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PARAMETER ESTIMATION FOR THE EULER-BERNOULLI-BEAM

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ABSTRACT

An approximation involving cubic spline functions for parameter estimation problems in the Euler-Bernoulli-beam equation (phrased as an optimization problem with respect to the parameters) is described and convergence is proved. The resulting algorithm was implemented and several of the test examples are documented. It is observed that the use of penalty terms in the cost functional can improve the rate of convergence.

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1. Introduction: Statement of the Problem and Notation.

In this investigation we are concerned with the parameter dependent equation

$$(1.1) \qquad y_{tt} = - D^2(q_1 D^2 y) - D^2(q_2 D^2 y_t) - q_3 y_t + q_4 y \, ,$$

$$(1.2) \qquad y(0,\cdot) = y_o, \ y_t(0,\cdot) = y_1 \, ,$$

$$(1.3) \qquad \text{boundary condition} \, ,$$

where $y = y(t,x)$, $0 < x < 1$ and where D stands for differentiation with respect to x. Together with appropriate boundary conditions, equations (1.1) − (1.2) arise in modelling the tranverse vibrations of a thin and elastic beam, where structural $(q_2 \neq 0)$ or viscous $(q_3 \neq 0)$ dampling may be present [6, Chapter 17]. In an effort to provide a mathematical description of the motion of large flexible systems, continuum models of the type (1.1) − (1.3) are receiving anew much attention in the present literature [2, 3, 7, 10, 18].

Different boundary conditions generally require a somewhat different analysis and we therefore restrict ourselves to the following boundary conditions of type k, k=1 or 2 or 3:

(k=1): $y(t,0) = y_{xx}(t,0) = y(t,1) = y_{xx}(t,1) = 0$

(simply supported beam),

(k=2): $y(t,0) = y_x(t,0) = y(t,1) = y_x(t,1) = 0$

(beam clamped at both ends),

(k=3): $y(t,0) = y_x(t,0) = y_{xx}(t,0) = y_{xxx}(t,1) = 0$

(cantilevered beam),

for $t > 0$.

Let us assume that (1.1), (1.2) together with boundary conditions of type k are the mathematical model of a physical system, for which observations $\hat{y}(t_i,x_j)$, $i=1,\ldots,l$ and $j=1,\ldots,m$, are available and which are expected to correspond to the solution $y(t_i,x_j;q)$ of (1.1) – (1.3) evaluated at (t_i,x_j) , with the parameter

$$q = (q_1,q_2,q_3,q_4,y_0,y_1)$$

chosen "correctly". Since it is mathematically untractable and practicably not feasable to require $\hat{y}(t_i,x_j) = y(t_i,x_j;q)$ for $i=1,\ldots,l$ and $j=1,\ldots,m$ we formulate the following optimization problem

(P)   minimize $\sum\limits_{i,j} |\hat{y}(t_i,x_j) - y(t_i,x_j;q)|^2$ over

     $q \in Q$ subject to y satisfying (1.1) – (1.3) ,

and refer to (P) as parameter estimation problem. Here $y(\cdot,\cdot;q)$ denotes the solution of (1.1) – (1.3) in a sense that will be specified below and Q, the set of admissable parameters is given by

$$Q \subset Q_1 \times Q_2 \times L^\infty \times L^\infty \times H^2_{(k)} \times H^0 ,$$

where $Q_1 = \{q_1 \in H^2 : q_1(x) \geq \alpha \}$, with $\alpha > 0$,

$$Q_2 = \{q_2 \in H^2 : q_2(x) \geq 0\}$$

and
$$H^2_{(1)}(q_1) = \{v \in H^2 : v(0) = v(1) = 0\} \ ,$$

$$H^2_{(2)}(q_1) = \{v \in H^2 : v(0) = v(1) = v'(0) = v'(1) = 0\} \ ,$$

$$H^2_{(3)}(q_1) = \{v \in H^2 : v(0) = v'(0) = 0\} \ .$$

The set Q is given a topology by taking

$$Q \subset \tilde{Q} = H^2_{weak} \times H^2_{weak} \times L^\infty \times L^\infty \times H^2 \times H^0 \quad .$$

Here H^2_{weak} denotes the space H^2 endowed with the weak topology.

The parameter estimation problem (P) is an infinite dimensional problem, and its approximation by a sequence of finite dimensional problems that are numerically implementable is the focus of this paper. Here we study approximation of (P) by projection on finite dimensional subspaces. By factoring the operators $D^2(q_1D^2)$ and $D^2(q_2D^2)$ into second order operators, the smoothness requirement on the functions in the subspaces will be that of H^2-regularity. In this respect our treatment differs from earlier work (see [2,3,7]), where it is assumed that the functions in the subspaces are sufficiently smooth, so that they lie in the domain of the generator of the semigroup associated with (1.1) - (1.3), in particular that they are H^4-functions. In special cases this requirement can be circumvented in [2,3,7] by transforming the equation into a higher dimensional system of equations. The present approach has the advantage that we can give a complete convergence

theory for the approximation of (P), which includes, for example, cubic spline approximations.

We also study, theoretically and numerically, the effect of adding a penalty term to the quadratic fit-to-data criterion involved in defining (P). The use of such a penalty term seems to be justified for practical problems and proved to be numerically useful, since it provides some robustness in the search for a minimum of (P). In some examples the computing time was reduced and in others less care was required to tune the parameters in the minimization routine when a penalty term was added. This will be illustrated with an example at the end of the paper.

The "observations" for our test problems were obtained by a generalized Crank-Nicolson-scheme, exept in very special cases, when the solution could be calculated analytically. Since numerical approximations for the trajectory of (1.1) - (1.3) do not seem to be readily available in the literature, we also compared the Crank-Nicolson-scheme and a cubic spline-projection algorithm for approximation of the trajectory with an example where the analytical solution is available. It turns out that the projection algorithm is superior to the Crank-Nicolson-scheme, both with respect to time and accuracy.

In our model equation we have not included a forcing function f(t,x). The results of this paper can easily be extended to include such a term, with techniques that are fairly routine and are discussed in detail in [5] for second order equations.

In section 2 we formulate (1.1) - (1.3) as an abstract differential
equation in a Hilbert space and prove a general convergence
result for projection schemes that is uniform in the parameters.
This parameter dependent state convergence result is used in
section 3 to study approximation of (P). Section 4 is devoted
to a short discussion of the use of penalization techniques
that proved to be numerically very useful for the present problem.
We have carried out extensive tests for a cubic spline scheme
and an example is discussed in section 5. A detailed description
of the scheme and of the phenomena that were observed will appear
elsewhere.

The notation that we employ is rather standard and we only make
a few comments. By $|\cdot|$ and $(\cdot,\cdot)$ we denote the usual norm and
inner product in L^2. Other norms are denoted by a subscript and
$\|\cdot\|_X$ is used to specify the operator norm in the space X.
We will not distinguish between column and row vectors. As usual,
$\sigma(A)$ and $\rho(A)$ stand for spectrum and resolvent set of a
closed operator A and A$|$D denotes the restriction of A to the
set D.

2. Abstract Formulation and Parameter Dependent State Convergence.

In the first part of this section we restrict our attention to
the equation

(2.1) $y_{tt} = - D^2(q_1 D^2 y) - D^2(q_2 D^2 y_t)$,

    $y(0,\cdot) = y_o$, $y_t(0,\cdot) = y_1$,

    boundary conditions of type k ,

where $0 \le x \le 1$, $t \ge 0$, $y_o \in H^2_{(k)}$ $y_1 \in H^o$ and k=1 or 2 or 3.
The general case given in (1.1) - (1.3) will follow from the
results concerning (2.1) by adding a suitably defined bounded
perturbation. At first we also assume y_o and y_1 to
be fixed. We reformulate (2.1) in a function space setting and
give the convergence result that is the essential tool for the
approximation of the parameter estimation problem (P). Several
preliminaries are required. For $q_1 \in Q_1$ the spaces $H^2_{(k)}(q_1)$
that were introduced in the previous section are endowed with the
inner product

$$\langle v,w \rangle_{q_1} = (q_1 D^2 v, D^2 w) .$$

In this way $H^2_{(k)}(q_1)$ become Hilbert spaces and the norm $|v|^2_{q_1} = \langle v,v \rangle_{q_1}$ is equivalent to the usual H^2-norm on $H^2_{(k)}$;
moreover this equivalence is uniform as q_1 varies in bounded
subsets of Q_1; for details see [7 , Theorem 1.8], for example.

Further we introduce the sets

$$H^4_{(1)} = \{v \in H^4 : v(0) = v''(0) = v(1) = v''(1) = 0\} \ ,$$

$$H^4_{(2)} = \{v \in H^4 : v(0) = v'(0) = v(1) = v'(1) = 0\} \ ,$$

$$H^4_{(3)} = \{v \in H^4 : v(0) = v'(0) = v''(1) = v'''(1) = 0\} \ .$$

Let Y be the space H^O endowed with the weighted inner product

$$(v,w)_Y = (q_1^{-1}v,w) \ ,$$

with $q_1 \in Q_1$ and define closed linear operators $C(q_1)$ from H^O to Y by

$$\text{dom } C(q_1) = H^2_{(k)} \ ,$$

$$C(q_1)v = q_1 D^2 v \ .$$

Note that $<v,w>_{q_1} = (C(q_1)v, C(q_1)w)_Y$ for $v,w \in H^2_{(k)}$ and that

$$(2.2) \qquad |C(q_1)v|_Y \geq \tilde{k} \min_{x \in [0,1]} q_1(x)|v|_{H^2} \ ,$$

for all $v \in H^2_{(k)}$ and a constant $\tilde{k}$ independent of q_1 and v.

Lemma 2.1. The adjoint $C^*_{(k)}(q_1)$ of $C(q_1)$ mapping from Y to H^O is given by

$$\text{dom } C^*_{(1)}(q_1) = \{v \in H^2 : v(0) = v(1) = 0\} \text{ ,}$$

$$\text{dom } C^*_{(2)}(q_1) = H^2 \text{ ,}$$

$$\text{dom } C^*_{(3)}(q_1) = \{v \in H^2 : v(1) = v'(1) = 0\} \text{ ,}$$

and

$$C^*_{(k)}(q_1)v = D^2 v \text{ .}$$

Proof. This technical result follows, with minor modifications due to the weighted H^0-norm of Y, from the calculations in [11, pg. 169-171].

Lemma 2.2. Let $A_{(k)}(q_1) = C^*_{(k)}(q_1)C(q_1)$. Then $A_{(k)}(q_1)$ is selfadjoint and strictly positive in H^0 and $\text{dom } A_{(k)}(q_1) = H^4_{(k)}$ is a core for $C(q_1)$.

Proof. By von Neumann's theorem [11, pg. 275], $A_{(k)}(q_1)$ is selfadjoint and $\text{dom } A_{(k)}(q_1)$ is a core for $C(q_1)$. The remaining assertions can easily be checked.

Corollary 2.1. The inclusions of $H^4_{(k)}$ in $H^2_{(k)}$ (in the $H^2_{(k)}$-norm) and of $H^4_{(k)}$ in H^0 are dense.

Proof. Recall that by the definition of a core, the set $\{(v,C(q_1)v) : v \in H^4_{(k)}\}$ is dense in the graph of $C(q_1)$. For each $(v,C(q_1)v)$ with $v \in H^4_{(k)}$ there exists a sequence $v_n \in H^2_{(k)}$ with $v_n \to v$ and $C(q_1)v_n \to C(q_1)v$ in H^0. Since the $H^2_{(k)}$-norm and the graph norm of $C(q_1)$ are equivalent, $H^4_{(k)}$

is dense in $H^2_{(k)}$. Finally, $H^2_{(k)}$ is clearly dense in H^0 and the claim follows.

Next we define for $q_2 \in Q_2$ a family of operators $A(q_2)$ by dom $A(q_2) = H^4_{(k)}$ and

$$A_{(k)}(q_2)v = D^2(q_2 D^2 v) .$$

To write (2.1) as a system of first order equations, we introduce the Hilbert spaces

$$\mathfrak{X} = H^2_{(k)}(q_1) \times H^0 ,$$

with the inner product denoted by $(\cdot,\cdot)_{\mathfrak{X}}$. In the notation of $\mathfrak{X}$ as well as with other symbols we now adopt the convention that contents permitting the indices k and q may be dropped. For $q \in Q$ let $\mathcal{A}_{(k)}(q)$ be given by dom $\mathcal{A}_{(k)}(q) = H^4_{(k)} \times H^4_{(k)}$ and define

$$\mathcal{A}_{(k)}(q) = \begin{pmatrix} 0 & 1 \\ -A_{(k)}(q_1) & -A_{(k)}(q_2) \end{pmatrix} .$$

We shall need the following technical lemma.

<u>Lemma 2.3.</u> For each $q_1 \in Q_1$ there exists a constant K such that

$$|A(q_1)| \geq K|v|_{H^4} ,$$

for all $v \in H^4_{(k)}$. If, moreover, q_1 varies in a bounded (with

respect to the H^2-norm) subset of Q_1 then K is independent of q_1.

Remark 2.1. In this lemma the requirement $q_1 \in H^2$ is needed in an essential way.

Proof of Lemma 2.3. The proof of such an inequality is fairly standard and we shall therefore only sketch it. Since $|A(q_1)v| \geq K_1|v|$ for some constant K_1 depending only on $|q_1|_C$ it suffices to show that $|A(q_1)v| \geq K_2|D^4v|$, where K_2 satisfies the specified properties. We have

$$\alpha^2|D^4v|^2 \leq |q_1D^4v|^2 = |A(q_1)v|^2 - |D^2q_1D^2v + 2Dq_1D^3v|^2 -$$

$$2\int_0^1 (q_1D^4v)(D^2q_1D^2v + 2Dq_1D^3v)dx \ .$$

For any $\epsilon > 0$ we find

$$I = 2\int_0^1 |q_1D^4v| \ |D^2q_1D^2v + 2Dq_1D^3v|dx$$

$$\leq 2|q_1|_{H^1}(\int_0^1 |D^4v| \ |D^2q_1D^2v|dx + 2\int_0^1 |Dq_1| \ |D^3v|dx)$$

$$\leq 2|q_1|_{H^1}(2\epsilon|D^4v|^2 + \frac{|q_1|_{H^2}^2}{4\epsilon}|D^2v|_{L^\infty}^2 + \frac{|q_1|_{H^1}^2}{2\epsilon}|D^3v|^2) \ .$$

Using a basic estimate [17, pg.18] we have for any $0 < \epsilon_1 < 1$ and an appropriately defined constant $\tilde{k}$ independend of q_1:

$$I \leq 2|q_1|_{H^1}[2\epsilon|D^4v|^2 + \tilde{k}|q_1|_{H^2}\epsilon^{-1}(\epsilon_1^{-2}|v|^2 + \epsilon_1|D^4v|^2)] \ .$$

In a similar way we find that

$$II = |D^2 q_1 D^2 v + 2 D q_1 D^3 v|^2 \leq |q_1|^2_{H^2} \hat{k}(\varepsilon_1^{-2}|v| + \varepsilon_1|D^4 v|) ,$$

with $\hat{k}$ independent of q_1. Those two estimates are used in (2.3) and for appropriately chosen constants ε and ε_1 we conclude that $|A(q_1)v| \geq K_2|D^4 v|$. This ends the proof.

__Proposition 2.1.__ For $q \in Q$ the operator $\mathcal{A}_{(k)}(q)$ is densely defined and dissipative. Its __closure__ $\bar{\mathcal{A}}_k(q)$ is maximal dissipative and generates a C_o-contraction semigroup $T(t;q)$ on X.

__Proof.__ By Corollary 2.1 $\mathcal{A}_{(k)}(q)$ is densely defined. Dissipativity of $\mathcal{A}_{(k)}(q)$ is easily verified. We note that for real $\lambda > 0$ and λ sufficiently small, $(\lambda - \mathcal{A}_{(k)}(q))(H^4_{(k)} \times H^4_{(k)})$ is dense in $H^4_{(k)} \times H^4_{(k)}$. In fact, for all $(a,b) \in H^4_{(k)} \times H^4_{(k)}$ we find that $(\lambda - \mathcal{A}_{(k)}(q))\binom{u}{v} = \binom{a}{b}$ is equivalent to $v = \lambda u - a$ and

$$(2.4) \qquad (\lambda^2 + A(q_1))u = -\lambda A(q_2)u + f ,$$

with $f = \lambda a + A(q_2)a + b \in H^o$. Let $(\lambda^2 + A(q_1))u = w$. Then (2.4) becomes

$$(2.5) \qquad w = -\lambda A(q_2)(\lambda^2 + A(q_1))^{-1}w + f ,$$

or $w = -\lambda A(q_2)A(q_1)^{-1}A(q_1)(\lambda^2 + A(q_1))^{-1}w + f$. We solve for $w \in H^o$ in (2.5). Note the $A(q_1)(\lambda^2 + A(q_1))^{-1}$ is a bounded operator in H^o uniformly in $\lambda > 0$, and by Lemma 2.3 $A(q_2)A(q_1)^{-1}$ is a bounded operator as well. Consequently a fixed point argument guarantees

the existence of a solution $w \in H^0$ of (2.5) for some λ_0 sufficiently small. We put $u = (\lambda_0^2 + A(q_1))^{-1} w \in H^4_{(k)}$ and the required density of the range of $(\lambda_0 - \mathcal{A}_{(k)}(q))$ is shown.

Now the dissipative operator $\mathcal{A}_{(k)}(q)$ admits a closure $\bar{\mathcal{A}}_{(k)}(q)$ [12, pg. 86]. Moreover the range of $(\lambda_0 - \mathcal{A}_{(k)}(q))$ is necessarily closed in X [12, pg. 86]; but it is also dense and so it coincides with X. This implies that the closure $\bar{\mathcal{A}}_{(k)}(q)$ of $\mathcal{A}_{(k)}(q)$ is a maximal dissipative operator and that it generates the semigroup $T(t;q)$ on $\mathfrak{X}$.

Remark 2.2. If $q_2 = 0$, then dom $\bar{\mathcal{A}}_{(k)}(q) = H^4_{(k)} \times H^2_{(k)}$ and the action of $\bar{\mathcal{A}}_{(k)}(q)$ coincides with that of $\mathcal{A}_k(q)$. Moreover $\bar{\mathcal{A}}_{(k)}(q)$ is skew adjoint in this case, as can be seen by employing von Neumann's theorem [11], for example. - For arbitrary $q \in Q$ it is simple to verify that dom $\bar{\mathcal{A}}_{(k)}(q) \subset H^2_{(k)} \times H^2_{(k)}$ and that

$$\langle \bar{\mathcal{A}}_{(k)}(q)\binom{u}{v}, \binom{a}{b} \rangle_{\mathfrak{X}} = \langle v, a \rangle_{q_1} - (q_1 D^2 u, D^2 b) - (q_2 D^2 v, D^2 b)$$

holds for every $(a,b) \in H^2_{(k)} \times H^2_{(k)}$.

Recall that for $z \in$ dom $\bar{\mathcal{A}}_{(k)}$ we have

(2.6) $\qquad \dfrac{d}{dt} T(t;q)z = \bar{\mathcal{A}}_{(k)}(q) T(t;q)z$, for $t > 0$.

This equation is formally equivalent to (2.1) with $T(t;q)(y_0, y_1) = (y, y_t)$.

We proceed to describe an operator theoretic form of a Galerkin approximation to (2.1). Let X^N be a sequence of finite dimensional subspaces of $H^2_{(k)}$ and denote by X^N_0 (X^N_1) the space X^N endowed with the H^0-($H^2_{(k)}(q_1)$-) topology. Note that X^N depends on k as well. Further we introduce the orthogonal projections $P^N_0 : H^0 \to X^N_0$ and $P^N_1 : H^2_{(k)}(q_1) \to X^N_1$. We put $X^N = X^N_1 \times X^N_0$ and define $\wp^N : X \to X^N$ by

$$\wp^N = \begin{pmatrix} P^N_1 & 0 \\ 0 & P^N_0 \end{pmatrix} \quad .$$

Whenever the dependence of P^N_1 and $\wp^N$ on q is relevant we write $P^N_1(q_1)$ and $\wp^N(q)$.

For $q_1 \in Q_1$ let

$$C^N(q_1) = C(q_1) | X^N \quad .$$

Clearly $C^N(q_1) \in \mathcal{L}(X^N_0, Y)$ and the adjoint $C^N(q_1)^*$ satisfies

$$(C^N(q_1)u, v)_Y = (u, C^N(q_1)^* v)_{H^0} \quad , \quad \text{for all} \quad u \in X^N_0 \quad , \quad v \in Y \quad .$$

We next define

$$A^N(q_1) = C^N(q_1)^* C^N(q_1)$$

and note that $A^N(q_1) \in L(X^N)$ and

$$(2.7) \quad (A^N(q_1)u, v) = (C^N(q_1)u, C^N(q_1)v)_Y = (C(q_1)u, C(q_1)v)_Y \quad ,$$

for all $u, v \in X^N$. Here we defined the approximating operators $A^N(q_1)$ on X^N only, but there is an obvious extension to all

of H^O given by $A_{ext}^N(q_1) = (C(q_1)P_o^N)^* C(q_1)P_o^N$. For $q_2 \in Q_2$ define operators $A^N(q_2) : X^N \rightarrow X^N$ by

$$(A^N(q_2)u,v) = (q_2 D^2 u, D^2 v) \quad \text{for all} \quad u,v \in X^N .$$

Finally we introduce the operator $\mathcal{A}_{(k)}^N(q) \in \mathcal{L}(X^N)$ approximating $\bar{\mathcal{A}}_{(k)}(q)$:

$$\mathcal{A}_{(k)}^N(q) = \begin{pmatrix} O & I \\ -A_{(k)}^N(q_1) & -A_{(k)}^N(q_2) \end{pmatrix} .$$

The Galerkin approximation to (2.1) can now be introduced as

$$\frac{d}{dt} z^N(t;q) = \mathcal{A}_{(k)}^N(q) z^N(t;q) ,$$

(2.8)

$$z^N(0;q) = \mathbb{P}^N(y_o, y_1) .$$

Note that (2.8) is an equation in the finite dimensional space X^N.

Lemma 2.4. The operators $\mathcal{A}_{(k)}^N(q)$, $q \in Q$, generate contraction semigroups $T^N(t;q)$ on X^N for each N. We have $T^N(t;q)\mathbb{P}^N(y_o,y_1) = z^N(t;q)$.

Proof. From the construction of $\mathcal{A}_{(k)}^N(q)$ it follows that $(\mathcal{A}_{(k)}^N(q)z,z)_X \leq 0$ for all $z \in X^N$. This and the finite dimensionality of X^N imply the result, see [12, pg. 90].

<u>Theorem 2.1.</u> Let q^N and $q^O \in Q$, N=1,2..., and assume that $P_O^N \rightarrow I$ strongly in H^O and $P_1^N(q_1^O) \rightarrow I$ strongly in $H_{(k)}^2(q_1^O)$. If moreover $(q_1^N, q_2^N) \rightarrow (q_1^O, q_2^O)$ weakly in $H^2 \times H^2$, then

$$T^N(t;q^N)P^N(q^N)z \rightarrow T(t;q^O)z \quad \text{in } \mathfrak{X} ,$$

uniformly in t as t varies in bounded subsets of $[0,\infty)$ and for every $z \in \mathfrak{X}$.

The approximation of the general equation (1.1) - (1.3) will follow trivially from this theorem, see Corollary 2.1 below. Before we give the proof of Theorem 2.1 we establish some technical lemmas.

<u>Remark 2.3.</u> Strictly speaking the above convergence result is not stated precisely, since the spaces in which the operators are defined vary topologically. A more precise statement would be given by

$$|T^N(t;q^N)P^N(q^N)\mathfrak{J}^N z - \mathfrak{J}^N T(t;q^O)z|_{\mathfrak{X}(q^N)} \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

uniformly in t in bounded intervals of $[0,\infty)$. Here $\mathfrak{J}^N : \mathfrak{X}(q^O) \rightarrow \mathfrak{X}(q^N)$ is the canonical isomorphism. Since the set $\{|q_1^N|_C : N=1,2,...\}$ is bounded, the spaces $\mathfrak{X}(q^N)$ are topologically equivalent and no confusion should arise by the abbreviated statement of the theorem.

<u>Lemma 2.5.</u> For $q \in Q$, the resolvent set of $\bar{\mathcal{A}}_{(k)}(q)$ as well as that of $\mathcal{A}_{(k)}^N(q)$ contains O.

__Proof.__ Let $(u,v) \in \mathfrak{X}^N$ such that $\mathcal{A}^N(q)(u,v) = (0,0)$. Then

$$\langle \mathcal{A}^N(q)\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle_{\mathfrak{X}} = 0 \quad \text{for all} \quad (x,y) \in \mathfrak{X}^N .$$

This is equivalent to $\langle v,x \rangle_{q_1} - \langle u,y \rangle_{q_1} - (A^N(q_2)v,y) = 0$. First let $y = 0$, and conclude that $v = 0$. Next putting $x = 0$ with y arbitrary implies $u = 0$ and in particular $0 \in \rho(\mathcal{A}^N(q))$.

Similarily, since $0 \in \rho(A(q_1))$ it is simple to show that 0 is not an eigenvalue of $\mathcal{A}(q)$ and that the range of $\mathcal{A}(q)$ includes $H^4_{(k)} \times H^4_{(k)}$ and is therefore dense in $\mathfrak{X}$. We will show that $\mathcal{A}(q)^{-1} \mid H^4_{(k)} \times H^4_{(k)}$ is continuous (in $\mathfrak{X}$). Let $(x,y) \in H^4_{(k)} \times H^4_{(k)}$. Then $\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A}^{-1}(q)\begin{pmatrix} x \\ y \end{pmatrix}$ is equivalent to

$$(2.9) \qquad v = x \quad \text{and} \quad u = -A(q_1)^{-1}(y + A(q_2)x) .$$

We have

$$(2.10) \quad |A(q_1)^{-1}y|^2_{q_1} = (C(q_1)A(q_1)^{-1}y, C(q_1)A(q_1)^{-1}y)_Y = (A(q_1)^{-1}y, y) \leq$$

$$\text{const}\,(q_1)|y|^2 .$$

Using [11, pg. 169] we find in a similar manner

$$(2.11) \quad |A(q_1)^{-1}A(q_2)x|_{q_1} = |C(q_1)A(q_1)^{-1}A(q_2)x|_Y = |C(q_1)C(q_1)^{-1}(C^*(q_1))^{-1}A(q_2)x|_Y$$

$$= |(C^*(q_1))^{-1}D^2(q_2D^2x)|_Y = |\frac{1}{\sqrt{q_1}}\,q_2D^2x| \leq \text{const}\,(q)|x|_{q_1} .$$

The estimates (2.10), (2.11) together with (2.9) imply

$|(u,v)|_X \leq K|(x,y)|_X$, for an appropriately defined constant

depending on q and indepent of (x,y) and the claim is verified.

Lemma 2.6. For $q \in Q$ and $z \in \dot{X}^N$ we have $\mathcal{A}^N_{(k)}(q)^{-1}z = P^N \mathcal{A}_{(k)}(q)^{-1}z$.

Proof. Let $z = (x,y)$ and define $\begin{pmatrix} u^N \\ v^N \end{pmatrix} = \mathcal{A}^N(q)^{-1}z$ and $\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A}(q)^{-1}z$. We need to show that

$$(2.12) \qquad u^N = P^N_1 u \quad \text{and} \quad v^N = P^N_o v .$$

For every N we have

$$\mathcal{A}^N(q)\begin{pmatrix} u^N \\ v^N \end{pmatrix} = \begin{pmatrix} v^N \\ -A^N(q_1)u^N - A^N(q_2)v^N \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} ,$$

and in particular $v^N = x$. Next we use Remark 2.2 and find that for every $(a,b) \in H^2_{(k)} \times H^2_{(k)}$

$$(2.13) \quad \langle \tilde{\mathcal{A}}(q)\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\rangle = \langle v,a \rangle_{q_1} - (q_1 D^2 u, D^2 b) - (q_2 D^2 v, D^2 b) = \langle x,a \rangle_{q_1} + (y,b) .$$

This implies $v = x$, so that the second equality in (2.12) is satisfied. Since $\cdot 0 \in \rho(\mathcal{A}(q))$ the first equality in (2.12) will be verified, once we show that $P^N_1 u$ is the (unique) solution of $A^N(q_1)u^N = -A^N(q_2)x - y$ in X^N . In the following calculation we use (2.13) with $a = 0$ and $b \in X^N$:

$$-(A^N(q_1)P_1^N u, b) = -<P_1^N u, b>_{q_1} = -<u, b>_{q_1} = (y, b) + (q_2 D^2 x, D^2 b) = (y, b) + (A^N(q_2)x, b) .$$

Since $b \in X^N$ is arbitrary this equality implies the result.

The convergence statement in the following lemma has to be interpreted in the sense explained in Remark 2.3.

<u>Lemma 2.7.</u> Let q^N, $q^o \in Q$ with $q^N \to q^o$ weakly in $\tilde{Q}$. Then

$$\mathcal{A}_{(k)}(q^N)^{-1} \to \mathcal{A}_{(k)}(q^o)^{-1} \quad \text{strongly in } X .$$

<u>Proof.</u> Let $(x, y) \in H^4_{(k)} \times H^4_{(k)}$ and define

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A}(q)^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u^N \\ v^N \end{pmatrix} = \mathcal{A}(q^N)^{-1} \begin{pmatrix} x \\ y \end{pmatrix} .$$

Then $v = v^N = x$, $u^N = -A(q_1^N)^{-1}(A(q_2^N)x + y)$ and $u = -A(q_1^o)(A(q_2^N)x + y)$. From Lemma 2.3 we conclude that $\{ |u^N|_{H^4} : N = 1, 2, \ldots \}$ is bounded. Therefore there exists a subsequence of u^N again denoted by u^N, converging weakly in H^4 and strongly in H^2 to an element $z \in H^4_{(k)}$. We have for every $w \in H^2_{(k)}$

$$-(q_1^N D^2 u^N, D^2 w) - (q_2^N D^2 x, D^2 w) = (y, w) .$$

Taking the limit in this last inequality we get

$$-(q_1^o D^2 z, D^2 w) - (q_2 D^2 x, D^2 w) = (y, w) .$$

This implies $z = u$. The usual subsequence argument can be used to show that the original sequence $u^N \to u$ in $H^2_{(k)}(q_1)$. Since

$\{\|\mathcal{A}^{-1}(q^N)\|_{\mathcal{X}(q^N)} : N=1,2,\ldots\}$ is uniformly bounded in N the
result follows from density of $H^4_{(k)} \times H^4_{(k)}$ in $\mathcal{X}$.

For the ease of the reader we state a version of the Trotter-Kato
approximation theorem that is readily applicable for the proof
of Theorem 2.1.

Lemma 2.8. [19] Let $q^N \in Q$, $q^0 \in Q$ and assume that
$|P^N(q^N)z|_{\mathcal{X}(q^N)} \to |z|_{\mathcal{X}(q^0)}$ for every $z \in \mathcal{X}$. Then

$$(2.14) \qquad |P^N(q^N)T(t;q^0)z - T^N(t;q^N)P^N(q^N)z|_{\mathcal{X}(q^N)} \to 0$$

for every $z \in \mathcal{X}$ if and only if there exists a $\lambda \in \cap \; \rho(\mathcal{A}^N_{(k)}(q^N)) \cap$
$\rho(\bar{\mathcal{A}}_{(k)}(q^0))$ such that

$$(2.15) \quad |(\lambda - \mathcal{A}^N_{(k)}(q^N))^{-1}P^N(q^N)z - P^N(q^N)(\lambda - \bar{\mathcal{A}}_{(k)}(q^0))^{-1}z|_{\mathcal{X}(q^N)} \to 0$$

for every $z \in \mathcal{X}$ and

$$(2.16) \qquad \|T^N(t;q^N)\|_{\mathcal{X}(q^N)} \leq Me^{\omega t}$$

for positive constants M and ω independent of N.

Again, a remark analogous to Remark 2.3 has to be made; (2.14)
for example, should read

$$|P^N(q^N)\mathcal{J}^N T(t;q^0)z - T^N(t;q^N)P^N(q^N)\mathcal{J}^N z|_{\mathcal{X}(q^N)} \to 0 \; .$$

Proof of Theorem 2.1. First we note that weak convergence of q_1^N implies that q_1^N is bounded in H^2 and therefore the $|\cdot|_{q_1^N}$-norm is uniformly in N equivalent to the H^2-norm. Since

$$|P_1^N(q_1^N)v-v|_{q_1^N} \leq |P_1^N(q_1^o)v-v|_{q_1^N} \leq K_3|P_1^N(q_1^o)v-v|_{q_1^o}$$

for some constant K_3 independent of N and $v \in H_{(k)}^2$, it follows that $|P_1^N(q_1^N)v-v|_{q_1^N} \to 0$ as $N \to \infty$ and

$$(2.17) \qquad |\mathbf{P}^N(q^N)z-z|_{\mathfrak{X}(q^o)} \to 0 \quad \text{as} \quad N \to \infty, \text{ for every } z \in \mathfrak{X} .$$

Moreover, for $v \in H_{(k)}^2$ we find

$$\left| |P_1^N(q_1^N)v|_{q_1^N} - |v|_{q_0^N} \right| = \left| |P_1^N(q_1^N)v|_{q_1^N} - |\frac{q_1^o}{q_1^N}v|_{q_1^N} \right| \leq |P_1^N(q_1^N)v - \frac{q_1^o}{q_1^N}v|_{q_1^N}$$

and this last term converges to 0 as $N \to \infty$.

Therefore $|\mathbf{P}^N(q^N)z|_{\mathfrak{X}(q^N)} \to |z|_{\mathfrak{X}(q^o)}$ for every $z \in \mathfrak{X}$. To verify the convergence result we use the estimate

$$(2.18) \qquad |T^N(t;q^N)\mathbf{P}^N(q^N)z - T(t;q^o)z|_{\mathfrak{X}} \leq$$

$$|T^N(t;q^N)\mathbf{P}^N(q^N)z - \mathbf{P}^N T(t;q^o)z|_{\mathfrak{X}} + |\mathbf{P}^N T(t;q^o)z - T(t;q^o)z|_{\mathfrak{X}} .$$

The set $\{T(t;q^o) : t \in [0,\tau]\}$ is compact for every $\tau > 0$ and therefore (2.17) implies that the second term on the right hand

side of (2.18) converges to zero uniformly in t on bounded intervals. To show convergence of the first term on the right hand side of (2.18) we use Lemma 2.8. In view of the previous calculations it suffices to verify (2.15) with $\lambda = 0$. By Lemma 2.5 and 2.6 we have for $z \in \mathfrak{X}$

$$
|\mathcal{A}^N(q^N)^{-1}\mathcal{P}^N(q^N)z - \mathcal{P}^N(q^N)\bar{\mathcal{A}}(q^0)^{-1}z|_{\mathfrak{X}(q^N)} \leq
$$

$$
\text{(2.19)} \quad |\mathcal{P}^N(q^N)\bar{\mathcal{A}}(q^N)^{-1}\mathcal{P}^N(q^N)z - \mathcal{P}^N(q^N)\mathcal{A}(q^N)^{-1}z|_{\dot{\mathfrak{X}}(q^N)} +
$$

$$
|\mathcal{P}^N(q^N)\mathcal{A}(q^N)^{-1}z - \mathcal{P}^N(q^N)\bar{\mathcal{A}}(q^0)^{-1}z|_{\mathfrak{X}(q^N)} \quad .
$$

This last term converges to 0 by Lemma 2.7. Since $\{\|\mathcal{P}^N(q^N)\mathcal{A}(q^N)^{-1}\| : N=1,2,\ldots\}$ is bounded, (2.17) implies convergence to 0 of the first term on the right hand side of (2.19) and the result is proved.

We now return to equation (1.1) - (1.3) and define operator an $\bar{\mathcal{A}}^B_{(k)}(q)$ by dom $\bar{\mathcal{A}}^B_{(k)}(q) = $ dom $\bar{\mathcal{A}}_{(k)}(q)$ and

$$
\bar{\mathcal{A}}^B_{(k)}(q) = \bar{\mathcal{A}}_{(k)}(q) + B(q)
$$

where

$$
B(q) = \begin{pmatrix} 0 & 0 \\ -q_3 & q_4 \end{pmatrix} .
$$

Since $\bar{\mathcal{A}}^B_{(k)}$ is a bounded perturbation of $\bar{\mathcal{A}}_{(k)}(q)$ it generates a semigroup, denoted by $T^B(t;q)$ for every $q \in Q$. Further we define approximating operators $\mathcal{A}^{N,B}_{(k)}(q)$ on $\mathfrak{X}^N$ by

$$(2.20) \qquad \mathcal{A}_{(k)}^{N,B}(q) = \mathcal{A}_{(k)}^{N}(q) + \mathcal{P}^{N}B \ ,$$

so that the Galerkin approximations to (1.1) - (1.3) become

$$\frac{d}{dt}z^{N}(t;q) = \mathcal{A}_{(k)}^{N,B}(q)z^{N}(t;q) \ ,$$

$$(2.21)$$

$$z^{N}(0;q) = \mathcal{P}^{N}(y_{o},y_{1}) \ .$$

The solution to (2.21) is given by the semigroup $T_{B}^{N}(t;q)$ generated by $\mathcal{A}_{(k)}^{N,B}(q)$.

<u>Corollary 2.1.</u> Let q^{N}, $q^{o} \in Q$ with $q^{N} \to q^{o}$ weakly in $\tilde{Q}$, and assume that $P_{o}^{N} \to I$ strongly in H^{o} and $P_{1}^{N}(q_{1}^{o}) \to I$ strongly in $H_{(k)}^{2}(q_{1}^{o})$. Then

$$T_{B}^{N}(t;q^{N})\mathcal{P}^{N}(q^{N})(y_{o}^{N},y_{1}^{N}) \to T^{B}(t;q^{o})(y_{o},y_{1}) \quad \text{in } \mathcal{X} \ ,$$

uniformly in t in bounded subsets of $[0,\infty)$.

<u>Proof.</u> To verify the claim we employ the triangle inequality:

$$|T_{B}^{N}(t;q^{N})\mathcal{P}^{N}(q^{N})(y_{o}^{N},y_{1}^{N}) - T^{B}(t;q^{o})(y_{o},y_{1})|_{\mathcal{X}} \leq$$

$$|T_{B}^{N}(t;q^{N})\mathcal{P}^{N}(q^{N})(y_{o}^{N},y_{1}^{N}) - T_{B}^{N}(t;q^{N})\mathcal{P}^{N}(q^{N})(y_{o},y_{1})|_{\mathcal{X}} +$$

$$|T_{B}^{N}(t;q^{N})\mathcal{P}^{N}(q^{N})(y_{o},y_{1}) - T^{B}(t;q^{o})(y_{o},y_{1})|_{\mathcal{X}} \ .$$

The operator norms $\|\mathcal{P}^{N}(q^{N})B(q^{N})\|_{\mathcal{X}(q^{N})}$ are uniformly bounded in N.

Consequently (see e.g. [15, Theorem 3.1.1]) $\|T_B^N(t;q^N)\mathcal{P}^N(q^N)\|$ is bounded uniformly in N and t in bounded intervals of $[0,\infty)$. This implies convergence to zero of the first term on the right hand side of (2.22). As for the second, we note that for $z \in \mathbb{X}$

$$|\mathcal{P}^N(q^N)B(q^N)z - B^N(q^0)\mathcal{P}^N(q^N)z|_{\mathbb{X}(q^N)} \leq |B(q^N)z - B(q^0)\mathcal{P}^N(q^N)z|_{\mathbb{X}(q^N)}$$

$$\leq |B(q^N)z - B(q^0)z|_{\mathbb{X}(q^N)} + |B(q^0)z - B(q^0)\mathcal{P}^N(q^N)z|_{\mathbb{X}(q^N)} \, |$$

and these two terms converge to zero by assumption. A simple perturbation result as stated in [19] implies the convergence of the last term in (2.22) and the corollary is verified.

Remark 2.4. In [19] the author applies the Trotter-Kato theorem to demonstrate the convergence of Galerkin and finite element approximations to second order hyperbolic systems. Our analysis here is similar, but more complicated due to the parameter dependence of the operators (and spaces) and the perturbation term $D^2(q_2D^2y_t)$, which is of maximal order. The main step in the proof of Theorem 2.1, the convergence of the resolvents, is reduced to convergence of the projections and the results of Lemma 2.6 and 2.7 .

Remark 2.5. Let us briefly compare the convergence result of this paper to the results of [2,3,7]. The main difference is given by the fact that Banks and Crowley in their work take the finite dimensional subspaces as subsets of the domain of the

generator. This generally requires more smoothness of the elements in the subspace than our approach does and in addition more boundary conditions have to be met, (compare $H^4_{(k)}$ to $H^2_{(k)}$). However the original operator $\mathcal{A}_{(k)}(q)$ can be factored so that this smoothness requirement can be relaxed, see [2,7]. In that case parameter dependent convergence was only shown for $q_2 = q_3 = 0$ in [7]. Furthermore in their application of the Trotter-Kato theorem, Banks and Crowley show convergence of the approximating generators on dense subsets of $\mathcal{X}$, which implies convergence of the resolvents, whereas we show convergence of the resolvents directly. This allows for the generality that q^N_1, q^N_2 need only to converge weakly as opposed to strong convergence of q^N_1, q^N_2 required in [3]. As a consequence, the usual compactness assumption on Q in parameter estimation problems (see (H1) below) is fairly weak in the present case.

Remark 2.6. We have chosen three specific boundary conditions here. Other "natural" boundary conditions, as for example combining a simply supported beam with a clamped end, can be treated by the technique that we used here, provided that it is guaranteed that ‖Cv‖ defines a norm on dom C which is equivalent to the H^2-norm and that the result of Lemma 2.3 holds. For a useful discussion of various boundary conditions see [8], for example.

3. Approximation of the Parameter Estimation Problem.

We return to the optimization problem (P) of Section 1. It will be convenient to introduce the notation

$$(u(t;q), v(t;q)) = T^B(t;q)z \ , \quad (u^N(t;q), v^N(t;q)) = T_B^N(t;q)\mathcal{P}^N z \ ,$$

for $z = (y_o, y_1) \in X$. The abstract formulation of (P) is given by

$$\text{(AP)} \quad \text{minimize} \ \sum_{i,j} |\hat{y}(t_i, x_j) - u(t_i, x_j; q)|^2 \quad \text{as q varies in Q} \ ,$$

and a sequence of optimization problems which will be shown to approximate (AP) is defined by

$$\text{(AP)}^N \quad \text{minimize} \ \sum_{i,j} |\hat{y}(t_i, x_j) - u^N(t_i, x_j; q)|^2 \quad \text{as q varies in Q} \ .$$

As in Section 2 we take $X^N \subset H^2_{(k)}$ and $X^N = X_1^N \times X_o^N$. The following hypotheses will be used:

(H1)   Q is a compact subset of $\tilde{Q}$ (in the norm of $H^2_{weak} \times H^2_{weak} \times L^\infty \times L^\infty \times H^2_{(k)} \times H^o$) ,

(H2) $P_o^N \to I$ strongly in H^o and $P_1^N(q_1) \to I$ strongly in $H^2_{(k)}$ for every q_1 contained in the first component of Q.

For the fit-to-data criteria in (AP) and (AP)^N we also use the notation

$$J(q) = \sum_{i,j} |\hat{y}(t_i,x_j) - u(t_i,x_j;q)|^2$$

and

$$J^N(q) = \sum_{i,j} |\hat{y}(t_i,x_j) - u^N(t_i,x_j;q)|^2 .$$

<u>Lemma 3.1.</u> Let (H1) hold. Then $(AP)^N$ has a solution $\bar{q}^N \in Q$ for each N.

<u>Proof.</u> From (2.21) and Gronwall's lemma it follows that $q^n \to q^0$ in Q implies $T^{N,B}(t,q^n)z \to T^{N,B}(t;q^0)z$ in $\mathfrak{X}$ and consequently $u^N(t;q^n) \to u^N(t;q^0)$ in C, uniformly in bounded intervals of t. The result then follows from the special form of the fit-to-data criterion and (H1).

Below N_p and M_p denote subsequences of N that tend to ∞ as $p \to \infty$.

<u>Theorem 3.1.</u> Assume that (H1) and (H2) hold and let $q^* \in Q$ be a limit point of $\{\bar{q}^N\}$, so that $\lim_{p \to \infty} \bar{q}^{N_p} = q^*$ is $\tilde{Q}$. Then q^* is a solution of (AP) and

(3.1) $T^{N_p}(t,\bar{q}^{N_p})\mathbf{P}^N(\bar{y}_0^{N_p},\bar{y}_1^{N_p}) \to T(t;q^*)(y_0^*,y_1^*)$ uniformly in

bounded intervals of $[0,\infty)$,

and

(3.2) $J^{N_p}(\bar{q}^{N_p}) \to J(q^*) .$

<u>Proof.</u>Compactness of Q implies the existence of a subsequence with the specified properties. For every p we have

(3.3) $J^{N_p}(\bar{q}^{N_p}) \le J^{N_p}(q) ,$

for every $q \in Q$. We use Corollary 2.1 and take the limit on both sides of (3.3). This implies $J(q^*) \leq J(q)$ for all $q \in Q$. The remaining assertions follow trivially from Corollary 2.1 .

Remark 3.1. It is easily seen that a more general fit to data criterion could be employed as well. In fact, let
$J(q) = I(u(\cdot;q),v(\cdot;q),\hat{y})$ and $J^N(q) = I(u^N(\cdot;q),v^N(\cdot;q),\hat{y})$
where $I(\cdot,\hat{y}):C(0,T;\mathcal{X}) \to R$ is continuous and $\hat{y}$ denotes some appropriate measurements. Then the results of Theorem 3.1 remain unchanged.

Remark 3.2. The numerically inclined reader might ask why we did not attempt to demonstrate any rate of convergence, provided the projections converge to the identity with a certain rate. Certainly the methods of section 2 allow to demonstrate a rate of convergence of the approximating semigroups to the semigroup $T^B(t;q)$ on $\mathcal{X}$, uniformly in $q \in Q$. It seems to be difficult, however, to obtain any estimate on the rate of convergence of the solutions $\bar{q}^N$ of $(AP)^N$ to q^*. Such a rate was only proved in a special case of an elliptic equation so far [9]. The technique used in [9] does not generalize to other types of equations and moreover it requires that every solution of the infinite dimensional optimization problem satisfies a certain regularity property.*

Remark 3.3. If the set Q in Theorem 3.1 is finite dimensional

---

*K. Kunisch and L. W. White have studied this regularity assumption recently in a forthcoming paper "Regularity of the diffusion coefficient in parameter estimation problems."

(e.g. a set of cubic polynomials with unknown coefficients as parameters), then $(AP)^N$ is a finite dimensional problem. If Q itself is infinite dimensional, then a further approximation of the elements in Q is required. We shall not pursue this problem since the techniques and difficulties that arise do not differ essentially from those for parabolic problems of second order which have been investigated in some detail in [4,13]. For a result on the approximation of variable coefficients in beam equations see [3].

4. The penalized fit-to-data criterion.

The set Q in Theorem 3.1 is constrained in two distinct ways. First there are *pointwise constraints* given in Q_1, Q_2 which guarantee wellposedness of the equation (1.1) $\div$ (1.3) and secondly there are constraints inherent in the *compactness* assumption that give existence of solutions of (AP) and $(AP)^N$. In our numerical implementation of the problems $(AP)^N$, however, we use unconstrained optimization algorithms because they are more simple to handle, - and because they work. To parallel at least partially what is done in practice we briefly discuss the use of a simple penalization method. Another aspect calls for the use of a penalization method as well: In practical examples it is likely that a priori knowledge of the system that is modelled leads us to expect the unknown parameter to

lie within certain bounds and this knowledge should be used in
the fit-to-data criterion. In all the numerical tests that we
carried out a deliberate use of a penalty term decreased the
cost of the calculations. For a discussion of the penalty method
for parameter estimation problems in second order parabolic
equations we refer to [14].

To simplify the presentation *we assume the parameters $(q_2, q_3, q_4,
y_o, y_1)$ to be known* and restrict our attention to estimating q_1.
We agree to drop the index "1" henceforth so that $q = q_1$. For $\gamma > \alpha$
we introduce the set

$$(4.1) \qquad \hat{Q}_1 = \{q \in H^2 : |q|_{H^2} \leq \gamma\} .$$

Note that $Q_1 \cap \hat{Q}_1$ is nonempty closed and convex and therefore
a weakly closed subset of H^2. Since $Q_1 \cap \hat{Q}_1$ is also bounded,
this set is weakly compact in H^2 and Theorem 3.1 is applicable,
provided (H2) holds. We need the following hypothesis:

(H3) The functional $\psi : H^2 \to \mathbb{R}^+$ is weakly lower semicontinuous
       and satisfies $\psi(q) = 0$ if and only if $q \in \hat{Q}_1$ and
       $\psi(q) \to \infty$ if $|q|_{H^2} \to \infty$.

In addition to (AP) and $(AP)^N$ we introduce the problems

$(APP)^M \qquad$ minimize $J(q) + M\psi(q)$ as q varies in Q_1 ,

and

$(APP)^{M,N} \qquad$ minimize $J^N(q) + M\psi(q)$ as q varies in Q_1 .

<u>Lemma 4.1.</u> Let (H3) hold. Then $(APP)^M$ and $(APP)^{M,N}$ have solutions $\bar{q}_M$ and $\bar{q}_M^N$ in Q_1. The set $\{|\bar{q}_M^N|_{H^2} : N=1,\ldots; M=1,\ldots\}$ is bounded.

<u>Proof.</u> Since ψ is radially unbounded, the existence of $\bar{q}_M^N$ and $\bar{q}_M$ trivially follows. Moreover for every $q \in \hat{Q}_1 \cap Q_1$ we have

$$(4.2) \qquad 0 \leq J^N(\bar{q}_M^N) + M\psi(\bar{q}_M^N) \leq J^N(q) .$$

But $\{J^N(q) : N=1,\ldots\}$ is bounded by Theorem 3.1 and therefore $\{\psi(\bar{q}_M^N) : N=1,\ldots; M=1,\ldots\}$ is bounded. Using (H3) this implies the claim.

<u>Theorem 4.1.</u> Assume that (H2) and (H3) hold and let q^* be a weak limit point of $\{\bar{q}_M^N\}$ so that $\bar{q}_{M_p}^{N_p} \to q^*$ weakly in H^2 with $N_p \to \infty$, $M_p \to \infty$. Then $q^* \in \hat{Q}_1 \cap Q_1$ is a solution of (AP) and

$$T^{N_p}(t;\bar{q}_{M_p}^{N_p})P^{N_p}(y_o,y_1) \to T(t;q^*)(y_o,y_1) \quad \text{in } \mathfrak{X} ,$$

uniformly in bounded intervals of $[0,\infty)$, and

$$J^{N_p}(\bar{q}_{M_p}^{N_p}) + M_p\psi(\bar{q}_{M_p}^{N_p}) \to J(q^*) \quad \text{as } p \to \infty .$$

<u>Proof.</u> The short proof is quite standard but it is included here for the sake of completeness. From (4.2) it follows that $\lim_{p\to\infty} \psi(\bar{q}_{M_p}^{N_p}) = 0$. Since $0 \leq \psi(q^*) \leq \lim_{p\to\infty} \inf \psi(\bar{q}_{M_p}^{N_p}) = 0$ we have $q^* \in \hat{Q}_1 \cap Q_1$. The following inequality holds for each $q \in \hat{Q}_1 \cap Q_1$

$$(4.3) \quad J^{N_p}(\bar{q}_{M_p}^{N_p}) \leq J^{N_p}(\bar{q}_{M_p}^{N_p}) + M_p(\bar{q}_{M_p}^{N_p}) \leq J^{N_p}(q) + M_p(q) = J^{N_p}(q) \ .$$

Taking the limit $p \to \infty$ in (4.3) and using Theorem 3.1 we arrive at

$$(4.4) \qquad J(q^*) \leq J(q) \quad \text{for every} \quad q \in \hat{Q}_1 \cap Q_1 \ .$$

This implies that q^* is a solution of (AP). The final assertion in the statement of the theorem follows from (4.3) with $q = q^*$.

5. An Example.

In this section we present numerical data for the approximation of unknown coefficients, where the exact solution of the differential equation is known. The subspaces $X^N \subset H^2_{(k)}$ are chosen as cubic spline functions, see [16, Chapter 4] with the basis reduced by two degrees of freedom for k=1 or 3 (four degrees for k=2) to account for the boundary conditions. By using elementary estimates on spline interpolation, it is simple to show that (H2) holds for boundary conditions of type k=1,2 or 3. Once a basis for X^N is chosen a matrix representation for the ordinary differential equation corresponding to (2.21) is readily derived. Depending on the parameters, this ordinary differential equation may be stiff or also highly oscillatory. In the first

case the NAG-implementation of the Gear algorithm was used to
solve the ordinary differential equation, in the latter we
either used the Gear algorithm or the NAG-implementation of
the Adams-Bashford algorithm. The minimization problem $(AP)^N$
was solved by the IMSL-implementation of the Levenberg-Marquardt
algorithm. Some experience has to be obtained in tuning the
parameters of the Levenverg-Marquardt algorithm as applied to
the present problem. We had very good success using a three-
step procedure: from one step to the next the convergence
criteria for the Levenberg-Marquardt algorithm as well as the
tolerance for the error test determining the variable stepsize
and order of the Gear (or Adams-Bashford) algorithm are sharpened,
and the approximated parameters of the previous step are used
as start up values in the succeeding one. All computations were
carried out in double precision in FORTRAN on the UNIVAC 1100/81
computer at the computing center of the Technical University of
Graz. - Examples similar to the one presented here involving
quintic and cubic spline approximations are described in [2,3,7].
As mentioned in the introduction, cubic spline approximation
could be used in [2,3,7] only after transforming (1.1) - (1.3)
to a system of three equations. In this case the structure of
the approximating system of the ordinary differential equations
is different from ours.

In our numerical experiments we took synthetic data as our
measurements $\hat{y}(t_i, x_j)$. For the example shown below the exact
value of the analytical solution was used, in all other examples
the data were generated by a finite difference scheme.

We now turn to the specific equation

$$y_{tt} = -q_1 D^4 y - q_2 D^4 y_t ,$$

$$y(0,x) = q_6 \sin \pi x ,$$

(5.1)

$$y_t(0,x) = 0 ,$$

boundary conditions of type 1 ,

where q_1, q_2 and q_6 are taken to be constants.

The solution of (5.1) for $t \geq 0$ and $0 \leq x \leq 1$ is given by

(5.2a) $u(t,x) = \exp(-a_1 t)\{C_1^1 \exp(a_2 t) + C_2^1 \exp(-a_2 t)\}\sin \pi x ,$

if $\Delta = \pi^4 q_2^2 - 4q_1 > 0$,

(5.2b) $u(t,x) = \exp(-a_1 t)\{C_1^2 \sin a_2 t + C_2^2 \cos a_2 t\}\sin \pi x ,$

if $\Delta < 0$, and

(5.2c) $u(t,x) = \{C_1^3 \exp(-a_2 t) + C_2^3 t \exp(-a_1 t)\}\sin \pi x ,$

if $\Delta = 0$. Here $a_1 = (\pi^4 q_2)/2$, $a_2 = (\pi^4 \sqrt{|\Delta|})/2$,
$C_1^1 = q_6(a_1 + a_2)/(2a_2)$, $C_2^1 = q_6(a_2 - a_1)/2a_2$, $C_1^2 = a_1 q_6 / a_2$,
$C_2^2 = q_6 = C_1^3$ and $C_2^3 = q_6(a_1 - 1)$.

In the numerical experiments the "correct" value of the parameter vector was taken to be

$$(\hat{q}_1, \hat{q}_2, \hat{q}_6) = (.1, .1, 2.) .$$

Each solution of $(AP)^N$ requires the approximate solution of (5.1) via (2.21) many times (see "function evaluations" below). It is therefore important that the algorithm for the approximation of the partial differential equation is a powerful one. We compared the numerical data given by the implementation of (2.21) with X^N taken as cubic spline functions as mentioned above, to a finite difference scheme which is constructed by generalizing an explicit scheme discussed in [1, pg. 280]. For (5.1) the projection-spline-scheme was superior with regards to accuracy as well as computing time. (The experience with all other numerical examples showed superiority of the projection - over the finite difference scheme as well.)

In the tables below we show the relative error in percent of the observed data, which is defined as

$$\text{relative error in percent } = \left| \frac{\text{exact value } - \text{ approximation}}{\text{exact value}} \right| 100 \ .$$

Table 1 shows the relative error (in percent) of the solution calculated by a finite difference technique in the time interval $[0,2]$. We took an equidistant grid of $[0,1]$ of grid length N^{-1}, and a grid along the t-axis of grid size $2J^{-1}$. The numerical solution is given at $(t,x) = (2,.5)$ where the relative error in percent $E^N(2.,.5)$ reached its maximum.

Table 1. Relative error i.p. of state at (2.,.5) (finite difference).

N/J	1oo	1ooo	2ooo	4ooo	CPU-time for J=4ooo
4	2.759	1.716	1.659	1.631	1.o74 sec.
8	1.535	o.4933	o.4366	o.4o82	2.563
16	1.249	o.2o71	o.15o4	o.1221	5.686
32	1.179	o.1367	o.o8oo	o.o5166	11.929
64	1.161	o.1192	o.o6247	o.o3414	24.417
128	1.157	o.1148	o.o58o9	o.o2976	49.394

Table 2 gives the data for the same forward problem calculated by means of the projection method with cubic spline subspaces, using the Gear algorithm to solve the ordinary differential equations and with N defined as above. The data are significantly better. The rate of convergence seems to be like N^{-4} in this example.

Table 2. Relative error i.p. of the state at (2.,.5) (cubic splines).

N	$E^N(2.,.5)$	maximum of absolute error	CPU-time
4	$.8455 \times 10^{-2}$	$.7885 \times 10^{-4}$	1.998 sec.
8	$.4959 \times 10^{-3}$	$.4616 \times 10^{-5}$	4.263
16	$.3o17 \times 10^{-4}$	$.2827 \times 10^{-6}$	1o.252

Next we show some results on the approximation of the "unknown" parameter values $(\hat{q}_1, \hat{q}_2, \hat{q}_6)$. The Levenberg-Marquardt minimization

algorithm requires a start-up value and it was taken to be $(0,0,0)$ for every N, although in practical examples one would generally take the solution of $(AP)^{N-1}$ as the start-up value for $(AP)^N$.

Table 3 contains the relative error in percent when each of the parameters is searched for separately.

Table 3. Relative error i.p. for q_1, q_2, q_3 separately.

	E^4	E^8	E^{16}	CPU-time for $N=4,16$
q_1	$.17 \times 10^{-2}$	$.78 \times 10^{-4}$	$.27 \times 10^{-4}$	1o.6, 6o.7 sec
q_2	$.11 \times 10^{-2}$	$.54 \times 10^{-4}$	$.24 \times 10^{-4}$	11.4, 65.8
q_6	$.24 \times 10^{-2}$	$.18 \times 10^{-3}$	$.92 \times 10^{-4}$	8.8, 5o.1

For each value of N the total number of iteration for the "three step" Levenberg-Marquardt algorithm was 7 for q_1, 9 or 10 for q_2 and 5 for q_6. A total of 19 function evaluations was necessary for the estimation of q_1, 22 for q_2 and 15 for q_6. When pairs of parameter values were searched simultaneously the results were not significantly different and the calculations were roughly twice as costly. Finally we give the data, when (q_1,q_2,q_6) were searched simultaneously. Again the start-up value is $(0,0,0)$.

Table 4. Relative error i.p. for (q_1, q_2, q_6) simultaneously.

	E^4	E^8	E^{16}	CPU-time for
q_1	$.57 \times 10^{-1}$	$.31 \times 10^{-2}$	$.11 \times 10^{-2}$	N=4: 36.692
q_2	$.57 \times 10^{-1}$	$.31 \times 10^{-2}$	$.11 \times 10^{-2}$	N=8: 5o.649
q_6	$.26 \times 10^{-4}$	$.4o \times 10^{-4}$	$.52 \times 10^{-4}$	N=16: 137.oo1

Depending on N there were 52-59 function evaluations necessary and 16-17 iterations of the Levenberg-Marquardt algorithm. When the start-up values in the three parameter search were taken as large as $(.5, .5, 5.)$ the algorithm was shown to be still convergent.

We also made experiments to study the effect of noise-corrupted observations. Table 5 shows the relative error in percent of the estimated parameters for N=4, when Gaussian noise with standard deviation σ is added to the exact solution $\hat{y}(t_i, x_j)$. For N=8 or 16 the relative error was not significantly different from that for N=4. Moreover, comparing the relative error of the estimated parameters under exact measurements with those of the corrupted ones, we observe that for $\sigma = .01$ the relative error in percent increases roughly speaking hundred times and for $\sigma = .1$ about thousand times. Rows 1-3 give the relative errors in percent when the parameters are estimated separately, rows 4-9, when they are searched for in pairs and rows 10-12 show the results for simultaneous estimation of (q_1, q_2, q_6). A cross in the table indicates that the results were obviously worthless.

Table 5. Relative error i.p. for $N=4$ and noise-corrupted data.

	$\sigma = .00$	$\sigma = .01$	$\sigma = .1$
q_1	$.17 \times 10^{-2}$	.12	2.4
q_2	$.11 \times 10^{-2}$	.12	2.8
q_6	$.24 \times 10^{-2}$	.27	1.6
q_1	$.57 \times 10^{-1}$	1.1	
q_2	$.57 \times 10^{-1}$	1.1	
q_1	$.12 \times 10^{-2}$	.24	1.3
q_6	$.28 \times 10^{-2}$	.35	1.7
q_2	$.16 \times 10^{-2}$	.20	1.9
q_6	$.28 \times 10^{-2}$	.33	.98
q_1	$.57 \times 10^{-1}$	4.4	
q_2	$.57 \times 10^{-1}$	4.1	
q_6	$.26 \times 10^{-4}$	.55	

Finally we give an example in which the penalty method was used effectively. Again we consider (5.1) and the "observations" are calculated from (5.2). In this case we take as initial guess (.3,0,0). This leads to a highly oscillatory behaviour of the trajectories of the differential equation and requires to decrease the tolerance in the Gear algorithm to 10^{-5}. With this small value for the tolerance the three-level Levenberg-Marquardt became ineffective and in Table 6 we therefore show the numerical data that were obtained after the first level. The penalty function ψ was taken to be $\psi = \psi_1 + \psi_2 + \psi_6$, where

$$\psi_1(q_1) = \begin{cases} 0 & \text{for} \quad a \leq q_1 \leq b \\ (q_1-a)^2 & \text{for} \quad q_1 < a \\ (q_1-b)^2 & \text{for} \quad q_1 > b \end{cases} \quad ,$$

with ψ_2 and ψ_6 defined analoguously. The "correct value" for the parameter vector is $(\hat{q}_1, \hat{q}_2, \hat{q}_6) = (.1, .1, 2)$, as before.

Table 6. Optimal parameter and relative error i.p. for simultaneous (q_1, q_2, q_6)-search with use of penalty functional (N=4).

	optim.param.	E^4	a	b	M	CPU-time
q_1	-	-	$-\infty$	∞	-	-
q_2	-	-	$-\infty$	∞	-	
q_3	-	-	$-\infty$	∞	-	
q_1	.100049	$.49 \times 10^{-1}$	10^{-5}	.35	100	23.9
q_2	.100049	$.49 \times 10^{-1}$	-.1	.2	100	
q_6	2.000153	$.77 \times 10^{-2}$	.0	2.5	100	
q_1	.100014	$.14 \times 10^{-1}$	10^{-5}	.35	400	25.4
q_2	.100014	$.14 \times 10^{-1}$	-.1	.15	400	
q_6	2.000074	$.37 \times 10^{-2}$	.0	2.5	400	

The numerical results of Table 6 were obtained with the scaling factor (modifying the Marquardt parameter) set initially at the same value (2.0) as in all other examples. In this case the scheme without penalization term does not converge. Convergence of the scheme without penalization can be obtained by taking the scaling factor as small as 1.1; in this case the optimal parameters do not differ essentially from those of Table 6 when penalization is used. This and other examples demonstrate that by using a penalty functional in the fit to data criterion some robustness in the choice of the parameters in the Levenberg-Marquardt algorithm is gained.
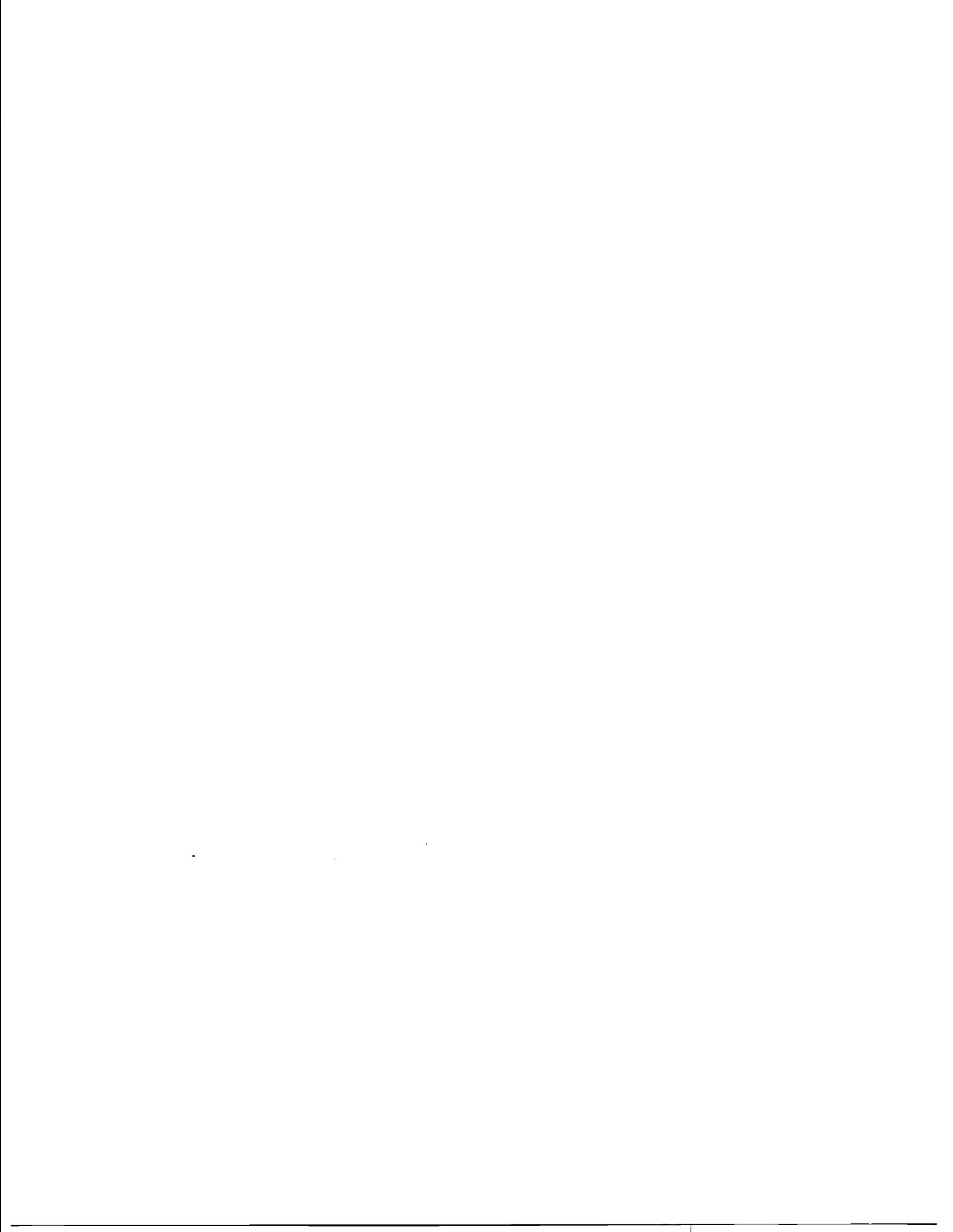
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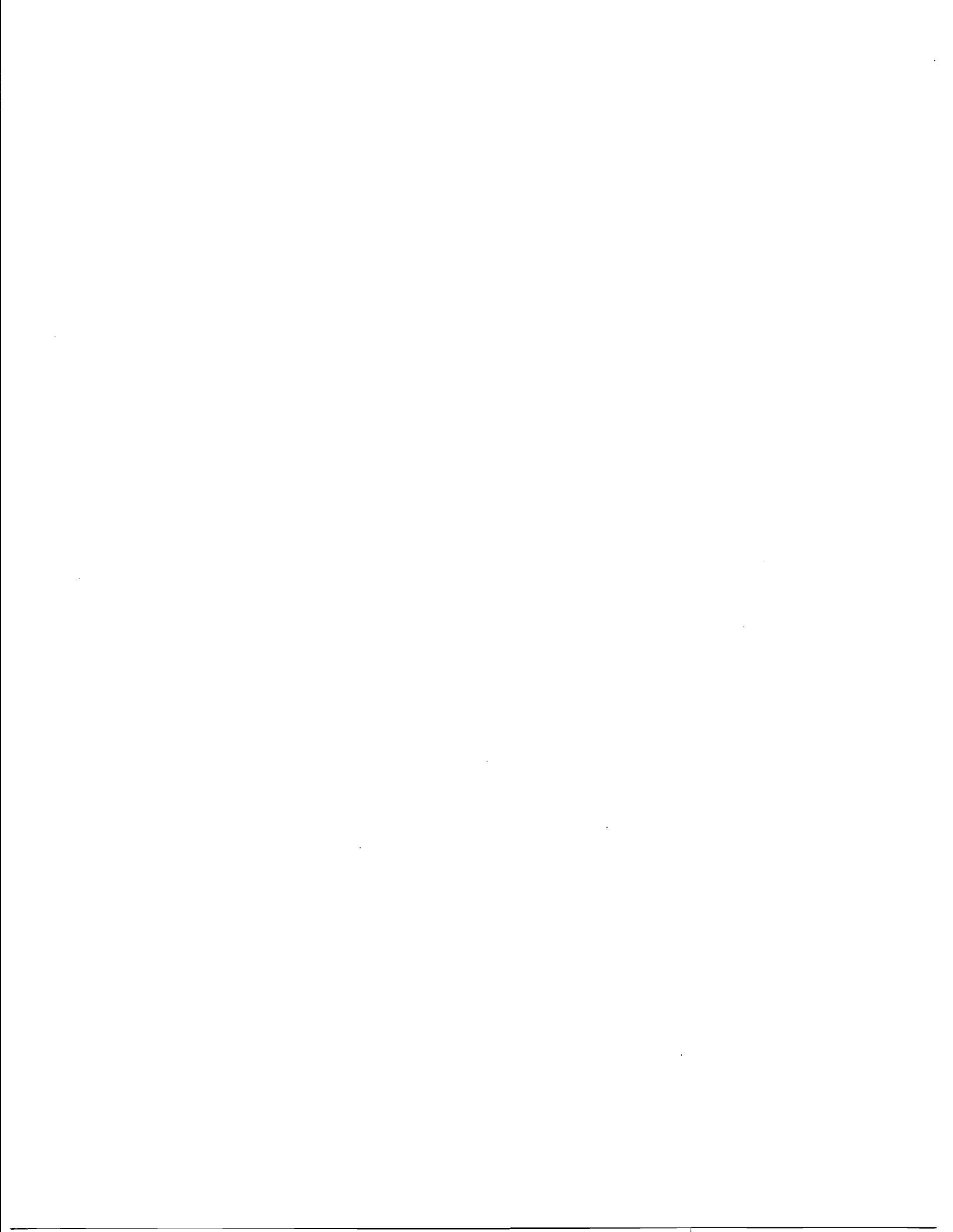## References

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16. Abstract

An approximation involving cubic spline functions for parameter estimation problems in the Euler-Bernoulli-beam equation (phrased as an optimization problem with respect to the parameters) is described and convergence is proved. The resulting algorithm was implemented and several of the test examples are documented. It is observed that the use of penalty terms in the cost functional can improve the rate of convergence.

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