$

where $\tau^*$ satisfies (5.14) and, as $\gamma \to 0$, $V(\gamma,\varepsilon) = O(\gamma^2), \Lambda(\gamma,\varepsilon) = o(\gamma^2)$, $\Lambda(\gamma,\varepsilon) = o(\gamma^2)$.

According to Theorem 5.2 and (i) and (ii) of Lemma 5.1 we can generate a solution of $(*)_\pi$ by finding a global minimum of $f$ on $\mathbb{R}^N$. If $\beta = \rho\hat{\beta}$ with $|\hat{\beta}| = 1$, note that

$$(5.17) \qquad f(\beta) = c(\hat{\beta}) \left[ \rho + \frac{q(\hat{\beta})}{2c(\hat{\beta})} \right]^2 - g(\hat{\beta}).$$

We minimize $f(\beta)$ on $\mathbb{R}^N$ by choosing $\rho = -q(\hat{\beta})/[2c(\hat{\beta})]$ and maximizing $g(\hat{\beta})$ on $|\hat{\beta}| = 1$. If $q(\beta) \not\equiv 0$ then we generate in this way at least one nontrivial solution of (5.5), say $(\beta^*, \tau^*)$, with $\tau^*$ satisfying (5.14). If we differentiate (5.8) and make use of (5.8) and (5.9), then we find that

$$(5.18) \qquad f''(\beta^*) = |\beta^*|^{-2}[-\tau^* I + q''(\beta^*) + c''(\beta^*)] = |\beta^*|^{-2} E \ .$$

Since $f$ has a minimum at $\beta^*$, we know that $f''(\beta^*)$, hence $E$, is at least positive semi-definite, (5.19) $0 \leq \beta E \beta$.

Thus, if E is nonsingular at a minimum of f, it must be positive definite; we shall see that the solution of $(*)_\pi$ generated (as in Theorem 5.2) from $(\beta^*, \tau^*)$ is then stable in $S_\pi$.

The relationship of the critical points of f to the generalized dissipation $V$ is given in the following remark.

Remark 5.2. From (5.3), (5.5) and Lemma 5.1 one sees that if $\beta_0$ is a critical point of $f(\beta)$ and $\tau_0 = 2f(\beta_0)$, then $\beta_0$ is also a critical point of $V(\tau_0,\beta)$ and $V(\tau_0,\beta_0) = 0$. If $\tau_0$ is also the absolute minimum of $2f(\beta)$ then $V = 0$ is the absolute minimum of $V(\tau_0,\beta)$.

We turn, then, to the question of stability of a solution $(v^*,\lambda^*) = (v^*(\gamma,\varepsilon),\lambda^*(\gamma,\varepsilon))$ of $(*)_\pi$ having the form (5.16). For small $\gamma$ and $\varepsilon$, the derived operator, $\mathcal{D}$, of (*) at $(v^*,\lambda^*)$ is a linear Fredholm operator of index zero, the perturbation by a small bounded linear operator of the self-adjoint operator $I - \mu_1 L$. As observed in [17], because of the invariance of the equations under translations of the (x,y)-plane, the stability of solutions of (*) in $H$ is always indeterminate. In the case of $S_\pi$, however, we have the following result. (The notion of stability here is "linearized stability" as in [16;17].)

Theorem 5.3. For $\gamma,\varepsilon$ sufficiently small, a solution $v(\gamma,\varepsilon)$ of $(*)_\pi$ obtained from Theorem 5.2 is stable in $S_\pi$ at $\lambda = \lambda(\gamma,\varepsilon)$ if all eigenvalues of the matrix E in (5.15) are positive, and unstable if some eigenvalue of E is negative. In particular, if $v(\gamma,\varepsilon)$ is generated from $(\beta^*,\tau^*)$ corresponding to a minimum of f and such that E is nonsingular, then $v(\gamma,\varepsilon)$ is stable in $S_\pi$ at $\lambda = \lambda(\gamma,\varepsilon)$.

To prove Theorem 5.3, one proceeds as in [16] to determine a subspace of $S_\pi$ invariant under $\mathcal{D}$ and corresponding to the N critical eigenvalues of $\mathcal{D}$ for sufficiently small $\gamma$ and $\varepsilon$. This subspace has a basis of the form

(5.20) $$z^i = (\psi^i + \psi^{-i}) + \gamma Z^i, \quad Z^i \in M_\pi^\perp, \quad i = 1,2,\ldots,N$$

satisfying

(5.21) $$\mathcal{D}z^i = \sum_{j=1}^{N} \gamma^2 b_{ij} z^j, \qquad i = 1,2,\ldots,N.$$

To establish the existence of the basis $\{z^1,\ldots,z^N\}$ in (5.20) one needs to show, in particular, that $T_\pi Z^i = Z^i$ so that $z^i$ belongs to $S_\pi$. The proof that $T_\pi Z^i = Z^i$ makes use of the fact that $(\psi^i + \psi^{-i})$ belongs to $M_\pi$ and follows along the lines of the derivation of (3.17) and the proof of Lemma 4.3. Since we assume in Theorem 5.2 that $E$ is nonsingular, (5.19) implies that all eigenvalues of $E$ are positive if $\beta^*$ minimizes $f$. In this case $v(\gamma,\varepsilon)$ is stable in $S_\pi$ at $\lambda = \lambda(\gamma,\varepsilon)$ for $(\gamma,\varepsilon)$ sufficiently small.

6. Subcritical hexagonal cellular solutions. We now restrict the problem to the hexagonal lattice and prove a general result about stable subcritical solutions which yields a selection principle for hexagonal cellular solutions. To fix the ideas we treat also the special case of $\dim M = 12$ in Remarks 6.1, 6.2 and 6.4; this case is the setting in which "exotic" solutions of the Bénard problem were originally studied in [10].

We begin by showing, for an unbounded sequence of integers $N$, that one can determine $N$ sextuples of critical wave vectors corresponding to the critical wave number, $\sigma_0$. These $6N$ vectors generate a nullspace, $M$, with $\dim M = 6N$. Take $\alpha_1 = \sqrt{3}\alpha$, $\alpha_2 = \alpha$ and choose $\alpha$ so that (2.13) has exactly $N$ distinct solutions $(n_0,m_0) = (n_j,m_j)_{j=1}^{N}$, where $n_j$ and $m_j$ are nonnegative integers of like parity for which the critical wave vector $\underset{\sim}{k}_j \equiv \alpha(\sqrt{3}n_j,m_j,0)$ makes an angle $\theta_j$, $0 \le \theta_j < \pi/3$, with $(1,0,0)$. I.e., take $\alpha = \sigma_0/\sqrt{M_0}$ where the integer $M_0$ is chosen so that the equation $3n^2 + m^2 = M_0$ has exactly $N$ solutions satisfying the above conditions. (It is well-known that such pairs $(N,M_0)$ exist for an unbounded sequence of integers $N$ (e.g., see [20, p.345,ex.5]).) We suppose the $N$ vectors, $\underset{\sim}{k}_j$, are ordered so that $0 \le \theta_1 < \theta_2 < \ldots < \theta_N < \pi/3$. Define the $N$ triples, $T_j \equiv (\underset{\sim}{k}_j,\underset{\sim}{k}_{j+N},\underset{\sim}{k}_{j+2N})$ where, for $j = 1,2,\ldots,N$, $k_{j+N}$ (resp., $\underset{\sim}{k}_{j+2N}$) is obtained by rotating $\underset{\sim}{k}_j$ counterclockwise through $\pi/3$ (resp., $2\pi/3$) radians. Note that the $3N$ vectors, $\underset{\sim}{k}_j$, have lengths $\sigma_0$ and direction angles $\theta_j$ satisfying $0 \le \theta_1 < \theta_2 < \ldots < \theta_{3N} < \pi$. Each of the $N$ triples, $T_j$, can now be extended to a sextuple, $(T_j, -T_j)$, if we define $\underset{\sim}{k}_{-j} = -\underset{\sim}{k}_j$ $(j = 1,\ldots,3N)$ in accordance with (2.16). In the above context there are infinitely many possible period rectangles corresponding to values of $\alpha = \sigma_0/\sqrt{M_0}$, however, the critical wave number,

$\eta_0$, and the "size" of the basic hexagonal cell remain <u>fixed</u> throughout the following discussion.

Remark 6.1. If in the above $(N,M_0) = (1,6)$ then $n_0 = 1$ and $m_0 = 1$ in (2.13) and $\dim \mathcal{M} = 6$. In this case we have one triple $(k_1,k_2,k_3)$ and one sextuple $(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3,-\underset{\sim}{k}_1,-\underset{\sim}{k}_2,-\underset{\sim}{k}_3)$, where $\underset{\sim}{k}_1 = \alpha(\sqrt{3},1,0)$, $k_2 = \alpha(0,2,0)$, and $\underset{\sim}{k}_3 = \alpha(-\sqrt{3},1,0)$. If $(N,M_0) = (2,28)$ then $n_0 = 3$ and $m_0 = 1$ in (2.13) and $\dim \mathcal{M} = 12$. In this case we have two triples $(\underset{\sim}{k}_1,\underset{\sim}{k}_3,\underset{\sim}{k}_5)$ and $(\underset{\sim}{k}_2,\underset{\sim}{k}_4,\underset{\sim}{k}_6)$, where

$$
\begin{aligned}
&\underset{\sim}{k}_1 = \alpha(3\sqrt{3},1,0) && \underset{\sim}{k}_2 = \alpha(2\sqrt{3},4,0)\\
(6.1)\qquad &\underset{\sim}{k}_3 = \alpha(\sqrt{3},5,0) && \underset{\sim}{k}_4 = \alpha(-\sqrt{3},5,0)\\
&\underset{\sim}{k}_5 = \alpha(-2\sqrt{3},4,0) && \underset{\sim}{k}_6 = \alpha(-3\sqrt{3},1,0).
\end{aligned}
$$

The first of these special cases, $\dim \mathcal{M} = 6$, was studied in [2; 5; 9; 17] in the context of classical hexagonal solutions. The second case, $\dim \mathcal{M} = 12$, was studied in [10] in the context of "exotic" solutions.

We now define a basis $\{\psi_j\}_{j=1}^{6N}$ for $\mathcal{M}$ in accordance with (3.1) and proceed as in Sections 3 through 5. To make use of Theorem 5.2 in the present setting, one needs to minimize $f$ on $\mathbb{R}^{3N}$, where $f$ is defined as in (5.6). Thus, we require, in particular, the coefficients in the functionals $q$ and $c$ defined by (5.4) with $N$ replaced by $3N$.

The coefficients of $q$ are given by

$$
(6.2)\qquad \Lambda_{ijm} = \begin{cases} 1, & \text{if } (i,j,m) \text{ is a permutation of } (n,n+N,\ n+2N) \text{ for some } n \in \{1,2,\dots,N\} \\ 0, & \text{otherwise.} \end{cases}
$$

Thus, setting $B = 2b$ with $b$ defined as in (3.21), we find

$$(6.3) \qquad q(\beta) = B\sum_{j=1}^{N} \beta_j\beta_{j+N}\beta_{j+2N}.$$

Note that if $b \neq 0$ then $q(\beta) \not\equiv 0$ since, e.g., $\underset{\sim}{k}_j - \underset{\sim}{k}_{j+N} + \underset{\sim}{k}_{j+2N} = 0$ for $j = 1,2,\ldots,N$.

We next discuss the coefficients $a_{ij}$ and $A_{ij}$ required to determine $e(\beta)$ (see (4.10),(5.4) and (A.37)). For our purposes it suffices to evaluate $a_{ij}$ and $a_{i(-j)}$ when $\underset{\sim}{k}_i$ and $\underset{\sim}{k}_j$ lie in the same triple $T_n$, $n = 1,2,\ldots,N$. Recall that $a_{ij}$ depends only on $|\underset{\sim}{k}_i + \underset{\sim}{k}_j|$, i.e., only on the angle between $\underset{\sim}{k}_i$ and $\underset{\sim}{k}_j$ (see (A.37) & ff.). When $\underset{\sim}{k}_i, \underset{\sim}{k}_j$ lie in the same triple and $\underset{\sim}{k}_i \neq \underset{\sim}{k}_j$ this angle is either $\pi/3$ or $2\pi/3$. We denote the corresponding values of $a_{ij}$ by $a(\pi/3)$ and $a(2\pi/3)$, respectively. It is now easily seen that if $a_{ij} = a(\pi/3)$ then $a_{i(-j)} = a(2\pi/3)$ and if $a_{ij} = a(2\pi/3)$ then $a_{i(-j)} = a(\pi/3)$. Since $A_{ij} = 2(a_{ij} + a_{i(-j)})$ when $i \neq \pm j$, and since $a_{p(-q)} = a_{i(-j)}$ when $a_{pq} = a_{ij}$, it follows that the $A_{ij}$ have a common value, $A = 2(a(\pi/3) + a(2\pi/3))$, when $i \neq \pm j$ and $\underset{\sim}{k}_i, \underset{\sim}{k}_j$ lie in the same sextuple. Similarly, when $i = j$, $|k_i + k_j| = 2\alpha_0$ so that the $A_{ii}$ have a common value, $C$, $i = 1,2,\ldots,3N$. Thus

$$(6.4) \qquad A_{ij} = \begin{cases} C, & \text{if } i = j \\ A, & \text{if } i \neq j \text{ and } \underset{\sim}{k}_i, \underset{\sim}{k}_j \in T_n, \quad n = 1,2,\ldots,N. \end{cases}$$

It follows from (4.10) and (A.37) that all $A_{ij} \geq 0$, hence $A \geq 0$; furthermore, hypothesis $(H_c)$ is equivalent to $C > 0$.

It is also possible to determine other relationships among the $A_{ij}$ when $\underset{\sim}{k}_i, \underset{\sim}{k}_j$ lie in different triples. Such relationships are not required to study the classical hexagonal cells but are given in (6.6b) below when $N = 2$.

<u>Remark 6.2</u>. In the context of $(N,M_0) = (2,28)$ in Remark 6.1 there are nine distinct positive values of $|\underset{\sim}{k}_i + \underset{\sim}{k}_j|$ for $i,j \in \{\pm1,\pm2,\ldots,\pm6\}$. Therefore, there are at

most nine distinct positive $a_{ij} \equiv a_{p_0 ij}$. One finds that $A_{ii} = a_{ii} = C$, $i = 1,2,\ldots,6$, and

$$
\text{(6.5)}\quad
\begin{aligned}
&a_{12} = a_{34} = a_{56},\ a_{1-6} = a_{23} = a_{45},\ a_{14} = a_{36} = a_{2-5},\\
&a_{1-4} = a_{25} = a_{3-6},\ a_{16} = a_{2-3} = a_{4-5},\ a_{1-2} = a_{3-4} = a_{5-6},\\
&a_{13} = a_{1-5} = a_{35} = a_{24} = a_{46} = a_{2-6}\\
&a_{15} = a_{1-3} = a_{26} = a_{2-4} = a_{3-5} = a_{4-6},\ \text{and}\ a_{-i-j} = a_{ij}\ .
\end{aligned}
$$

It follows that the $A_{ij}$ satisfy

$$
\text{(6.6)}\quad
\begin{aligned}
&\text{(a)}\ A_{13} = A_{15} = A_{24} = A_{26} = A_{35} = A_{46} = A\\
&\text{(b)}\ A_{12} = A_{34} = A_{56},\ A_{14} = A_{25} = A_{36},\ A_{16} = A_{23} = A_{45}\ .
\end{aligned}
$$

The relationships (6.6b) are needed for a complete analysis of "exotic" solutions when $N = 2$.

From (5.4b) and (6.4) we get $c(\beta) = \tilde{c}(\beta) + d(\beta)$, where

$$\text{(6.7)}\quad \tilde{c}(\beta) = \frac{1}{4}C\sum_{i=1}^{3N}\beta_i^4 + \frac{1}{2}A\sum_{i=1}^{N}(\beta_i^2\beta_{i+N}^2 + \beta_i^2\beta_{i+2N}^2 + \beta_{i+N}^2\beta_{i+2N}^2)$$

and $d(\beta)$ denotes the contribution to the sum in (5.4b) of terms $A_{ij}\beta_i^2\beta_j^2$ for which $\underset{\sim}{k}_i$ and $\underset{\sim}{k}_j$ lie in different triples. Note that $d(\beta) \geq 0$, $\beta \in R^{3N}$, since $A_{ij} \geq 0$. Thus, $f(\beta) \geq \tilde{f}(\beta)$, where $f(\beta)$ is defined in (5.6) and

$$\text{(6.8)}\quad \tilde{f}(\beta) \equiv \begin{cases} (q(\beta) + \tilde{c}(\beta))/|\beta|^2, & \text{if } \beta \neq 0\\ 0, & \text{if } \beta = 0\ . \end{cases}$$

The functional $\tilde{f}$ and its critical points play a key role in the determination of stable, subcritical hexagonal solutions. Since $(H_c)$ is equivalent to $C > 0$ in (6.4), the functional $\tilde{c}$ also satisfies $(H_c)$, so that Lemma 5.1 is applicable to both $f$ and $\tilde{f}$.

Lemma 6.1. The nontrivial critical points, $\beta$, of $\tilde{f}$ satisfy $\beta_n^2 = \beta_{n+N}^2 = \beta_{n+2N}^2$, $n = 1,2,\ldots,N$. Moreover, $\tilde{f}$ assumes its absolute minimum,

(6.9) $$f_0 \equiv -B^2/9C_1 \quad \text{with} \quad C_1 \equiv C + 2A,$$

at those critical points for which all nonzero $\beta_n$ satisfy $\beta_n^2 = 4B^2/9C_1^2$.

Proof. At a nontrivial critical point we have (see (5.8), (5.9))

(6.10) $$0 = |\beta|^2 \frac{\partial f}{\partial \beta_i}(\beta) = -2\tilde{f}(\beta)\beta_i + \frac{\partial q}{\partial \beta_i} + \frac{\partial c}{\partial \beta_i}, \quad i = 1,2,\ldots,3N.$$

Let $T_n = (k_n, k_{n+N}, k_{n+2N})$ be any triple and let $i,j,m$ be the indices $(n, n + N, n + 2N)$ written in any order. Multiply the $i^{\underline{th}}$ equation in (6.10) by $\beta_i$, the $j^{\underline{th}}$ equation by $\beta_j$ and subtract to get

(6.11) $$0 = (\beta_i^2 - \beta_j^2)[-2\tilde{f}(\beta) + C(\beta_i^2 + \beta_j^2) + A\beta_m^2] \ .$$

By making use of the equivalence of (i) and (ii) of Lemma 5.1 applied to $\tilde{f}$, one sees as in (5.14) that $\tilde{f}(\beta) \leq 0$. Hence (6.11) and $(H_c)$ imply that $\beta_i^2 = \beta_j^2$. Since $n$ and the order of $i,j,m$ are arbitrary, we have $\beta_n^2 = \beta_{n+N}^2 = \beta_{n+2N}^2$, $n = 1,2,\ldots,N$. Observe that if $k_i \in T_n$ the $i^{\underline{th}}$ equation in (6.10) involves only $\beta_n$, $\beta_{n+N}$ and $\beta_{n+2N}$. Since we may change the signs of any pair of these three $\beta_j$'s without changing the $i^{\underline{th}}$ equation, we may suppose at a critical point of $\tilde{f}$ that $\beta_{n+2N} = \beta_{n+N} = \beta_n$, $n = 1,2,\ldots,N$. Then the three equations in (6.10) corresponding to each $T_n$ become identical and (6.10) reduces to $N$ equations for $\beta_n$, $n = 1,2,\ldots,N$. We suppose that exactly $M_0$ of the $\beta_n$ are nonzero and reorder the indices so that $\beta_n \neq 0$ if $n = 1,2,\ldots,M_0$, and $\beta_n = 0$ if $n = M_0 + 1,\ldots,N$. Then (6.10) may be replaced by

(6.12) $$0 = -2\tilde{f}(\beta) + B\beta_i + C_1\beta_i^2, \quad i = 1,\ldots,M_0,$$

where $C_1 = C + 2A$. When $M_0 = 1$ one solves (6.8) and (6.12) to obtain $\beta_1^2 = 4B^2/9C_1^2$, $\tilde{f}(\beta) = f_0 \equiv -B^2/9C_1$. When $M_0 \geq 2$ one subtracts the equation for $\beta_j$ from that for $\beta_i$ to get the $M_0(M_0 - 1)/2$ equations

(6.13) $$0 = (\beta_i - \beta_j)[B + C_1(\beta_i + \beta_j)], \quad j = i+1,\ldots,M_0; i = 1,\ldots,M_0.$$

It is easy to deduce from (6.13) that the $\beta_j$'s either are all equal or assume exactly two distinct values. When the $\beta_j$'s are all equal, the system (6.8), (6.12) becomes a pair of equations for $\beta_1$, $\tilde{f}(\beta)$ and one finds that $\beta_1^2 = 4B^2/9C_1^2$, $\tilde{f}(\beta) = f_0$. In the case of exactly two distinct $\beta_j$, we suppose $\beta_1 \neq \beta_2$ with $p_1$ of the $\beta_j$'s equal to $\beta_1$ and $p_2$ of the $\beta_j$'s equal to $\beta_2$, $p_1 + p_2 = M_0$. Then the system (6.8), (6.12) reduces to

(6.14) (a) $$\tilde{f}(\beta) = [B(p_1\beta_1^3 + p_2\beta_2^3) + \tfrac{3}{4}C_1(p_1\beta_1^4 + p_2\beta_2^4)]\Big/3(p_1\beta_1^2 + p_2\beta_2^2)$$

(b) $$2\tilde{f}(\beta) = B\beta_i + C_1\beta_i^2, \quad i = 1,2.$$

Since $\beta_1, \beta_2$ are different and nonzero we seek a solution in the form $\beta_2 = s\beta_1$, $s \neq 0,1$. Using (6.14) to express $\tilde{f}(\beta)$ and $\beta_1$ in terms of $s$, one finds that $\beta_1 = -B\big/C_1(1+s)$, $\tilde{f}(\beta) = B^2 s\big/2C_1(1+s)^2$ and the solutions are determined by the roots, $s$, of

$$0 = -p_2 s^4 + 2p_2 s^3 + 2p_1 s - p_1 \quad .$$

The latter equation has exactly two real roots $s_1$, $s_2$, which satisfy $0 \leq s_1 < \frac{1}{2}$, $2 < s_2$. If $f_i$ is the value of $\tilde{f}$ corresponding to $s_i$, $i = 1,2$, then one shows that $f_i > f_0$ so that these solutions do not give the absolute minimum of $\tilde{f}$.

Recall that $f(\beta) \geq \tilde{f}(\beta)$ for all $\beta \in \mathbf{R}^{3N}$ and, in addition, observe that $f(\beta) = \tilde{f}(\beta) = f_0 \equiv -B^2/9C_1$ at points, $\beta$, of the form

$$(6.15)\qquad \begin{cases} \beta_i = 0 \quad \text{if} \quad i \neq N,\ n+N,\ n+2N \\ \beta_n = \beta_{n+N} = \beta_{n+2N} = 2B/3C_1, \quad n = 1,2,\ldots,N . \end{cases}$$

Since $f_0$ is the absolute minimum of $\tilde{f}$, it is also the absolute minimum of $f$, i.e.

$$(6.16)\qquad f_0 = \min_{\beta \in R^{3N}} f(\beta) .$$

It follows from (6.16) that each point of the form (6.15) is a critical point of $f$. Moreover, since $C_1 > 0$, one can show that the matrix $E$ in (5.15) is nonsingular at these points; in fact, $\det E \geq (\beta_1^6 C_1/2)^N$. Thus, according to Theorem 5.3 each of the points (6.15) generates a subcritical solution, $v = v(n,N)$, of $(*)_\pi$ stable in $S_\pi$. Note that because of (6.16) and Lemma 5.1, there are no other solutions in $S_\pi$ generated by solutions $(\beta,\tau)$ of (5.5) with $\tau < \tau_0 \equiv 2f_0$.

Remark 6.3. One can, of course, also consider the solutions $v(n,N)$ as solutions of (*) in $H$. The stability of the $v(n,N)$ in $H$ is determined to lowest order by the eigenvalues of the $6N \times 6N$ Jacobian matrix of the full selection equations (3.24) at $\gamma = \epsilon = 0$. One finds as in [T, pp. 642-643] that all but two of these critical eigenvalues are positive and, because of the invariance of the equations (2.1) under translations of the $(x,y)$-plane, the remaining two are 0. Thus, the stability arguments in [1] apply also to the hexagonal solution $v(n,N)$.

We shall call a solution, $v$, of (*) a hexagonal cellular solution if the leading term in $v$ has zero component across the vertical faces of a right hexagonal cylinder $Z$ and also across the vertical faces of cells obtained from $Z$ by repeated reflection across the vertical faces (the axis of $Z$ is parallel to the $z$-axis and the cross sections $z = z_0$, $-\frac{1}{2} \le z_0 \le \frac{1}{2}$ are regular hexagons). For example, the solution $v(n,N)$ generated by (6.15) is a hexagonal solution (note the shape of the streamlines in [10, Fig. 1]; see also [3, §16]). One can show (e.g., see [3, §16]) that $\psi = \psi^n + \psi^{n+N} + \psi^{n+2N}$ has zero component across the vertical faces of $Z$ whose cross section $z = 0$ is the hexagon with center at $(x,y) = (0,0)$ and vertices at $\pm(4\pi/3\gamma_0^2)k_i$, $k_i \in T_n$. Clearly, the same is true of $\bar{\psi}$, corresponding to $k_i \in -T_n$, hence of $\zeta = \psi + \bar{\psi}$. Furthermore, the flow $\zeta$ has the positive $z$-direction along the $z$-axis. Thus, we see that the leading term in $v(n,N)$ has this hexagonal structure and, since $\beta_3 < 0$, $\rho > 0$, the flow is upward along the $z$-axis when $\gamma > 0$ and downward when $\gamma < 0$.

One may also investigate the existence of exotic solutions in $S_\gamma$ for general $N$ by the methods of the present section. To determine the stability of exotic solutions, however, requires the verification of certain inequalities among the coefficients of the functional $f$ in (5.6a). This is illustrated in the following remark for the case $N = 2$.

Remark 6.4. Besides the simple hexagonal solutions determined above, one obtains in the case $N = 2$ additional solutions corresponding to

(6.17) (a) $\alpha = -2B^2/9(C_1 + A_1), \beta_i = -2B/3(C_1 + A_1)$, $i = 1,\dots,6$

(b) $\alpha = [-s_1C_1 + (1 + s_1 + s_1^2)A_1]\beta_1^2, \beta_1 = \beta_3 = \beta_5 = -B/(1+s_1)(C_1-A_1)$,

$\beta_2 = \beta_4 = \beta_6 = s_1\beta_1$,

where $A_1 = A_{12} + A_{14} + A_{16}$ (see (6.6)). Here $s_1$, $0 < s_1 < 1$, is a root of

$$0 = 2(C_1 - A_1)(s^3 + s^2 + s) - (C_1 + 2A_1)(s^2 + 1)^2. \tag{6.18}$$

One finds that the existence of $s_1$, hence of (6.17b), as well as the stability of both solutions in (6.17) depends on the sign of $C_1 - 7A_1$.

The solution (6.17a) corresponds to the first exotic solution (case 2a) in [10].

Observe that in each of the solutions (6.15), all $\beta_i$'s corresponding to a given triple, $T_n$, are equal. The functional $f$, however, does not change if we change the signs of any two $\beta_i$'s corresponding to the same triple. Thus, each of the hexagonal solutions generated by (6.15) yields three additional hexagonal solutions. One can show that the four solutions obtained this way are translations of one another. Moreover, all of the solutions, $v(n,N)$, generated by (6.15) (for $n = 1,2,\ldots,N$; $N$ in a suitable, unbounded sequence) are, at least to first order, rotations of $v(1,1)$.

Our main results for classical hexagonal cellular solutions are summarized in the next paragraph and hold under the hypotheses $C \neq 0$, $B \neq 0$ (see Remark 5.1). These hypotheses are independent of $N$ and are analogous to the minimum hypotheses required for a bifurcation analysis at $\lambda_c$ when $N = 1$.

Hexagonal cellular solutions. For each $N$ in a suitable unbounded sequence there are $4N$ solutions of (*) generated by absolute minima of the selection functional, $f$. These solutions are subcritical and stable in $S_\pi = S_\pi(N)$. Each of these solutions exhibits the classical hexagonal cellular form with size independent of $N$. The stability of, e.g., $v(1,N)$ in $S_\pi(N)$ shows that the hexagonal cellular solutions are, in particular, stable to perturbations in "directions" corresponding to $N$ critical wave vectors. Thus, letting $N$ range over the unbounded sequence, we obtain, in a sense, the stability of the classical hexagonal cells in infinitely many such critical directions.

7. Concluding remarks. There is no attempt in the present paper to obtain "all of the local solutions near $\lambda = \mu_1$" of the Bénard problem with symmetric boundary conditions even in the simplest of cases. The motivation has been rather to provide a first step toward showing that the hexagonal cellular solutions are the "preferred" subcritical solutions of the Bénard problem in physical situations with temperature dependent material properties. In fact, the recent results of Buzano and Golubitsky [2] and Golubitsky, Swift and Knobloch [5] indicate how difficult it would be to obtain "all of the local solutions near $\lambda = \mu_1$" even in the case in Section 6 when $\dim M = 12$. In [2], [5] those authors consider situations corresponding here to the case in Section 6 of one triple of critical wave vectors, i.e., $\dim M = 6$ and, by an application of group theory and, in [2], also singularity theory, they obtain "all of the local solutions" of a six-dimensional problem $P$. (One assumes that $P$ corresponds to the finite-dimensional problem generated from the Bénard problem by means of the Lyapunov-Schmidt method relative to the first eigenvalue of the linearized problem.) The detailed results in [2] are of particular interest because they show for the Bénard problem that the mathematical possibility exists of having stable subcritical hexagonal-type solutions, stable supercritical roll-type solutions, and a third type of solution that provides a transition between rolls and hexagons. There are, of course, some difficulties encountered in carrying over the finite-dimensional results in [2], [5] to an infinite-dimensional mathematical model and many such difficulties and their interpretations for the Bénard problem are discussed in [2, §11]. The most pertinent such difficulty relative to the method presented here is the fact that the detailed nature

of the results in [2], [5] are highly dependent upon the relatively low dimension of the problem $P$ whereas the basic results of Busse [1] are essentially independent of the dimension of any underlying finite-dimensional problem. One of the main goals of our study of the Bénard problem was to develop a rigorous stability method useful in a setting that also is independent of the dimension of any underlying finite-dimensional problem. The results of Section 6 show that this goal has been achieved and that in our approach the selection of stable subcritical hexagonal cellular solutions is closely related to a minimization condition on the generalized dissipation. As in earlier work on the Bénard problem (e.g., see [1; 15]), there remains in the case of temperature-dependent material properties the problems of finding a strict physical interpretation of the generalized dissipation and a description of the actual selection mechanism. Finally, we note that the methods introduced here can be modified to yield also the description of stable supercritical states and the stability relationships between roll-type solutions and hexagonal cellular solutions.

Appendix. Here we justify equation (2.23) and prove Lemma 3.1.

First we show that the eigenfunctions $\{\psi^{pqj}\}$ in (2.20) can be scaled with constants independent of $j$ so that (2.23) holds. In fact, we may assume that each $\phi^{pq}$ has been scaled by a constant depending only on $p$ and $q$ such that

(A.1) $$\frac{2}{\sigma_p^2}\int_{-1/2}^{1/2}[D^2\phi_3^{pq}]^2dz = \frac{\alpha_1\alpha_2}{4\pi^2} \quad ,$$

where $D \equiv \frac{d^2}{dz^2} - \sigma_p^2$. (The integrand on the left in (A.1) is not zero, by uniqueness of the initial-value problem $D^2\phi = 0$, $\phi(\frac{1}{2}) = \phi'(\frac{1}{2}) = 0$ (see (2.12e)).) From (2.3) and (2.21) we get

(A.2) $$(\psi^{pqj},\psi^{rst}) = J(p,q,j;r,s,t)\int_0^{2\pi/\alpha_1}\int_0^{2\pi/\alpha_2} e^{i(\underset{\sim}{k}_{pj} - \underset{\sim}{k}_{rt})\cdot \underset{\sim}{x}}dydx$$

$$= \delta_{pr}\delta_{jt}\frac{4\pi^2}{\alpha_1\alpha_2}J(p,q,j;r,s,t).$$

Here, since $\phi_3^{pqj} = \phi_3^{pq}$ and $\phi_4^{pqj} = \phi_4^{pq}$ are real and independent of $j$,

(A.3) $$J(p,q,j;r,s,t) \equiv \int_{-1/2}^{1/2}\Bigg\{(\underset{\sim}{k}_{pj}\cdot\underset{\sim}{k}_{rt})\sum_{m=1}^{3}\phi_m^{pqj}\bar{\phi}_m^{rst}$$
$$+\sum_{m=1}^{3}\frac{d}{dz}\phi_m^{pqj}\frac{d}{dz}\bar{\phi}_m^{rst} + \frac{1}{Pr}[(\underset{\sim}{k}_{pj}\cdot\underset{\sim}{k}_{rt})\phi_4^{pq}\phi_4^{rs} + \frac{d}{dz}\phi_4^{pq}\frac{d}{dz}\phi_4^{rs}]\Bigg\}dz \ .$$

From (A.2) we see that $J$ is needed only when $r = p$ and $t = j$. Then we may integrate by parts in (A.3) making use of (2.12) to show that

(A.4) $$J(p,q,j;p,s,j) = \mu_{pq}\int_{-1/2}^{1/2}(\phi_4^{pq}\phi_3^{ps} + \phi_3^{pq}\phi_4^{ps})dz \quad .$$

Since both $\phi^{pq}$ and $\phi^{ps}$ satisfy (2.12) we have, after integrating by parts,

(A.5) $$0 = (\mu_{ps} - \mu_{pq})\sigma_p^2\int_{-1/2}^{1/2}(\phi_4^{pq}\phi_3^{ps} + \phi_3^{pq}\phi_4^{ps})dz \ .$$

Thus (A.5) shows that if $\mu_{ps} \neq \mu_{pq}$ (i.e., if $s \neq q$) then $\phi^{pqj}$ and $\phi^{psj}$ are orthogonal in the sense that

$$(A.6) \qquad 0 = \int_{-1/2}^{1/2} (\phi_4^{pq}\phi_3^{ps} + \phi_3^{pq}\phi_4^{ps})dz .$$

In particular,

$$(A.7) \qquad J(p,q,j;p,s,j) = \delta_{qs} J(p,q,j;p,q,j).$$

But from (2.12a), integration by parts and (A.1) we get

$$(A.8) \qquad J(p,q,j;p,q,j) = 2\sigma_p^{-2} \int_{-1/2}^{1/2} \phi_3^{pq} D^4 \phi_3^{pq} dz = \frac{\alpha_1\alpha_2}{4\pi^2} .$$

Combining (A.2), (A.7) and (A.8), we obtain (2.23).

Next we give a proof of Lemma 3.1; some aspects of the work is closely related to corresponding steps in [1] or in [10]. According to Lemma 2.1 the operators L and M are bounded on H. Since L is also compact, it is easy to see that K is bounded on $M^{\perp}$. If $v \in H$ has the form (3.3) and A is any bounded linear operator on $H$, then Av may be computed term by term in the sum so that the formulas (3.7) follow easily. The positivity and self-adjointness of K are simple consequences of (2.23), (3.7) and the fact that the $\mu_{pq}$ are real.

Part (ii) of Lemma 3.1 follows easily from the definition of M if we show that

$$(A.9) \qquad 0 = (M\psi^m, \bar{\psi}^j), \qquad \text{for } |m|,|j| = 1,2,\ldots,N .$$

Since $\phi_4 = \phi_4^{p_0 1}$ and $\phi_3 = \phi_3^{p_0 1}$ are even functions of $z$ (see [10]), (A.9) follows from

$$(M\psi^m, \bar{\psi}^j) = 2\int_{\Omega} z\psi_4^m \psi_3^j = \frac{8\pi^2}{\alpha_1\alpha_2}\, \delta(k_m + k_j) \int_{-1/2}^{1/2} z\phi_4\phi_3 dz = 0 .$$

The assertions (iii) of the lemma are obtained from (3.2), (3.3) and the identity

$$\int_{\mathcal{D}} (u\cdot\nabla v)\cdot w = \int_{\mathcal{D}} \{\nabla\cdot[u(v\cdot w)] - (u\cdot\nabla w)\cdot v\} = -\int_{\mathcal{D}} (u\cdot\nabla w)\cdot v .$$

This last identity is easily verified for smooth $u,v,w \in H$ and is proved in general by a standard limiting argument using the boundedness, in $u,v,w$, of the functional $(\Phi(u,v),w)$:

(A.10) $$|(\Phi(u,v),w)| \leq \text{const}\, \|u\|\,\|v\|\,\|w\|.$$

(The inequality (A.10) follows from (3.2) by application of the Schwarz and Poincare inequalities.) In addition, one may use (A.10) to show that all assertions in part (iv) of Lemma 3.1 are consequences of (A.11) and (A.13), below.

On substituting (2.20) and (2.21) into (3.2) we are led to (recall that $p_0$ and $p_0 1$ are suppressed: $\sigma_{p_0} = \sigma$, $\underset{\sim}{k}_{p_0 j} = \underset{\sim}{k}_j$, etc.)

(A.11) $$(\Phi(\psi^{pqm},\psi^j),\bar{\psi}^n) = -\frac{4\pi^2}{\alpha_1\alpha_2}\,\delta(\underset{\sim}{k}_{pm} + \underset{\sim}{k}_j + \underset{\sim}{k}_n)\,I(p,q,m;p_0,j,n)$$

where

(A.12) $$I(p,q,m;p_0,j,n) \equiv \int_{-1/2}^{1/2} \left\{ -\sigma_p^{-2}(\underset{\sim}{k}_{pm}\cdot\underset{\sim}{k}_j)\,\frac{d\phi_3^{pq}}{dz}\,(\phi^j\cdot\phi^n) + \phi_3^{pq}\left[\sigma^{-4}(\underset{\sim}{k}_j\cdot\underset{\sim}{k}_n)\,\frac{d^2\phi_3}{dz^2}\,\frac{d\phi_3}{dz} + \frac{d\phi_3}{dz}\,\phi_3 + \frac{d\phi_4}{dz}\,\phi_4\right]\right\} dz .$$

The right hand side of (A.11) is zero because of the $\delta$ term, except when $\underset{\sim}{k}_{pm} + \underset{\sim}{k}_j + \underset{\sim}{k}_n = 0$. In this exceptional case the vectors $\underset{\sim}{k}_j$, $\underset{\sim}{k}_n$ and $\underset{\sim}{k}_{pm}$ form an isosceles triangle so that $\underset{\sim}{k}_{pm}\cdot\underset{\sim}{k}_j = \underset{\sim}{k}_{pm}\cdot\underset{\sim}{k}_n$. Consequently $m$ and $j$ may be interchanged in (A.12) without changing I. In this case then, from (A.11), and part (iii) of Lemma 3.1 we have

(A.13) $$(\Phi(\psi^{pqm},\psi^j),\bar{\psi}^n) = (\Phi(\psi^{pqm},\psi^n),\bar{\psi}^j) = -(\Phi(\psi^{pqm},\psi^j),\bar{\psi}^n) = 0 .$$

To prove the formulas of part (v) of the lemma we take $\psi = \sum_{|j|=1}^{N} \beta_j \psi^j$ and calculate the various terms. Now

$$(A.14) \qquad (M\psi^j, \psi^{pqm}) = 2\int_\Omega z\psi_4^j \bar{\psi}_3^{pqm} dz = \delta_{pp_0} \delta_{jm} b_{0q} .$$

Here

$$(A.15) \qquad b_{0q} \equiv \frac{8\pi^2}{\alpha_1\alpha_2} \int_{-1/2}^{1/2} z\phi_4^{p_0 1} \phi_3^{p_0 q} dz, \qquad |q| = 1,2,\ldots$$

is real and $b_{01} = b_{0(-1)} = 0$ since $\phi_3$ and $\phi_4$ are even. Then

$$(A.16) \qquad M\psi = \sum_{|j|=1}^{N} \beta_j \left( \sum_{|q|=2}^{\infty} b_{0q} \psi^{p_0 qj} \right) .$$

Similarly, from (2.24) we are led to

$$(A.17) \qquad M^*\bar{\psi}^n = \sum_{|q|=2}^{\infty} b_{0q}^* \bar{\psi}^{p_0 qn} ,$$

where

$$(A.18) \qquad b_{0q}^* \equiv \frac{8\pi^2}{\alpha_1\alpha_2} \int_{-1/2}^{1/2} z\phi_4^{p_0 q} \phi_3^{p_0 1} dz, \qquad |q| = 1,2,\ldots$$

is real. Since $M\psi \subset M^\perp$, $KM\psi$ may be obtained from (A.16) and (3.7):

$$(A.19) \qquad KM\psi = \sum_{|j|=1}^{N} \beta_j \sum_{|q|=2}^{\infty} \tilde{b}_{0q} \psi^{p_0 qj} ,$$

where $\tilde{b}_{0q} = \mu_{p_0 q}(\mu_{p_0 q} - \mu_1)^{-1} b_{0q}$. From (A.17) and (A.19) we have

$$(A.20) \qquad (MKM\psi, \bar{\psi}^n) = \sum_{|j|=1}^{N} \beta_j \left\{ \sum_{|m|,|q|=2}^{\infty} \tilde{b}_{0m} b_{0q}^* (\psi^{p_0 mj}, \bar{\psi}^{p_0 qn}) \right\} = b_0 \beta_{-n} .$$

Here

$$(a.21) \qquad b_0 = \sum_{|q|=2}^{\infty} \tilde{b}_{0q} b_{0q}^*$$

is real. This proves (3.8).

We shall require

(A.22) $$(\Phi(\psi^j,\psi^m),\bar{\psi}^{pqn}) = \delta(\underset{\sim}{k}_j + \underset{\sim}{k}_m + \underset{\sim}{k}_{pn})I_2(p_0,j,m,p,q,n),$$

where, by (2.21),

(A.23) $$I_2(p_0,j,m,p,q,n) \equiv \frac{4\pi^2}{\alpha_1\alpha_2}\int_{-1/2}^{1/2}\left\{\sigma^{-2}(\underset{\sim}{k}_j\cdot\underset{\sim}{k}_m)\frac{d\phi_3}{dz}\phi^m\cdot\phi^{pqn} - \phi_3\left(\frac{d\phi^m}{dz}\cdot\phi^{pqn}\right)\right\}dz .$$

We are interested in $I_2$ only when the vectors $\underset{\sim}{k}_j$, $\underset{\sim}{k}_m$, $\underset{\sim}{k}_{pn}$ form an isosceles triangle, otherwise the $\delta$-factor in (A.22) is zero. From this triangle we see that $\underset{\sim}{k}_j\cdot\underset{\sim}{k}_{pn} = \underset{\sim}{k}_m\cdot\underset{\sim}{k}_{pn} = -\frac{1}{2}\sigma_p^2$ and $\underset{\sim}{k}_j\cdot\underset{\sim}{k}_m = \frac{1}{2}\sigma_p^2 - \sigma^2$ . In this case, (A.23) leads to $I_2(p_0,j,m,p,q,n) = I_3(p_0,p,q)$, where

(A.24) $$I_3(p_0,p,q) = -\frac{4\pi^2}{\alpha_1\alpha_2}\int_{-1/2}^{1/2}\left\{\left(1-\frac{1}{2}\frac{\sigma_p^2}{\sigma^2}\right)\frac{d\phi_3}{dz}\left[\frac{1}{2\sigma^2}\frac{d\phi_3}{dz}\frac{d\phi_3^{pq}}{dz} + \phi_3\phi_3^{pq} + \phi_4\phi_4^{pq}\right]\right.$$
$$\left. + \phi_3\left[\frac{1}{2\sigma^2}\frac{d^2\phi_3}{dz^2}\frac{d\phi_3^{pq}}{dz} + \frac{d\phi_3}{dz}\phi_3^{pq} + \frac{d\phi_4}{dz}\phi_4^{pq}\right]\right\}dz$$

is real and depends on $j$ and $m$ only through $\sigma_p = |\underset{\sim}{k}_j + \underset{\sim}{k}_m|$. On the basis of (A.22) and (A.24) we have

(A.25) $$(\Phi(\psi^j,\psi^m),\bar{\psi}^{pqn}) = \delta(\underset{\sim}{k}_j + \underset{\sim}{k}_m + \underset{\sim}{k}_{pn})I_3(p_0,p,q) .$$

From (3.6), (3.7) and (A.17) we have

(A.26) $$(MKF(\psi),\bar{\psi}^n) = (F(\psi),KM^*\bar{\psi}^n) = \sum_{|q|=2}^{\infty}\hat{b}_{0q}^*(F(\psi),\bar{\psi}^{p_0qn}),$$

where $\hat{b}_{0q}^* = \mu_{p_0q}(\mu_{p_0q}-\mu_1)^{-1}b_{0q}^*$ is real. Furthermore, from (A.25) with $p = p_0$ we get

(A.27) $$(F(\psi),\bar{\psi}^{p_0qn}) = \sum_{|j|,|m|=1}^{N}\beta_j\beta_m\delta(\underset{\sim}{k}_j + \underset{\sim}{k}_m + \underset{\sim}{k}_n)I_3(p_0,p_0,q) .$$

Combining (A.26) with (A.27) we obtain (3.9) with real $b_1$ given by

(A.28) $$b_1 = \sum_{|q|=2}^{\infty} \bar{b}^*_{0q} I_3(p_0,p_0,q) \quad .$$

In a similar manner we may utilize (A.19), (A.25) and part (iii) of Lemma 3.1 to establish (3.10) with real constant $b_2$ given by

(A.29) $$b_2 = \sum_{|q|=2}^{\infty} \bar{b}_{0q} I_3(p_0,p_0,q) \quad .$$

Equation (3.11) follows easily from the observation that

(A.30) $$(\Gamma(\psi^i,\psi^m),\bar{\psi}^n) = \delta(k_i + k_m + k_n)b_3 \quad ,$$

where

(A.31) $$b_3 = \frac{4\pi^2}{\alpha_1\alpha_2}\int_{-1/2}^{1/2}(\phi_4)^2\phi_3 dz > 0 \ .$$

Next we consider part (vi) of the lemma. From (iii) of Lemma 3.1 and the bilinearity of $\Phi$ we have

(A.32) $$(\Phi(\psi,KF(\psi)),\bar{\psi}^n) = -(\Phi(\psi,\psi^n),\overline{KF(\psi)})$$

$$= - \sum_{|i|,|j|,|m|=1}^{N} \beta_i\beta_j\beta_m(\Phi(\psi^j,\psi^n),\overline{K\Phi(\psi^i,\psi^m)}) \ .$$

We may obtain the last inner product by means of Parseval's equation as follows. From (A.25) the coefficient of $\psi^{pqh}$ in the Fourier expansion of $\Phi(\psi^j,\psi^n)$ is $\delta(\underset{\sim}{k}_j + \underset{\sim}{k}_n + \underset{\sim}{k}_{ph})I_3(p_0,p,q)$, while the coefficient of $\psi^{pqh}$ in the Fourier expansion of $\overline{K\Phi(\psi^i,\psi^m)}$ is

(A.33) $$\begin{cases} \delta(\underset{\sim}{k}_i + \underset{\sim}{k}_m - \underset{\sim}{k}_{ph})\mu_{pq}(\mu_{pq} - \mu_1)^{-1}I_3(p_0,p,q), & \text{if} \quad (p,q) \neq (p_0,1) \\ 0, & \text{if} \quad (p,q) = (p_0,1) \quad . \end{cases}$$

Consequently,

$$(\Phi(\psi^j,\psi^n),\overline{K\Phi(\psi^i,\psi^m)}) = \Sigma_0\delta(\underset{\sim}{k}_j+\underset{\sim}{k}_n+\underset{\sim}{k}_{ph})\delta(\underset{\sim}{k}_i+\underset{\sim}{k}_m-\underset{\sim}{k}_{ph})\,\frac{\mu_{pq}I_3^2(p_0,p,q)}{(\mu_{pq}-\mu_1)} \tag{A.34}$$

where $\Sigma_0$ denotes summation over $(p,q,h)$ in the same set of integer triples as in (3.7).

Given $j$ and $n$, the only way a term in the sum on the right in (A.34) can be nonzero is for

$$\underset{\sim}{k}_{ph} = -\underset{\sim}{k}_j - \underset{\sim}{k}_n = \underset{\sim}{k}_i + \underset{\sim}{k}_m \; . \tag{A.35}$$

These relations determine $p$ and $h$ completely, in terms of $p_0, j$ and $n$ (or in terms of $p_0, i$ and $m$) so that only $q$ need be summed in (A.34). The relations (A.35) also require that either $\underset{\sim}{k}_i = -\underset{\sim}{k}_j$ and $\underset{\sim}{k}_m = -\underset{\sim}{k}_n$ (i.e., $i = -j$ and $m = -n$) or $\underset{\sim}{k}_i = -\underset{\sim}{k}_n$ and $\underset{\sim}{k}_m = -\underset{\sim}{k}_j$ (i.e., $i = -n$ and $m = -j$). It follows that

$$\begin{aligned}\delta(\underset{\sim}{k}_j+\underset{\sim}{k}_n+\underset{\sim}{k}_{ph})\delta(\underset{\sim}{k}_i+\underset{\sim}{k}_m-\underset{\sim}{k}_{ph}) &= \delta(i+j)\delta(n+m) + \delta(i+n)\delta(m+j)\\ &\quad - \delta(i+j)\delta(i+n)\delta(i-m) \quad .\end{aligned} \tag{A.36}$$

If we combine (A.32), (A.34) and (A.36), then we obtain (3.12) with non-negative constants $a_{p_0jn}$ given by

$$a_{p_0jn} = \sum_{q=q_1}^{\infty} \mu_{pq}(\mu_{pq}-\mu_1)^{-1}I_3^2(p_0,p,q), \quad |j|,|n| = 1,2,\ldots,N, \; j \neq -n \tag{A.37}$$

where $q_1 = 1$, if $p \neq p_0$ and $q_1 = 2$ if $p = p_0$. Note that $a_{p_0jn}$ depends on $j$ and $n$ only through $p$, i.e., through $\sigma_p = |\underset{\sim}{k}_j + \underset{\sim}{k}_n|$. In particular we have

$$a_{p_0jn} = a_{p_0nj} = a_{p_0(-j)(-n)} \; . \tag{A.38}$$

Furthermore, when $j = -n$ we may, for convenience, define

(A.39) $$a_{p_0 j(-j)} = 0.$$

(The sum in (A.37) is meaningless in this case, since (2.12) has no non-trivial solutions when $\sigma = 0$ $(= |k_j + k_{(-j)}|)$ so that $0 \neq \sigma_p$ for any $p$). Because $(p,q) \neq (p_0,1)$ in (A.37), we see when $n \neq -j$ that $a_{p_0 jn} = 0$ if and only if $I_3(p_0,p,q) = 0$ for all integers $q$ with $|q| \geq q_1$.

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